hexagon:2611.01126v1

NESTED-OVALS FEWNOMIAL SYSTEMS: A UNIVERSAL TWO-PARAMETER DISPROOF OF KOUCHNIRENKO'S CONJECTURE FOR ALL $(t_1, t_2) \geq (3, 3)$

IVAN BESEVIC

Abstract. For positive integers $t_1, \ldots, t_n \geq 2$, let $\mathcal{S}_n(t_1, \ldots, t_n)$ denote the maximum number of nondegenerate strictly positive real solutions $x \in \mathbb{R}^n_{>0}$ of a square polynomial system $f_1(x) = \cdots = f_n(x) = 0$ where each f_i has at most t_i monomials; write $\mathcal{S}(t_1, t_2) := \mathcal{S}_2(t_1, t_2)$. Kouchnirenko's 1975 conjecture posited the Bézout-type bound $\mathcal{S}_n(t_1, \ldots, t_n) \leq \prod_{i=1}^n (t_i - 1)$. While counterexamples were previously known for bivariate trinomials $(3, 3)$ (Haas, 2002), $(3, t)$ systems (Gomez–Niles–Rojas, 2007; Besevic, 2026), and isolated small supports, no unified constructive family was known to refute Kouchnirenko's conjecture simultaneously across *all* pairs $(t_1, t_2) \geq (3, 3)$ and every fixed t-nomial diagonal $(t, \ldots, t)$ in arbitrary dimension $n \geq 2$.

In this paper we introduce a *nested-ovals* curve family

$$f_1(u, v) = u + v - 1 + \sum_{k=1}^{t_1 - 3} (-1)^{k-1} L^{-k^2} (uv)^{-k} = 0$$

coupled to an inductively constructed lower-envelope t_2-nomial $F_{t_2}(u, v) = 0$ with an asymmetric tail. We prove four results, all completely formalized in Lean 4 (`v4.36.0`, Mathlib4) with zero `sorry` and only the three standard axioms:

(1) **Universal two-parameter lower bound:** For all integers $t_1 \geq 4$ and $t_2 \geq 3$,

$$\mathcal{S}(t_1, t_2) \geq t_1 t_2 - t_1 - 2 = (t_1 - 1)(t_2 - 1) + (t_2 - 3),$$

alongside $\mathcal{S}(3, t_2) \geq 3t_2 - 4$ for $t_1 = 3$. For every fixed $t_1 \geq 3$, this raises the asymptotic slope in t_2 from Kouchnirenko's $t_1 - 1$ to t_1, and strictly beats Kouchnirenko's bound for every (t_1, t_2) with $t_1 \geq 4, t_2 \geq 4$ (and $(3, t_2)$ with $t_2 \geq 3$).

(2) **All diagonal (t, t) systems:** For every integer $t \geq 4$,

$$\mathcal{S}(t, t) \geq t^2 - t - 2 = (t - 1)^2 + (t - 3) > (t - 1)^2,$$

providing a growing linear excess over Kouchnirenko's bound for every $t \geq 4$.

(3) **Exponential multivariate disproof for every fixed $t \geq 4$:** Taking disjoint products of diagonal bivariate blocks (with a univariate t-nomial when n is odd) yields, for all $n \geq 2$ and $t \geq 4$,

$$\mathcal{S}_n(\underbrace{t, \ldots, t}_{n}) \geq (t^2 - t - 2)^{\lfloor n/2 \rfloor} (t - 1)^{n \bmod 2} = \left(1 + \frac{t - 3}{(t - 1)^2}\right)^{\lfloor n/2 \rfloor} (t - 1)^n.$$

(4) **Explicit machine-certified quadrinomial record $\mathcal{S}(4, 4) \geq 10$ and $B(2, 4) \geq 10$:** The symmetric integer pair sharing $x^2 y^2$ (7 distinct monomials total, degrees 12 and 47)

$$10000x^{10}y^2 + 10000x^2y^{10} - 10000x^2y^2 + 1 = 0, \qquad 8x^{46}y + 8xy^{46} + 35x^4y^4 - 25x^2y^2 = 0$$

admits 10 nondegenerate positive solutions (Kouchnirenko = 9), certified via exact rational 2D Newton–Banach contraction in Lean 4, simultaneous Pandrosion–residue complex root certification (`pandsysroots v0.34.0`), and 520-digit Newton–Jacobian verification, together with explicit 520-digit certified systems for $\mathcal{S}(4, 5) \geq 14$, $\mathcal{S}(4, 6) \geq 18$, $\mathcal{S}(5, 5) \geq 17$, and $\mathcal{S}(5, 6) \geq 22$.

1. Introduction

Khovanskii's fewnomial theory [9, 10] established that the number of nondegenerate positive real solutions of a real polynomial system $f_1(x) = \cdots = f_n(x) = 0$ in $\mathbb{R}^n_{>0}$ is bounded solely in

2020 *Mathematics Subject Classification.* Primary 14P10; Secondary 14M25, 68V20.

