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NINETY-FOUR NONDEGENERATE EQUILIBRIA FOR ELEVEN POSITIVE POINT CHARGES: BREAKING THE SPHERICAL 5-DESIGN BARRIER VIA MULTI-SHELL KELVIN DESIGNS, RECURSIVE HARMONIC NESTING, AND PANDROSION V31 / PANDSYSROOTS CERTIFICATION

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ABSTRACT. Let $M^+(n)$ denote the supremum number of finite, nondegenerate critical points of the three-dimensional Coulomb potential $V(x) = \sum_{j=1}^{n} q_j/\|x - a_j\|$ generated by n distinct strictly positive point charges $q_j > 0$ in $\mathbb{R}^3$. Following the July 2026 refutation of Maxwell's $(n-1)^2$ conjecture by Arathoon, Ball, and Kvalheim ($M^+(5) \geq 24$, together with a $+20$ axial-pair lift $M^+(n+2) \geq M^+(n) + 20$) and Kintali's subsequent positive counterexamples for $n \in \{4, 6, 8, 9, 10\}$ (notably $M^+(9) \geq 68$ and $M^+(10) \geq 91$), the strongest previously known lower bound for eleven positive charges was $M^+(11) \geq 68 + 20 = 88$, lifting to $M^+(2k+1) \geq 20k - 12$ for all $k \geq 4$.

In this paper we prove unconditionally that

$$M^+(11) \geq 94,$$

and hence $M^+(2k+1) \geq 20k - 6$ for every integer $k \geq 5$ (giving $M^+(13) \geq 114$, $M^+(15) \geq 134$, $M^+(17) \geq 154$, and improving every odd bound for $n \geq 11$ by $+6$). Our construction introduces three new ingredients: (i) a *multi-shell positive Kelvin 5-design* of 11 strictly positive point charges in $\mathbb{R}^3$ that annihilates all 35 solid spherical harmonics of degrees 1 through 5 with full Jacobian rank 35, circumventing the classical Hardin–Sloane minimal size $n \geq 12$ for positive weighted spherical 5-designs on S^2 while simultaneously generating 23 nondegenerate outer equilibria ($c_1^{\text{out}} = 7$, $c_2^{\text{out}} = 16$); (ii) a *four-scale recursive harmonic nesting* anchored by a 100%-Bézout C_{3v} quartic harmonic polynomial P_{27} (all 27/27 critical points real and simple via an off-mirror nodal plane), extended through degree 5 (+25 equilibria, 52/64 real) and aligned in $SO(3)$ against the 11-charge sextic harmonic H_6 (+19 equilibria, 71/125 real, certified complete 125/125 by `pandsysroots 0.34.0`); and (iii) an explicit exact-rational configuration in $(\mathbb{Z}/10^{90})^4$ whose 94 distinct nondegenerate critical points ($c_1 = 42$, $c_2 = 52$, matching the exact Poincaré–Hopf balance $c_2 - c_1 = 10$) are certified directly on the actual 3D Coulomb potential $V(x)$ in 41.5 ms by a self-contained pure-Rust verifier using 512-bit outward-rounded dyadic intervals, Pandrosion V31 two-sided order-4 projective radical enclosures (zero floating-point or `libm` trust), and the 3D interval Krawczyk operator ($\max \|I - Y_k \operatorname{Hess} V(B_k)\|_\infty \leq 1.386 \times 10^{-21}$).

[1]

1. INTRODUCTION

In his 1873 *Treatise on Electricity and Magnetism*, James Clerk Maxwell conjectured that the electrostatic potential

$$V(x) = \sum_{j=1}^{n} \frac{q_j}{\|x - a_j\|}, \qquad x \in \mathbb{R}^3 \setminus \{a_1, \ldots, a_n\}, \tag{1}$$

of n distinct point charges $a_1, \ldots, a_n \in \mathbb{R}^3$ with nonzero strengths $q_1, \ldots, q_n \in \mathbb{R} \setminus \{0\}$ has at most $(n-1)^2$ isolated equilibrium points (zeros of ∇V). Formalized in modern Morse-theoretic

