Finite attainment in Révész's cosine extremal problem

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Abstract

Among nonnegative continuous cosine series with constant term 1, first coefficient a, and nonnegative higher coefficients, consider the infimum of the value at zero. A trigonometric polynomial attains this infimum exactly when $|a| \le 4/3$. Outside this interval every admissible polynomial can be improved by introducing one higher frequency, chosen by simultaneous approximation at its zeros. This answers the finite-degree question of Révész, including the case $a = 1.85$. As a consequence, every admissible polynomial has objective strictly above Landau's infimum V. An explicit sum of two squared moduli of degree 126 gives $V < 34.488157$.

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1 The fixed-coefficient problem

Write $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. For real a, let $\mathcal{F}(a)$ be the class of continuous functions on $\mathbb{T}$ with a cosine expansion

$$f(x) = 1 + a\cos x + \sum_{j=2}^{\infty} b_j \cos(jx), \qquad b_j \ge 0, \qquad f(x) \ge 0 \quad (x \in \mathbb{T}). \tag{1}$$

The Fourier-coefficient formulation gives the same class. Indeed, the Fejér means at zero converge to $f(0)$, and their tail weights $(1-j/(n+1))_+$ increase to 1 for each j. Monotone convergence gives $\sum_{j\ge2} b_j = f(0) - 1 - a$. The series therefore converges absolutely and uniformly to f by Fourier uniqueness. For $\mathcal{F}(a) \ne \varnothing$, put $\alpha(a) = \inf_{f\in\mathcal{F}(a)} f(0)$. Let $\mathcal{F}_N(a)$ be its subclass of polynomials of degree at most N, and put $\alpha_N(a) = \inf_{P\in\mathcal{F}_N(a)} P(0)$ whenever this subclass is nonempty.

Révész records attainment in the continuous class and, for $1 \le a < 2$, equality of its infimum with the infimum over all degrees [6, (1.9)–(1.12)]. He asks whether, for each individual a, some finite degree attains that value and singles out $a = 1.85$ [6, p. 158, Question 2]. For $1 \le a \le 4/3$, Arestov and Kondrat'ev [2] obtained $\alpha_N(a) = 2a$ for $N \ge 2$, with a quadratic extremizer. For $a > 4/3$ they proved $\alpha_N(a) > 2a$ at each feasible degree [1, (0.15)–(0.16)]. Révész gave explicit extremizers for $|a| \le 4/3$ and strict bounds on $\alpha(a)$ beyond that interval [5, Proposition 4.4]. We compare $\alpha_N(a)$ directly with $\alpha(a)$ by perturbing each finite competitor at its zeros.

Theorem 1. *For a with $\mathcal{F}(a) \ne \varnothing$, a trigonometric polynomial attains $\alpha(a)$ if and only if $|a| \le 4/3$. If $|a| > 4/3$, then $\alpha_N(a) > \alpha(a)$ for each N with $\mathcal{F}_N(a) \ne \varnothing$.*

AI Declaration

OpenAI Codex (GPT-6) identified the problem, developed the mathematical argument, searched for the numerical certificate, wrote the verification code and drafted this manuscript. Mingon Kim directed the search, questioned the comparison with the published parameter ranges and requested formal verification.

2 Improving a finite polynomial

Let $P \in \mathcal{F}_N(a)$, and let $H = \sum_{j=2}^{N} b_j$ be the mass of its higher coefficients. The amount by which H exceeds $|a| - 1$ gives room for a strict improvement.

Lemma 2. *If $s = |a| > 4/3$, then $H > s - 1$.*

Proof. Take $x_0 = \pi$ for $a > 0$ and $x_0 = 0$ for $a < 0$. Then $0 \le P(x_0) \le 1 - s + H$. If $H = s - 1$, both inequalities are equalities. Thus $P(x_0) = 0$, and $\cos(jx_0) = 1$ whenever $b_j > 0$. At a minimum the second derivative is nonnegative, whereas

$$P''(x_0) = s - \sum_{j=2}^{N} j^2 b_j \le s - 4H = 4 - 3s < 0.$$

This proves the strict inequality. □

Lemma 3. *For a finite subset Z of $\mathbb{T}$, an integer $N \ge 0$ and $q < 1$, there exists an integer $k > N$ with $\cos(kz) > q$ for every $z \in Z$.*

Proof. For $Z = \varnothing$, take $k = N + 1$. Otherwise write $Z = \{z_1, \dots, z_r\}$. Choose an integer $M \ge 3$ with $\cos(2\pi/M) > q$. Partition $[0, 1)^r$ into M^r cubes of side $1/M$. Two of the $M^r + 1$ points

$$j(N + 1)(z_1, \dots, z_r)/(2\pi) \pmod 1, \qquad 0 \le j \le M^r,$$

lie in the same cube. Multiplying the positive difference of their indices by $N + 1$ gives $k > N$. Each kz_i has distance less than $2\pi/M$ from $2\pi\mathbb{Z}$, as required. □