Key words and phrases. Fewnomials, Kouchnirenko's conjecture, bivariate sparse polynomial systems, nested ovals, tropical lower envelope, Lean 4 formalization.

terms of the number of variables and the number of monomials, independently of the degrees. Two closely related extremal functions govern sharp fewnomial lower bounds:

- $\mathcal{S}_n(t_1,\dots,t_n)$, the supremum of the number of nondegenerate solutions in $\mathbb{R}^n_{>0}$ when each polynomial $f_i \in \mathbb{R}[x_1,\dots,x_n]$ has at most t_i monomials (individual term-count budget, with $\mathcal{S}(t_1,t_2) := \mathcal{S}_2(t_1,t_2)$);
- $B(n,k)$, the supremum when all n polynomials are supported on a *common* set of at most $n+k+1$ distinct monomials (total support codimension k).

In 1975, A. G. Kouchnirenko conjectured that $\mathcal{S}_n(t_1,\dots,t_n) \le \prod_{i=1}^n (t_i-1)$, generalizing Descartes' rule of signs $\mathcal{S}_1(t) = t-1$. For $n=2$, the first counterexample was discovered by Haas [8], who exhibited a pair of bivariate trinomials with $5 > (3-1)^2 = 4$ positive roots; Li, Rojas, and Wang [11] subsequently proved that $\mathcal{S}(3,3) = 5$ is sharp. For trinomial–t-nomial pairs $(3,t)$, Gomez, Niles, and Rojas [7] constructed systems achieving $\mathcal{S}(3,t) \ge 2t-1$ for all $t \ge 3$ (strictly exceeding Kouchnirenko's $2t-2$), and in earlier work [1] we improved this linear bound to $\mathcal{S}(3,t) \ge 3t-4$ for all $t \ge 3$, raising the asymptotic slope from 2 to 3. On the upper-bound side, Bihan and Sottile [4] established sharp Gale-dual bounds in terms of total monomials, while El Hilany [5] and El Hilany–Tavenas [6] proved sharper upper bounds for $\mathcal{S}(3,t)$ (notably $\mathcal{S}(3,t) \le 3\cdot 2^{t-2} - 1$ for $t \le 9$). In the total-monomial setting $B(n,k)$, Bihan, Rojas, and Sottile [3] proved $B(n,k) \ge \lfloor (n+k)/k \rfloor^k$, which we recently improved exponentially for all $k \ge 2$ via Folded-Haas trinomial systems [2].

Despite these advances, two fundamental questions remained open for the individual term-count function $\mathcal{S}_n(t_1,\dots,t_n)$:

(1) Does $\mathcal{S}(t_1,t_2)$ strictly exceed Kouchnirenko's bound $(t_1-1)(t_2-1)$ for *every* pair (t_1,t_2) with $t_1,t_2 \ge 3$—and in particular, can one raise the asymptotic slope in t_2 from t_1-1 to t_1 for every fixed $t_1 \ge 3$?
(2) Does the diagonal (t,t) admit a uniform constructive counterexample $\mathcal{S}(t,t) > (t-1)^2$ for *every* $t \ge 3$ (yielding exponential multivariate disproofs for every fixed $t \ge 3$)?

The naive approach of lifting a trinomial $u+v=1$ to an open t_1-nomial curve $u+v-1-\varepsilon(uv)^{-p}=0$ fails to beat Kouchnirenko on $(4,4)$, because an open perturbation splits the single oval into unbounded branches that only add t_2-3 intersections instead of t_2-2. Our key geometric insight is to use an *alternating multiscale nested-ovals curve*

$$f_1(u,v) = u+v-1+\sum_{k=1}^{t_1-3} (-1)^{k-1} L^{-k^2} (uv)^{-k} = 0,$$

whose real positive locus consists of nested closed ovals (and an innermost open branch when t_1 is odd) indexed by $k=1,\dots,t_1-3$ at separated scales $uv \asymp L^{-(2k-1)}$. When paired with a rational-exponent t_2-nomial $F_{t_2}(u,v)$ engineered to possess *both* $3t_2-4$ simple roots on the outer diagonal $u+v=1$, a strict tropical lower envelope of t_2-2 alternating sign transitions along $v=(uv)^\lambda$ ($\lambda \in (0,1/2)$), and an asymmetric tail $-su^S + 2su^{S+3/64}v^{1/64}$ ($s \in \{\pm 1\}$), every arc $k \ge 1$ harvests t_2-2 roots on the upper branch $u<v$, and every inner arc $k \ge 2$ harvests an additional root on the lower branch $u>v$.

2. Main Theorems

We state our four main results precisely. All roots are required to be strictly positive ($x \in \mathbb{R}^n_{>0}$) and *nondegenerate* ($\det J_f(x) \ne 0$).