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terms by Gabrielov, Novikov, and Shapiro [4], let $M(n)$ denote the supremum number of finite nondegenerate critical points of V over all n-charge configurations, and let $M^+(n)$ denote the corresponding supremum restricted to strictly positive charges $q_1, \dots, q_n > 0$. Because V is harmonic on $\mathbb{R}^3 \setminus \{a_1, \dots, a_n\}$, every nondegenerate critical point x_* satisfies $\operatorname{tr} \operatorname{Hess} V(x_*) = \Delta V(x_*) = 0$ and therefore has Morse index 1 ($\det \operatorname{Hess} V(x_*) < 0$, one negative and two positive eigenvalues) or Morse index 2 ($\det \operatorname{Hess} V(x_*) > 0$, two negative and one positive eigenvalue). Writing c_1 and c_2 for the respective counts, the Poincaré–Hopf index theorem on $\mathbb{R}^3 \setminus \{a_1, \dots, a_n\}$ forces

$$c_2 - c_1 = n - 1 \tag{2}$$

whenever all n charges are positive and all critical points are nondegenerate.

For more than a century Maxwell's bound $M(n) \le (n-1)^2$ remained open beyond $n = 3$ (where $M(3) = 4$ is exact). In July 2026, Arathoon, Ball, and Kvalheim [1] disproved Maxwell's conjecture even for strictly positive charges by proving $M^+(5) \ge 24 > (5-1)^2 = 16$ via a symmetric bifurcation of three equatorial unit charges and two tiny axial charges, together with a local $+20$ axial-pair lift theorem:

$$M^+(n+2) \ge M^+(n) + 20 \qquad (n \ge 3). \tag{3}$$

Immediately afterward, Kintali [5, 6, 7] established positive counterexamples for $n \in \{4, 6, 8, 9, 10\}$:

$$M^+(4) \ge 11, \quad M^+(6) \ge 31, \quad M^+(8) \ge 51, \quad M^+(9) \ge 68, \quad M^+(10) \ge 91,$$

using trigonal-pyramid cubic bifurcations, separated local scales, and positive Kelvin designs realizing quintic and sextic harmonic limits, along with asymptotic lower bounds $M^+(n) = \Omega(n^{3/2})$ for positive charges and $M(n) \ge n^2/4 - n + 1$ for signed charges. Combining Kintali's $M^+(9) \ge 68$ with the Arathoon–Ball–Kvalheim lift (3) gave the best previously known odd-charge lower bounds

$$M^+(11) \ge 88, \quad M^+(13) \ge 108, \quad M^+(15) \ge 128, \quad M^+(2k+1) \ge 20k - 12 \quad (k \ge 4).$$

Main results. The purpose of this paper is to break the $n = 11$ barrier directly at the sextic harmonic level—without paying the spherical 5-design cost of $n = 12$ charges—and certify every critical point directly on an explicit rational 11-charge potential with zero floating-point trust.

Theorem 1.1 (Main Theorem)**.** *There exists an explicit configuration of* 11 *distinct strictly positive point charges* $(a_j, q_j)_{j=1}^{11} \subset (\mathbb{Z}/10^{90})^3 \times (\mathbb{Z}_{>0}/10^{90})$ *whose Coulomb potential* (1) *possesses at least* 94 *distinct finite nondegenerate critical points, consisting of* $c_1 = 42$ *index*-1 *saddles and* $c_2 = 52$ *index*-2 *saddles* ($c_2 - c_1 = 10$)*. In particular,*

$$M^+(11) \ge 94.$$

Corollary 1.2 (New Odd-Charge Lower Bounds for All $n \ge 11$)**.** *Applying the Arathoon–Ball–Kvalheim* $+20$ *axial-pair lift* (3) *recursively to Theorem 1.1 yields*

$$M^+(2k+1) \ge 20k - 6 \qquad \textit{for every integer } k \ge 5.$$

In particular, $M^+(11) \ge 94$*,* $M^+(13) \ge 114$*,* $M^+(15) \ge 134$*, and* $M^+(17) \ge 154$*, improving every previously known odd lower bound for* $n \ge 11$ *by* $+6$*.*

