HOLOMORPHY FOR HOMOGENEOUS POLYNOMIALS WITH ORDINARY SINGULARITIES

JASON TURNER

Abstract. We prove a vanishing theorem for twisted Igusa integrals of homogeneous polynomials whose projective hypersurfaces are reduced and have only isolated ordinary multiple points. Under Denef's monodromy exclusion, the integrals on the integer polydisc and its origin residue class vanish outside finitely many places, uniformly in the character. We also compute the integral on every residue class.

hexagon:2611.01118v1

1. Introduction

Let $F \subset \mathbb{C}$ be a number field, fix an algebraic closure $\overline{F} \subset \mathbb{C}$, and let $f \in F[x_1, \dots, x_n]$ be nonconstant. For a nonarchimedean completion K of F, choose a uniformizer π and write $R = \mathcal{O}_K$, $\mathfrak{m} = \pi R$, and $k = R/\mathfrak{m}$, with $\#k = q$. Normalize the valuation $v_K : K^\times \to \mathbb{Z}$ by $v_K(\pi) = 1$, put $|z| = q^{-v_K(z)}$, and define $\mathrm{ac}(z) = \pi^{-v_K(z)} z$ for $z \neq 0$. For a continuous finite-order character $\chi : R^\times \to \mathbb{C}^\times$, consider the twisted Igusa integrals

$$Z_f(\chi, s) = \int_{R^n} \chi(\mathrm{ac}\, f(x)) |f(x)|^s \, dx, \tag{1.1}$$

$$Z_{f,0}(\chi, s) = \int_{\mathfrak{m}^n} \chi(\mathrm{ac}\, f(x)) |f(x)|^s \, dx. \tag{1.2}$$

Additive Haar measure gives R volume 1; products carry the product measure. The integrand is set equal to zero on $f^{-1}(0)$, and initially $\Re(s) > 0$. For integral polynomials and points, a bar denotes reduction modulo $\mathfrak{m}$. Igusa proved that these integrals extend meromorphically as rational functions of $T = q^{-s}$ [7]. We also use T as the argument of this rational continuation. The integrals encode weighted counts of solutions of congruences; Denef's report [3, Sections 1–3] explains this arithmetic interpretation and the formulas obtained from embedded resolution.

For a component E_i of the total transform in an embedded resolution h, let N_i be its multiplicity in $\mathrm{div}(f \circ h)$ and let $\nu_i - 1$ be its multiplicity in $\mathrm{div}(h^*(dx_1 \wedge \cdots \wedge dx_n))$. For a residue-field character of order d, the good reduction formula has denominator factors

$$1 - q^{-\nu_i} T^{N_i} \qquad (d \mid N_i).$$

Thus divisibility of the resolution multiplicities controls the possible poles. Denef's holomorphy conjecture replaces this resolution-dependent condition by one expressed through local monodromy.

For $a \in f^{-1}(0)(\mathbb{C})$, let $M_{f,a}$ be the local Milnor fibre, with the action of a counterclockwise generator of the punctured target disc. Write $\mathrm{Eig}(f, a)$ for the eigenvalues on $\bigoplus_j H^j(M_{f,a}, \mathbb{C})$, including H^0. They are roots of unity. We use $\mathrm{ord}(\lambda)$ for the multiplicative order of a root of unity and $\mathrm{ord}\,\chi$ for the order of a finite-order character. For an integer $d \geq 1$, set

$$\mathcal{H}_f(d): \qquad d \nmid \mathrm{ord}(\lambda) \quad \text{for every } a \in f^{-1}(0)(\mathbb{C}) \text{ and every } \lambda \in \mathrm{Eig}(f, a). \tag{1.3}$$

Denef formulated the conjecture in [3, Conjecture 4.4.2]: outside finitely many places of F, $\mathcal{H}_f(\mathrm{ord}\,\chi)$ implies holomorphy of both (1.1) and (1.2) on $\mathbb{C}$. This divisibility condition

Date: November 26, 2026.

2020 *Mathematics Subject Classification.* 11S40, 14B05, 32S40.

Key words and phrases. Igusa local zeta function, monodromy, ordinary multiple point.

excludes all eigenvalue orders that are multiples of d. The constant class in H^0 has eigenvalue 1, so $\mathcal{H}_f(d)$ implies $d > 1$.

The relation between resolution and monodromy is expressed by A'Campo's formula [1], whose exponents are Euler characteristics of resolution strata. Denef connected these strata with arithmetic character sums in [4]. His non-contribution theorem, stated also in [3, Section 4.6], applies to a proper divisor whose multiplicity is divisible by d and which meets no other such divisor. If its open stratum has Euler characteristic zero, its contribution to the twisted integral vanishes. The cohomological proof in [5, Section 5] embeds the relevant compactly supported cohomology into generalized local monodromy eigenspaces. A separate result of that paper, Theorem 1.2, gives strictly negative degree in T for the origin integral of a residue-field character when local monodromy at the origin has no eigenvalue of the exact character order. The non-contribution and degree theorems control the local terms and the final radial expression, respectively.

Veys proved holomorphy for plane curves in [9]. He subsequently extended the non-contribution argument to certain connected unions of exceptional divisors [10, Theorem 5.11]. Denef and Veys proved a Thom–Sebastiani theorem: holomorphy for a polynomial in $n-1$ variables implies holomorphy after adding x_n^k for $k \geq 3$, with further results for $k = 2$ [6, Theorem 3.1 and Remarks 3.2–3.3].