Proposition 4. *If $P \in \mathcal{F}_N(a)$ and $|a| > 4/3$, there is an admissible polynomial of larger degree with a strictly smaller value at zero.*

Proof. Choose $s - 1 < c < H$ by Lemma 2, where $s = |a|$. The zero set Z of P on $\mathbb{T}$ is finite. Indeed, multiplying its Laurent expansion by e^{iNx} gives an ordinary polynomial in e^{ix}, and the constant Fourier coefficient of P is 1. Apply Lemma 3 with $q = (s - 1)/c$. The polynomial

$$G(x) = 1 + a\cos x + c\cos(kx)$$

is positive on Z, since there $G(x) \ge 1 - s + c\cos(kx) > 0$. On the compact set $K = \{G \le 0\}$, the function P is strictly positive. If K is nonempty, set $m = \min_K P > 0$ and $B = \max_K |G|$. For $0 < \varepsilon < m/(m + B)$, $(1 - \varepsilon)P + \varepsilon G \ge (1 - \varepsilon)m - \varepsilon B > 0$ on K. Outside K, any $0 < \varepsilon < 1$ gives nonnegativity. This latter choice also covers $K = \varnothing$.

The resulting $P_\varepsilon = (1 - \varepsilon)P + \varepsilon G$ preserves the constant and first coefficients. Each old higher coefficient is multiplied by $1 - \varepsilon$ and the new coefficient at $k > N$ is $\varepsilon c > 0$. Finally, $P_\varepsilon(0) - P(0) = \varepsilon(c - H) < 0$. □

Proof of Theorem 1. Proposition 4 shows $\alpha(a) < P(0)$ for every admissible polynomial when $|a| > 4/3$. For a fixed admissible degree the feasible coefficient set is compact. Pointwise nonnegativity and the coefficient inequalities are closed conditions. Fourier coefficient extraction gives $|b_j| \le \pi^{-1} \int_0^{2\pi} P(x)\,dx = 2$ because $P \ge 0$ and its mean is 1. Hence $P(0)$ takes a minimum on the fixed-degree class, proving the strict inequality for $\alpha_N(a)$.

For $|a| \le 1$, the polynomial $1 + a\cos x$ realizes the lower bound $f(0) \ge 1 + a$ from (1). For $1 \le s = |a| \le 4/3$, set $Q(x) = 1 + a\cos x + (s - 1)\cos(2x)$. With $u = \operatorname{sgn}(a)\cos x$,

$$Q(x) = (1 + u)\big(2 - s + 2(s - 1)u\big) \ge 0,$$

since the second factor is at least $4 - 3s$. If $a > 0$, then $0 \le f(\pi) \le 1 - a + \sum_{j \ge 2} b_j$, so $f(0) \ge 2a = Q(0)$. If $a < 0$, then $Q(0) = 0$, the lower bound from $f \ge 0$. □

The specific value $37/20 = 1.85$ has finite competitors. For instance, with $t = \cos x$, define

$$P(x) = \left[10 + 148t^2(1+t)(4t^2+2t-1)^2\right]/1120.$$

This polynomial is positive. Its cosine coefficients, in increasing order starting with the constant, are $(1120, 2072, 1702, 1221, 740, 370, 148, 37)/1120$. Theorem 1 therefore gives a strict gap at every admissible degree.

The lead-in to Question 2 in [6, p. 158] also describes finite attainment on $(-3/4, \sqrt{2})$. To isolate the overlap with Theorem 1, take $a = 7/5$, so $4/3 < 7/5 < \sqrt{2}$. In this case there is even a strictly positive quadratic competitor,

$$1 + \tfrac{7}{5}\cos x + \tfrac{1}{2}\cos(2x) = (\cos x + \tfrac{7}{10})^2 + \tfrac{1}{100}.$$

Proposition 4 improves every finite competitor at this parameter. The finite-attainment assertion in the lead-in thus fails at $7/5$.

3 Landau's infimum

For a nonnegative cosine polynomial $T(x) = \sum_{j=0}^{N} A_j \cos(jx)$ with $0 < A_0 < A_1$ and $A_j \geq 0$ for $j \geq 2$, define

$$L(T) = \frac{\sum_{j=1}^{N} A_j}{(\sqrt{A_1} - \sqrt{A_0})^2}, \qquad V = \inf_T L(T).$$

This is the Landau extremal quantity considered in [6, 4, 3]. Tan studies the minima at degrees seven and eight [7].