Theorem 2.1 (Universal Two-Parameter Lower Bound)**.** *For every integer $t_1 \ge 4$ and every integer $t_2 \ge 3$, there exists a bivariate sparse polynomial system $f_1(x,y) = f_2(x,y) = 0$ with integer exponents in $\mathbb{N}^2$ where f_1 has at most t_1 monomials and f_2 has at most t_2 monomials, admitting at least*

$$t_1t_2 - t_1 - 2 = (t_1-1)(t_2-1) + (t_2-3)$$

pairwise distinct nondegenerate strictly positive real solutions. For $t_1 = 3$ and every $t_2 \ge 3$, the same construction yields $\mathcal{S}(3,t_2) \ge 3t_2-4$. In particular, $\mathcal{S}(t_1,t_2) > (t_1-1)(t_2-1)$ holds for every *pair (t_1,t_2) with $t_1,t_2 \ge 3$ (using symmetry $\mathcal{S}(t_1,t_2) = \mathcal{S}(t_2,t_1)$ for $(t_1,3)$ with $t_1 \ge 4$).*

TABLE 1. Comparison of our lower bound $\mathcal{S}(t_1,t_2) \geq t_1t_2-t_1-2$ ($t_1 \geq 4, t_2 \geq 3$; and $3t_2-4$ for $t_1=3$) against Kouchnirenko's bound $(t_1-1)(t_2-1)$. Bold entries are additionally certified as explicit low-complexity systems in Rust and 520-digit Python.

(t_1,t_2)	Kouchnirenko	Ours	Excess	Slope in t_2	Certification
$(3,3)$	4	5	$+1$	$2 \to 3$	Haas [8]; Lean 4
$(3,4)$	6	**8**	$+2$	$2 \to 3$	Rust (`pandsysroots`); Lean 4
$(3,t_2)$	$2t_2-2$	$3t_2-4$	$+(t_2-2)$	$2 \to 3$	Lean 4 ($\forall\, t_2 \geq 3$)
$(4,4)$	9	**10**	$+1$	$3 \to 4$	Lean 4 (expl.) + Rust + 520-d.
$(4,5)$	12	**14**	$+2$	$3 \to 4$	520-digit + Lean 4 (general)
$(4,6)$	15	**18**	$+3$	$3 \to 4$	520-digit + Lean 4 (general)
$(4,t_2)$	$3t_2-3$	$4t_2-6$	$+(t_2-3)$	$3 \to 4$	Lean 4 ($\forall\, t_2 \geq 3$)
$(5,5)$	16	**17**	$+1$	$4 \to 5$	520-digit + Lean 4 (general)
$(5,6)$	20	**22**	$+2$	$4 \to 5$	520-digit + Lean 4 (general)
(t_1,t_2)	$(t_1-1)(t_2-1)$	$t_1t_2-t_1-2$	$+(t_2-3)$	$t_1-1 \to t_1$	Lean 4 ($\forall\, t_1 \geq 4,\, t_2 \geq 3$)
(t,t)	$(t-1)^2$	t^2-t-2	$+(t-3)$	—	Lean 4 ($\forall\, t \geq 4$)

Theorem 2.2 (All Diagonal (t,t) Systems)**.** *For every integer $t \geq 4$,*

$$\mathcal{S}(t,t) \geq t^2-t-2 = (t-1)^2+(t-3) > (t-1)^2.$$

Combined with Haas's $\mathcal{S}(3,3)=5>4$, Kouchnirenko's conjecture fails on the diagonal (t,t) for every $t \geq 3$.

Theorem 2.3 (Exponential Multivariate Disproof for Every Fixed $t \geq 4$)**.** *For every dimension $n \geq 2$ and every term budget $t \geq 4$, there exists an $n \times n$ polynomial system $f_1(x) = \cdots = f_n(x) = 0$ where each f_i has at most t monomials, admitting at least*

$$(t^2-t-2)^{\lfloor n/2 \rfloor}(t-1)^{n \bmod 2} = \left(1+\frac{t-3}{(t-1)^2}\right)^{\lfloor n/2 \rfloor}(t-1)^n$$

pairwise distinct nondegenerate strictly positive real solutions.

Theorem 2.4 (Explicit Certified Record $\mathcal{S}(4,4) \geq 10$ and $B(2,4) \geq 10$)**.** *The symmetric bivariate integer quadrinomial system*

$$f_1(x,y) = 10000x^{10}y^2 + 10000x^2y^{10} - 10000x^2y^2 + 1 = 0, \tag{1}$$

$$f_2(x,y) = 8x^{46}y + 8xy^{46} + 35x^4y^4 - 25x^2y^2 = 0, \tag{2}$$

whose polynomials have 4 monomials each and share x^2y^2 (7 distinct monomials total, codimension $k = 7-2-1 = 4$), admits at least 10 pairwise distinct nondegenerate strictly positive real solutions in $(0,1)^2$. Consequently, $\mathcal{S}(4,4) \geq 10 > 9$ and $B(2,4) \geq 10$.

Remark 2.5 (Jordan Curve Parity Check). The root counts produced by Theorem 2.1 satisfy an exact topological consistency check. For $t_1 = 4$, the positive locus of $f_1(u,v) = u+v-1+L^{-1}(uv)^{-1} = 0$ is a single closed oval inside $(0,1)^2$, so Bézout/Jordan intersection parity requires an *even* number of transverse intersections with $F_{t_2} = 0$; indeed $(3t_2-4)+(t_2-2) = 4t_2-6$ is even for all t_2. Increasing t_1 by 2 adds one nested closed oval (two consecutive arcs k even and $k+1$ odd), contributing $(t_2-1)+(t_2-1) = 2t_2-2$ new roots—again strictly even.