For homogeneous polynomials, Rodrigues and Veys introduced the corresponding projective integral and related it to the affine integral [8, Section 3]. They proved the conjecture when the Euler characteristic of the projective complement is nonzero [8, Theorem 2.2], and for all homogeneous polynomials in three variables [8, Theorem 3.7]. Their projective reduction [8, Theorem 3.5] applies in every dimension. Once the projective integral is constant, homogeneity gives a radial denominator and Denef's degree theorem forces the constant to vanish. Later, Castryck, Ibadula and Lemahieu proved arithmetic holomorphy for Newton nondegenerate surface singularities [2]. Veys surveys these results and their relation to the monodromy conjecture in [11, Section 6.1].

Definition 1.1. A singular point of a reduced hypersurface in a smooth r-dimensional variety is an *ordinary multiple point of multiplicity m* if, in local coordinates centred there, its equation has the form

$$g(y) = g_m(y) + g_{m+1}(y) + \cdots, \qquad m \geq 2,$$

where g_j is homogeneous of degree j, $g_m \neq 0$, and $\{g_m = 0\} \subset \mathbb{P}^{r-1}$ is a smooth hypersurface of degree m.

The ordinary-point case below follows from Denef's non-contribution theorem and the projective reduction of Rodrigues and Veys. Blowing up the ordinary points in the projective coordinate charts gives pairwise disjoint proper exceptional divisors, while the strict transform has multiplicity 1. For $d > 1$, each exceptional divisor whose multiplicity is divisible by d therefore satisfies Denef's isolation condition. A'Campo's formula and (1.3) force its open stratum to have Euler characteristic zero, so its contribution vanishes. This also specializes [10, Theorem 5.11] to a single exceptional divisor. We give a direct proof by calculating the local character sums, the integrals on residue classes, and the remaining radial term.

Theorem 1.2. *Let $n \geq 3$ and let $f \in F[x_1, \ldots, x_n]$ be homogeneous of degree $D \geq 1$. Assume that f is squarefree over $\overline{F}$ and that the projective hypersurface*

$$X = \operatorname{Proj}(F[x_1, \ldots, x_n]/(f)) \subset \mathbb{P}^{n-1}_F$$

has only isolated ordinary multiple points after extension to $\overline{F}$. There is a finite set S of nonarchimedean places of F such that the following holds for every $K = F_v$, $v \notin S$. For every continuous finite-order character $\chi : R^\times \to \mathbb{C}^\times$ with $\mathcal{H}_f(\operatorname{ord}\chi)$,

$$Z_f(\chi, s) = Z_{f,0}(\chi, s) = 0. \tag{1.4}$$

For $a \in k^n$, let $Z_{f,a}$ denote the integral over the residue class reducing to a. If χ is induced by a character $\bar{\chi} : k^\times \to \mathbb{C}^\times$, then

$$Z_{f,a}(\chi, s) = \begin{cases} 0, & \bar{f}(a) = 0, \\ q^{-n}\bar{\chi}(\bar{f}(a)), & \bar{f}(a) \neq 0. \end{cases} \tag{1.5}$$

If χ is nontrivial on $1 + \mathfrak{m}$, every $Z_{f,a}$ is zero. In particular all these residue-class integrals are holomorphic.

The theorem includes projective complements with Euler characteristic zero. Section 2 computes the ordinary-point monodromy; Section 3 evaluates the local integrals, including the even-variable quadratic case. Section 4 proves the uniform arithmetic statement by good reduction and radial integration.

2. Ordinary points and local monodromy

Throughout this section $r \geq 2$ and $m \geq 2$. For a local complex hypersurface equation g at a zero b, let M_j be the monodromy automorphism of $H^j(M_{g,b}, \mathbb{C})$. We use the convention

$$\zeta_{g,b}(t) = \prod_j \det(1 - tM_j \mid H^j(M_{g,b}, \mathbb{C}))^{(-1)^{j+1}}. \tag{2.1}$$

For an embedded resolution h, let E_i run over all irreducible components of $\operatorname{div}(g \circ h)$, including the strict transform, and let N_i be their multiplicities. A'Campo's formula [1, Théorème 3] reads

$$\zeta_{g,b}(t) = \prod_i (1 - t^{N_i})^{-\chi_c(E_i^\circ \cap h^{-1}(b))}, \qquad E_i^\circ = E_i \setminus \bigcup_{j \neq i} E_j. \tag{2.2}$$

Here χ_c is the compactly supported topological Euler characteristic. For the complex algebraic varieties used below it equals the usual Euler characteristic. We use this formula for both isolated and nonisolated singularities.

Lemma 2.1. *One blowup resolves a hypersurface germ with an ordinary multiple point. The exceptional divisor is $\mathbb{P}^{r-1}$ with multiplicity m. The reduced strict transform is smooth, has multiplicity 1, and meets the exceptional divisor transversally along the smooth hypersurface $H = \{g_m = 0\} \subset \mathbb{P}^{r-1}$.*

Proof. In a blowup chart write $y_1 = t$ and $y_i = tz_i$ for $2 \leq i \leq r$. The pullback equation is

$$g \circ h = t^m\big(g_m(1, z) + tg_{m+1}(1, z) + t^2 g_{m+2}(1, z) + \cdots\big). \tag{2.3}$$

The exceptional divisor is $t = 0$. Its intersection with the strict transform is $g_m(1, z) = 0$. At every point of this intersection some derivative of $g_m(1, z)$ in a z direction is nonzero, by smoothness of H. Thus the parenthesized factor and t form part of a system of local coordinates. This proves smoothness and transversality near the exceptional divisor. Let U be a neighbourhood of the compact exceptional fibre on which these conclusions hold. Properness of h makes the image of the complement of U closed and disjoint from the centre. Shrink the original neighbourhood to avoid that image. Away from the exceptional fibre, h is an isomorphism, so the original hypersurface is smooth away from the centre there. This gives the embedded resolution. When $r = 2$, H consists of m distinct points of $\mathbb{P}^1$; the strict branches meet the exceptional curve at distinct points. □

Lemma 2.2. *For a smooth hypersurface H of degree m in $\mathbb{P}^{r-1}$,*

$$e(r, m) := \chi_c(\mathbb{P}^{r-1} \setminus H) = \frac{1 - (1 - m)^r}{m}. \tag{2.4}$$

For $r, m \geq 2$, this number is zero precisely when $m = 2$ and r is even.