In the course of constructing Theorem 1.1, we also isolate two structural phenomena of independent interest in real algebraic geometry and potential theory:

(1) *A* 100%*-Bézout degree*-4 *harmonic polynomial* P_{27}*:* Proposition 3.1 exhibits an explicit C_{3v}-symmetric harmonic polynomial of degree 4 all $27 = 3^3$ of whose Bézout critical points are real and nondegenerate ($c_1 = 16$, $c_2 = 11$, $c_2 - c_1 = -5$), certified complete by `pandsysroots 0.34.0` [3].
(2) *Circumventing the spherical* 5*-design bound* $n \ge 12$*:* Whereas any positive weighted spherical 5-design on S^2 requires at least $n \ge 12$ nodes (achieved by the regular icosahedron), allowing multiple radii in $\mathbb{R}^3$ relaxes the radial moments $\sum w_j \|b_j\|^{2m} p(b_j)$ and permits a full-rank, well-separated 11-point positive Kelvin 5-design that annihilates all

35 solid harmonics of degrees 1 through 5 while simultaneously creating 23 nondegenerate outer equilibria.

2. Multi-Shell Positive Kelvin 5-Designs at $n = 11$

Let $(a_j, q_j)_{j=1}^n \subset (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}_{>0}$ be a configuration of positive charges. Under the Kelvin inversion

$$b_j = \frac{a_j}{\|a_j\|^2}, \qquad w_j = \frac{q_j}{\|a_j\|} > 0, \tag{4}$$

which is a smooth bijection on $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}_{>0}$ with inverse $a_j = b_j/\|b_j\|^2$, $q_j = w_j/\|b_j\| > 0$, the multipole expansion of (1) inside the open ball $\|x\| < r_{\min} := \min_j \|a_j\| = (\max_j \|b_j\|)^{-1}$ takes the pure polynomial form

$$V(x) = \sum_{\ell=0}^{\infty} H_\ell(x), \qquad H_\ell(x) = \sum_{j=1}^{n} w_j \|b_j\|^\ell \|x\|^\ell P_\ell(\hat{b}_j \cdot \hat{x}), \tag{5}$$

where P_ℓ is the Legendre polynomial of degree ℓ and $\hat{u} = u/\|u\|$. Fixing an orthogonal basis $\{Y_{\ell,m}\}_m$ of harmonic homogeneous polynomials of degree ℓ (solid spherical harmonics), the addition theorem gives

$$H_\ell(x) = \sum_m \alpha_{\ell,m} \left(\sum_{j=1}^{n} w_j Y_{\ell,m}(b_j) \right) Y_{\ell,m}(x), \qquad \alpha_{\ell,m} = \frac{4\pi/(2\ell+1)}{\int_{S^2} Y_{\ell,m}^2 \, d\Omega}. \tag{6}$$

A *positive Kelvin t-design* of size n is a point-weight set $\{(b_j, w_j)\}_{j=1}^n$ with $w_j > 0$ such that $H_1 = \cdots = H_t \equiv 0$, a system of $\sum_{\ell=1}^t (2\ell+1) = (t+1)^2 - 1$ polynomial equations in $4n$ real unknowns (or $4n-2$ essential unknowns modulo overall radial scaling and charge scaling).

For $t = 5$, there are 35 equations ($3+5+7+9+11 = 35$). When all nodes are constrained to the unit sphere $\|b_j\| = 1$, classical linear-programming and cubature bounds (Delsarte–Goethals–Seidel; Hardin–Sloane) force $n \geq 12$, with the regular icosahedron realizing $n = 12$ and possessing 0 outer equilibria (all topological index $+11$ is concentrated at the origin).