3. Inductive Lower-Envelope Family $F_{t_2}(u,v)$

For a rational-exponent term $(c,a,b) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, define its tropical valuation along the ray $v = t^\lambda$, $u = t^{1-\lambda}$ ($\lambda \in (0,1/2)$) by

$$E_{a,b}(\lambda) := a + \lambda(b-a).$$

Note that shifting all u-exponents by a constant $-a_0$ shifts every $E_{a,b}(\lambda)$ by the uniform constant $-(1-\lambda)a_0$, leaving all pairwise comparisons and transition slopes invariant.

Proposition 3.1 (Constructive Lower-Envelope Family)**.** *For every integer $t \geq 2$, there exists an explicit t-term rational-exponent generalized polynomial*

$$F_t(u,v) = \sum_{j=0}^{t-1} c_j u^{a_j} v^{b_j} \qquad (c_j \in \mathbb{Q} \setminus \{0\},\ a_j, b_j \in \mathbb{Q})$$

with pairwise distinct exponent pairs (a_j, b_j), satisfying:

(1) Diagonal simple roots: *$u \mapsto F_t(u, 1-u)$ has $3t-4$ strictly increasing simple roots in $(0,1)$.*

(2) Strict lower-envelope chain: *There exist $\lambda_{\min} \in (0, 1/2]$ and $t-2$ strictly increasing transition slopes $\lambda_{\min} \leq \lambda_0 < \lambda_1 < \cdots < \lambda_{t-3} < 1/2$ such that:*
- *for every $\lambda \in (0, \lambda_{\min})$ and every $i \in \{1, \ldots, t-1\}$, $E_{a_0,b_0}(\lambda) < E_{a_i,b_i}(\lambda)$;*
- *for every $j \in \{0, \ldots, t-3\}$, $c_j c_{j+1} < 0$, $a_j < a_{j+1}$, $b_{j+1} < b_j$, $E_{a_j,b_j}(\lambda_j) = E_{a_{j+1},b_{j+1}}(\lambda_j)$, and for all $i \notin \{j, j+1\}$,*

$$E_{a_j,b_j}(\lambda_j) < E_{a_i,b_i}(\lambda_j).$$

(3) Asymmetric tail: *There exist $s \in \{-1, +1\}$ and $S \in \mathbb{Q}$ with $a_0 \leq S$ such that the last two terms are $(-s, S, 0)$ and $(2s, S + \frac{3}{64}, \frac{1}{64})$, while every earlier term $i < t-2$ satisfies $a_i < S$ and $b_i > 0$.*

Proof outline. We induct on $t \geq 2$, strengthening the boundary-pole insertion construction of [1].

Base case $t = 2$: Set

$$F_2(u,v) = -1 + 2u^{3/64} v^{1/64}.$$

On $v = 1-u$, its derivative is $F_2'(u, 1-u) = \frac{1}{32}(3-4u)u^{-61/64}(1-u)^{-63/64}$, with unique peak at $u = 3/4$ where $2(3/4)^{3/64}(1/4)^{1/64} > 2(1/4)^{1/16} > 1$, and negative values at $\varepsilon = (1/4)^{64}$ and $1-\varepsilon$. Thus $F_2(u, 1-u)$ has $2 = 3(2) - 4$ simple roots. Moreover, $a_1 - a_0 = 3/64 < 1/16$, $c_0 c_1 = -2 < 0$, and for all $\lambda \in (0, 1/2)$,

$$E_{0,0}(\lambda) = 0 < \frac{3}{64} - \frac{\lambda}{32} = E_{3/64,1/64}(\lambda),$$

so $\lambda_{\min} = 1/2$, $s = 1$, and $S = 0$ satisfy all invariants.

Inductive step $t \to t+1$: Given F_t, multiply by $\sigma = \operatorname{sgn}(c_0) \in \{\pm 1\}$ and shift u-exponents by $-a_0$ so the leading two terms become $C_1 v^{b_0} - C_2 u^{\gamma} v^{b_1}$ with $C_1, C_2 > 0$ and $\gamma = a_1 - a_0 \in (0, 1/16)$. As in [1], we prepend a new term $-c_{\text{new}} u^{-\delta} v^M$ with integer M sufficiently large, rational $c_{\text{new}} \in (0, C_1)$ chosen between the two boundary-layer bounds at u_B, u_3, and rational $\delta \in (0, \gamma)$ sufficiently small; this creates 3 new simple roots near $u = 0$ while persisting all $3t-4$ old simple roots on $[u_C, 1)$, giving $3(t+1) - 4$ simple roots in total.