Proof. Write $h = c_1(\mathcal{O}_H(1))$ for the hyperplane class on H and $c(TH)$ for the total Chern class of its tangent bundle. The Euler sequence restricted to H and the tangent–normal sequence are

$$0 \longrightarrow \mathcal{O}_H \longrightarrow \mathcal{O}_H(1)^{\oplus r} \longrightarrow T\mathbb{P}^{r-1}|_H \longrightarrow 0,$$
$$0 \longrightarrow TH \longrightarrow T\mathbb{P}^{r-1}|_H \longrightarrow \mathcal{O}_H(m) \longrightarrow 0.$$

Multiplicativity of total Chern classes in these sequences gives

$$c(TH) = \frac{(1+h)^r}{1+mh}.$$

The degree of H gives $\int_H h^{r-2} = m$. The Chern–Gauss–Bonnet formula therefore gives

$$\chi_c(H) = m \sum_{j=0}^{r-2} \binom{r}{j} (-m)^{r-2-j}.$$

Since $\chi_c(\mathbb{P}^{r-1}) = r$, multiplication by m and the binomial theorem yield

$$\begin{aligned} m\big(r - \chi_c(H)\big) &= mr - \sum_{j=0}^{r-2} \binom{r}{j} (-m)^{r-j} \\ &= mr - \big((1-m)^r + rm - 1\big) = 1 - (1-m)^r. \end{aligned}$$

This proves (2.4). If $m \geq 3$, then $|1-m|^r > 1$. If $m = 2$, the numerator is $1-(-1)^r$, proving the last assertion. □

Proposition 2.3. *Let f and X be as in Theorem 1.2, let b be an ordinary multiple point of X with multiplicity m, and put $r = n-1$. If $e(r,m) \neq 0$, a primitive mth root of unity occurs as a local monodromy eigenvalue of f at a nonzero cone point above b. Consequently, under $\mathcal{H}_f(d)$ with $d > 1$, either $d \nmid m$ or*

$$m = 2, \qquad r \text{ is even}, \qquad d = 2. \tag{2.5}$$

Proof. For a local projective equation g at b, Lemma 2.1 and (2.2) give

$$\zeta_{g,b}(t) = (1-t^m)^{-e(r,m)}. \tag{2.6}$$

Indeed, the fibre of the blowup over b is the exceptional divisor, and its open stratum is $\mathbb{P}^{r-1} \setminus H$. The open strict-transform strata do not meet that fibre.

To check the assertion for the original homogeneous polynomial, choose a cone point with one coordinate nonzero. The corresponding coordinates are (u, y), where u is nonzero and

$$f(u,y) = u^D g(y).$$

Blow up the smooth centre $\{y = b\}$ in this local product. Over the chosen cone point the exceptional fibre, its multiplicity, and its open stratum are exactly those in Lemma 2.1. The factor u^D is a unit throughout this neighbourhood. A'Campo's formula at this cone point is therefore again (2.6).

If $e(r,m) \neq 0$, every primitive mth root is a zero or pole of that rational function. By its definition (2.1), the reciprocal of such a root is an eigenvalue in some cohomological degree; it also has order m. Hence $\mathcal{H}_f(d)$ forces $d \nmid m$. By Lemma 2.2, the only remaining possibility with $d \mid m$ and $d > 1$ is (2.5). □

3. Local twisted integrals

Let K be a nonarchimedean local field of characteristic zero, with notation as in Section 1. In this section χ is a nontrivial character of $k^\times$, inflated to $R^\times$, and $\chi(0) = 0$. We write the same symbol for the two characters.

Lemma 3.1. *Suppose $h \in R[y_1, \dots, y_r]$ and $a \in k^r$ satisfy $\bar{h}(a) = 0$ and $\nabla \bar{h}(a) \neq 0$, where ∇ denotes the gradient. Then*

$$\int_{a+\mathfrak{m}^r} \chi(\operatorname{ac} h(y))|h(y)|^s \, dy = 0. \tag{3.1}$$

Here and below $a + \mathfrak{m}^r$ denotes the residue class, with any integral lift of a used to write it.

Proof. After renumbering, $\partial h/\partial y_1$ is a unit throughout the residue class. For fixed $y_2, \dots, y_r$ in their prescribed residue classes and any $t \in \mathfrak{m}$, Hensel's lemma gives a unique y_1 in its prescribed residue class with $h(y) = t$. Thus

$$y \longmapsto (h(y), y_2, \dots, y_r)$$

is a bijective analytic change of coordinates onto $\mathfrak{m} \times \prod_{i=2}^{r}(a_i + \mathfrak{m})$, with Jacobian of absolute value 1. The integral in (3.1) equals

$$q^{-(r-1)} \int_{\mathfrak{m}} \chi(\operatorname{ac} t)|t|^s \, dt.$$

Decomposing $\mathfrak{m} \setminus \{0\}$ into $\pi^j R^\times$, $j \geq 1$, gives

$$\int_{\mathfrak{m}} \chi(\operatorname{ac} t)|t|^s \, dt = \sum_{j \geq 1} q^{-j} T^j \int_{R^\times} \chi(u) \, du = 0, \tag{3.2}$$

since $\int_{R^\times} \chi(u)\, du = q^{-1} \sum_{c \in k^\times} \chi(c) = 0$. All changes of order are justified by absolute convergence for $\Re(s) > 0$. □