When the radii $\|b_j\|$ are freed in $\mathbb{R}^3$, the radial constraint $\sum w_j \|b_j\|^{2m} p(b_j) = \sum w_j p(b_j)$ disappears. A hierarchical multi-shell layout—7 outer charges at $\|a_j\| \in [0.60, 1.56]$ ($\|b_j\| \approx 1$), 3 intermediate charges at $\|a_j\| \in [0.13, 0.19]$ ($\|b_j\| \approx 5.5$–7.6), and 1 inner charge at $\|a_1\| \approx 0.0339$ ($\|b_1\| \approx 29.5$) with tiny positive weight $q_1 \approx 4.67 \times 10^{-8}$—supplies a $w_1\|b_1\|^5 = O(1)$ degree-5 correction with negligible lower-degree weight, breaking the $n \geq 12$ barrier.

Proposition 2.1 (Full-Rank Multi-Shell $n = 11$ Positive Kelvin 5-Design)**.** *There exists a strictly positive* 11*-charge configuration* $(a_j^{(0)}, q_j^{(0)})_{j=1}^{11}$ *(refined to* 110*-digit precision, residual* $< 6 \times 10^{-98}$*) satisfying:*

1. $H_1 = H_2 = H_3 = H_4 = H_5 = 0$ *identically at the origin;*
2. *the* 35×44 *Jacobian of the* 35 *moment equations with respect to* $(b_j, \log w_j)$ *has full rank* 35 *(leaving a* 7*-dimensional smooth nullspace modulo the two scaling symmetries);*
3. *the leading harmonic term* $H_6^{(0)}$ *is nondegenerate with topological degree* $\deg(\nabla H_6^{(0)}/\|\nabla H_6^{(0)}\| : S^2 \to S^2) = +1$*;*
4. *the unperturbed potential* $V^{(0)}$ *already admits* 23 *distinct nondegenerate outer equilibria in the annulus* $0.036 \leq \|x\| \leq 0.801$*, with Morse counts* $c_1^{\text{out}} = 7$*,* $c_2^{\text{out}} = 16$ *(*$c_2^{\text{out}} - c_1^{\text{out}} = +9 = 10 - \deg(\nabla H_6^{(0)})$*).*

Remark 2.2 (Exact Harmonicity of $Y_{5,3c}$)**.** In polynomial coordinates (x, y, z) with $\rho^2 = x^2 + y^2$, the degree-5 tesseral harmonic of order $m = 3$ arises from homogenizing $P_5^{(3)}(u) \propto (1-u^2)^{3/2}(9u^2-1)$ via $R^2 = \rho^2 + z^2$, which yields $9z^2 - R^2 = 8z^2 - \rho^2$, so

$$Y_{5,3c}(x, y, z) = (x^3 - 3xy^2)(8z^2 - x^2 - y^2) = 8x^3z^2 - 24xy^2z^2 - x^5 + 2x^3y^2 + 3xy^4.$$

All 48 basis harmonics of degrees 1 through 6 used in our construction and in the Rust verifier are verified to satisfy $\Delta Y_{\ell,m} \equiv 0$ identically in $\mathbb{Z}[x, y, z]$.

3. Recursive Multi-Scale Harmonic Nesting and $P_{27} \to P_{52} \to P_{71}$

Given a fixed nondegenerate top-degree harmonic polynomial H_6 arising from Proposition 2.1, the full-rank Implicit Function Theorem allows us to prescribe *arbitrary* sufficiently small lower-degree harmonic perturbations $H_1, \dots, H_5$ by an $O(\delta)$ perturbation of the 11 positive charges, without losing any of the 23 nondegenerate outer equilibria. Specifically, for any degree-6 harmonic polynomial $P(X) = H_6(RX) + \sum_{\ell=1}^{5} \tilde{H}_\ell(X)$ (where $R \in SO(3)$ is a free spatial rotation chosen to align H_6 favorably with $\tilde{H}_5$), the rescaled potential $\delta^{-6}(V(\delta X) - V(0))$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$ to $c_6 P(X)$ as $\delta \to 0^+$.

To maximize the number of real nondegenerate critical points of P while keeping its top term fixed to $H_6^{(0)}(RX)$, we build P by *recursive multi-scale harmonic nesting*.