To preserve the lower-envelope and tail invariants, we impose just two additional conditions when choosing M and δ:

$$M > b_0 + 1 \qquad \text{and} \qquad \delta < \frac{\lambda_{\min}(M - b_0)}{2}.$$

The new transition slope between the prepended term $(-\delta, M)$ and the shifted old leading term $(0, b_0)$ is

$$\lambda_{\text{new}} := \frac{\delta}{(M - b_0) + \delta} \in (0, \lambda_{\min}).$$

For $\lambda \in (0, \lambda_{\text{new}})$, $E_{-\delta,M}(\lambda) < E_{0,b_0}(\lambda) < E_{a_i - a_0, b_i}(\lambda)$ for all $i \geq 1$ by the inductive hypothesis on $(0, \lambda_{\min})$. At $\lambda = \lambda_{\text{new}}$, terms 0 and 1 are equal and strictly beat all terms $i \geq 2$ since $\lambda_{\text{new}} < \lambda_{\min}$. At each old transition slope $\lambda_j \geq \lambda_{\min} > \lambda_{\text{new}}$, the shifted old minimum satisfies $E_{\min}(\lambda_j) \leq E_{0,b_0}(\lambda_j) < E_{-\delta,M}(\lambda_j)$, so the prepended term never disrupts any existing transition. Finally, $s' = \sigma s$, $S' = S - a_0 \geq 0 > -\delta$, and $M \geq 1 > 0$ preserve the asymmetric tail verbatim. □

4. Nested-Ovals Geometry and Proof of Theorems 2.1–2.3

Fix $t_1 \geq 4$, $t_2 \geq 3$, set $m := t_1 - 3 \geq 1$, and let $F_{t_2}(u, v)$ be the t_2-term polynomial from Proposition 3.1. For a large parameter $L \geq 4$, consider the alternating multiscale curve

$$f_1(u, v) := u + v - s_L(uv) = 0, \qquad s_L(t) := 1 - \sum_{r=1}^{m} (-1)^{r-1} L^{-r^2} t^{-r}. \tag{3}$$

Whenever $t > 0$, $s_L(t) > 0$, and $4t < s_L(t)^2$, the quadratic $u+v = s_L(t)$, $uv = t$ solves explicitly to two smooth positive branches:

$$v_{\mathrm{big}}(t) := \frac{s_L(t) + \sqrt{s_L(t)^2 - 4t}}{2}, \qquad u_{\mathrm{sml}}(t) := \frac{s_L(t) - \sqrt{s_L(t)^2 - 4t}}{2},$$

giving two symmetric points on $f_1 = 0$: $\mathsf{halfA}(t) = (u_{\mathrm{sml}}(t), v_{\mathrm{big}}(t))$ (where $u < v$) and $\mathsf{halfB}(t) = (v_{\mathrm{big}}(t), u_{\mathrm{sml}}(t))$ (where $u > v$).

4.1. **Arc Monotonicity and Universal Log-Derivative Estimate.** For each arc index $k \in \{1, \ldots, m\}$, set $t_k := L^{-(2k-1)}$, $\sigma_k := (-1)^{k-1}$, and parameterize $t = wt_k$ on the compact window $I_k := [1 - \frac{1}{2k}, 1 + \frac{1}{2k}] \subset [\frac{1}{2}, \frac{3}{2}]$. Rewriting $s_L(wt_k) = \sum_{r=0}^{m} (-1)^r w^{-r} L^{r(2k-1)-r^2}$, observe the exact quadratic identity

$$r(2k-1) - r^2 = (k^2 - k) - (r - k)(r - k + 1).$$

Hence the exponent of L equals $k^2 - k$ for $r \in \{k-1, k\}$ and is at most $(k^2 - k) - 2$ for every other $r \in \{0, \ldots, m\} \setminus \{k-1, k\}$. Separating the two leading terms yields:

$$\sigma_k s_L(wt_k) = L^{k^2-k} w^{-k}(w - 1) + O_m\Big(L^{k^2-k-2}\Big),$$

$$\sigma_k \frac{d}{dw} s_L(wt_k) = L^{k^2-k} w^{-(k+1)}\big(k - (k-1)w\big) + O_m\Big(L^{k^2-k-2}\Big).$$

On I_k, we have $k - (k-1)w \geq \frac{k+1}{2k} \geq \frac{1}{2}$, so for a constant $c_A = c_A(m) > 0$ and all sufficiently large L:

$$\sigma_k s_L(w_- t_k) \leq -c_A L^{k^2-k}, \quad \sigma_k s_L(w_+ t_k) \geq c_A L^{k^2-k}, \quad \sigma_k \frac{d}{dw} s_L(wt_k) \geq c_A L^{k^2-k} \ (\forall\, w \in I_k). \tag{4}$$

Thus $s_L(wt_k)$ sweeps every interval $[s_1, s_2] \subset (0, c_A L^{k^2-k}]$ strictly monotonically as w traverses a unique subinterval $[w_a, w_b] \subset I_k$.

Moreover, differentiating $u + v = s(w)$ and $uv = wt_k$ gives, on any interval where $16wt_k \leq s(w)^2$ (so $v_{\mathrm{big}} - u_{\mathrm{sml}} \geq s/2$):

Lemma 4.1 (Universal Log-Derivative Estimate)**.** *Along* $\mathsf{halfA}(wt_k)$ *($u = u_{\mathrm{sml}}, v = v_{\mathrm{big}}$) with* $w \geq 1/2$ *and* $16wt_k \leq s^2$*, let* $\Omega := s'(w)/(v - u)$*. Then for every* $(a, b) \in \mathbb{R}^2$*,*

$$\left| a\frac{u'}{u} + b\frac{v'}{v} - (b - a)\Omega \right| \leq 4(|a| + |b|).$$

Along $\mathsf{halfB}(wt_k)$ *($u = v_{\mathrm{big}}, v = u_{\mathrm{sml}}$), the same bound holds with* $(a - b)\Omega$ *in place of* $(b - a)\Omega$*.*

Finally, note the elementary Jacobian criterion: along any smooth curve $(u(\tau), v(\tau))$ contained in $f_1 = 0$ where $\nabla f_1 \neq 0$, if $\frac{d}{d\tau} F_{t_2}(u(\tau), v(\tau)) \neq 0$ at a zero, then $\det J(f_1, F_{t_2}) \neq 0$. For (3), the identity $u\partial_u f_1 - v\partial_v f_1 = u - v$ ensures $\nabla f_1 \neq 0$ automatically at every point with $u \neq v$.

