hexagon:2611.01123v1

ALGEBRAIC COMPARISON OF TORUS BOUNDARY STATES AND AN OBSTRUCTION TO COMPLETION

HAO-YU SUN

Abstract. Torus boundary states admit both root-of-unity coordinates and formal-series descriptions, but identifying finite states need not identify their completions. We construct an injective algebraic comparison between compatible Newton interpolation and Hopf-pairing data and the coefficient models of Gukov–Manolescu and Guicardi–Jagadale. The common domain is $\mathbb{Z}[A, A^{-1}][v]$, with three copies indexed by the nonzero mod-two residues of half-integral exponent lattices. A chosen filling functional takes value one on its normalized reference state and zero on an independent state. The domain exhausts the finite states generated from the reference state by the specified spectral coordinate over the common coefficient ring. We then impose uniform formal convergence together with discrete root coordinates on the first model and specified uniform Laurent-series topologies on the other two. A cyclotomic interpolation sequence converges to zero in the first model and to the reference state in the other two. Every coordinate projection of the resulting closed relation is therefore noninjective. The obstruction persists in extensions retaining these topologies and the scalar multiples of the reference state. The same sequence produces a zero-framed trefoil limit, whose relation to a negative-definite plumbing retains an explicitly computed framing scalar in the target coefficient field.

1. Polynomial agreement and incompatible limits

Write quantum-torus monomials in the order $Y^n X^m$, with $X^m Y^n = q^{-2mn} Y^n X^m$, where q is a formal parameter. The exponents m and n are called the electric and magnetic coordinates, respectively. Consider the torus state

$$\Omega_0 = \sum_{n \in \mathbb{Z}} Y^n (X^{1/2} - X^{-1/2}).$$

Its magnetic coordinate ranges through all integers, while its electric support consists of two half-integers. Multiplying on the right by a polynomial in

$$\mathsf{v}_X = \frac{2 - X - X^{-1}}{1 - q}$$

still gives a well-defined coefficient state. The same polynomial has a cyclotomic description: at an odd prime, the spectral variable takes the values $(2 - q^{i+1} - q^{-i-1})/(1 - q)$ in a finite quotient. This common polynomial calculus is a natural place to compare boundary states before asking whether their infinite expansions have the same meaning.

Homological blocks arose from the fivebrane construction of Gukov–Putrov–Vafa [6, Section 6.2] and the reconstruction of Witten–Reshetikhin–Turaev (WRT) invariants proposed by Gukov–Pei–Putrov–Vafa [5, Conjecture 2.1]. Extending such series to manifolds with boundary raises a separate question: which operations on finite states remain continuous after a completion is chosen?

The relevant coefficient models come from the two-variable knot series of Gukov–Manolescu (GM) [4, Sections 6–7] and the ordered quantum-torus states of Guicardi–Jagadale (GJ) [2, Sections 2–4]. The cyclotomic model uses a strict completion in which one finite approximation must satisfy both a formal congruence at $A = -1$ and finitely many exact root conditions,

2020 *Mathematics Subject Classification.* Primary 57K16; Secondary 57K31, 13J10.
Key words and phrases. torus boundary state, cyclotomic interpolation, Hopf pairing, completion obstruction.

following the companion paper on integral root fibers and closed factorizations [10, Sections 6–7]. Appendix A gives the genus-one quotient, Newton-pullback, Hopf and cap arguments; it retains the saturated-lattice comparison of [10, Theorem 4.7 and equation (2)]. We use the local name *Newton–Hopf model* for the construction that records a state's Newton interpolation coordinates, monomial coordinates, and Hopf-pairing image as a compatible triple. A finite state can be identified in these models without identifying their completions. The distinguished state corresponding to the constant polynomial 1 is called the *primitive*; the root-side cap, or filling functional, is normalized to take value one on it.

All numbered GJ formula references below use the PDF equation numbers in [2], version 1. Version 2 is also available [3]; its Sections 3.2–4.1 retain the displayed pairing and two-twist formulas discussed in Appendix B. We therefore retain the explicit pairing and phase conventions used here, as well as the original proof locators. Neither edition supplies the simultaneous root and uniform coefficient topologies imposed in our completed comparison problem.

Related comparisons between knot series and cyclotomic expansions use different completions. Park's inverted Habiro series [8, Section 2] uses reciprocal cyclotomic factors that expand in powers of the spectral variable or its inverse; Svoboda's theta-function expression requires a lower bound on its coefficients [11, Theorem 1.2]. Park also constructs skein-to-series maps for nice knot complements and compatible convergent surgery transforms [9, Theorems B–C]. These results provide neither the simultaneous strict root approximation nor the uniform coefficient topology imposed below. Our obstruction concerns that specified comparison and does not exclude these other realizations.

Write $R_0 = \mathbb{Z}[A, A^{-1}]$, put $\lambda = 1+A$, and use the coefficient substitution $A \mapsto -q$. For transported states we choose coherently $t^4 = q$, so the coefficient maps are $R_0 \to \mathbb{Q}((q)) \to \mathbb{C}((t))$. The three nonzero residues $\epsilon \in (\mathbb{Z}/2)^2$ index the affine lattices $D_\epsilon = \mathbb{Z}^2 + \epsilon/2$. They are characteristic residues of exponent lattices; they are not spin structures. The precise transported zero labels are given in Section 3.

Theorem 1.1 (Algebraic comparison)**.** *In each of the three characteristic sectors there is an injective R_0-linear comparison, parameterized by $R_0[v]$, between the strict Newton–Hopf compatible triples and the two external coefficient models. In the base sector its external expression is*

$$P(v) \longmapsto \Omega_0 P(\mathsf{v}_X).$$

The coefficient functions are globally well-behaved in the sense of Gukov–Manolescu, and the external maps have explicit algebraic inverses on their images. The domain gives the full primitive-generated comparison in the class $\mathbf{SPol}_{R_0}$ of Definition 3.2: coefficients come from R_0, and the prescribed spectral operation is multiplication by v.

For example, the primitive $P_0 = 1$ and the polynomial

$$P_1 = \lambda v^2 - 4v - \lambda(1 - A^{-1})^2$$

are independent. Under the comparison,

$$P_1(-q, \mathsf{v}_X) = \frac{X^2 + X^{-2} - q^2 - q^{-2}}{1 - q}.$$

The outer electric coefficient detects P_1, although its primitive cap vanishes. Thus the comparison already distinguishes an uncontracted state that a single cap cannot see. Section 3 proves the theorem and gives the inverse coefficient formulas.

To compare infinite states, retain the strict topology on the cyclotomic model and choose uniform q-valuation topologies on proper external coefficient spaces. In the base chart, convergence means that the lower valuation of the coefficient error tends to infinity uniformly over $D_{(1,0)}$. The other two charts carry the transported topology. These are choices made here, not topologies attributed to either external paper. The simultaneous relation is obtained by

forming the three states associated to each polynomial and taking the closure of these triples in the product topology. The *primitive scalar orbit* consists of all R_0-multiples of the primitive.

Theorem 1.2 (Obstruction to completion)**.** *With these topologies, the simultaneous closure of the algebraic comparison is complete and Hausdorff, but each of its three coordinate projections has a nonzero kernel. Consequently it cannot give a common complete state module with continuous injective comparison maps whose inverses on their images are continuous. This obstruction persists in every R_0-linear extension retaining the same target topologies and the primitive scalar orbit, including the allowed homeomorphic gauges and fixed invertible topological lines of Definition 4.2.*

The mechanism is a single sequence, not a comparison of finitely many root values. Write Φ_m for the m-th cyclotomic polynomial, enumerate all odd primes $p_i \geq 5$, and set

$$D_N(A) = \lambda^N \prod_{i \leq N} \Phi_{2p_i}(A).$$

This polynomial tends to zero formally and eventually vanishes at each selected physical root. On substituting $A = -q$, however, $D_N(-q) = \prod_{i \leq N}(1 - q^{p_i})$ has constant coefficient one. Its inverse modulo q^N produces a polynomial $E_N \in R_0$ with the incompatible limits

$$\begin{aligned} E_N\Omega_0 \text{ in the strict root model} &\longrightarrow 0, \\ E_N\Omega_0 \text{ in either external chart} &\longrightarrow \Omega_0. \end{aligned}$$

The full argument in Section 5 includes uniform convergence and both off-diagonal points of the closed relation.

The right-handed trefoil (RHT) gives a geometric application of the same mechanism. Section 6 applies E_N to explicit polynomial approximants of its two-variable series. Their strict root coordinates converge to zero, while their external limits are the nonzero multiple $2q^{-1}$ of the zero-framed trefoil state. The corresponding -7-framed negative-definite plumbing satisfies

$$\rho^{\mathrm{GJ}}_{T_+} Z^{\mathrm{GJ}}_{-7} = q^{-1/2} |\mathrm{RHT}, 0\rangle.$$

Here $q^{-1/2}$ is retained in the target field. It supplies a modular orbit relation there, without constructing a common modular operator or a common framing line.

The contribution is the explicit polynomial comparison together with this obstruction for the declared topologies. The supporting three-sector construction records the exact $\Gamma_0(2)$ stabilizer of the original Gukov–Manolescu lattice. Appendix B proves the uniform plumbing support criterion and records the pairing and twist conventions needed to interpret the imported formulas. The knot series and modular monomial formulas themselves remain attributed to their sources. No maximality over arbitrary localizations or additional target-dependent generators is asserted. Common pairings, gluing, and framing structures would require the continuous comparison that fails here.

2. The three affine sectors

The primitive Ω_0 is supported on $D_{(1,0)} = (\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z}$. Exchanging meridian and longitude sends this lattice to $D_{(0,1)}$. Thus modular transport is naturally a family of maps between affine lattices. We construct that family before using it to transport the polynomial comparison.

