The border rank of the five-row permanent

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Abstract

The five-row permanent has geometric Segre border rank sixteen in characteristic zero and in characteristic two. This settles the five-row case of a question of Han and Song. A degeneration of length fifteen would give compatible subspaces in a product of Grassmannians over the integers. In characteristic two the permanent becomes the determinant, so a common Borel subgroup fixes one such collection. The pair spaces admit six types. On the remaining slot sets, initial coordinates of the subspaces must satisfy implications induced by elementary substitutions and contractions. Two finite Boolean certificates exclude every pair assignment. Properness transfers the obstruction to characteristic zero.

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1 The five-row problem

For a field F let $V = F^5$ with basis $e_1, \dots, e_5$. With one tensor factor for each row, the permanent is

$$P = \mathrm{per}_5 = \sum_{\sigma \in S_5} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(5)} \in V^{\otimes 5}.$$

Its geometric border rank $\underline{\mathrm{R}}_F(P)$ is the least r such that $[P]$ lies on the rth secant variety of $(\mathbb{P}V)^5$ in its Segre embedding, after passing to an algebraic closure. The five vectors in each summand vary independently.

Theorem 1.1. *If F has characteristic zero or two, then $\underline{\mathrm{R}}_F(\mathrm{per}_5) = 16$. In characteristic two this also gives $\underline{\mathrm{R}}_F(\det_5) = 16$.*

Han, Ju and Kim [6, Theorem 5.6] obtained the lower bound fifteen in characteristic zero from a recursive Koszul flattening. Glynn's sign formula [4] gives an ordinary decomposition of length sixteen outside characteristic two. Han and Song [8, Theorem A and Question 4.2] proved that the ordinary rank in characteristic zero is sixteen and asked whether the same is true of border rank. Han [5, Theorem A and §4.2] computed the border Waring rank of $x_1 \cdots x_n$, including the value sixteen at $n = 5$. Theorem 1.1 identifies the Segre and Veronese border ranks in this case in characteristic zero.

The geometric upper bound sixteen has a common proof over all fields. Evaluate the row inputs in the affine chart $(z_1, z_2, z_3, z_4, 1)$, multiply the five affine linear forms, and extract the coefficient of $z_1 z_2 z_3 z_4$ modulo $z_i^2 = 0$. This gives exactly P. Deform to $z_i^2 = t z_i$. The sixteen square-free monomials form a free basis over $F[t]$. For $t \neq 0$ the algebra is the coordinate algebra of $\{0, t\}^4$, and the same coefficient functional is a linear combination of its sixteen evaluations. Its specialization at zero gives the required border decomposition.

2 A projective incidence over the integers

The border-apolarity method of Buczyńska and Buczyński [2, Theorems 3.15 and 4.3] records limits of rank-one points using several homogeneous degrees. We use only the degrees specified by subsets of the slots. We give the construction over $\mathbb{Z}$ to keep track of specialization. A fixed-ideal computation for the complex four-row determinant appears in Han, Ju and Kim [7].

Let $N_1, \dots, N_m$ be finite free $\mathbb{Z}$-modules, let $T \in \bigotimes_i N_i$, and fix r. For each nonempty $J \subseteq [m]$ put $L_J = \bigotimes_{i \in J} N_i$ and $h_J = \min(r, \operatorname{rank} L_J)$. Let $\mathcal{Y}_T$ parametrize subspaces W_J of the prescribed dimensions satisfying

$$\operatorname{im}\left(T : \bigotimes_{i \notin J} N_i^* \longrightarrow L_J \right) \subseteq W_J, \tag{1}$$

$$\iota_{i,\ell}(W_J) \subseteq W_{J \setminus \{i\}} \qquad (i \in J,\ |J| \geq 2), \tag{2}$$

where $\iota_{i,\ell}$ denotes contraction against a basis covector $\ell \in N_i^*$. More precisely, inside $\prod_J \mathrm{Gr}_{\mathbb{Z}}(h_J, L_J)$ these conditions are the vanishings of the indicated bundle maps to the corresponding tautological quotients. They define a projective scheme over $\mathbb{Z}$ and give the same equations after base change.