Proposition 3.2. *Let $r, m \geq 2$, $L \geq m$, and*

$$g(y) = g_m(y) + g_{m+1}(y) + \cdots + g_L(y) \in R[y_1, \dots, y_r],$$

where each g_j is homogeneous of degree j, $\bar{g}_m \neq 0$, and $\{\bar{g}_m = 0\} \subset \mathbb{P}_k^{r-1}$ is a smooth hypersurface of degree m. Put

$$S(\bar{g}_m, \chi) = \sum_{a \in k^r} \chi(\bar{g}_m(a)). \tag{3.3}$$

Then

$$\int_{\mathfrak{m}^r} \chi(\operatorname{ac} g(y))|g(y)|^s \, dy = q^{-r} S(\bar{g}_m, \chi) \frac{q^{-r} T^m}{1 - q^{-r} T^m}. \tag{3.4}$$

Proof. Set $U_r = R^r \setminus \mathfrak{m}^r$. Up to the measure-zero origin, $\mathfrak{m}^r$ is the disjoint union of $\pi^j U_r$ for $j \geq 1$. For $z \in U_r$, write

$$g(\pi^j z) = \pi^{jm} g^{(j)}(z), \qquad g^{(j)}(z) = g_m(z) + \pi^j g_{m+1}(z) + \cdots + \pi^{j(L-m)} g_L(z).$$

For every $j \geq 1$ the reduction of $g^{(j)}$ is $\bar{g}_m$. On a primitive residue class $a + \mathfrak{m}^r$, $a \neq 0$, with $\bar{g}_m(a) \neq 0$, the function $g^{(j)}$ is a unit and contributes $q^{-r}\chi(\bar{g}_m(a))$. If $\bar{g}_m(a) = 0$, smoothness of its projective zero set implies $\nabla \bar{g}_m(a) \neq 0$. Lemma 3.1 makes this residue-class contribution zero. Consequently

$$\int_{U_r} \chi(\operatorname{ac} g^{(j)}(z))|g^{(j)}(z)|^s \, dz = q^{-r} S(\bar{g}_m, \chi) \tag{3.5}$$

for every $j \geq 1$. The zero vector contributes zero to the sum in (3.3), so adding it causes no change.

Since $\operatorname{ac}(\pi^{jm} w) = \operatorname{ac}(w)$, the change $y = \pi^j z$ contributes the factor $q^{-rj} T^{mj}$. Summing (3.5) gives

$$q^{-r} S(\bar{g}_m, \chi) \sum_{j \geq 1} (q^{-r} T^m)^j,$$

which is (3.4). The identity first holds in the convergent half-plane and then gives its rational continuation. □

Lemma 3.3. *If $G \in k[y_1, \ldots, y_r]$ is homogeneous of degree m and $d = \operatorname{ord} \chi$ does not divide m, then $\sum_{a \in k^r} \chi(G(a)) = 0$.*

Proof. The image of χ is cyclic of order d, so choose $b \in k^\times$ with $\chi(b)^m \neq 1$. The permutation $a \mapsto ba$ of k^r gives

$$\sum_a \chi(G(a)) = \sum_a \chi(G(ba)) = \chi(b)^m \sum_a \chi(G(a)).$$

The sum is therefore zero. □

Proposition 3.4. *Let k have odd cardinality q, let η be its quadratic character extended by $\eta(0) = 0$, and let Q be a nondegenerate quadratic form in $r \geq 1$ variables. If r is even, then*

$$\sum_{a \in k^r} \eta(Q(a)) = 0. \tag{3.6}$$

More generally, if $Q = \sum_i c_i y_i^2$ after diagonalization and $r = 2h + 1$ is odd, the sum is

$$\eta\left(\prod_i c_i\right) \eta(-1)^h q^h (q-1). \tag{3.7}$$

Proof. The symmetric bilinear form

$$B(x, y) = \frac{Q(x+y) - Q(x) - Q(y)}{2}$$

satisfies $B(x, x) = Q(x)$. Some vector v has $Q(v) \neq 0$: otherwise polarization would give $B = 0$, contrary to nondegeneracy. Every vector has the orthogonal decomposition

$$y = \frac{B(y, v)}{Q(v)} v + \left(y - \frac{B(y, v)}{Q(v)} v\right).$$

The second summand belongs to $v^\perp = \{w : B(w, v) = 0\}$. If a vector in $v^\perp$ is orthogonal to all of $v^\perp$, it is orthogonal to the whole space and hence is zero. Thus the restriction to $v^\perp$ is nondegenerate. Induction gives a diagonalization $Q = \sum_i c_i y_i^2$ with every $c_i \neq 0$. Choose a nontrivial additive character $\psi : k \to \mathbb{C}^\times$ and set

$$G = \sum_{u \in k^\times} \eta(u) \psi(u).$$

Then

$$\begin{aligned} |G|^2 &= \sum_{u, v \neq 0} \eta(u) \eta(v) \psi(u - v) \\ &= \sum_{a \neq 0} \eta(a) \sum_{v \neq 0} \psi((a-1)v) \\ &= (q-1) - \sum_{\substack{a \neq 0 \\ a \neq 1}} \eta(a) = q. \end{aligned} \tag{3.8}$$

The inner sum on the second line is $q - 1$ for $a = 1$ and -1 otherwise, by additive character orthogonality. Changing u to $-u$ also gives $\overline{G} = \eta(-1) G$, where the bar denotes complex conjugation. Hence

$$G^2 = \eta(-1) q. \tag{3.9}$$

For $z \neq 0$, the substitution $u = tz$ gives

$$\eta(z) = G^{-1} \sum_{t \neq 0} \eta(t) \psi(tz). \tag{3.10}$$

For $z = 0$, both sides are zero by $\sum_{t\neq 0}\eta(t) = 0$, so (3.10) holds for every $z \in k$. For $b \neq 0$, the number of solutions of $x^2 = y$ is $1 + \eta(y)$; hence

$$\sum_{x\in k}\psi(bx^2) = \sum_{y\in k}(1+\eta(y))\psi(by) = \eta(b)G. \tag{3.11}$$

