A COUNTEREXAMPLE TO A FOLKLORE CONJECTURE ON GELFAND PAIRS

RUI CHEN AND JIEWEN WANG

Abstract. We construct a connected symmetric pair over $\mathbb{Q}_5$ for which every irreducible unramified principal series has at most one invariant linear form, whereas an explicit irreducible admissible representation has exactly two. The ambient group is the subgroup of $\mathrm{GL}_2 \times \mathrm{GL}_2$ defined by $\det(g_1)\det(g_2) = 1$, and the symmetric subgroup is a twisted diagonal copy of SL_2. The construction uses the two constituents of the restriction to $\mathrm{SL}_2(\mathbb{Q}_5)$ of a quadratic principal series of $\mathrm{GL}_2(\mathbb{Q}_5)$. It gives a counterexample to the formulation of a folklore conjecture on Gelfand pairs recorded by Aizenbud and Sayag in terms of generic unramified principal series.

hexagon:2611.01125v2

1. Introduction and the symmetric pair

For a locally compact totally disconnected group G and a closed subgroup H, write

$$m_H(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \mathbb{C}).$$

All representations in this paper are smooth complex representations; $\pi^\vee$ denotes the smooth contragredient. We call (G, H) a *Gelfand pair* if $m_H(\pi) \leq 1$ for every irreducible admissible representation π of G. A representation is H-distinguished if $m_H(\pi) > 0$.

Aizenbud and Sayag record the following folklore assertion [1, Conjecture I], which we paraphrase here.

Conjecture 1.1. For a spherical pair $\mathbf{H} \subset \mathbf{G}$ over a non-Archimedean local field, generic multiplicity one among the H-distinguished irreducible unramified principal series implies that (G, H) is a Gelfand pair.

Here $G = \mathbf{G}(k)$ and $H = \mathbf{H}(k)$. By an *irreducible unramified principal series* we mean a full normalized induction from an unramified character of a Borel subgroup that is irreducible. This is distinct from an arbitrary irreducible subquotient of a reducible unramified induction. We prove multiplicity one for every distinguished irreducible full unramified principal series in our example, so no choice of a generic subset within that class is needed. We also describe an explicit nonempty generic family.

Set $k = \mathbb{Q}_5$, let $v_5(5) = 1$, and normalize $|5| = 5^{-1}$. Define the split algebraic group

$$\mathbf{G} = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det(g_1)\det(g_2) = 1\}. \tag{1.1}$$

Put $t = \mathrm{diag}(5, 1)$ and define a k-involution by

$$\theta(g_1, g_2) = (tg_2t^{-1}, t^{-1}g_1t), \qquad \mathbf{H} = (\mathbf{G}^\theta)^\circ.$$

Throughout the paper,

$$G = \mathbf{G}(k), \qquad S = \mathrm{SL}_2(k), \qquad N = S \times S, \qquad H = \mathbf{H}(k) = \{(tst^{-1}, s) : s \in S\}. \tag{1.2}$$

2020 *Mathematics Subject Classification.* Primary 22E50.

Key words and phrases. p-adic representations, symmetric pairs, Gelfand pairs, principal series.

Theorem 1.2. *The group* $\mathbf{G}$ *is connected, split and reductive, and* $\mathbf{H} \subset \mathbf{G}$ *is a connected symmetric spherical pair. It has the following properties.*

(1) *Every irreducible unramified principal series* π *of* G *satisfies* $m_H(\pi) \leq 1$. *There is a nonempty family of such representations with* $m_H(\pi) = 1$.
(2) *There is an explicit irreducible admissible representation* Π *of* G *with* $m_H(\Pi) = 2$.

Consequently, (G, H) *is not a Gelfand pair and gives a counterexample to Conjecture 1.1 in the formulation above.*

2. Rank-one facts and restriction to the derived group

For smooth characters α, β of $k^\times$ write

$$R(\alpha, \beta) = \mathrm{Ind}_{B_2(k)}^{\mathrm{GL}_2(k)}(\alpha \otimes \beta), \qquad I(\chi) = \mathrm{Ind}_{B_S}^{S}(\chi),$$

where both inductions are normalized, $B_S = B_2(k) \cap S$, and $\chi(\mathrm{diag}(a, a^{-1})) = \chi(a)$. Restriction of functions gives

$$R(\alpha, \beta)|_S \simeq I(\alpha\beta^{-1}). \tag{2.1}$$

Indeed, $\mathrm{GL}_2(k) = B_2(k)S$, and the square root of the Borel modulus restricts to the square root of the modulus of B_S.