Let $G = SL(2, \mathbb{Z})$, $P = (\mathbb{Z}/2)^2 \setminus \{0\}$, and $\epsilon_0 = (1, 0)$. A bar on an integral matrix denotes reduction modulo two. Coordinates are columns $z = (m, n)^T$; the oriented boundary basis (μ, λ_∂) has intersection $\mu \cdot \lambda_\partial = 1$. The subscript on λ_∂ distinguishes a longitude from the coefficient $\lambda = 1 + A$. Fix the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad T_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Use the field $\mathbf{k}_{\mathrm{GM}} = \bigcup_{N\geq 1} \mathbb{Q}((q^{1/N}))$. For $b : D_\epsilon \to \mathbf{k}_{\mathrm{GM}}$, let $\Omega(b)$ be the union of all q-exponent supports of its values, and define

$$V_\epsilon = \{b : \Omega(b) \text{ is bounded below and has finite image in } \mathbb{Q}/\mathbb{Z}\}.$$

We call this the *global support condition*: one lower bound and one finite set of fractional exponent classes must work for all (m, n). Only $V_{(1,0)}$ is the function space of [4, Definition 6.5]. The remaining two fibers and their transports are the construction below. The uniform support condition is over all boundary fluxes, not separately for each one.

Lemma 2.1 (Affine transport)**.** *For $M \in G$, one has $MD_\epsilon = D_{\bar{M}\epsilon}$. Consequently*

$$(\rho^{\mathrm{GM}}_{M,\epsilon} b)(z) = b(M^{-1}z) \tag{1}$$

defines a linear isomorphism $V_\epsilon \to V_{\bar{M}\epsilon}$, preserving $\Omega(b)$ exactly. These maps satisfy

$$\rho^{\mathrm{GM}}_{M,\bar{N}\epsilon}\rho^{\mathrm{GM}}_{N,\epsilon} = \rho^{\mathrm{GM}}_{MN,\epsilon}.$$

Proof. For an integral lift $\widetilde{\epsilon}$, $M(\mathbb{Z}^2 + \widetilde{\epsilon}/2) = \mathbb{Z}^2 + M\widetilde{\epsilon}/2$. The resulting affine set depends only on reduction modulo two. Inverse precomposition reindexes the values bijectively, so it preserves total support. Finally, $b(N^{-1}M^{-1}z) = b((MN)^{-1}z)$, and the inverse is the transport by M^{-1}. □

The transformation groupoid $G \ltimes P$ has the three residues as objects; a matrix M gives an arrow from ϵ to $\bar{M}\epsilon$. Thus a representation assigns a vector space to each sector and compatible linear maps to these arrows.

Theorem 2.2 (Canonical extension of the original sector)**.** *The stabilizer of $D_{(1,0)}$ is $H = \Gamma_0(2)$, and its G-orbit consists of all three sectors. The family V_ϵ is the canonical extension of its H-action to the transformation groupoid $G \ltimes P$: every representation of this groupoid agreeing with the given fiber and stabilizer action has a unique natural isomorphism from this family that is the identity on $V_{(1,0)}$.*

Proof. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the equality $\bar{M}(1,0)^T = (1,0)^T$ says that a is odd and c is even. Since $ad - bc = 1$, this is equivalent to c even, which gives H. The matrices I, S, T_- reach respectively $(1,0), (0,1), (1,1)$, proving the orbit assertion.

For each ϵ, set $\mathcal{T}_\epsilon = \{M : \bar{M}\epsilon_0 = \epsilon\}$. The associated fiber is

$$(\mathcal{T}_\epsilon \times V_{\epsilon_0})/\!\sim, \qquad (Mh, b) \sim (M, \rho_h b), \quad h \in H.$$

Its vector-space structure is obtained using any transporter, and the equivalence relation makes it independent of that choice. The map $[M, b] \mapsto \rho_M b$ is well-defined by the group law and surjective because ρ_M is an isomorphism. If $\rho_M b = \rho_N c$, then $h = M^{-1}N \in H$ and $b = \rho_h c$; hence the two representatives are equivalent. This proves injectivity. In another groupoid representation W the map is necessarily $[M, b] \mapsto W(M)b$, and the same argument proves well-definedness and invertibility. Naturality and identity on the original fiber force this formula, proving uniqueness. □

The three generators have the following sector actions:

ϵ	S	T_+	T_-
$(1,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(1,0)$	$(1,1)$	$(0,1)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$.

This table explains why the three fibers are retained separately. The literal convention $\sigma_M b = b \circ M$ in the source becomes $\rho_M = \sigma_{M^{-1}}$ on the stabilizer; inverse precomposition expresses a left action, so composition follows matrix multiplication in the order displayed above.

For the ordered quantum-torus symbols $e_{(m,n)} = Y^n X^m$, with $X^m Y^n = q^{-2mn} Y^n X^m$, use the common left convention $\rho_M^{\mathrm{GJ}} = \widetilde{\rho}_{M^T}^{\mathrm{GJ}}$, where the tilde denotes the source convention in [2, Lemma 1]. Explicitly,

$$\rho_M^{\mathrm{GJ}}(e_{(m,n)}) = q^{-acm^2-2bcmn-bdn^2} e_{(am+bn,cm+dn)}. \tag{2}$$

This is a termwise map of formal sums. To verify its left composition, write $F_M(m,n) = mn - (am+bn)(cm+dn)$, using $ad-bc=1$. Then $F_N(z) + F_M(Nz) = F_{MN}(z)$; the phases telescope and the exponent columns compose by MN. Thus common T_+ uses the source operator at T_-, common T_- uses that at T_+, and common S uses that at S^{-1}.

The same groupoid carries the original partial GM pairing. On pairs for which the coefficientwise locally finite sum belongs to $\mathbf{k}_{\mathrm{GM}}$, put

$$B_\epsilon(b^-, b^+) = \sum_{z \in D_\epsilon} b^-(z) b^+(z).$$

A change of index gives

$$B_{\bar{M}\epsilon}(\rho_M b^-, \rho_M b^+) = B_\epsilon(b^-, b^+).$$

It preserves the summation domain. Reflection by $R = \mathrm{diag}(-1,1)$ fixes every sector and gives the involution $(\mathcal{R}b)(z) = b(Rz)$, satisfying $\mathcal{R}\rho_M = \rho_{RMR}\mathcal{R}$. These statements concern the separate GM model. They do not assert a common pairing with the Newton–Hopf or GJ models.

3. The polynomial comparison

A single polynomial will now specify three states. The Newton–Hopf model records its root and formal coordinates, while the two external models record its ordered Laurent coefficients. The outer electric coefficient gives a direct inverse and makes the comparison independent of a choice of cap.

3.1. Coefficients and the transported zero label.

The common source ring is the discrete Laurent polynomial ring

$$R_0 = \mathbb{Z}[A, A^{-1}], \qquad q_A = -A, \qquad \lambda = 1 + A = 1 - q_A. \tag{3}$$

Only integral powers of q occur in the base algebraic core and in the interpolation sequence. The transported GJ sectors require a coherent quarter-power branch. Choose once and for all $t = q^{1/4}$ and use

$$R_0 \longrightarrow \mathbb{Q}((q)) \longrightarrow \mathbb{C}((t)) \subset \mathbb{C}_q, \qquad A \longmapsto -q, \qquad t^4 = q. \tag{4}$$

The first target is the integral-power subfield of $\mathbf{k}_{\mathrm{GM}}$; the second map is the chosen coherent Puiseux embedding into the algebraic closure $\mathbb{C}_q = \mathbb{C}((q))^{\mathrm{alg}}$. Write β_{GM} and β_{GJ} for the two maps from R_0. Thus the base GM and GJ coefficient rows below become equal after applying the chosen coefficient embedding. Before that embedding, they belong to different coefficient fields.

For an oriented three-manifold M with a chosen boundary framing, a GJ label is a pair $(\mathfrak{s}, h)$, where $\mathfrak{s} \in \mathrm{Spin}(M)$ and $h \in H^1(M; \mathbb{Q}/\mathbb{Z})$. The set $\mathrm{Spin}(M)$ is a torsor for $H^1(M; \mathbb{Z}/2)$. Changing its origin identifies

$$(\mathfrak{s}_1, h) \sim (\mathfrak{s}_2, h + \mathfrak{s}_1 - \mathfrak{s}_2),$$

where the difference is included in $H^1(M; \mathbb{Q}/\mathbb{Z})$ by $1 \bmod 2 \mapsto 1/2 \bmod \mathbb{Z}$. The boundary grade is the restriction of h plus the framing-dependent Spin contribution. On a parameterized torus it belongs to $H^1(T^2; \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^2$, written here in (X, Y) exponent order. These are the label and grading conventions of [2, Sections 2.1–2.2].

The ordinary Bockstein in this convention is the connecting homomorphism

$$\mathrm{Bk} : H^1(M; \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(M; \mathbb{Z}) \quad \text{from} \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \tag{5}$$

For an invertible integral surgery matrix $L \in \mathrm{Mat}_b(\mathbb{Z})$, the source's coordinates identify it with $h \mapsto Lh$ from $L^{-1}\mathbb{Z}^b/\mathbb{Z}^b$ to $\mathbb{Z}^b/L\mathbb{Z}^b$. In those coordinates the Spin-induced affine origin is $[2L\mathfrak{s}]$ modulo $2L\mathbb{Z}^b$, and the associated Spin^c label is $[2L\mathfrak{s} + 2\,\mathrm{Bk}(h)]$. The comparison below uses only the zero-decoration orbit and the specified plumbing identification with the GM relative label; it requires a transported Spin-induced origin before calling that relative label zero.