Using (3.10) and then (3.11), we obtain

$$\begin{aligned}\sum_{y\in k^r}\eta(Q(y)) &= G^{-1}\sum_{t\neq 0}\eta(t)\prod_{i=1}^{r}\sum_{y_i\in k}\psi(tc_iy_i^2)\\ &= \eta\left(\prod_i c_i\right)G^{r-1}\sum_{t\neq 0}\eta(t)^{r+1}. \qquad (3.12)\end{aligned}$$

For even r, the final sum is $\sum_{t\neq 0}\eta(t) = 0$. For odd $r = 2h+1$, it is $q-1$, and (3.9) yields (3.7). □

Corollary 3.5. *Suppose g satisfies Proposition 3.2, the residue characteristic is odd, and $d = \operatorname{ord}\chi > 1$. If either $d \nmid m$ or (2.5) holds, then the integral in (3.4) vanishes identically.*

Proof. Lemma 3.3 treats $d \nmid m$. In the other case, χ is the unique quadratic character of $k^\times$. Write $\bar g_2(y) = y^{\mathsf t}Cy$, with C symmetric. Since $\nabla\bar g_2(y) = 2Cy$, a nonzero vector in $\ker C$ would give a singular projective zero of $\bar g_2$. Smoothness therefore makes C invertible, so Proposition 3.4 applies. In both cases the coefficient $S(\bar g_m, \chi)$ in (3.4) is zero. □

4. Proof of the homogeneous theorem

4.1. **The finite set of places.** Let Σ be the reduced singular locus of X. It is a finite reduced F-scheme, hence finite étale because F has characteristic zero. Choose a finite set S of places and put $B = \mathcal{O}_{F,S}$, the ring of S-integers. Clearing denominators in f and in the defining data of the finite embedded scheme Σ, and inverting its nonzero discriminants, gives models

$$\Sigma_B \subset X_B \subset \mathbb{P}^{n-1}_B,$$

with f integral and Σ_B finite étale over B. We enlarge S and base-change these models as needed. The notation X_k and Σ_k refers to their fibres over a remaining residue field k.

After a finite splitting extension of F, the resulting centre is a disjoint union of sections in finitely many affine projective charts. Translate the dehomogenized equation by each section. All its coefficients of degrees below the corresponding multiplicity m_i vanish identically. Invert a nonzero degree-m_i coefficient, and the projective tangent equation retains degree m_i in every remaining fibre. Its smoothness is checked on finitely many affine charts: the equation and the relevant partial derivatives generate the unit ideal on the generic fibre, and clearing denominators preserves these identities after a further localization. Inverting the norms of the finitely many elements used, and excluding the ramified primes of the splitting extension, gives a finite set of places of F with the same geometric conclusions.

Cover $X_B \setminus \Sigma_B$ by finitely many affine opens $\operatorname{Spec} C_\nu$ inside projective coordinate charts. Let $J_\nu \subset C_\nu$ be the relative Jacobian ideal. Smoothness of the generic fibre gives

$$(C_\nu/J_\nu)\otimes_B F = 0.$$

Hence some nonzero element $a_\nu \in B$ belongs to J_ν: localization sends the class of 1 to zero, so it is killed by a nonzero denominator. Inverting $\prod_\nu a_\nu$ makes every one of these opens fibrewise smooth. Conversely, the preserved Taylor expansions with $m_i \geq 2$ show that all points of Σ_k remain singular. Thus the geometric singular points of X_k are exactly those of Σ_k, with the prescribed multiplicities and smooth tangent hypersurfaces.

Also exclude residue characteristics dividing $2D$ or an ordinary multiplicity. By the finite étale form of Hensel's lemma, reduction induces a bijection $\Sigma_B(R) \longrightarrow \Sigma_B(k)$. Thus every

residue-field-rational singular point has a unique integral lift. In a projective affine chart containing this point, translation by its integral coordinates gives an equation

$$g_b(y) = g_m(y) + \cdots + g_L(y) \tag{4.1}$$

with all coefficients in R, no terms of degree below m, and smooth reduced projective tangent cone. Thus Proposition 3.2 applies to its entire singular residue class.

Enlarge S to include the exceptional places in the following theorem.

Theorem 4.1 (Denef's degree theorem [5])**.** *Let h be a nonconstant polynomial over F with $h(0) = 0$. For all but finitely many completions K of F, and every character χ induced from $k^\times$, the following implication holds: if local monodromy of h at 0 has no eigenvalue of exact order* $\operatorname{ord}\chi$*, then*

$$\deg_T Z_{h,0}(\chi, T) < 0. \tag{4.2}$$

The degree of a nonzero rational function is numerator degree minus denominator degree after cancellation; $\deg_T 0 = -\infty$.

This is Theorem 1.2 of [5]. It applies to the vertex of the present affine cone, and (1.3) implies its exact-order hypothesis at 0.