We use the usual rank-one principal-series facts. For $\mathrm{GL}_2(k)$, $R(\alpha, \beta)$ is irreducible unless $\alpha\beta^{-1} = |\cdot|^{\pm 1}$, and two irreducible such inductions are isomorphic exactly when their inducing unordered pairs of characters agree; see [2, Section 4.5]. For unramified χ, the representation $I(\chi)$ is irreducible exactly when

$$\chi \notin \{|\cdot|, |\cdot|^{-1}, \eta\}, \qquad \eta(x) = (-1)^{v_5(x)}. \tag{2.2}$$

This includes the irreducibility of $I(1)$; see [3, Corollaries 9.3.3 and 9.4.6]. For irreducible unramified $I(\chi)$ and $I(\psi)$ one has

$$I(\chi)^\vee \simeq I(\chi^{-1}), \qquad I(\chi) \simeq I(\psi) \iff \psi \in \{\chi, \chi^{-1}\}. \tag{2.3}$$

The forward implication in the second assertion follows by comparing normalized Jacquet modules, whose semisimplifications have characters χ, χ^{-1}. The reverse implication is the rank-one intertwining isomorphism. These are also part of the calculation in [3, Section 9].

For a representation U of S and $g \in \mathrm{GL}_2(k)$, set $U^g(s) = U(gsg^{-1})$. In particular, any restriction $U = R(\alpha, \beta)|_S$ satisfies $U^g \simeq U$, with intertwiner supplied by $R(\alpha, \beta)(g)$.

Lemma 2.1. *Let* ψ *be a nontrivial quadratic character of* $k^\times$. *Then* $R_\psi = R(1, \psi)$ *is irreducible and*

$$R_\psi|_S = W_+ \oplus W_-,$$

where W_+ *and* W_- *are nonisomorphic irreducible admissible representations of* S. *The operator* $R_\psi(g)$ *preserves both constituent subspaces if* $\psi(\det g) = 1$ *and exchanges them if* $\psi(\det g) = -1$. *The same assertion holds for the corresponding dual subspaces in* $R_\psi^\vee$.

Proof. The GL_2 irreducibility criterion applies because a nontrivial quadratic character is not $|\cdot|^{\pm 1}$. Let Z be the center of $\mathrm{GL}_2(k)$ and put $L = ZS$. Then

$$L = \{g : \det g \in (k^\times)^2\}, \qquad \mathrm{GL}_2(k)/L \simeq k^\times/(k^\times)^2,$$

so L is open and normal with finite abelian quotient. Since Z acts by scalars, the L-invariant subspaces and intertwiners of R_ψ are exactly the S-invariant ones. Finite-index Clifford theory [4, Lemma 2.1] gives a finite semisimple restriction with r distinct constituents of a common multiplicity e, and

$$e^2 r = \#\{\mu : \mu^2 = 1,\ R_\psi \otimes (\mu \circ \det) \simeq R_\psi\}. \tag{2.4}$$

By the classification of irreducible GL_2 principal series, a self-twist in (2.4) is equivalent to

$$\{\mu, \psi\mu\} = \{1, \psi\}.$$

The two solutions are $\mu = 1$ and $\mu = \psi$. Thus $e^2 r = 2$, which forces $e = 1$ and $r = 2$. The stabilizer of either constituent is the intersection of the kernels of these self-twists [4, Corollary 2.2], namely $\ker(\psi \circ \det)$. This proves the preservation and exchange assertions. Passing to the contragredient preserves this permutation of the two subspaces. □

Lemma 2.2. *For smooth admissible representations U_1, U_2 of S, with the action of H on $U_1 \boxtimes U_2$ induced by* (1.2)*, there is a natural isomorphism*

$$\mathrm{Hom}_H(U_1 \boxtimes U_2, \mathbb{C}) \simeq \mathrm{Hom}_S(U_1^t, U_2^\vee). \tag{2.5}$$

Proof. An invariant bilinear form b corresponds to the map $u \mapsto b(u, -)$. If an open subgroup fixes u under U_1^t, invariance implies that it fixes $b(u,-)$ in the dual of U_2. Thus the map takes values in the smooth contragredient. The invariance identity is exactly S-equivariance, and the construction is reversible. □