Let $a_{10} = (\mathfrak{s}_0, h = 0)$ be this zero knot label. In the source's zero-framed knot convention, $\mathfrak{s}_0(\text{meridian}) = 0$ and $\mathfrak{s}_0(\text{longitude}) = 1/2$ [2, Sections 2.1 and 4.1]. Its exponent grade is

$$r_{10} = (1/2, 0) = \epsilon_0/2,$$

so the separate external shift is $\ell(a_{10}) = 0$. With

$$g_{10} = I, \qquad g_{01} = S, \qquad g_{11} = T_-,$$

put $a_\epsilon = g_\epsilon a_{10}$. Here $g_\epsilon a_{10}$ denotes mapping-cylinder transport of the boundary-parameterized Spin structure, cohomology decoration, and chosen framing, with the left convention of Section 2. It transports the affine origin as well as the zero cohomology class. The exact rational lifts of the boundary grades are

(6)

ϵ	g_ϵ	r_ϵ	$\ell(a_\epsilon)$
$(1,0)$	I	$(1/2,0)$	$(0,0)$
$(0,1)$	S	$(0,1/2)$	$(0,0)$
$(1,1)$	T_-	$(1/2,1/2)$	$(0,0)$.

Hence

$$r_\epsilon = \epsilon/2 + \ell(a_\epsilon), \qquad \ell(g_\epsilon a_{10}) = g_\epsilon \ell(a_{10}) = 0.$$

Writing $a_\epsilon = (\mathfrak{s}_\epsilon, h_\epsilon)$, transport gives $h_\epsilon = 0$, and the ordinary Bockstein satisfies $\mathrm{Bk}(h_\epsilon) = 0$ relative to the transported Spin-induced affine basepoint. The Bockstein is applied to h_ϵ, not to the combined pair $(\mathfrak{s}_\epsilon, h_\epsilon)$. The Newton–Hopf model receives three externally indexed copies; it does not acquire an intrinsic topological label.

3.2. **The strict Newton–Hopf model and its spectral coordinate.** The Newton–Hopf state module used here consists of compatible triples

$$\mathfrak{G}_{\mathrm{N}}^T = \{(u, T_{\mathrm{pair}}u, S_{\mathrm{pair}}^+u) : u \in \mathscr{C}_W^T\}.$$

The superscript T records the maximal Newton pullback. The map T_{pair} is the Newton change of frame, not a mapping-class Dehn twist. The third coordinate retains its boundary orientation and its one-dimensional normalization line. Appendix A defines this model and proves the required Newton pullback and line-preserving Hopf identifications.

At an odd prime $p \geq 5$, the Newton–Hopf physical monomial nodes are

$$\rho_{i,p} = \frac{2 - q_p^{i+1} - q_p^{-i-1}}{\lambda_p}, \qquad q_p = -A_p.$$

Together with the primitive monomial cap, these nodes force the external spectral substitution

$$\mathsf{v}_X = \frac{2 - X - X^{-1}}{1 - q}, \qquad \mathsf{v}_X\big|_{X=q} = 1 - q^{-1} = \lambda/A. \tag{7}$$

Set

$$\Omega_0 = \sum_{n \in \mathbb{Z}} Y^n (X^{1/2} - X^{-1/2}), \qquad \mathcal{B} = R_0[v].$$

For $P \in \mathcal{B}$, define

$$K_{\mathrm{N}}(P) = \big(T_{\mathrm{pair}}^{-1}P,\ P,\ S_{\mathrm{pair}}^+ T_{\mathrm{pair}}^{-1} P\big), \tag{8}$$

$$K_{\mathrm{GM}}(P)(m,n) = [Y^n X^m]\, \Omega_0 P(\mathsf{v}_X), \tag{9}$$

$$K_{\mathrm{GJ}}(P) = \Omega_0 P(\mathsf{v}_X), \tag{10}$$

where the external expressions first apply $A \mapsto -q$ to the coefficients of P, and the last expression is written with every Y to the left. Right multiplication by the displayed Laurent polynomial in X introduces no quantum-torus phase.

Proposition 3.1 (State-level algebraic comparison)**.** *The maps* (8)–(10) *are injective after the base changes* (4)*. Every GM row is globally well-behaved, and both external maps have algebraic inverses on their images.*

Proof. If P has degree d, then $P(\mathsf{v}_X)$ is a finite Laurent polynomial in X with coefficients in $\mathbb{Q}((q))$: each power of $(1-q)^{-1}$ is a bounded-below integral-power series. Multiplication by Ω_0 leaves finite electric support. Its lower q-exponent bound is independent of the magnetic integer n, which is precisely the global GM support condition.

If the leading term of P is $a_d v^d$, then the coefficient of $X^{d+1/2}$ in its external row is

$$(-1)^d (1-q)^{-d} \beta_{\mathrm{GM}}(a_d).$$

Lower powers cannot reach this electric exponent, so the displayed coefficient determines a_d. Subtracting the recovered leading term leaves a polynomial of smaller degree; descending induction therefore recovers every coefficient of P. Newton–Hopf injectivity is the content of the maximal Newton-pullback isomorphism (Appendix A). □

3.3. **A primitive and an independent non-vacuum state.** Take

$$P_0 = 1, \qquad P_1 = W_1(A, v) = \lambda v^2 - 4v - \lambda(1 - A^{-1})^2.$$

The external specialization simplifies exactly to

$$W_1(-q, \mathsf{v}_X) = \frac{X^2 + X^{-2} - q^2 - q^{-2}}{1-q}. \tag{11}$$

The primitive GM row is

$$K_{\mathrm{GM}}(P_0)(m,n) = \delta_{m,1/2} - \delta_{m,-1/2}. \tag{12}$$

The non-vacuum row is independent of n and equals

$$\frac{1}{1-q} \begin{cases} 1, & m = 5/2, \\ -1, & m = 3/2, \\ -(q^2+q^{-2}), & m = 1/2, \\ q^2+q^{-2}, & m = -1/2, \\ 1, & m = -3/2, \\ -1, & m = -5/2, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Its Newton–Hopf triple is (w_1, W_1, f_1), and the primitive cap kills it because (11) vanishes at $X = q$. If an external row is $aK(P_0) + bK(P_1)$, then, for any magnetic index n,

$$b = (1-q)c_{5/2,n}, \qquad a = c_{1/2,n} + (q^2+q^{-2})c_{5/2,n}. \tag{14}$$

Thus the two uncontracted states are independent and the comparison is not a vacuum-line construction.

For every $\epsilon \in P$, let $\mathcal{B}_\epsilon$ be an external copy of $\mathcal{B}$, and define the target rows

$$K_{\mathrm{GM},\epsilon} = \rho^{\mathrm{GM}}_{g_\epsilon} K_{\mathrm{GM}}, \qquad K_{\mathrm{GJ},\epsilon} = \rho^{\mathrm{GJ}}_{g_\epsilon} K_{\mathrm{GJ}}. \tag{15}$$

These maps transport the external states into the sectors of (6). Equation (15) is not a common modular action on the Newton–Hopf model.

3.4. **The exact algebraic maximality claim.** We retain the two distinguished seeds

$$\Sigma = \{P_0, P_1\}.$$

The generation rule closes this set under finite R_0-linear combinations and the declared spectral endomorphism v. It does not say that $R_0[v]$ is finitely generated merely as an R_0-module.

Definition 3.2 (The polynomial comparison class over R_0)**.** We write $\mathbf{SPol}_{R_0}$ for the class of algebraic comparisons with the following properties: a primitive is distinguished; finite Newton–Hopf monomial coordinates have coefficients in R_0; multiplication by v is intertwined with multiplication by v_X; and external rows have finite electric support in the base chart before sector transport or completion. The primitive-generated part consists of the finite R_0-linear combinations of iterates of v on that primitive. All maximality assertions concern these vectors and their concrete target coordinates; no morphisms between different comparisons are needed.

Proposition 3.3 (Maximality in the polynomial comparison class)**.** *The family $(\mathcal{B}_\epsilon)_{\epsilon\in P}$, with $\mathcal{B}_\epsilon \cong R_0[v]$, is the full primitive-generated comparison in $\mathbf{SPol}_{R_0}$.*

Proof. Repeated application of v to the primitive gives every polynomial in $R_0[v]$. Conversely, every coefficient-origin finite monomial coordinate is such a polynomial, and spectral intertwining forces the external row (9)–(10). No additional primitive-generated finite vector remains. □

The qualifier *coefficient-origin* means that all finite coefficients come from the specified common ring. A target-dependent vector such as $\Phi_4(A)^{-1}\mathbf{1}$ may be a legitimate Newton–Hopf vector but is not generated over the common source ring (3). We make no absolute maximality claim after arbitrary localization or after adding noncyclic target-specific generators. The later no-extension theorem is nevertheless hereditary under allowed finite seed enlargements, because its witness already lies in the scalar orbit of the primitive.

4. The completed comparison problem

The polynomial maps identify their images algebraically. A completed comparison would also have to recover a state continuously from any one of its coordinates. We now specify the three topologies for this question. The external choices are made on proper coefficient subspaces; neither [4] nor [2] specifies them on its full state module.

4.1. **The strict root topology.** The monomial coordinate of the Newton–Hopf triple is $\mathscr{C}_v^{\mathrm{base}}$ from Appendix A. It is a closed submodule of a uniform formal λ-adic c_0 space times a product of finite-rank root modules, each with the discrete topology. Here c_0 means that, for each power of λ, all but finitely many coefficients are divisible by that power. A finite polynomial is first reduced in each physical quotient. The completion is formed from common finite approximants and is not defined by evaluating arbitrary formal series at roots.

In particular, a sequence of finite polynomials tends to zero if its formal coefficients have λ-order tending uniformly to infinity and its coordinate at each fixed prime is eventually zero. The maximal Newton pullback and the compatible-triple chart are homeomorphisms, so the same criterion applies to the triples. Denote this complete triple space by $\widehat{\mathscr{W}_{\mathrm{N}}} = \mathfrak{G}_{\mathrm{N}}^{T}$.