4.2. **Harvesting All $t_1 t_2 - t_1 - 2$ Disjoint Nondegenerate Roots.** For sufficiently large L, we harvest three pairwise disjoint root families:

Family 1: Outer diagonal (Side 1, $3t_2 - 4$ roots). On a compact square $K = [\mu, 1 - \mu]^2$ containing all $3t_2 - 4$ simple roots $(r_i, 1 - r_i)$ of $F_{t_2}(u, 1 - u)$, we have $uv \geq \mu^2 > 0$, so $|\phi_L(uv)| = O(L^{-1})$ and $|\phi_L'(uv)| = O(L^{-1})$ uniformly on K. Thus $f_1(u, v) = 0$ defines a smooth graph $v = v_L(u) = 1 - u + O(L^{-1})$ with $v_L'(u) = -1 + O(L^{-1})$, and C^1-persistence preserves all $3t_2 - 4$ simple roots with $uv \geq \mu^2$ and $\det J \neq 0$.

Family 2: Upper arcs halfA ($m(t_2-2)$ roots). Fix $k \in \{1,\dots,m\}$ and $j \in \{0,\dots,t_2-3\}$. By Proposition 3.1, choose uniform $\eta, \Delta > 0$ so the windows $[\lambda_j - \eta, \lambda_j + \eta] \subset (0, 1/2)$ are pairwise disjoint (separated by $> 2\eta$) and every term $i \notin \{j, j+1\}$ satisfies

$$E_{a_i,b_i}(\lambda') \geq \max\big(E_{a_j,b_j}(\lambda'),\, E_{a_{j+1},b_{j+1}}(\lambda')\big) + \Delta \qquad \forall\, \lambda' \in [\lambda_j - \eta, \lambda_j + \eta].$$

By (4), extract $[w_a, w_b] \subset I_k$ where $s_L(wt_k)$ sweeps $[s_1, s_2] = [t_k^{\lambda_j+\eta}, t_k^{\lambda_j-\eta}]$. Consider the normalized function $H(w) := u^{-a_{j+1}} v^{-b_{j+1}} F_{t_2}(u, v)$ on $\mathsf{halfA}(wt_k)$:

$$H(w) = c_{j+1} + c_j u^{a_j - a_{j+1}} v^{b_j - b_{j+1}} + \sum_{i \notin \{j,j+1\}} c_i u^{a_i - a_{j+1}} v^{b_i - b_{j+1}}.$$

Let $D_j := (b_j - b_{j+1}) + (a_{j+1} - a_j) > 0$. At $s = s_1 = t_k^{\lambda_j+\eta}$, the j-th term decays as $\Theta(t_k^{\eta D_j}) \to 0$ and the tail is $O(t_k^{\Delta}) \to 0$, so $\operatorname{sgn}(H) = \operatorname{sgn}(c_{j+1})$. At $s = s_2 = t_k^{\lambda_j-\eta}$, the j-th term grows as $\Theta(t_k^{-\eta D_j}) \to \infty$ while the tail is $O(t_k^{\Delta} \cdot t_k^{-\eta D_j})$, so $\operatorname{sgn}(H) = \operatorname{sgn}(c_j) = -\operatorname{sgn}(c_{j+1})$. By the Intermediate Value Theorem, there exists $w_0 \in [w_a, w_b]$ with $H(w_0) = 0$. At w_0, the j-th term balances $-c_{j+1}$ up to $O(t_k^{\Delta})$, while $|\Omega| \geq c_A / s_2 = c_A t_k^{-(\lambda_j - \eta)} \to \infty$. By Lemma 4.1, $|H'(w_0)| \geq \frac{1}{4}|c_{j+1}| D_j |\Omega| > 0$, proving $\det J \neq 0$.

Family 3: Lower arcs halfB ($m-1$ roots, for $k = 2,\dots,m$). Fix $k \in \{2,\dots,m\}$ and set $[s_1, s_2] := [2^{-64} t_k^{-1/2}, t_k^{-1/2}]$. Crucially, because $k \geq 2$,

$$(k^2 - k) - \left(k - \frac{1}{2}\right) = (k-1)^2 - \frac{1}{2} \geq \frac{1}{2} > 0,$$

so $s_2 = L^{k-1/2} \leq c_A L^{k^2-k}$ for all large L, and (4) yields $[w_a, w_b] \subset I_k$ sweeping $[s_1, s_2]$. On $\mathsf{halfB}(wt_k)$, define $H_B(w) := s\, u^{-S} F_{t_2}(u, v)$ using the asymmetric tail of Proposition 3.1:

$$H_B(w) = -1 + 2u^{3/64} v^{1/64} + \sum_{i=0}^{t_2-3} (s c_i) u^{a_i - S} v^{b_i}.$$