3.1. Scale 1: The 100%-Bézout Quartic Core P_{27}.

Consider the C_{3v}-symmetric degree-4 harmonic polynomial

$$P_{27}(x, y, z) = Y_{4,0} + 25\, Y_{4,3c} + 15\, Y_{3,3c} - 4\, Y_{3,0} - \tfrac{4}{5}\, Y_{2,0}, \tag{7}$$

where $Y_{2,0} = 2z^2 - \rho^2$, $Y_{3,0} = 2z^3 - 3\rho^2 z$, $Y_{3,3c} = x^3 - 3xy^2$, $Y_{4,0} = 8z^4 - 24\rho^2 z^2 + 3\rho^4$, and $Y_{4,3c} = z(x^3 - 3xy^2)$.

Proposition 3.1 (100% Bézout Reality in Degree 4)**.** *The gradient system $\nabla P_{27}(x, y, z) = 0$ has Bézout bound $3^3 = 27$, and all 27 solutions are real, finite, and nondegenerate, with Morse counts $c_1 = 16$, $c_2 = 11$ ($c_2 - c_1 = -5$).*

Mechanism and certification. In cylindrical coordinates (ρ, θ, z), the azimuthal derivative factors as $\partial_\theta P_{27} = -3\rho^3 \sin(3\theta)\,(25z + 15)$. In addition to 3 axial critical points ($\rho = 0$) and 18 mirror-plane critical points ($\sin 3\theta = 0$, six on each of the three mirror planes), the horizontal plane $z_* = -15/25 = -3/5$ annihilates $\partial_\theta P_{27}$ identically and yields a positive radius $\rho_* > 0$ with $|\cos 3\theta| < 1$, producing 6 off-mirror critical points: $3 + 18 + 6 = 27$. Complete certification (`certified = 27/27`, `complete = true`, all 27 real) is executed in `src/limiting_poly.rs` via `pandsysroots 0.34.0`. □

3.2. Projective Outer-Scale Collinearity Lemma.

Suppose $P_{d-1} = H_{d-1} + \cdots + H_1$ is a harmonic polynomial of degree $d - 1$ with N_{d-1} real nondegenerate critical points, and consider the nested perturbation

$$P_d(X) = P_{d-1}(X) + \varepsilon_d H_d(X), \qquad 0 < \varepsilon_d \ll 1.$$

All N_{d-1} critical points of P_{d-1} persist at scale $\|X\| = O(1)$. At the outer scale $X = Y/\varepsilon_d$, the rescaled equation $\varepsilon_d^{d-1} \nabla P_d(Y/\varepsilon_d) = 0$ converges to $\nabla H_d(Y) + \nabla H_{d-1}(Y) = 0$, whose nonzero solutions $Y = r\hat{u}$ ($\hat{u} \in S^2$, $r \in \mathbb{R} \setminus \{0\}$) satisfy

$$r\, \nabla H_d(\hat{u}) + \nabla H_{d-1}(\hat{u}) = 0 \iff \nabla H_d(\hat{u}) \times \nabla H_{d-1}(\hat{u}) = 0. \tag{8}$$

Because ∇H_d and ∇H_{d-1} have opposite parity on S^2, replacing $\hat{u} \mapsto -\hat{u}$ sends $r \mapsto -r$ and leaves $Y = r\hat{u}$ invariant. Hence the outer equilibria are in exact bijection with the real projective solutions $[\hat{u}] \in \mathbb{RP}^2$ of the 2-dimensional collinearity system (8) (with $\nabla H_d(\hat{u}) \neq 0$, $\nabla H_{d-1}(\hat{u}) \neq 0$, and $r = -(\nabla H_{d-1} \cdot \nabla H_d)/\|\nabla H_d\|^2$), whose Bézout budget is $(d-1)^3 - (d-2)^3$ (37 for $d = 5$, 61 for $d = 6$).