The selected scalar $A_p = e^{\pi i/p}$ has order $2p$, with $q_p = -A_p$ and $\lambda_p = 1 + A_p$. Thus $\Phi_{2p}(A_p) = 0$. In particular this A_p is not the order-$4p$ bracket root.

4.2. **Uniform external coefficient charts.** Let $\mathscr{W}_{10}$ be the space of functions $b : D_{(1,0)} \to \mathbb{Q}((q))$ for which

$$\nu(b) := \inf_{z\in D_{(1,0)}} \nu_q(b(z)) > -\infty, \qquad \nu(0) = +\infty. \tag{16}$$

Here ν_q is the lowest exponent of a nonzero Laurent series, with $\nu_q(0) = +\infty$. The sets $\{b : \nu(b) \geq r\}$ form a neighborhood basis at zero. Convergence in this topology controls every coefficient at the same valuation order.

Lemma 4.1 (Complete coefficient charts)**.** *The space $\mathscr{W}_{10}$ is complete and Hausdorff, and every element is a globally well-behaved GM function. The analogous space $\mathscr{W}_{10}^{\mathbb{C}}$ with values in $\mathbb{C}((t))$, normalized by $\nu_q(t) = 1/4$, is also complete and Hausdorff. The coefficient embedding $\mathscr{W}_{10} \to \mathscr{W}_{10}^{\mathbb{C}}$ is isometric.*

Proof. Let (b_α) be Cauchy. Beyond some index it differs uniformly from one fixed member by valuation at least zero; hence its tail has a common lower bound. For each z, completeness of the Laurent-series field gives a limit $b(z)$ with the same bound. If $\nu(b_\alpha - b_\beta) \geq r$ for all sufficiently large indices, passing to the pointwise limit gives $\nu(b_\alpha - b) \geq r$. Thus convergence is uniform and the limit belongs to the space. Uniqueness follows coefficientwise. Integral exponents and the uniform bound verify GM's global support condition. The argument is unchanged for quarter-integral exponents, and the chosen embedding preserves valuations. □

Its ordered GJ realization is

$$Jb = \sum_{(m,n)\in D_{(1,0)}} b(m,n) Y^n X^m, \qquad b \in \mathscr{W}_{10}^{\mathbb{C}},$$

with the topology transported by J. For $g_{10} = I, g_{01} = S, g_{11} = T_-$, define the other charts by

$$\mathscr{W}_\epsilon = \rho_{g_\epsilon}^{\mathrm{GM}} \mathscr{W}_{10}, \qquad \mathscr{W}_\epsilon^{\mathrm{ord}} = \rho_{g_\epsilon}^{\mathrm{GJ}} J \mathscr{W}_{10}^{\mathbb{C}},$$

again transporting topology. A Gaussian coefficient in the last chart may be unbounded when expressed in the original coordinates. It is part of the chart map; there is no additional ambient uniform norm after that transport. The polynomial rows lie in these charts, and the trefoil series below converges there by a uniform tail estimate.

4.3. **Allowed continuous comparisons.** We use the local name **CoeffGraph** for the following class of solutions to an extension problem. Its comparison arrows are the coordinate maps to the three targets. A *cotest* is our term for an opposite-boundary test vector on the domain of a specified partial pairing. A *growth-dual* cotest has separately specified coefficient-growth conditions that make that pairing meaningful; no such domain is added by the state comparison itself. A *gauge* below is a permitted change of state coordinates. Positive and opposite-boundary states refer to boundary orientation, not positivity of coefficients. The extension problem requires the following conditions.

Definition 4.2 (The completed comparison problem)**.** A member of **CoeffGraph** would extend the three-sector algebraic comparison subject to these requirements:

(1) The common scalar ring is discrete R_0, with exactly $R_0 \to \mathbb{Q}((q)) \to \mathbb{C}((t))$, $A \mapsto -q$, $t^4 = q$.
(2) The finite seeds are P_0, P_1 in every sector. Closure under finite R_0-linear combinations and multiplication by v gives $\mathcal{B}_\epsilon = R_0[v]$. No new localization is part of this rule.
(3) The common fiber is the simultaneous relation closure in the three specified target topologies and must be complete and Hausdorff. Every coordinate projection must be a continuous injection whose inverse on its image is continuous.
(4) The sector charts, left action convention and transported zero labels are those already specified. Further labels require their own affine data.
(5) Allowed diagonal gauges are state-independent R_0-linear homeomorphisms with homeomorphic inverses, preserving the primitive normalization. Allowed line corrections tensor with a fixed invertible one-dimensional topological line.

(6) Opposite-boundary vectors, pairings and growth-dual cotests may be added only on their separately specified partial domains after a positive common state module exists. A contraction does not change equality or convergence of positive states.

For each ϵ, set

$$\Gamma_\epsilon = \{(K_{\mathrm{N}}(P), K_{\mathrm{GM},\epsilon}(P), K_{\mathrm{GJ},\epsilon}(P)) : P \in R_0[v]\}, \qquad \mathcal{R}_\epsilon = \overline{\Gamma_\epsilon} \subset \widehat{\mathscr{W}_{\mathrm{N}}} \times \mathscr{W}_\epsilon \times \mathscr{W}_\epsilon^{\mathrm{ord}}. \tag{17}$$

This closure is always a complete Hausdorff submodule, because the ambient product is complete and Hausdorff. It is the minimal closed relation containing the algebraic comparison. The next section tests the remaining requirement: whether its coordinate projections are injective.

5. A sequence with incompatible limits

The obstruction already occurs among scalar multiples of the primitive. Cyclotomic factors make each fixed discrete-root coordinate eventually zero. After external substitution those same factors have constant coefficient one, so a truncated inverse makes their product approach the unit in the Laurent-series topology. Both operations remain in R_0.

Enumerate all odd primes at least five as $p_1 < p_2 < \cdots$, and put

$$D_N(A) = \lambda^N \prod_{i=1}^{N} \Phi_{2p_i}(A), \qquad d_N(q) = D_N(-q) = \prod_{i=1}^{N} (1 - q^{p_i}). \tag{18}$$

The last identity uses $\Phi_{2p}(-q) = \Phi_p(q) = (1-q^p)/(1-q)$. Since $d_N(0) = 1$, there is a unique polynomial $U_N \in \mathbb{Z}[q]$ of degree less than N with $d_N U_N \equiv 1 \pmod{q^N}$. Define

$$E_N(A) = D_N(A) U_N(-A) \in R_0. \tag{19}$$

For example, when $N = 1$ and $p_1 = 5$, one has $U_1 = 1$ and $E_1(-q) = 1 - q^5$. The factor already kills the $p = 5$ root coordinate, is divisible by λ formally, and agrees with one to positive q-order. The construction repeats all three features at arbitrarily high order and over every fixed prime.

Lemma 5.1 (Incompatible primitive limits)**.** *For every ϵ,*

$$K_{\mathrm{N}}(E_N) \longrightarrow 0, \tag{20}$$

$$K_{\mathrm{GM},\epsilon}(E_N) \longrightarrow K_{\mathrm{GM},\epsilon}(1), \qquad K_{\mathrm{GJ},\epsilon}(E_N) \longrightarrow K_{\mathrm{GJ},\epsilon}(1). \tag{21}$$

The complementary sequence $1 - E_N$ tends to the primitive in the Newton–Hopf coordinate and to zero in both external coordinates.

Proof. Every formal coefficient of E_N is divisible by λ^N. At the fixed root A_{p_i}, its Φ_{2p_i} factor makes the entire root state zero for all $N \geq i$. These are precisely the strict convergence conditions from Appendix A. The bicontinuous triple chart gives (20).

In the base external chart, $E_N(-q) = 1 + q^N H_N(q)$ for a polynomial $H_N \in \mathbb{Z}[q]$. Since the primitive row has valuation zero uniformly in (m, n), its coefficient error has valuation at least N. Both base limits follow; transport by the fixed chart homeomorphisms gives all sectors. Subtracting from the primitive proves the complementary limits. □

The cyclotomic factors are necessary to this argument. λ^N alone is formally small but does not eventually vanish in any fixed discrete root coordinate.

Theorem 5.2 (Noninjective completed comparison)**.** *For every sector, the complete Hausdorff relation $\mathcal{R}_\epsilon$ of (17) contains the two nonzero points*

$$(0, K_{\mathrm{GM},\epsilon}(1), K_{\mathrm{GJ},\epsilon}(1)), \qquad (K_{\mathrm{N}}(1), 0, 0). \tag{22}$$

Every coordinate projection therefore has a nonzero kernel. No member of **CoeffGraph** *exists. More generally, no R_0-linear common state module containing the same primitive scalar orbit*

has comparison embeddings into these targets that are continuous and inverse-continuous on their images.

Proof. The first point in (22) is the limit of the graph points parameterized by E_N; the second comes from $1 - E_N$. Their indicated coordinates are nonzero by the algebraic injectivity theorem. The first is in the kernel of the Newton–Hopf projection, and the second is in the kernel of either external projection. Hence the relation is not a comparison graph.

For the stronger formulation, let $\mathbf{1}$ be the common primitive. External inverse continuity would imply $E_N\mathbf{1} \to \mathbf{1}$. Continuity of its Newton–Hopf map would then imply $K_{\mathrm{N}}(E_N) \to K_{\mathrm{N}}(1)$. The lemma makes the same sequence converge to zero. Hausdorffness and the nonzero primitive give a contradiction. This argument uses only the primitive scalar orbit, not a claim that every common state is cyclic. □

Corollary 5.3 (Stability of the obstruction)**.** *Adding finite seeds, sector copies, opposite-boundary vectors or cotests cannot remove this obstruction while R_0-linearity, the primitive orbit and the target topologies are retained. The allowed homeomorphic gauges and fixed invertible topological lines also preserve it.*

Proof. The same scalar sequence remains a subdiagram of each enlargement. A permitted gauge commutes with its scalars and carries its two limits to the corresponding two limits under a homeomorphism. Tensoring with an invertible fixed line is reversible and has the same property. Neither operation can identify a nonzero primitive with zero. □

Theorem 1.2 is Theorem 5.2 and Corollary 5.3 together with the completeness construction. The topology exists; injectivity of the completed coordinate maps is the first failed condition. A topology finer than the specified external valuation topology, or a category of noninjective correspondences, is a different comparison problem. The algebraic theorem places no obstruction on those different choices.