Because $u \geq \frac{1}{2} s_1 = 2^{-65} t_k^{-1/2}$, $v = wt_k/u = O(t_k^{3/2})$, $a_i < S$, and $b_i > 0$ for all $i < t_2 - 2$, every term in the sum decays uniformly as $O(t_k^{\delta_B}) \to 0$ ($\delta_B > 0$). Writing $2u^{3/64} v^{1/64} = 2u^{1/32} (wt_k)^{1/64}$: at $s = s_1$, this term is $\leq 2 \cdot 2^{-2} \cdot 2^{1/64} < 3/4$, so $H_B < -1/8 < 0$; at $s = s_2$, it is $\geq 2(1/2)^{1/32}(1/2)^{1/64} = 2(1/2)^{3/64} > 3/2$, so $H_B > 1/4 > 0$. Thus $H_B(w_0) = 0$ at some $w_0 \in [w_a, w_b]$, where $2u_0^{3/64} v_0^{1/64} \geq 1/2$ and $|\Omega| \geq c_A L^{k^2-k}/s_2 \geq c_A L^{1/2} \to \infty$. By Lemma 4.1, the main derivative coefficient is $\frac{3}{64} - \frac{1}{64} = +\frac{1}{32} > 0$, giving $|H_B'(w_0)| \geq \frac{1}{128}|\Omega| > 0$, hence $\det J \neq 0$.

Disjointness, clearing denominators, and multivariate product. For large L: (i) Family 1 has $uv \geq \mu^2 > \frac{3}{2} L^{-1} \geq uv$ on all arcs $k \geq 1$; (ii) distinct arcs $k_1 < k_2$ have disjoint uv-intervals $[\frac{1}{2} L^{-(2k-1)}, \frac{3}{2} L^{-(2k-1)}]$; (iii) on a fixed arc k, halfA has $u < v$ while halfB has $v < u$; (iv) distinct transitions $j_1 < j_2$ on halfA have disjoint v-intervals since $t_k^{2\eta} < 1/2$. The total root count is

$$(3t_2 - 4) + (t_1 - 3)(t_2 - 2) + (t_1 - 4) = t_1 t_2 - t_1 - 2.$$

Setting $u = x^D, v = y^D$ where $D = \operatorname{lcm}_j(\operatorname{den}(a_j), \operatorname{den}(b_j)) \geq 1$ and clearing negative powers via monomial multipliers $(xy)^{Dm}$ and $x^P y^Q$ produces genuine integer-exponent polynomials with at most t_1 and t_2 monomials and multiplies $\det J$ by a strictly positive factor, proving Theorems 2.1 and 2.2. Theorem 2.3 then follows immediately by taking the disjoint-variable product of $\lfloor n/2 \rfloor$ copies of the bivariate diagonal system from Theorem 2.2, together with the univariate t-nomial $\prod_{r=1}^{t-1}(z - r) = 0$ when n is odd.

5. Explicit Certified Counterexamples

To complement the asymptotic theorems, we construct explicit low-degree systems for small (t_1, t_2) and certify them across three independent verification layers.

5.1. **Explicit Proof of Theorem 2.4 ($\mathcal{S}(4,4) \geq 10$ and $B(2,4) \geq 10$).** Starting from our symmetric $(3,4)$ degree-45 trinomial–quadrinomial system [1] with 8 positive roots, replacing the trinomial $u+v=1$ by the single closed oval $u+v-1+10^{-4}(uv)^{-1/4}=0$ and setting $u=x^8, v=y^8$ yields the symmetric quadrinomial system (1)–(2) of degrees 12 and 47 sharing the monomial x^2y^2 (7 distinct monomials total). By symmetry $(x,y) \leftrightarrow (y,x)$, it suffices to certify 5 nondegenerate roots in the strict lower triangle $0<y<x<1$:

$$(x_1,y_1) \approx (0.93492845,\ 0.01657847), \qquad (x_2,y_2) \approx (0.99980078,\ 0.42351006),$$
$$(x_3,y_3) \approx (0.99540057,\ 0.65993746), \qquad (x_4,y_4) \approx (0.97244674,\ 0.81783841),$$
$$(x_5,y_5) \approx (0.95886883,\ 0.85487007).$$

In `ExplicitS44.lean`, each root is enclosed inside an exact rational box of radius $r=10^{-6}$ using an exact rational approximate inverse Jacobian $A \in \mathbb{Q}^{2\times 2}$, uniform $[0,1]^2$ Lipschitz constants $L_1 = 1.4\times 10^6$ and $L_2 = 1.9\times 10^4$, and the Banach contraction mapping theorem verified entirely by exact rational arithmetic (`norm_num`), yielding $5+5=10$ pairwise distinct nondegenerate positive roots with zero floating-point trust.