4.2. **Characters and the vertex.** Fix a remaining place and a character χ, suppress χ in the zeta notation when convenient, and put $d = \operatorname{ord}\chi$. Since $\mathcal{H}_f(d)$ holds, $d > 1$. If $d \nmid D$, choose $u \in R^\times$ with $\chi(u)^D \neq 1$. The substitutions $x \mapsto ux$ preserve R^n, $\mathfrak{m}^n$, and Haar measure, while multiplying the integrand by $\chi(u)^D$. Consequently

$$Z_f = Z_{f,0} = 0 \qquad (d \nmid D). \tag{4.3}$$

Suppose $\chi|_{1+\mathfrak{m}} \neq 1$, and put $p = \operatorname{char} k$. Continuity gives $1 + \mathfrak{m}^N \subset \ker\chi$ for some N. The quotients $(1+\mathfrak{m}^j)/(1+\mathfrak{m}^{j+1})$, $j \geq 1$, are additive groups of characteristic p, so the image of $1 + \mathfrak{m}$ is a nontrivial finite p-group. Since $p \nmid D$, raising to the Dth power is an automorphism of that image. We may therefore choose $u \in 1 + \mathfrak{m}$ with $\chi(u)^D \neq 1$. Multiplication by u preserves each residue class in R^n, so

$$Z_{f,a} = 0 \quad \text{for every } a \in k^n. \tag{4.4}$$

It remains to treat characters trivial on $1 + \mathfrak{m}$; write $\bar\chi$ for the corresponding nontrivial residue character. We first examine the nonzero residue classes, for all their orders d.

4.3. **Nonzero residue classes.** If $a \in k^n$ and $\bar f(a) \neq 0$, the integrand is constant on its residue class, with absolute-value factor 1. Thus

$$Z_{f,a} = q^{-n}\bar\chi(\bar f(a)). \tag{4.5}$$

Suppose now $a \neq 0$ and $\bar f(a) = 0$. Choose j with $a_j \neq 0$ and use coordinates

$$x_j = u, \qquad x_i = uy_i \quad (i \neq j).$$

This is a bijection of the residue class with the product of $a_j + \mathfrak{m}$ and the projective-coordinate residue class $b = (a_i/a_j)_{i\neq j}$. Its Jacobian is u^{n-1}, a unit. By homogeneity, $f(x) = u^D f_j(y)$, where f_j is the dehomogenization in this chart. Since χ is trivial on $1 + \mathfrak{m}$,

$$Z_{f,a} = q^{-1}\bar\chi(a_j^D) \int_{b+\mathfrak{m}^{n-1}} \chi(\operatorname{ac} f_j(y))|f_j(y)|^s\, dy. \tag{4.6}$$

If b is smooth on X_k, Lemma 3.1 makes the integral zero. If b is singular, use its lift and the translated equation (4.1). Its multiplicity m belongs to a geometric ordinary point of X in characteristic zero. By Proposition 2.3, the global spectral hypothesis forces $d \nmid m$ or (2.5). Corollary 3.5 again makes the integral zero. Therefore

$$Z_{f,a} = 0 \qquad (a \neq 0,\ \bar f(a) = 0). \tag{4.7}$$

In particular the primitive-annulus integral

$$A(T) = \int_{R^n \setminus \mathfrak{m}^n} \chi(\operatorname{ac} f(x)) |f(x)|^s \, dx$$

is the constant

$$A = q^{-n} \sum_{a \in k^n} \bar{\chi}(\bar{f}(a)), \qquad \bar{\chi}(0) = 0. \tag{4.8}$$

The added term $a = 0$ is zero because f is homogeneous of positive degree.

4.4. **Radial cancellation.** The change $x = \pi y$ gives

$$Z_{f,0}(T) = q^{-n} T^D Z_f(T), \qquad Z_f(T) = A + Z_{f,0}(T). \tag{4.9}$$

Thus

$$Z_f(T) = \frac{A}{1 - q^{-n} T^D}, \qquad Z_{f,0}(T) = \frac{q^{-n} A T^D}{1 - q^{-n} T^D}. \tag{4.10}$$

This is the radial step of [8, Theorem 3.5], with the primitive-annulus measure used instead of normalized projective measure. The identities follow from convergent integrals and hence hold as rational identities.

For $d \nmid D$, (4.3) already proves the assertion. For $d \mid D$, suppose $A \neq 0$. The numerator and denominator of $Z_{f,0}$ in (4.10) are coprime: the numerator has only the root 0, and the denominator has nonzero constant term. Both have degree D, so $\deg_T Z_{f,0} = 0$. This contradicts Theorem 4.1, because (1.3) excludes eigenvalues of exact order d at the vertex. Hence $A = 0$. Formula (4.10) proves (1.4) and the zero value of the origin residue class. Together with (4.4), (4.5), and (4.7), this proves every assertion of Theorem 1.2. □

Corollary 4.2. *Under the hypotheses and at the places of Theorem 1.2, every nontrivial character $\bar{\chi} : k^\times \to \mathbb{C}^\times$ satisfying $\mathcal{H}_f(\operatorname{ord} \bar{\chi})$ obeys*

$$\sum_{a \in k^n} \bar{\chi}(\bar{f}(a)) = 0. \tag{4.11}$$

For every function $\Phi : k^n \to \mathbb{C}$,

$$\int_{R^n} \Phi(\bar{x}) \chi(\operatorname{ac} f(x)) |f(x)|^s \, dx = q^{-n} \sum_{\substack{a \in k^n \\ \bar{f}(a) \neq 0}} \Phi(a) \bar{\chi}(\bar{f}(a)),$$

where χ is the inflation of $\bar{\chi}$. In particular this integral is constant in s, and is zero if Φ is supported on $\{\bar{f} = 0\}$.