6. The trefoil and the framing scalar

The scalar sequence applies to the trefoil boundary state through finite polynomial approximants. Their degree grows with the truncation, so the formal estimate must control all coefficients at once; the external tail estimate must also be uniform in the magnetic coordinate. The proof below establishes both bounds for the same sequence of states. Taking one magnetic coefficient of a hypothetical common state would instead require a comparison operation that has not been constructed.

6.1. Polynomial approximants of the zero-framed state.

The right-handed trefoil (RHT) series is

$$F_{\mathrm{RHT}}(X,q) = \frac{q}{2}\sum_{r\geq 1}\epsilon_r(X^{r/2} - X^{-r/2})q^{(r^2-1)/24}, \tag{23}$$

where $\epsilon_r = -1$ for $r \equiv 1, 11 \pmod{12}$, $\epsilon_r = 1$ for $r \equiv 5, 7 \pmod{12}$, and zero otherwise [4, Section 7.3]. Its zero-label ordered boundary state is

$$|\mathrm{RHT}, 0\rangle = \sum_{n\in\mathbb{Z}} Y^n F_{\mathrm{RHT}}(X,q), \qquad b_{\mathrm{RHT},0}(m,n) = [X^m]F_{\mathrm{RHT}}(X,q). \tag{24}$$

The GJ plumbed-knot formula supplies this ordered placement [2, Section 4.3]. The coefficient rows in (24) are equal after the chosen coefficient embedding.

Put

$$J_{\mathrm{tr}} = \{j \geq 0 : j \equiv 0, 2, 3, 5 \pmod{6}\}, \qquad N_j = j(j+1)/6.$$

The exponent N_j is integral on this set and tends to infinity. Define the integral character polynomial C_j by

$$C_j(X+X^{-1}) = \frac{X^{j+1/2} - X^{-j-1/2}}{X^{1/2} - X^{-1/2}} = \sum_{k=-j}^{j} X^k.$$

The last expression is symmetric and hence is a polynomial in $X+X^{-1}$ with integer coefficients. Its corresponding common polynomial is

$$y_j = \epsilon_{2j+1}(-A)^{N_j} C_j(2-\lambda v) \in R_0[v]. \tag{25}$$

Since $2-\lambda \mathsf{v}_X = X + X^{-1}$ under $A = -q$,

$$K_{\mathrm{GJ}}(y_j) = \epsilon_{2j+1} q^{N_j} \sum_{n\in\mathbb{Z}} Y^n (X^{j+1/2} - X^{-j-1/2}),$$

with the same GM coefficient row. Thus

$$\sum_{j\in J_{\mathrm{tr}}} K_{\mathrm{GJ}}(y_j) = 2q^{-1}|\mathrm{RHT}, 0\rangle. \tag{26}$$

The common-ring scalar is $2q^{-1} = \beta(-2A^{-1})$. It is nonzero but is not an invertible gauge over R_0.

Proposition 6.1 (A trefoil point in the kernel of the root projection)**.** *List J_{tr} increasingly as $j_1, j_2, \ldots,$ and let $s_N = \sum_{a=1}^{N} y_{j_a}$. Then*

$$K_{\mathrm{N}}(E_N s_N) \longrightarrow 0, \tag{27}$$

$$K_{\mathrm{GM}}(E_N s_N) \longrightarrow 2q^{-1} b_{\mathrm{RHT},0}, \qquad K_{\mathrm{GJ}}(E_N s_N) \longrightarrow 2q^{-1}|\mathrm{RHT}, 0\rangle. \tag{28}$$

In particular the closed relation contains this nonzero trefoil point with zero Newton–Hopf coordinate. The same statements hold in every transported sector chart.

Proof. All monomial coefficients of s_N are λ-integral. After multiplication by E_N, each has order at least N, uniformly over the growing polynomial degree. The first N physical prime coordinates vanish identically. The strict convergence criterion proves (27).

For either base external coefficient chart, let S_{RHT} denote the corresponding right side of (26). Each term has minimum valuation N_j, uniformly over the magnetic coordinate. Consequently

$$\nu(K(s_N)) \geq 0, \qquad \nu(S_{\mathrm{RHT}} - K(s_N)) \geq N_{j_{N+1}} \longrightarrow \infty.$$

The error has two sources: truncating the trefoil series and multiplying its truncation by the interpolator. Since $\nu(\beta(E_N) - 1) \geq N$, the nonarchimedean inequality gives

$$\nu(\beta(E_N)K(s_N) - S_{\mathrm{RHT}}) \geq \min\{N, N_{j_{N+1}}\} \longrightarrow \infty.$$

This proves both external limits. The first supported term has nonzero coefficient, so the limit is nonzero. Transport homeomorphisms preserve these limits in the other sectors. □

The proposition also directly excludes a common lift of the zero-framed state under the comparison requirements. If such a lift existed, R_0-linearity would give the lift of its multiple $2q^{-1}$. External inverse continuity would make the common approximants converge to that lift, whereas continuity and injectivity of the Newton–Hopf map would force it to be zero. The nonzero external limit is a contradiction.

6.2. **The negative-definite plumbing.** The rooted plumbing with ordered vertices $(-2,-3,-1,\text{root})$ and matrix

$$M_{-7}=\begin{pmatrix}-2&0&1&0\\0&-3&1&0\\1&1&-1&1\\0&0&1&-7\end{pmatrix} \tag{29}$$

has leading principal minors $-2,6,-1,1$, so it is negative definite. Deleting the distinguished root gives the trefoil plumbing in S^3 of [4, Section 5.1]. The relative self-conjugate label is represented by $a_0=(1,1,-1,1)^T$. We record the normalization because it is needed for the comparison with the zero-framed series.

Let M_{-6} be (29) with the last diagonal entry changed to -6. For $e_3=(0,0,1,0)^T$ and $k=(3,2,6,1)^T$,

$$M_{-6}k=0,\quad \operatorname{rank}M_{-6}=3,\quad M_{-6}e_3=M_{-7}e_3=a_0,$$
$$e_3^TM_{-6}e_3=a_0^TM_{-7}^{-1}a_0=-1.$$

The leading three principal minors of M_{-6} are $-2,6,-1$, so its inertia is three negative and one zero direction. The signatures are therefore -3 and -4. In the distinguished-vertex formula and degenerate normalization of [4, Sections 6.1 and 6.6–6.7], changing the root weight from -6 to -7 contributes $q^{n^2+1/4}X^{-n}$. The signature normalization contributes $q^{3(-4+3)/4}=q^{-3/4}$; the displayed quadratic normalization is unchanged. Hence the exact coefficient formula is

$$\widehat{Z}_{0,\Gamma_{2,3}^{-7}}(X,n,q)=q^{n^2-1/2}X^{-n}F_{\mathrm{RHT}}(X,q). \tag{30}$$

It has a global lower exponent bound $1/2$, so it is a GM state, not only a formula for each fixed n.

Proposition 6.2 (The target modular orbit)**.** *If $f_m=[X^m]F_{\mathrm{RHT}}$, the ordered placement of the -7-framed state is*

$$Z_{-7}^{\mathrm{GJ}}=\sum_{m,n}q^{n^2-1/2}f_mY^nX^{m-n},$$

and the common left convention gives

$$\rho_{T_+}^{\mathrm{GJ}}Z_{-7}^{\mathrm{GJ}}=q^{-1/2}|\mathrm{RHT},0\rangle. \tag{31}$$

Proof. For common T_+, equation (2) says $\rho_{T_+}(Y^nX^{m-n})=q^{-n^2}Y^nX^m$. It cancels the magnetic Gaussian in every term, leaving exactly $q^{-1/2}\sum_{m,n}f_mY^nX^m$. □

The scalar in (31) is invertible in $\mathbb{C}((t))$, where $q^{-1/2}=t^{-2}$. It is not in the image of R_0. This target-side equation locates the forbidden zero-framed lift on the named plumbing's modular orbit; it does not transfer a lift through a common T_+ or replace the scalar by a common framing line. On the GM side, the framing change in (30) retains its magnetic Gaussian and is not the inverse-precomposition sector transport.

6.3. **Why the unmodified Wilson series does not converge strictly.** The character polynomials C_j represent the Wilson-line terms; we refer to the corresponding series of states y_j as the Wilson series. The multiplying sequence E_N is doing real work. The literal terms y_j satisfy $y_j \bmod \lambda=\epsilon_{2j+1}(2j+1)\neq 0$. Even the natural four-term blocks fail to tend to zero.

Proposition 6.3 (Wilson terms and four-term blocks)**.** *For $r\geq 0$,*

$$B_r=y_{6r}+y_{6r+2}+y_{6r+3}+y_{6r+5}=36(2r+1)(1+v)\lambda+O(\lambda^2).$$

Thus neither the individual supported terms nor these blocks tend to zero in the formal strict coordinate.

Proof. The character formula gives $C_j(2) = 2j + 1$ and $C'_j(2) = j(j + 1)(2j + 1)/6$, the latter by summing the squares of the integer weights in $\sum_{k=-j}^{j} X^k$. Since $-A = 1 - \lambda$,

$$y_j = \epsilon_{2j+1}(2j + 1)[1 - N_j(1 + v)\lambda] + O(\lambda^2).$$

The signs in a block are $-, +, +, -$. Their constant terms sum to zero, and substituting its four values of j gives $36(2r + 1)(1 + v)$ for the coefficient of λ. This coefficient is nonzero for every r. □

These identities explain the distinction between external convergence of the knot series and strict convergence of its polynomial representatives. They are not needed for the uniform obstruction theorem, which uses the entire primitive scalar orbit instead of an ordering of the Wilson series.