3. Multiplicity one for irreducible unramified principal series

Every unramified character λ of $T = \mathbf{T}(k)$, where $\mathbf{T}$ is the diagonal torus in $\mathbf{G}$, extends to an unramified character $(\alpha_1 \otimes \beta_1) \boxtimes (\alpha_2 \otimes \beta_2)$ of the diagonal torus of $\mathrm{GL}_2 \times \mathrm{GL}_2$. For example, write an element of T as

$$\big(\mathrm{diag}(a,b), \mathrm{diag}(c, (abc)^{-1})\big).$$

An unramified character in the three independent coordinates a, b, c extends by taking $\beta_2 = 1$. If $\widetilde{G} = \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ and $\widetilde{B} = B_2(k) \times B_2(k)$, then $\widetilde{G} = \widetilde{B}G$ and the Borel moduli agree on $B = \mathbf{B}(k)$. Consequently

$$\pi(\lambda) := \mathrm{Ind}_B^G(\lambda) \simeq (R_1 \boxtimes R_2)|_G, \qquad R_j = R(\alpha_j, \beta_j). \tag{3.1}$$

Set $\chi_j = \alpha_j \beta_j^{-1}$ and $U_j = I(\chi_j)$. Then

$$\pi(\lambda)|_N \simeq U_1 \boxtimes U_2. \tag{3.2}$$

Proposition 3.1. *If $\pi(\lambda)$ is irreducible, then $m_H(\pi(\lambda)) \le 1$.*

Proof. First, each R_j must be irreducible: a nonzero proper invariant subspace in one factor would give such a subspace in $(R_1 \boxtimes R_2)|_G$. Hence $\chi_j \neq |\cdot|^{\pm 1}$.

If neither χ_j equals η, both U_j are irreducible by (2.2). Since $U_1^t \simeq U_1$, Lemma 2.2 and Schur's lemma give the required bound.

Suppose exactly one of the characters equals η. The corresponding $U_j = I(\eta)$ is a direct sum of two non-isomorphic irreducibles by Lemma 2.1, and the other U_j is irreducible. No constituent of $I(\eta)$ is isomorphic to the other U_j or its contragredient. Indeed, the semisimplified normalized Jacquet module of $I(\eta)$ is $\eta \oplus \eta$, whereas that of the other U_j has characters χ_j, χ_j^{-1}, neither equal to η. Exactness of the Jacquet functor rules out such an isomorphism. Conjugation by t only permutes the constituents of $I(\eta)$. It follows from (2.5) that $m_H(\pi(\lambda)) = 0$.

Finally suppose $\chi_1 = \chi_2 = \eta$. Write $R_j = R(\eta, 1) \otimes (\beta_j \circ \det)$ and choose the two constituent subspaces U_j^+, U_j^- of its restriction to S. Lemma 2.1 applies, since $R(\eta,1) \simeq R(1,\eta)$; a scalar twist does not change the permutation of subspaces. For $(g_1, g_2) \in G$ one has $\eta(\det g_1) = \eta(\det g_2)$. Thus

$$(U_1^+ \boxtimes U_2^+) \oplus (U_1^- \boxtimes U_2^-), \qquad (U_1^+ \boxtimes U_2^-) \oplus (U_1^- \boxtimes U_2^+)$$

are complementary nonzero G-stable subspaces of $\pi(\lambda)$. This contradicts irreducibility and excludes the last case. □

Proposition 3.2. *If χ_1 and χ_2 both avoid the three characters in* (2.2)*, then $\pi(\lambda)$ is irreducible and*

$$m_H(\pi(\lambda)) = \begin{cases} 1, & \chi_2 = \chi_1 \text{ or } \chi_1^{-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Proof. The restriction (3.2) is an irreducible representation of the product N, because both factors are irreducible admissible representations. Any G-invariant subspace is N-invariant, so $\pi(\lambda)$ is irreducible. Formula (3.3) follows from (2.5), (2.3), and Schur's lemma. □

For completeness, the parameters have a concrete algebraic description. Two extensions of λ in (3.1) differ by the same unramified determinant character in both factors, so the ratios χ_1, χ_2 are independent of the extension. The unramified parameter torus is $(\mathbb{C}^\times)^3$, with coordinates $z_1 = \chi_1(5)$, $z_2 = \chi_2(5)$ and a third independent coordinate. On the open set $z_1, z_2 \notin \{5^{-1}, 5, -1\}$, the distinguished locus is precisely

$$z_2 = z_1 \quad \text{or} \quad z_2 = z_1^{-1}.$$

The open set meets each of these two subtori densely. For example, $z_1 = z_2 = 2$ gives an irreducible distinguished principal series. This verifies the nonemptiness and the generic multiplicity-one assertion explicitly. Proposition 3.1 also handles every irreducible full unramified induction outside this open set.