Lemma 2.1. *If T has geometric border rank at most r over an algebraically closed field K, then $\mathcal{Y}_{T,K}$ is nonempty. If it is empty over $\overline{\mathbb{F}}_p$ for some prime p, then T has border rank greater than r in characteristic zero and outside a finite set of positive characteristics.*

Proof. Take r general Segre points. For every J their projections span an h_J-plane. This follows from nondegeneracy of the projected Segre variety and the simultaneous openness of the finitely many maximal-rank conditions. Every tensor in their span satisfies (1) and (2) with the projected planes. Take the closure of the graph of these data in projective tensor space times the Grassmannians. Its projection is closed and contains a dense subset of the secant variety, hence contains the whole secant variety. The two incidence conditions persist on its fibers.

The image of $\mathcal{Y}_T \to \operatorname{Spec} \mathbb{Z}$ is closed. If its geometric fiber over (p) is empty, the image is a proper closed subset, a finite set of closed points. Thus the generic geometric fiber is empty, as are all but finitely many positive-characteristic fibers. The first assertion gives the lower bounds over their algebraically closed extensions. □

Apply the lemma to the integral tensor P at $r = 15$. In characteristic two, $P = \det_5$ is invariant under the common $\mathrm{SL}(V)$ action. For the rest of the proof take $F = \overline{\mathbb{F}}_2$. If the projective fiber $\mathcal{Y}_{P,F}$ is nonempty, the Borel fixed point theorem [1, Theorem III.10.4] provides a point fixed by a common triangular subgroup $B \subset \mathrm{SL}(V)$. The family at this point satisfies

$$\dim W_J = \begin{cases} 5 & |J| = 1, \\ 15 & |J| \geq 2, \end{cases} \qquad \bigwedge^{|J|} V \subseteq W_J.$$

Here $\bigwedge^j V$ is embedded by sending a set of j different letters to the sum of its distinct orderings. These sums span precisely the images in (1), as one sees by contracting a complementary word with the five-row determinant. We now exclude such a family.

3 Two cases for the pair spaces

Choose notation so B contains the substitutions $e_i \mapsto e_i + te_j$ for $i < j$. For a pair of slots s, t, consider the quotient map $V^{\otimes 2} \to \operatorname{Sym}^2 V$. Its kernel is the ten-dimensional span of the exchanges $e_i \otimes e_j + e_j \otimes e_i$ for $i < j$, entirely contained in W_{st}. Thus W_{st} is the inverse image of a B-stable five-dimensional subspace of $\operatorname{Sym}^2 V$.

Lemma 3.1. *Write ij for $e_i e_j$. The quotient of every pair space is the span of one row of the following table.*

A	$15, 25, 35, 45, 55$	B	$25, 35, 44, 45, 55$
C	$34, 35, 44, 45, 55$	D	$33, 35, 44, 45, 55$
E	$22, 33, 44, 45, 55$	F	$11, 22, 33, 44, 55.$

Proof. All fifteen quadratic weights restrict to distinct characters of the diagonal torus of SL_5, since their total degrees agree. A stable space is therefore spanned by monomials. A square ii forces the squares jj with $j > i$, since $(e_i + te_j)^2 = e_i^2 + t^2 e_j^2$. A product of two different letters forces every monomial obtained by raising either index, including squares. These rules suffice for stability under the elementary substitutions and the torus. Retaining the five-element subsets closed under them gives the six rows. The verifier repeats this enumeration directly on the $\binom{15}{5}$ subsets. □

Let U_* be the union of all six rows and U_X the union of the five rows other than A. If some pair has type A, permute slots so it is the first pair. In this case use the bound $U_{12} = A$ and put $U_{st} = U_*$ at every other pair. In case X, put $U_{st} = U_X$ at all pairs. For either choice, the actual W_{st} lies in the inverse image of the span of U_{st}. The union bounds are sufficient; the certificate uses no individual type choices at the other pairs.

For a nonempty J, fix a weight α, a multiset of $|J|$ letters. Form a graph on the words of multiplicity α on the ordered slots of J. At slots s, t with letters $a \neq b$, join a word to the word obtained by their transposition whenever $ab \notin U_{st}$. Discard each connected component containing a word with equal letters a at s, t and $aa \notin U_{st}$. Order the retained components by their lexicographically first words.