Corollary 6.4 (Restriction of any higher-genus comparison)**.** *A boundary theory in higher genus cannot have a genus-one restriction with the same primitive scalar orbit, the same strict and external coefficient topologies, and the same injective bicontinuous comparison maps. This conclusion is conditional on that precise restriction.*

Proof. Such a restriction would provide the common module excluded by Theorem 5.2. □

The polynomial comparison, the separate target transports and the complete closed relation all survive. The obstruction concerns making that relation a common completed state module. It consequently supplies no further common cap, pairing, gluing or framing structure.

Appendix A. The strict Newton–Hopf model

This appendix gives the genus-one Newton-pullback, Hopf-image and cap arguments used in the comparison, following the companion paper on integral root fibers and closed factorizations [10, Sections 6–7]. The quotient-basis and Hopf calculations below are given directly. The finite saturated lattice is identified using the transported closed-torus orthogonal basis in [1, Definition 2.1 and Theorem 2.2, with $c = 0$] and its global integral comparison with the square-node frame in [10, Theorem 4.7 and equation (2)]. The lowered Hopf pairing is the even-color restriction of [7, Proposition 3.13]. The root variables and the normalization line are retained explicitly.

A.1. **Root modules and finite Newton coordinates.** Let $A_p = e^{\pi i/p}$, $q_p = -A_p$, $\lambda_p = 1 + A_p$, $\mathcal{O}_p = \mathbb{Z}[A_p]$, and $d_p = (p-1)/2$, for odd primes $p \geq 5$. Then q_p is a primitive p-th root. The selected variable A_p equals the square of the bracket root $\widetilde{A}_p = e^{\pi i/(2p)}$. This transports Marché's bracket quantum integer to

$$[r]_p = \frac{A_p^r - A_p^{-r}}{A_p - A_p^{-1}}.$$

For the invariant torus lattice, transport the Gilmer–Masbaum parameter by $q_{\mathrm{GM}} = q_p = -A_p$. In this one parameter name, GM denotes Gilmer–Masbaum; on the comparison maps in the body it denotes Gukov–Manolescu. The change of variable is made before identifying the polynomial and integral lattices.

Define the finite monomial quotient

$$V_p = \mathcal{O}_p[v]/(Q_{d_p,p}(v)), \qquad Q_{d_p,p}(v) = \prod_{i=0}^{d_p-1}(v - \rho_{i,p}), \quad \rho_{i,p} = \frac{2 - q_p^{i+1} - q_p^{-i-1}}{\lambda_p}. \tag{32}$$

The roots lie in $\mathcal{O}_p$, so reduction is over this ring. Put, over R_0,

$$r_i(A) = \left(\frac{A^{2i+1} - A^{-2i-1}}{\lambda}\right)^2, \qquad W_n(A, v) = \prod_{i=0}^{n-1}(\lambda v^2 - 4v - \lambda r_i(A)).$$

Each r_i is a Laurent polynomial because its numerator vanishes at $A=-1$. The finite Newton states are represented by $W_n(A_p,v)$ for $0\le n<d_p$. In the closed skein variable $z=\lambda_p v-2$, they are

$$w_{n,p}=\lambda_p^{-n}c_{n,p},\qquad c_{n,p}=\prod_{i<n}\bigl(z^2-(A_p^{2i+1}+A_p^{-2i-1})^2\bigr).$$

These formulas are equal term by term. Thus $c_{n,p}$ is a Newton interpolation product in z^2 over $\mathcal{O}_p$, whereas $w_{n,p}$ is its rescaled lattice vector after $z=\lambda_p v-2$. Polynomial triangularity and the integral saturated-lattice comparison are separate steps in the proof below.

We index the closed even core basis so that e_a has color $2a$. The matrix H_p below is the Hopf-link pairing matrix before its normalizing line factor is inserted; this is the meaning of *lowered*.

Lemma A.1 (The finite Newton and Hopf frames)**.** *The states $w_{n,p}$, $0\le n<d_p$, form a basis of V_p and, under the closed-skein identification, of the invariant saturated lattice $\mathcal{S}_p^+$. In the even core basis e_a, the lowered Hopf matrix is*

$$(H_p)_{ab}=[(2a+1)(2b+1)]_p,\qquad H_p^2=\delta_p I,\qquad \delta_p=-\frac{p}{(A_p-A_p^{-1})^2}.$$

If $C_p e_n=c_{n,p}$ and $\Lambda_p=\operatorname{diag}(1,\lambda_p,\dots,\lambda_p^{d_p-1})$, then

$$H_pC_p=L_p\Lambda_p^2,\qquad C_p,L_p\in GL_{d_p}(\mathcal{O}_p).$$

Proof. The closed even core polynomials are monic in z^2; consequently C_p is unitriangular. The transported orthogonal basis of Gilmer–Masbaum uses its linear-node Newton numerators divided by factorials whose ideals are (λ_p^n). The displayed square-node numerators have twice their degree and define a different frame. The global integral comparison in [10, Theorem 4.7 and equation (2)] places these normalized columns in the Gilmer–Masbaum lattice. Its determinant is a unit, with each factorial divided by λ_p^n contributing a unit, so the adjugate gives an integral inverse over the full coefficient ring. This identifies the saturated lattice and its w-basis.

The basis property in the monomial quotient can also be seen directly in (32). Write $z_a=A_p^{2a+1}+A_p^{-2a-1}$. The values $(z_a+2)/\lambda_p$ are precisely the nodes $\rho_{i,p}$, in a permuted order, since odd residues give one representative of each nonzero residue modulo sign. Evaluation of W_n at these nodes is triangular. Dividing its determinant by the monomial Vandermonde leaves $\prod_{i<a}(z_a+z_i)$. Each factor is a unit: it factors as

$$z_a+z_i=(A_p^{a+i+1}+A_p^{-a-i-1})(A_p^{a-i}+A_p^{-a+i}),$$

and the nonzero indices here are less than p; the corresponding cyclotomic plus-factors are units. Thus the transition to the monomial basis is unimodular.

The classical Hopf formula gives H_p, with the sign equal to one on even colors. Equivalently, its row a is evaluation at z_a multiplied by $[2a+1]_p$. It follows that

$$(H_pC_p)_{an}=[2a+1]_p\prod_{i<n}(z_a^2-z_i^2).$$

For $a<n$ this is zero. For $a\ge n$, each difference factors as

$$z_a^2-z_i^2=(A_p^{2(a-i)}-A_p^{-2(a-i)})(A_p^{2(a+i+1)}-A_p^{-2(a+i+1)}).$$

Each factor is λ_p times a unit. After removing λ_p^{2n}, the resulting lower triangular matrix has unit diagonal; this is L_p. Finally, the elementary odd sine sum

$$\sum_{b=0}^{d_p-1}\sin\frac{(2a+1)(2b+1)\pi}{p}\sin\frac{(2c+1)(2b+1)\pi}{p}=\frac{p}{4}\delta_{ac}$$

follows from the cosine product identity. Dividing by $\sin^2(\pi/p)$ proves the square formula for H_p. □

Introduce a one-dimensional normalization line generated by $\mathbf{d}_p$, with $\mathbf{d}_p^2 = \delta_p$, and let $\mathbf{d}_p^\vee = \mathbf{d}_p^{-1}$. Set

$$g_{n,p} = \lambda_p^n L_p e_n, \qquad f_{n,p} = \mathbf{d}_p^\vee g_{n,p}.$$

Multiplying the Smith factorization by Λ_p^{-1} proves

$$H_p w_{n,p} = g_{n,p}, \qquad S_p^+ w_{n,p} = f_{n,p}, \qquad S_p^+ = \mathbf{d}_p^\vee H_p. \tag{33}$$

Both maps preserve the coefficient vector in the displayed frames. The inverse cylinder has the opposite boundary orientation and sends $f_{n,p}$ to $w_{n,p}$ after the line contraction. This follows from $(\mathbf{d}_p^\vee)^2 H_p^2 = 1$.

A.2. **Common approximants.** Let S_{ev} be the multiplicative set generated by the cyclotomic factors $\Phi_m(A)$ that are units at A_p for every sufficiently large selected prime, excluding $\lambda = \Phi_2(A)$. Put

$$R^{\mathrm{ev}} = S_{\mathrm{ev}}^{-1} R_0, \qquad R_\lambda = \varprojlim_N R^{\mathrm{ev}}/(\lambda^N).$$

In particular $\Phi_4(A) = A^2 + 1$ is inverted and is congruent to two modulo λ; hence two is a unit in R_λ. This is a coefficient ring for the target formal chart, not a new common scalar ring for the comparison.

Let R^{all} be the fractions in R^{ev} whose reduced denominators are units at every selected root. For the Newton label n, let R_n^{act} consist of fractions whose reduced denominators are units at every prime with $n < d_p$. Only finitely many primes are excluded for a fixed n. Define

$$\mathcal{A}_W^{\mathrm{fin}} = \bigoplus_{n \geq 0} R_n^{\mathrm{act}} e_n, \qquad \mathcal{A}_v^{\mathrm{fin}} = R^{\mathrm{all}}[v].$$

For a finite Newton coordinate $a = (a_n)$, put

$$j_W(a) = \left((\mathrm{Taylor}_\lambda a_n)_n, \left(\sum_{n < d_p} \mathrm{Ev}_p(a_n) w_{n,p} \right)_p \right).$$

For a finite monomial polynomial, define j_v by its Taylor coefficient family and its quotient-first reductions in V_p. Set

$$\mathscr{C}_W = \overline{j_W(\mathcal{A}_W^{\mathrm{fin}})} \subset c_0(\mathbb{Z}_{\geq 0}, R_\lambda) \times \prod_p \mathcal{S}_p^+,$$

$$\mathscr{C}_v^{\mathrm{base}} = \overline{j_v(\mathcal{A}_v^{\mathrm{fin}})} \subset c_0(\mathbb{Z}_{\geq 0}, R_\lambda) \times \prod_p V_p.$$

The formal factor has the uniform λ-adic topology; each physical factor is discrete. Here c_0 means that outside a finite set all coefficients are divisible by any prescribed power of λ.