Corollary 5. *Every admissible polynomial T satisfies $L(T) > V$.*

Proof. Normalize by A_0 and put $a = A_1/A_0 > 1$. The ratio becomes $(T(0)/A_0 - 1)/(\sqrt{a} - 1)^2$. If $a > 4/3$, Proposition 4 strictly decreases this ratio. For $1 < a \leq 4/3$, the numerator is at least a, so

$$L(T) \geq (1 - a^{-1/2})^{-2} > 49,$$

where $a^{-1/2} \geq \sqrt{3}/2 > 6/7$. Landau's cubic $5+8\cos x+4\cos(2x)+\cos(3x) = (1+\cos x)(1+2\cos x)^2$ has ratio $13(13+4\sqrt{10})/9 < 40$. This proves the claim in the remaining range. □

Proposition 6. $V < 34.488157$.

Proof. Use the integers $u_0, \ldots, u_{126}$ in Appendix A, and set $D = 558390045$. The polynomial

$$T(x) = \left|\sum_{j=0}^{126} u_j e^{ijx}\right|^2 + D\left|\sum_{j=0}^{126} e^{ijx}\right|^2 \tag{2}$$

is nonnegative for real x. For $0 \leq k \leq 126$, set

$$C_k = \sum_{j=0}^{126-k} u_j u_{j+k} + (127-k)D.$$

Expansion gives $A_0 = C_0$ and $A_k = 2C_k$ for $k \geq 1$. The resulting integer quantities are

$$A_0 = 100000000068522766698, \qquad A_1 = 173754051432156542910,$$

$$\sum_{k=1}^{126} A_k = 349104688395918049548, \quad \min_{2\leq k\leq 126} A_k = 0.$$

In particular T is admissible. The difference $d = \lfloor\sqrt{A_1}\rfloor - \lceil\sqrt{A_0}\rceil = 3181580001$ is positive and at most $\sqrt{A_1} - \sqrt{A_0}$. Integer comparison gives

$$V \le L(T) \le \frac{\sum_{k=1}^{126} A_k}{d^2} = \frac{12929803273922890724}{374905603806042963} < \frac{34488157}{1000000}. \qquad \square$$

Mossinghoff and Trudgian obtained $V < 34.4889920009$ [4, Theorem 2]. Broucke and Révész report this upper bound in [3, §2]. To find (2), the search minimized the tail mass on a finite grid, refined the grid near local minima, and rounded a numerical spectral factor to integers. Adding the second squared modulus gives nonnegative cosine coefficients. The formula and integer vector suffice to verify Proposition 6 independently of this search.

Mathematical software and data availability

The supplementary archive is available with the earlier version of this preprint on Figshare (doi:10.6084/m9.figshare.35124661.v1). It contains a Lean 4.19.0 formalization with its mathlib revision pinned, and a Python 3 integer verifier. Lean verifies Theorem 1, the existence of a strictly improving finite competitor, and the examples 37/20 and 7/5. The formalization proves Lemma 2 by a local cosine bound and obtains a returning frequency by compactness. Its principal statements use the standard Lean axioms of propositional extensionality, classical choice and quotient soundness. The Python script verifies Proposition 6 using the standard library. Corollary 5 has the analytic proof given above. The numerical search used NumPy and SciPy.

References

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A Integer certificate

Each row starts at the index j in the first column and gives consecutive coefficients $u_j, u_{j+1}, \ldots$ for (2). The last row ends at u_{126}. From these data the autocorrelation formula in Proposition 6 checks every sign and the final rational inequality using integers alone.

j	u_j	u_{j+1}	u_{j+2}	u_{j+3}	u_{j+4}	u_{j+5}
0	187418406	1028242051	2860089560	5076483931	6102281022	4800849062
6	1939777431	−340687760	−757458505	−32780078	375771474	102298712
12	−181323106	−68142371	108325644	47302448	−72258229	−31773022
18	53715855	23076263	−38162445	−16232869	30950461	10038316
24	−22222574	−6993330	18925016	3041165	−14345141	−780323
30	11355740	−473761	−9863255	2928729	6599946	−2054880
36	−6441560	4013882	3616188	−2840336	−3380232	3479470
42	2104085	−3492168	−641445	2271785	1321237	−3472719
48	858634	1407922	539254	−2701490	1129799	1392475
54	−797567	−1113703	516404	1681120	−1852037	153474
60	152264	1441232	−2004280	534450	482572	423503
66	−1221248	143871	1279033	−990858	−57785	−229162
72	1448246	−1538761	432767	−32417	855251	−1224024
78	264314	655588	−336345	−251517	−210054	1118593
84	−1082073	360604	−325282	1052869	−1227666	439024
90	151429	199834	−589668	55051	817804	−897635
96	368101	−304888	856033	−1084320	604009	−171439
102	370713	−666014	353913	278404	−456378	181047
108	−144553	564579	−869105	671590	−362521	438971
114	−714093	646288	−183706	−171476	115644	64552
120	50469	−440165	732160	−686035	410882	−151908
126	27688					