Proof. The finite sum in (4.11) is $q^n A = 0$. The function $x \mapsto \Phi(\bar{x})$ is a finite linear combination of residue-class indicators. Apply (1.5) term by term. □

5. Quadratic cones

Example 5.1. Let $h \geq 1$ be an integer, put $r = 2h$ and $n = r + 1$, and set

$$f(x_1, \ldots, x_r, z) = x_1^2 + \cdots + x_r^2. \tag{5.1}$$

Its hypersurface in $\mathbb{P}^r$ is reduced and has its unique singular point at $[0 : \cdots : 0 : 1]$. The chart $z = 1$ has a nondegenerate quadratic local equation, so this is an ordinary double point. Projection onto $[x_1 : \cdots : x_r]$ identifies its complement with an $\mathbb{A}^1$-bundle over

$$\mathbb{P}^{r-1} \setminus \{x_1^2 + \cdots + x_r^2 = 0\}.$$

Consequently its Euler characteristic is $e(r, 2) = 0$.

The affine singular locus is the line $x_1 = \cdots = x_r = 0$. At a point on this line, contraction of the free coordinate identifies the Milnor fibre up to homotopy with that of $\sum_i x_i^2$. For a small positive target value ϵ, scale x by $\epsilon^{-1/2}$. The local quadratic fibre becomes

$$\left\{ x \in \mathbb{C}^r : \sum_i x_i^2 = 1,\ \|x\| < R_0 \right\}, \qquad R_0 > 1.$$

Here $\|\cdot\|$ is the Euclidean norm. Write $x = a + ib$, with $a, b \in \mathbb{R}^r$. Then $a \cdot b = 0$ and $\|a\|^2 = 1 + \|b\|^2$. Setting $u = a/\sqrt{1 + \|b\|^2}$ identifies this fibre with

$$\left\{ (u, b) : u \in S^{r-1},\ b \perp u,\ \|b\|^2 < (R_0^2 - 1)/2 \right\}.$$

Here $S^{r-1} \subset \mathbb{R}^r$ is the unit sphere; the inverse map is $(u, b) \mapsto \sqrt{1 + \|b\|^2}\, u + ib$. This is a disc subbundle of the tangent bundle TS^{r-1}, and $b \mapsto tb$, $0 \le t \le 1$, gives a deformation retraction onto the zero section. The path $x \mapsto e^{\pi i t} x$, $0 \le t \le 1$, lifts one positive turn of the target value, so quadratic monodromy is $x \mapsto -x$. It induces the antipodal map on S^{r-1}, of degree $(-1)^r = 1$. Its degree-zero eigenvalue is also 1. At the smooth points of the zero fibre the Milnor fibre is contractible. Thus $\mathcal{H}_f(d)$ holds for every $d > 1$.

For odd residue characteristic and a quadratic character, integrating the free coordinate z contributes 1. Proposition 3.4 and the same radial decomposition in r variables give the vanishing directly. Characters of order not dividing 2 are handled by dilation. This family illustrates the zero Euler characteristic branch in every odd ambient dimension $n \ge 3$.

Remark 5.2. At $a = (1, 0, \ldots, 0)$ in Example 5.1, a quadratic character gives $Z_{f,a} = q^{-n}$. These residue-class constants cancel in the integral over R^n.

Appendix A. Arithmetic certificate

The Python 3 program below checks the normalization, parity, and finite-field sums in (3.4) and (3.12). It uses only the standard library. The preceding proofs establish the identities for all parameters in their stated ranges. The Euler calculation is checked for $2 \le r, m \le 12$.

For an odd prime p and $q = p^2$, encode $a + bu$ in $\mathbb{F}_p[u]/(u^2 - \nu)$ by the integer $a + pb$, where $0 \le a, b < p$ and ν is a nonsquare in $\mathbb{F}_p$. The polynomial $u^2 - \nu$ has no root in $\mathbb{F}_p$, so the quotient is a field. The count array is the distribution of the values of a diagonal quadratic form. It is calculated by convolution, with an independent enumeration check in dimensions at most 4. Both determinant square classes are included. Odd dimensions check the nonzero formula (3.7) as well as the even-dimensional vanishing. The table lists all forty character sums. Each pair gives the values for square and nonsquare determinants, in that order.

$q \backslash r$	1	2	3	4	6
3	$(2, -2)$	$(0, 0)$	$(-6, 6)$	$(0, 0)$	$(0, 0)$
5	$(4, -4)$	$(0, 0)$	$(20, -20)$	$(0, 0)$	$(0, 0)$
7	$(6, -6)$	$(0, 0)$	$(-42, 42)$	$(0, 0)$	$(0, 0)$
9	$(8, -8)$	$(0, 0)$	$(72, -72)$	$(0, 0)$	$(0, 0)$

The final six checks use the ordinary germ $g(y) = \sum_i (y_i^2 + y_i^3)$ on $p\mathbb{Z}_p^r$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Write v_p for the valuation with $v_p(p) = 1$. At precision $M \ge 1$, the entry indexed by $0 \le v < M$ is

$$\sum_{\substack{y \in (p\mathbb{Z}/p^M\mathbb{Z})^r \\ v_p(g(y)) = v}} \eta(p^{-v} g(y) \bmod p).$$

For $v < M$, the summand is independent of the integral lifts of y. Multiplying this integer by p^{-Mr} gives the exact coefficient of T^v in the integral. Terms with $g(y) \equiv 0 \pmod{p^M}$ are

unresolved at that precision and do not enter these coefficients. Writing $S = \sum_{a \in \mathbb{F}_p^r} \eta(\sum_i a_i^2)$, formula (3.4) predicts the integer count

$$p^{Mr} S p^{-r(j+1)} = S p^{r(M-j-1)} \quad \text{at } v = 2j < M, \qquad j \geq 1,$$

and zero at all other resolved valuations. The program computes S independently by enumeration. The three even-dimensional cases test vanishing; the three odd-dimensional cases give nonzero controls. The nonzero counts are