Lemma A.2 (Strict approximation criterion)**.** *These two completed modules are complete and Hausdorff. Membership in either closure means that for each N a single finite approximant agrees with the formal coordinate modulo λ^N, uniformly in the label, and agrees exactly with the physical coordinates at $p_1, \ldots, p_N$. In particular a sequence of finite states whose formal orders tend uniformly to infinity and whose fixed root coordinates are eventually zero tends to zero in the strict module.*

Proof. The formal c_0 module is complete: a uniform Cauchy family has a coordinatewise limit, the common error bound passes to that limit, and its tail is small at every fixed order. It is Hausdorff, as are the discrete physical modules and their product. The two modules are closed submodules of these complete Hausdorff products. The displayed conditions are precisely a cofinal neighborhood basis in the products, which proves the approximation and convergence assertions. □

No root evaluation of every element of R_λ is used here. For example, $\Phi_{2p}(A)$ is inverted in R^{ev} but vanishes at A_p, preventing an extension of that evaluation to the whole ring.

A.3. **The maximal Newton pullback.** Write $W_n = \sum_k t_{kn}(A)v^k$, and let T_{pair} act by this Newton matrix on formal coefficients and by the finite matrices in Lemma A.1 on physical coordinates.

Lemma A.3 (Formal Newton inverse)**.** *The support is $0 \le k \le 2n$, and on that support*

$$\mathrm{ord}_\lambda(t_{kn}) = |k-n|, \qquad t_{nn} \equiv (-4)^n \pmod{\lambda^2}.$$

The Newton matrix is a bicontinuous automorphism of the uniform formal product and of its c_0 submodule over R_λ.

Proof. Choose constant, linear and quadratic terms when expanding W_n. To obtain degree $k \le n$ at smallest order, choose k linear terms and $n-k$ constant terms; for $k \ge n$, choose $k-n$ quadratic and $2n-k$ linear terms. Their leading coefficients have the same sign, so they cannot cancel. All other choices raise the order by an even positive amount. At $k=n$ the leading term is $(-4v)^n$, and the next correction has order at least two. Normalize column n by the unit $(-4)^{-n}$. The matrix is $I+\mathsf{N}$, where N raises uniform order by at least one. The bound away from the diagonal also preserves c_0. The uniformly convergent Neumann series $\sum_{r\ge0}(-\mathsf{N})^r$ is the two-sided continuous inverse. □

Proposition A.4 (Maximal two-sided Newton coordinate)**.** *The module*

$$\mathscr{C}_W^T = \mathscr{C}_W \cap T_{\mathrm{pair}}^{-1}(\mathscr{C}_v^{\mathrm{base}})$$

is closed, complete and Hausdorff, and

$$T_{\mathrm{pair}} : \mathscr{C}_W^T \xrightarrow{\sim} \mathscr{C}_v^{\mathrm{base}}$$

is a homeomorphism. It is maximal among submodules of $\mathscr{C}_W$ whose paired image lies in the monomial module. Every finite polynomial in $R_0[v]$ therefore has a unique preimage.

Proof. The ambient formal map is an isomorphism by Lemma A.3; the physical maps are isomorphisms by Lemma A.1. Thus their product is a homeomorphism. Closedness and maximality follow from the pullback condition. It remains to show that the inverse of every target point belongs to $\mathscr{C}_W$. We first approximate inverse images of the finite target core, then use its density and the ambient homeomorphism. This proves surjectivity on the pullback without requiring invariance of the entire source module.

For a finite target monomial cv^k, Newton interpolation gives the inverse coefficient in row n as

$$c\lambda^{-n}(g^k)[r_0,\dots,r_n], \qquad g(y) = 2\lambda^{-1}\bigl(1-\sqrt{1+\lambda^2 y/4}\bigr),$$

where brackets denote divided differences and the square root has constant term one. The selected values are $g(r_i) = (2+A^{2i+1}+A^{-2i-1})/\lambda$, so the divided differences are fractions in the original variable. The node-difference factorization uses only nonzero quantum integers of indices below p when row n is active at p. After the displayed powers of λ are canceled, their denominators are units at each such incidence. Thus finite truncations are in $\mathcal{A}_W^{\mathrm{fin}}$. The formal inverse is in c_0 by the preceding lemma. At a fixed prime the inverse finite matrix uses exactly the active rows, so truncations become exact once those rows are included. The strict

approximation criterion puts the inverse finite monomial in $\mathscr{C}_W$. These monomials span a dense target core. Applying the ambient inverse homeomorphism and then closedness of $\mathscr{C}_W$ proves the assertion for the full target. □

The pullback matters because a Newton coefficient only needs a denominator that is a unit at primes where its label is active. After the change to monomial coordinates, denominators must be units at every selected prime. The first condition need not imply the second. The proposition makes no assertion that the entire strict Newton module is invariant under that change.

A.4. **The compatible triple and its cap.** On formal coefficient copies, let S^{+}_{pair} preserve coefficients; on physical coordinates use (33). Its image is given the corresponding product topology. The coefficient maps are isomorphisms in each root frame and on the formal factor, so S^{+}_{pair} is a product homeomorphism onto its image, with the normalization line and boundary orientation retained. Define

$$\mathfrak{G}^{T}_{\mathrm{N}} = \{(u, T_{\mathrm{pair}}u, S^{+}_{\mathrm{pair}}u) : u \in \mathscr{C}^{T}_{W}\}.$$

It is complete and Hausdorff, and projection to its monomial coordinate is a homeomorphism by Proposition A.4. This is exactly the strict compatible-triple model used in the body. The third coordinate retains its normalization line and orientation; T_{pair} is a Newton change of frame, not a Dehn twist.

Proposition A.5 (Primitive and monomial caps)**.** *With $y_p = e_0^T$, $h_p = y_p H_p$ and $\widehat{y}_p = \mathbf{d}_p y_p$, one has*

$$h_p(w_{n,p}) = \widehat{y}_p(f_{n,p}) = \delta_{n0}.$$

The caps extend continuously to the strict images. In the monomial coordinate the cap on finite polynomials is

$$\varepsilon_v(P) = P(\lambda/A).$$

Formally it extends by its convergent coefficient series; physically it evaluates the already formed quotient at $v = \rho_{0,p} = \lambda_p/A_p$.

Proof. The zeroth row of $L_p\Lambda_p$ is e_0^T: in the Hopf row-evaluation formula every $c_{n,p}$ with $n > 0$ vanishes at the first Newton node. Equation (33) then gives both cap identities, with the line factors contracted in the second. On formal coordinates these maps select the zeroth Newton coefficient; on each physical coordinate they are maps of finite-rank discrete modules. Finite common approximants therefore map to the rank-one scalar closure of the Taylor/root values of R^{all}, and continuity extends the cap to the closures.

The row-zero inverse divided difference in the preceding proof is $g(r_0)^k = (\lambda/A)^k$. This proves the formula for every finite polynomial. Since λ/A has positive formal order, the series is convergent on c_0. The physical value $\rho_{0,p}$ is a root of $Q_{d_p,p}$, so evaluation factors through that quotient. The formal and physical maps agree on the common finite core and hence on its closure. □

In particular the compatible triples of the two seeds are $(w_0, 1, f_0)$ and (w_1, W_1, f_1). The first cap value is one; the second is zero. No infinite root-product duality or full strict modular action is required for these statements.

Appendix B. Support, pairings and modular phases

The main comparison uses uniformly supported coefficient states and a specified monomial modular action. This appendix proves the additional support criterion and explains which pairing and phase formulas are used. All comparisons to the external papers are to their cited versions. These calculations concern their separate state models and add no common pairing after the obstruction theorem.

B.1. **Uniform support for rooted plumbing states.** The rooted-plumbing formula of [4, Definition 6.1 and Section 6.1] is defined under a weak-negative hypothesis: if M is the plumbing matrix, then M^{-1} is negative definite on the span of the non-distinguished vertices of degree at least three. This makes the open expression meaningful for each fixed boundary variable. Definition 6.5 of that source imposes a global condition over all boundary fluxes. Passing from the fixed-boundary expression to a coefficient state therefore requires a bound that also controls the boundary direction.

To obtain a bound uniform over boundary fluxes, we must include the unintegrated boundary coordinate in the directions of unbounded support. Let v_0 be the distinguished degree-one vertex, let

$$B = \{v \neq v_0 : \deg(v) \geq 3\}, \qquad W = \operatorname{span}_{\mathbb{R}}(e_{v_0}, e_v : v \in B), \qquad Q = M^{-1}.$$

In the principal-value constant-term expansion, an internal vertex of degree zero, one, or two contributes from the finite sets

$$\{-2, 0, 2\}, \qquad \{-1, 1\}, \qquad \{0\},$$

respectively. Branching coordinates have two-sided infinite parity support, and the boundary coordinate is not integrated. Therefore the recession subspace of the full coefficient support is exactly W.

For $\ell = a + 2Mz$, the variable exponent is, up to the fixed front factor,

$$-\frac{1}{4}\ell^T Q \ell. \tag{34}$$

After the finitely many low-degree internal coordinates are fixed, the remaining points lie, up to finite central gaps, in affine lattices $\ell_{a,c} + \Lambda$, where $\Lambda = (2M\mathbb{Z}^s) \cap W$ spans W, with s the number of vertices of the plumbing graph. Moreover,

$$\frac{1}{4}\ell^T M^{-1} \ell = z^T M z + a^T z + \frac{1}{4} a^T M^{-1} a.$$

The first two terms are integral, so the finite-image condition modulo $\mathbb{Z}$ is automatic for each fixed label. Uniform lower boundedness remains.