4. An irreducible representation with two periods

Define the ramified quadratic character

$$\xi(5^n u) = (-1)^n \left(\frac{\overline{u}}{5}\right), \qquad n \in \mathbb{Z}, \quad u \in \mathbb{Z}_5^\times, \tag{4.1}$$

where the last factor is the quadratic character of $\mathbb{F}_5^\times$. The unique decomposition $k^\times = 5^{\mathbb{Z}} \times \mathbb{Z}_5^\times$ shows that this defines a smooth character. It is quadratic and nontrivial on units, and $\xi(5) = -1$.

Let $R = R(1, \xi)$. By Lemma 2.1,

$$R|_S = V_+ \oplus V_-, \qquad V_+ \not\simeq V_-.$$

Its contragredient restricts as $R^\vee|_S = V_+^\vee \oplus V_-^\vee$. This follows either from the finite semisimple decomposition or directly from smooth duality: the center acts through a smooth character, and ZS is open in $\mathrm{GL}_2(k)$, so smoothness of a functional for S implies smoothness for $\mathrm{GL}_2(k)$. In the restriction $(R \boxtimes R^\vee)|_G$ define

$$\Pi = (V_+ \boxtimes V_-^\vee) \oplus (V_- \boxtimes V_+^\vee). \tag{4.2}$$

Proposition 4.1. *The subspace* (4.2) *is G-stable, is an irreducible admissible representation of G, and satisfies $m_H(\Pi) = 2$.*

Proof. For $(g_1, g_2) \in G$, the determinant relation and quadraticity give

$$\xi(\det g_1) = \xi(\det g_2).$$

If this common value is 1, the action preserves each of the two summands in (4.2); if it is -1, the action exchanges them, by Lemma 2.1. Thus Π is G-stable.

As an N-module, Π is the direct sum of two nonisomorphic irreducible modules: their first S-factors are nonisomorphic. Every N-submodule is therefore the sum of a subset of these two summands. The element

$$\gamma = (t, t^{-1}) \in G$$

exchanges the summands, since $\xi(\det t) = \xi(5) = -1$ and $\xi(\det t^{-1}) = -1$. A nonzero G-submodule must consequently contain both summands. This proves irreducibility. The ambient representation $(R \boxtimes R^\vee)|_G$ is a normalized principal series of G, by the same restriction argument as in (3.1), now allowing ramified characters. It is admissible; its subrepresentation Π is therefore admissible as well.

Because $\xi(\det t) = -1$, Lemma 2.1 also gives

$$V_+^t \simeq V_-, \qquad V_-^t \simeq V_+.$$

Apply Lemma 2.2 to the two summands of (4.2). Admissibility identifies $V_\pm$ with their smooth double duals, and Schur's lemma yields

$$\begin{aligned} m_H(\Pi) &= \dim \mathrm{Hom}_S(V_+^t, V_-) + \dim \mathrm{Hom}_S(V_-^t, V_+) \\ &= 1 + 1 = 2. \end{aligned}$$

Concretely, the two periods can be written as

$$\ell_+(v_+ \otimes f_-) = f_-(R(t)^{-1} v_+), \qquad \ell_-(v_- \otimes f_+) = f_+(R(t)^{-1} v_-),$$

each extended by zero on the other summand of (4.2). The identity $R(t)^{-1} R(tst^{-1}) = R(s) R(t)^{-1}$ verifies H-invariance directly. Both forms are nonzero, and their supports on the two summands show that they are independent. □

The geometric calculation in Section 1, Propositions 3.1 and 3.2, and Proposition 4.1 complete the proof of Theorem 1.2.

Remark 4.2. The conclusion concerns the implication from generic unramified principal-series multiplicity one to the Gelfand property. It does not assert that multiplicity one holds on all irreducible subquotients of reducible principal series, or on generic families in every Bernstein component.

Declaration of generative AI use

The initial counterexample was obtained with ChatGPT-6 Astra, which was also used to review the initial output and improve its exposition. OpenAI Codex was subsequently used to expand the proofs, check references, and revise the exposition and LaTeX source. Rui Chen checked the original draft step by step, Jiewen Wang understood the research problem and part of the technical approach, directed a self-developed Codex-based multi-agent system to advance the proof, and optimized the performance of AIGC-assisted writing used in the working process (not for the final manuscript).

References

1. Avraham Aizenbud and Eitan Sayag, *On a Cohen–Macaulay property of spherical varieties*, Unpublished manuscript, January 4, 2015.
2. Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997.
3. W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, Unpublished notes, draft of May 7, 1993.
4. S. S. Gelbart and A. W. Knapp, *L-indistinguishability and R groups for the special linear group*, Advances in Mathematics **43** (1982), no. 2, 101–121.

Email address: rchen69@jhu.edu

Email address: wan-jw16@tsinghua.org.cn