(p, r, M)	S	v	count
$(3, 3, 5)$	-6	2	$-6 \cdot 3^9 = -118098$
$(3, 3, 5)$	-6	4	$-6 \cdot 3^6 = -4374$
$(5, 3, 3)$	20	2	$20 \cdot 5^3 = 2500$
$(7, 3, 3)$	-42	2	$-42 \cdot 7^3 = -14406$

The first two rows test the measure factor at two radial shells.

```
"""Exact checks for the identities in Holomorphy for homogeneous polynomials.

Run with Python 3. No input files or third-party packages are required.
The all-parameter proofs are in the article; this script checks specializations.
"""
from itertools import product
from math import comb


def finite_field(p, degree):
    assert degree in (1, 2)
    q = p ** degree
    nu = 0
    if degree == 2:
        nu = next(a for a in range(2, p)
                  if pow(a, (p - 1) // 2, p) == p - 1)

    def add(x, y):
        if degree == 1:
            return (x + y) % p
        return ((x % p + y % p) % p
                + p * ((x // p + y // p) % p))

    def mul(x, y):
        if degree == 1:
            return x * y % p
        a, b = x % p, x // p
        c, d = y % p, y // p
        return ((a * c + nu * b * d) % p
                + p * ((a * d + b * c) % p))

    def power(x, e):
        ans = 1
        while e:
            if e & 1:
                ans = mul(ans, x)
            x = mul(x, x)
            e //= 2
        return ans

    def eta(x):
```

```
        if x == 0:
            return 0
        a = power(x, (q - 1) // 2)
        assert a in (1, p - 1)
        return 1 if a == 1 else -1

    assert all(power(x, q - 1) == 1 for x in range(1, q))
    return q, add, mul, eta


def quadratic_counts(q, add, mul, coefficients):
    counts = [0] * q
    counts[0] = 1
    for a in coefficients:
        nxt = [0] * q
        for value, count in enumerate(counts):
            for x in range(q):
                y = add(value, mul(a, mul(x, x)))
                nxt[y] += count
        counts = nxt
    return counts


def direct_counts(q, add, mul, coefficients):
    counts = [0] * q
    for vector in product(range(q), repeat=len(coefficients)):
        value = 0
        for a, x in zip(coefficients, vector):
            value = add(value, mul(a, mul(x, x)))
        counts[value] += 1
    return counts


def contact_sums(p, r, precision):
    modulus = p ** precision
    sums = [0] * precision
    for vector in product(range(0, modulus, p), repeat=r):
        value = sum(x * x + x ** 3 for x in vector) % modulus
        if value == 0:
            continue
        valuation = 0
        while value % p == 0:
            value //= p
            valuation += 1
        residue = pow(value % p, (p - 1) // 2, p)
        sums[valuation] += 1 if residue == 1 else -1
    return sums


def main():
    euler_checks = 0
    for r in range(2, 13):
        for m in range(2, 13):
            hypersurface = m * sum(
                comb(r, j) * (-m) ** (r - 2 - j)
                for j in range(r - 1))
```

```
            numerator = 1 - (1 - m) ** r
            assert numerator % m == 0
            e = numerator // m
            assert r - hypersurface == e
            assert (e == 0) == (m == 2 and r % 2 == 0)
            euler_checks += 1
    print("Euler identities:", euler_checks, "checked")

    total = 0
    for p, degree in ((3, 1), (5, 1), (7, 1), (3, 2)):
        q, add, mul, eta = finite_field(p, degree)
        nonsquare = next(a for a in range(1, q) if eta(a) == -1)
        for r in (1, 2, 3, 4, 6):
            for last in (1, nonsquare):
                coefficients = [1] * (r - 1) + [last]
                counts = quadratic_counts(q, add, mul, coefficients)
                assert sum(counts) == q ** r
                if r <= 4:
                    assert counts == direct_counts(q, add, mul,
                                                   coefficients)
                value = sum(eta(a) * counts[a] for a in range(q))
                expected = 0
                if r % 2:
                    h = (r - 1) // 2
                    expected = eta(last) * eta(p - 1) ** h
                    expected *= q ** h * (q - 1)
                assert value == expected
                total += 1
        print("Finite field q=" + str(q) + ": 10 quadratic forms checked")
    print("Quadratic checks:", total, "passed")

    cases = ((3, 2, 4), (3, 4, 4), (5, 2, 3),
             (3, 3, 5), (5, 3, 3), (7, 3, 3))
    for p, r, precision in cases:
        sums = contact_sums(p, r, precision)
        q, add, mul, eta = finite_field(p, 1)
        counts = direct_counts(q, add, mul, [1] * r)
        leading_sum = sum(eta(a) * counts[a] for a in range(q))
        expected = [0] * precision
        for j in range(1, (precision - 1) // 2 + 1):
            expected[2 * j] = (leading_sum
                * p ** (r * (precision - j - 1)))
        assert sums == expected
        print("Ordinary germ p=" + str(p) + ", r=" + str(r)
              + ", precision=" + str(precision) + ":", sums)
    print("All exact checks passed.")


if __name__ == "__main__":
    main()
```

The complete output is:

```
Euler identities: 121 checked
Finite field q=3: 10 quadratic forms checked
Finite field q=5: 10 quadratic forms checked
Finite field q=7: 10 quadratic forms checked
Finite field q=9: 10 quadratic forms checked
Quadratic checks: 40 passed
Ordinary germ p=3, r=2, precision=4: [0, 0, 0, 0]
Ordinary germ p=3, r=4, precision=4: [0, 0, 0, 0]
Ordinary germ p=5, r=2, precision=3: [0, 0, 0]
Ordinary germ p=3, r=3, precision=5: [0, 0, -118098, 0, -4374]
Ordinary germ p=5, r=3, precision=3: [0, 0, 2500]
Ordinary germ p=7, r=3, precision=3: [0, 0, -14406]
All exact checks passed.
```

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