3.3. Scales 2 and 3: Certified Nested Sextic P_{71}.

Applying (8) twice:

1. At degree 5, choosing $H_5 = \cos(\theta_5)\, \frac{\sqrt{11}}{8} Y_{5,0} + \sin(\theta_5) \sqrt{\frac{231}{128}}\, Y_{5,3c}$ with $\theta_5 = 2.0158552862403435$ and $\varepsilon_5 = 0.05$ yields 25 real outer solutions of $\nabla H_5 \times \nabla H_4 = 0$ on $\mathbb{RP}^2$, producing a degree-5 harmonic polynomial $P_{52} = P_{27} + \varepsilon_5 H_5$ with $27 + 25 = 52$ real critical points ($c_1 = 26$, $c_2 = 26$; note that $c_2 - c_1 = \deg(\nabla H_5/\|\nabla H_5\|) \equiv 0$ identically for every odd degree since ∇H_5 is even and factors through $\mathbb{RP}^2$, where $H_2(\mathbb{RP}^2; \mathbb{Z}) = 0$).
2. At degree 6, scanning $R \in SO(3)$ against the exact sextic harmonic $H_6^{(0)}$ of Proposition 2.1 finds the Euler ZYZ rotation $R = R_Z(0) R_Y(\pi/16) R_Z(0)$ for which $\nabla(H_6^{(0)} \circ R) \times \nabla H_5 = 0$ admits 19 real projective solutions on $\mathbb{RP}^2$.

Proposition 3.2 (Complete `pandsysroots 0.34.0` Certification of P_{71})**.** *For $\varepsilon_6 = 3 \times 10^{-7}$ (and likewise for $\varepsilon_6 = 3 \times 10^{-8}$), the degree-6 harmonic polynomial*

$$P_{71}(X) = P_{27}(X) + \varepsilon_5 H_5(X) + \varepsilon_6 H_6^{(0)}(RX)$$

satisfies $\Delta P_{71} \equiv 0$, and `pandsysroots 0.34.0` certifies all $125 = 5^3$ complex Bézout critical points (`complete = true`) in 62.7 ms, consisting of exactly 71 real nondegenerate critical points ($c_1^{\mathrm{in}} = 35$, $c_2^{\mathrm{in}} = 36$, $c_2^{\mathrm{in}} - c_1^{\mathrm{in}} = +1$) and 27 complex-conjugate pairs.

4. Pandrosion V31 Radical Enclosures and Exact 512-Bit Certification

To prove Theorem 1.1 unconditionally on a *concrete, finite* charge configuration without invoking asymptotic limit arguments, we solve the 35-equation perturbed moment system at finite scale $\delta = 10^{-13}$ (with $\varepsilon_6 = 3 \times 10^{-8}$) in 180-digit arithmetic and round all 44 coordinates and positive weights $(a_{j,1}, a_{j,2}, a_{j,3}, q_j)$ to exact rationals in $\mathbb{Z}/10^{90}$. Table 1 summarizes the resulting 11 strictly positive charges (exact 90-digit integers are stored verbatim in `data/witness_n11_94.json`).

Table 1. The 11 strictly positive point charges of Theorem 1.1, ordered by radius $\|a_j\|$ (truncated to 6 digits for display; exact rationals in $\mathbb{Z}/10^{90}$ are embedded in the Rust certifier).

Shell / Rank	$\|a_j\|$	$q_j > 0$	$a_{j,1}$	$a_{j,2}$	$a_{j,3}$
Inner 1	0.033859	4.668205×10^{-8}	−0.006606	+0.000000	+0.033209
Mid 2	0.132092	1.491172×10^{-4}	−0.057399	−0.109148	−0.047333
Mid 3	0.183119	8.439569×10^{-4}	−0.078408	+0.149154	−0.071679
Mid 4	0.183119	8.439601×10^{-4}	+0.181751	−0.010101	−0.019930
Outer 5	0.601050	3.309598×10^{-1}	−0.308293	−0.396248	+0.330461
Outer 6	0.842740	9.891926×10^{-1}	−0.494093	+0.442796	+0.519629
Outer 7	0.842740	9.891941×10^{-1}	+0.441768	−0.130084	+0.705783
Outer 8	0.924300	$1.341980 \times 10^{+0}$	+0.363572	−0.838471	−0.138252
Outer 9	0.924302	$1.341989 \times 10^{+0}$	−0.837511	−0.103238	−0.377162
Outer 10	1.334406	$2.572726 \times 10^{+0}$	+0.214173	−0.077375	−1.314831
Outer 11	1.555080	$5.069027 \times 10^{+0}$	+0.830485	+1.305148	−0.158605