5.2. **Independent Numerical Certifications.**

(1) **Simultaneous Pandrosion–residue complex root certification (`pandsysroots v0.34.0`):** Unlike homotopy continuation, `pandsysroots` isolates and certifies all complex roots via simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy. Running `cargo run --release` certifies:
   - $\mathcal{S}(3,4) \geq 8$ (deg 8 and 45, 6 monomials): 360/360 complex roots certified in 0.08 s, 8/8 positive, max radius 3.04×10^{-13}, $\min|\det J| = 6.26$;
   - $\mathcal{S}(4,4) \geq 10$ (deg 12 and 45, 8 monomials): 540/540 complex roots certified in 0.08 s, 10/10 positive, max radius 3.86×10^{-13}, $\min|\det J| = 1.97\times 10^3$;
   - $B(2,4) \geq 10$ (system (1)–(2), 7 shared monomials): 540/540 complex roots certified in 0.09 s, 10/10 positive, max radius 4.35×10^{-13}, min separation 0.0394, $\min|\det J| = 30.5$.

(2) **520-digit arbitrary-precision verification (`verify_240digits.py`):** Using `mpmath` at 520-digit working precision with 2D Newton refinement (in log-coordinates for multi-scale systems), we certify all five explicit systems in Table 1: $\mathcal{S}(4,4) \geq 10$ (max residual 6.77×10^{-518}), $\mathcal{S}(4,5) \geq 14$ (5.91×10^{-517}), $\mathcal{S}(4,6) \geq 18$ (5.91×10^{-517}), $\mathcal{S}(5,5) \geq 17$ (1.99×10^{-399}), and $\mathcal{S}(5,6) \geq 22$ (1.99×10^{-399}), all with $\min|\det J| \geq 0.4024$.

6. Lean 4.36.0 Formalization and Axiom Audit

The entire proof suite is formalized in Lean 4 (`v4.36.0`, Mathlib4) under `lean/BivariateFewnomials/` across five self-contained modules:

- `SparseSystem.lean`: Multivariate `SparsePoly n`, `SparseSystem n`, exact calculus partial derivatives, $n\times n$ Jacobian determinants, `HasTermBounds`, `HasAtLeastRoots`, disjoint products (`productSystems`), and univariate t-nomials.
- `ExplicitS44.lean`: Zero-float 2D Newton–Banach proof of Theorem 2.4 (`explicit_S44_and_B24_ge_10`).
- `InductiveFamily.lean`: Complete proof of Proposition 3.1 (`exists_genInv`).
- `NestedOvalsBridge.lean`: Generalized bivariate calculus, curve Jacobian criterion, uniform denominator clearing (`bridge_to_sparseSystem2`), and Side 1 persistence (`side1_roots`).
- `NestedOvals.lean`: Arc monotonicity (`arc_Sk_props`), universal log-derivative bounds, `halfA_root`, `halfB_root`, and complete proofs of Theorems 2.1–2.3.

Executing `#print axioms` in `BivariateFewnomials.lean` confirms zero `sorryAx`, zero `native_decide`, and exclusive dependence on the standard axioms `[propext, Classical.choice, Quot.sound]`. All Lean 4 proofs, Rust certification code (`pandsysroots v0.34.0`), and 520-digit Python verification scripts are publicly available at `https://github.com/ivan-fr/fewnomial-bivariate-kouchnirenko`.

References

[1] I. Besevic, *An improved lower bound $\mathcal{S}(3,t) \geq 3t-4$ for positive solutions of bivariate trinomial–t-nomial systems and an explicit degree-45 eight-root example*, preprint, 2026.

[2] I. Besevic, *Constructive Folded-Haas trinomial systems and exponential lower bounds on fewnomial root counts*, preprint, 2026.

[3] F. Bihan, J. M. Rojas, and F. Sottile, *On the sharpness of fewnomial bounds and the number of components of fewnomial hypersurfaces*, in *Algorithms in Algebraic Geometry*, IMA Vol. Math. Appl., vol. 146, Springer, New York, 2008, pp. 15–20.

[4] F. Bihan and F. Sottile, *New fewnomial upper bounds from Gale dual polynomial systems*, Moscow Math. J. **7** (2007), no. 3, 387–407.

[5] B. El Hilany, *Counting real roots of polynomial systems with few monomials*, J. Pure Appl. Algebra **222** (2018), no. 5, 1141–1157.

[6] B. El Hilany and S. Tavenas, *Improved fewnomial bounds for bivariate systems*, Discrete Comput. Geom. (2024).

[7] E. Gomez, A. Niles, and J. M. Rojas, *Beating Kouchnirenko's conjecture for bivariate fewnomials*, preprint, Texas A&M University, 2007.

[8] B. Haas, *A simple counterexample to Kouchnirenko's conjecture*, Beiträge Algebra Geom. **43** (2002), no. 1, 1–8.

[9] A. G. Khovanskii, *On a class of systems of transcendental equations*, Soviet Math. Dokl. **22** (1980), 762–765.

[10] A. G. Khovanskii, *Fewnomials*, Translations of Mathematical Monographs, vol. 88, American Mathematical Society, Providence, RI, 1991.

[11] T.-Y. Li, J. M. Rojas, and X. Wang, *Counting real connected components of trinomial curve intersections and m-nomial hypersurfaces*, Discrete Comput. Geom. **30** (2003), no. 3, 379–414.

ESIEE-IT, France
Email address: ivan.besevic@edu.esiee-it.fr