Proposition B.1 (GM support criterion and its domain)**.** *On a nonempty affine component, the displayed pre-cancellation support is bounded below exactly when*

$$Q|_W \preceq 0 \tag{35}$$

and

$$r^T Q \ell_{a,c} = 0 \quad \text{for every } r \in \operatorname{rad}(Q|_W). \tag{36}$$

These conditions are sufficient for the actual state. In particular, the strict boundary-augmented condition

$$M^{-1}|_W < 0 \tag{37}$$

guarantees the uniform support condition for every relative label. Full negative definiteness implies (37), but it is not necessary. Ordinary weak negativity alone does not imply membership in the state space.

Proof. Because the matrix and recession space are rational, positive directions can be chosen rational and then multiplied into the full recession lattice. Its radical is rational as well. A positive lattice direction makes $\ell^T Q \ell$ tend to $+\infty$, so (34) is unbounded below. A radical direction violating (36) gives the same behavior linearly in one orientation. Under (35)–(36), the negative-definite complement contributes a bounded-above concave quadratic, while the radical contributes no growth. This proves the pre-cancellation criterion and the stated sufficient conditions. □

The proposition deliberately does not assert a matrix-only converse for an individual state: cancellation could remove pre-cancellation terms, and no uniform noncancellation theorem is imported or proved here. Full negative definiteness remains the sufficient range established in the cited source; the boundary-augmented criterion is the additional sufficient criterion proved here.

For instance,

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

is indefinite, but with the first degree-one vertex distinguished and no branching vertices one has $W = \mathbb{R}e_1$ and $M^{-1}|_W = -1$. Thus the sufficient condition is genuinely weaker than full negative definiteness.

The one-vertex solid torus gives the decisive sign check. For a reduced slope p/r, $r > 0$, the variable part of the exponent is

$$-\frac{(pn+a)^2}{pr}.$$

Every nonempty congruence branch contains infinitely many n. Thus $p < 0$ gives exponents tending to $+\infty$, while $p > 0$ gives exponents tending to $-\infty$. Negative slopes define well-behaved states; positive slopes do not lie in these coefficient spaces. The general formula excludes $p = 0$, apart from the separately defined singular zero-label state. Because sector transport preserves $\Omega(b)$ exactly, it cannot repair the positive-slope failure. A convergent positive-slope contraction would require a separately constructed growth-dual cotest.

B.2. **The coefficient pairing and boundary reflection.** Write $e_{n,m} = Y^n X^m$ in the rational quantum torus, with

$$X^m Y^n = q^{-2mn} Y^n X^m.$$

Basis subscripts follow the written monomial order (Y, X), whereas coefficient subscripts retain the electric–magnetic order (m, n):

$$\psi = \sum_{m,n} \psi_{m,n} e_{n,m}.$$

For a column $z = (m, n)^T$, the shorthand e_z thus means $e_{n,m}$. In [2, Section 3.2], equation (22) and the dagger/constant-term description in equations (23)–(24) give different coefficient pairings:

$$B_{22}(e_{r,s}, e_{u,v}) = \delta_{u,-r}\delta_{v,-s}, \qquad B_\dagger(e_{r,s}, e_{u,v}) = \delta_{u,r}\delta_{v,s}.$$

If $w(e_{n,m}) = e_{-n,-m}$, then

$$B_{22}(\psi, \phi) = B_\dagger(w\psi, \phi). \tag{38}$$

The Weyl permutation in (38) is not a scalar gauge: for example,

$$B_\dagger(X, X) = 1, \quad B_{22}(X, X) = 0, \qquad B_\dagger(X, X^{-1}) = 0, \quad B_{22}(X, X^{-1}) = 1.$$

The Hopf calculation, Proposition 4.4 and rational-filling calculations in [2, Sections 4.2–4.3] use the operational form

$$G_{\mathrm{op}}(\psi, \phi) = B_\dagger(\psi, s_0\phi), \qquad s_0(e_{n,m}) = e_{n,-m}. \tag{39}$$

Accordingly, we use $B_\dagger$ and (39). The displayed equation (22) is not used as an equivalent definition.

The reflection s_0 is a linear positive-boundary-to-negative-cotest map. It is not an anti-algebra map. The genuine anti-map determined by $X \mapsto X^{-1}$, $Y \mapsto Y$ is

$$\sigma(e_{n,m}) = X^{-m}Y^n = q^{2mn} e_{n,-m}. \tag{40}$$

Substituting (40) for s_0 would alter the gluing kernel, so the theorem does not do so.

With no topology or summation law added to the full arbitrary-formal-sum module, the maximal canonical algebraic pairing domain is

$$D_\dagger^{\mathrm{alg}} = \{(\psi,\phi) : \#\{(n,m) : \psi_{m,n}\phi_{m,n} \neq 0\} < \infty\}.$$

Indeed, in the discrete algebraic category, a net of finite partial sums is eventually constant exactly in this situation. Larger infinite pairings remain object-specific and require a stated summation order, local-finiteness or convergence proof, codomain, denominator bound, and stability under every mapping word used.

The modular action has a finite-support algebraic adjoint rather than a simultaneous-invariance law. If $\rho_M e_z = q^{F_M(z)} e_{Mz}$, then

$$\rho_M^\star e_y = q^{F_M(M^{-1}y)} e_{M^{-1}y}$$

satisfies $B_\dagger(\psi, \rho_M\phi) = B_\dagger(\rho_M^\star\psi, \phi)$. The adjoint identity follows on monomials: both sides are zero unless $y = Mz$, and in that case both equal $q^{F_M(z)}$. Bilinearity extends it to the finite-support domain. Applying the same Gaussian operator to both entries is not pairing invariant. For the source-named T_+, the vectors X and X^2 give respectively $B_\dagger(T_+X, T_+X) = q^{-2}$ and $B_\dagger(T_+X^2, T_+X^2) = q^{-8}$, excluding a single scalar repair.

B.3. **The two-twist phase.** In this subsection only, $\widetilde{T}_+$ denotes the *source-named* GJ generator. Under the left action convention of Section 2, it is the operator assigned to common T_-. The proved monomial formula gives, for every rational α,

$$\widetilde{T}_+^k(Y^nX^\alpha) = q^{-k\alpha^2} Y^{n+k\alpha} X^\alpha. \tag{41}$$

Hence on $\alpha = m + 1/2$,

$$\widetilde{T}_+^2(Y^nX^{m+1/2}) = q^{-(2m+1)^2/2} Y^{n+2m+1} X^{m+1/2}.$$

For the primitive solid-torus state

$$\Omega_\delta = \sum_{n\in\mathbb{Z}} Y^{n+\delta}(X^{1/2} - X^{-1/2}),$$

this yields

$$\widetilde{T}_+\Omega_\delta = q^{-1/4}\Omega_{\delta+1/2}, \qquad \widetilde{T}_+^2\Omega_\delta = q^{-1/2}\Omega_\delta. \tag{42}$$

Equation (36) of [2, version 1] uses exponent $-(2m+1)^2/4$ for the two-step support shift. It is therefore smaller by a factor of two than (41). The later general modular formulas use the proved one-step action, so we use (41)–(42) and do not use equation (36) for the two-step phase.

The ratio between the two phases is $q^{-\alpha^2}$, which depends on the monomial. It cannot be absorbed into a single mapping-class scalar or a rank-one framing line. The allowed changes of trivialization likewise retain the discrepancy, for the following explicit reasons.

A simultaneous change of basis G conjugates the operators and carries the state to $G\Omega_\delta$. Equation (42) then gives

$$(G\widetilde{T}_+^2G^{-1})(G\Omega_\delta) = q^{-1/2}G\Omega_\delta.$$

Thus its closed-loop eigenvalue cannot become $q^{-1/4}$. A sectorwise scalar c_δ contributes $c_{\delta+1/2}/c_\delta$ and $c_\delta/c_{\delta+1/2}$ on the two steps; their product is one.

More generally, suppose a diagonal rescaling $D(e_{n,m}) = \chi(n,m)e_{n,m}$ preserves the primitive up to a fixed line scalar, $D\Omega_\delta = c_\delta\Omega_\delta$. Equality of its displayed coefficients forces $\chi(n+\delta, 1/2) = \chi(n+\delta, -1/2) = c_\delta$ for every $n \in \mathbb{Z}$. The rescaling ratio on the two-step shift is consequently

$$\frac{\chi(n+\delta\pm1, \pm1/2)}{\chi(n+\delta, \pm1/2)} = 1.$$

If the dagger pairing is retained, the corresponding opposite-cotest rescaling is forced to be inverse, so it supplies no additional loop factor. These arguments cover the simultaneous changes of trivialization, sector scalars, and primitive-preserving diagonal gauges with the stated pairing and line compatibility.

Finally, multiplying $\widetilde{T}_+$ by a new scalar u is not a conjugation: it sends the algebra unit to u and fails to be a unital algebra action unless $u = 1$. Declaring such a scalar to be a new projective multiplier changes the framing cocycle. Within the specified allowed changes, the printed two-step discrepancy therefore cannot be removed while retaining the proved one-step action and primitive cap.

The comparison throughout this paper therefore uses the proved one-step monomial action, its displayed iterates and the coefficient pairing on its declared domain. None of these separate-target calculations supplies the common continuous embeddings excluded by Theorem 5.2.

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Weinberg Institute, The University of Texas at Austin, 2515 Speedway, C1600, Austin, Texas 78712, USA

SpaceX, USA

Email address: hkdavidsun@utexas.edu