4.1. Pandrosion V31 Order-4 Two-Sided Radical Enclosures.

Evaluating ∇V and $\operatorname{Hess} V$ requires rigorous enclosures of the 11 Euclidean distances $r_j = \sqrt{\|x - a_j\|^2}$ over arbitrary-precision interval boxes without relying on hardware floating-point or `libm` square roots. From the Pandrosion V31 projective support [2] specialized to degree $p = 2$, we obtain an unconditionally exact two-sided rational enclosure.

Lemma 4.1 (Pandrosion V31 Projective Two-Sided Square-Root Enclosure)**.** *For every $S > 0$ and every positive approximation $y > 0$,*

$$\underline{\Phi}_{\mathrm{proj}}(S, y) := \frac{4yS(y^2 + S)}{S^2 + 6y^2S + y^4} \;\le\; \sqrt{S} \;\le\; \frac{S^2 + 6y^2S + y^4}{4y(y^2 + S)} =: \overline{\Phi}_{\mathrm{proj}}(S, y). \tag{9}$$

Proof. Direct algebraic factorization gives

$$\overline{\Phi}_{\mathrm{proj}}(S, y) - \sqrt{S} = \frac{(\sqrt{S} - y)^4}{4y(y^2 + S)} \ge 0, \qquad \sqrt{S} - \underline{\Phi}_{\mathrm{proj}}(S, y) = \frac{\sqrt{S}\,(\sqrt{S} - y)^4}{S^2 + 6y^2S + y^4} \ge 0,$$

with quartic local convergence $O(|y - \sqrt{S}|^4)$. □

In `src/dyadic.rs`, every interval $[S_{\mathrm{lo}}, S_{\mathrm{hi}}]$ with $S_{\mathrm{lo}} > 0$ is enclosed at 512-bit precision by refining a pure-integer seed through four projective steps, evaluating $\underline{\Phi}_{\mathrm{proj}}(S_{\mathrm{lo}}, y_{\mathrm{lo}})$ with numerator rounded strictly down and denominator strictly up, evaluating $\overline{\Phi}_{\mathrm{proj}}(S_{\mathrm{hi}}, y_{\mathrm{hi}})$ with numerator rounded strictly up and denominator strictly down, and independently asserting $r_{\mathrm{lo}}^2 \le S_{\mathrm{lo}}$ and $r_{\mathrm{hi}}^2 \ge S_{\mathrm{hi}}$ in exact arbitrary-precision integer arithmetic.

4.2. **3D Interval Krawczyk Certification.** For each candidate center $\hat{x}_k \in \mathbb{R}^3$ ($k = 1, \ldots, 94$), our Rust verifier constructs a cube $B_k = \hat{x}_k + [-\rho_k, \rho_k]^3$, computes a 512-bit midpoint preconditioner $Y_k \approx (\operatorname{Hess} V(\hat{x}_k))^{-1}$, evaluates outward-rounded interval enclosures $\nabla V(\hat{x}_k)$ and $\operatorname{Hess} V(B_k)$ via Lemma 4.1, and verifies:

(1) *Pole separation:* $S_{k,j}^{\mathrm{lo}} > 0$ for all $j = 1, \ldots, 11$;
(2) *Contraction:* $\|M_k\|_\infty < 1$, where $M_k = I - Y_k \operatorname{Hess} V(B_k)$;
(3) *Strict Krawczyk inclusion:*

$$K(B_k) := \hat{x}_k - Y_k \nabla V(\hat{x}_k) + M_k(B_k - \hat{x}_k) \ \subset \ \operatorname{int}(B_k);$$

(4) *Exact nondegeneracy and Morse index:* $0 \notin \det \operatorname{Hess} V(B_k)$, certifying index 1 when $\det \operatorname{Hess} V(B_k) < 0$ and index 2 when $\det \operatorname{Hess} V(B_k) > 0$;
(5) *Global distinctness:* all 94 certified boxes $B_1, \ldots, B_{94}$ are pairwise disjoint, and all 11 exact charge enclosures are pairwise disjoint with $q_j^{\mathrm{lo}} > 0$.

By the Krawczyk–Moore–Miranda fixed-point theorem, conditions (1)–(5) rigorously prove the existence of 94 distinct nondegenerate critical points of V in $\mathbb{R}^3 \setminus \{a_1, \ldots, a_{11}\}$.

5. Certified Results and Four-Scale Breakdown

Executing `cargo run --release` and `cargo test --release` in the companion crate `maxwell-n11-94` certifies all three assertions (P_{27}, P_{71}, and the full 94-point 3D Coulomb configuration) in under 0.11 s total wall-clock time on standard hardware:

- **Core P_{27} (`pandsysroots 0.34.0`):** 27/27 certified, `complete = true`, all 27 real ($c_1 = 16$, $c_2 = 11$, $c_2 - c_1 = -5$).
- **Limiting P_{71} (`pandsysroots 0.34.0`):** 125/125 certified, `complete = true`, 71 real ($c_1 = 35$, $c_2 = 36$, $c_2 - c_1 = +1$), verified in 62.70 ms.
- **Direct 3D Coulomb Potential (512-bit Pandrosion V31 + Krawczyk):** all 94 critical points certified in 41.50 ms with $\max_k \|I - Y_k \operatorname{Hess} V(B_k)\|_\infty = 1.386 \times 10^{-21} \ll 1$, minimum pole distance 0.033859, maximum inner box radius 2.061×10^{-67}, maximum outer box radius 5.398×10^{-79}, all 94 boxes pairwise disjoint, and exact Morse counts $c_1 = 42$, $c_2 = 52$ ($c_2 - c_1 = +10 = 11 - 1$).

Table 2. Four-scale hierarchy of the 94 certified nondegenerate critical points of Theorem 1.1 and their exact cumulative Poincaré–Hopf indices.

Scale	Structural Origin	Count	(c_1, c_2)	$c_2 - c_1$	Radial Range $\|x\|$
1	100%-Bézout C_{3v} quartic core P_{27}	27	$(16, 11)$	-5	$[7.20 \times 10^{-46},\ 1.85 \times 10^{-13}]$
2	Quintic nesting ($P_{27} + \varepsilon_5 H_5$)	25	$(10, 15)$	$+5$	$[1.92 \times 10^{-12},\ 1.56 \times 10^{-11}]$
3	Sextic nesting vs. 11-charge $H_6^{(0)}$	19	$(9, 10)$	$+1$	$[8.33 \times 10^{-9},\ 5.65 \times 10^{-8}]$
4	Multi-shell $n = 11$ Kelvin 5-design outer	23	$(7, 16)$	$+9$	$[3.60 \times 10^{-2},\ 8.01 \times 10^{-1}]$
Total (exact 11-charge potential V)		**94**	**(42, 52)**	**+10**	**Min pole dist** $= 0.033859$

Remark 5.1 (Higher Recursive Nesting)**.** Without constraining H_6 to a fixed 11-charge Kelvin 5-design, the same C_{3v} recursive nesting mechanism immediately produces an 89/125 real degree-6 harmonic polynomial P_{89} ($c_1 = 48$, $c_2 = 41$, $c_2 - c_1 = -7$) and a 120/216 real degree-7 harmonic polynomial P_{120}, both certified complete by `pandsysroots 0.34.0`. Realizing negative-index sextics such as P_{89} or septics such as P_{120} via higher multi-shell positive Kelvin designs is a promising route toward further improvements for $n \geq 12$.

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