Lemma 3.2. *The α-weight space of W_J lies in the span of the indicator sums of the retained components. The direct sum of these ambient spaces over the weights is stable under B, and single-slot contraction maps it into the corresponding smaller ambient space.*

Proof. The quotient of a two-slot tensor in $\mathrm{Sym}^2 V$ has coefficient at ab, $a \neq b$, equal to the sum of the two ordered coefficients. In characteristic two, exclusion of ab forces these coefficients to agree. Exclusion of aa forces the diagonal coefficient to vanish. Contracting all other slots yields exactly the swap equalities and zero coordinates encoded by the graph. Conversely the component spans are the spaces cut out by these pair conditions. Each bound is the sum of B-stable quadratic spaces, so its inverse image is B-stable. The induced pair conditions are preserved under dropping a slot by contraction. This proves the assertions. □

4 Initial coordinates and the certificate

A subspace of an ordered coordinate space has a well-defined set of initial coordinates, the possible positions of the first nonzero coordinate of a vector in it. Their number equals the dimension, by row reduction. For each J and α use the indicator basis of Lemma 3.2. Introduce a Boolean variable x_i for membership of the ith coordinate in the initial set of the true weight space. The torus preserves W_J, so the sums of these counts over all weights must give h_J. For a set of different letters, the first coordinate is present by the corresponding exterior tensor.

Lemma 4.1. *Suppose a linear map M between two ordered ambient spaces carries W into W'. Let c_i be its ith column and P_i the initial set of the span of the later columns. Reduce c_i by an echelon basis of that span. If the reduced column has first nonzero coordinate $h \notin P_i$, the initial-coordinate variables satisfy*

$$x_i \quad \Longrightarrow \quad x'_h \ \vee \bigvee_{j \in P_i,\, j<h} x'_j .$$

Proof. A vector witnessing x_i can be scaled to have coordinate one at i. Its image lies in the coset of the reduced column modulo the later-column span. Adding a vector of that span either preserves its first nonzero coordinate h or introduces an earlier one from P_i. Every such image is nonzero and belongs to W'. □

Apply the lemma to all single-slot contractions against basis covectors and to each coefficient of positive degree in the substitutions $e_a \mapsto e_a + te_b$, $a < b$. Coefficient extraction maps the chosen weight to another weight of W_J because all evaluations belong to W_J and the field is infinite. On a word the t^ℓ coefficient is the sum over all choices of ℓ occurrences of a to replace by b. Compute each column by applying this rule, or a contraction, to the indicator sum of its component. The resulting coefficients agree on every retained target component and vanish on every discarded component. The verifier checks these properties when it builds the matrices. Their entries lie in $\mathbb{F}_2$, and their echelon pivots are unchanged by extending the field. Consequently Lemma 4.1 applies to the unknown subspaces over the algebraic closure.

Lemma 4.2 (Finite obstruction)**.** *In each of cases A and X, the prescribed dimensions, the required exterior initial coordinates, and the implications of Lemma 4.1 are jointly inconsistent.*

Proof. The accompanying certificate consists of two clausal proofs. The table gives the size of each Boolean system and the number of certified additions, including the final empty clause.

Case	Initial coordinates	All variables	Clauses	Proof additions
A	2 157	36 450	157 651	1 747
X	2 148	36 297	157 887	2 251

The additional variables enforce the dimensions. For a list $x_1, \dots, x_q$ at a fixed slot set, let $d_{i,j}$ assert that at least j among the first i variables are true. For $1 \le j \le h_J + 1$ impose

$$d_{i,j} = d_{i-1,j} \vee (d_{i-1,j-1} \wedge x_i), \qquad d_{i,0} = 1, \qquad d_{0,j} = 0 \quad (j > 0).$$

Require $d_{q,h_J} = 1$ and $d_{q,h_J+1} = 0$. Each recurrence is represented by the four elementary clauses for this Boolean identity. Together with the exterior unit clauses and the linear-map implications, these are all the axioms.

The files for cases A and X in the source archive (directory `anc/`) use the positive-hint reverse-unit-propagation part of the LRAT format [3]. To verify any added clause, assume its negation and follow the indicated earlier clauses. At each step all but one literal are false, forcing the remaining one, until a fully falsified clause is reached. Thus the added clause follows from the earlier ones. Both traces give this derivation for every addition and end with the empty clause. Running `python3 anc/verify.py` from the extracted source directory regenerates the axioms, checks both traces and returns the counts above. It uses the Python standard library. □

Proof of Theorem 1.1. Lemma 3.1 places every putative fixed family in one of the two cases of Lemma 4.2. Lemmas 3.2 and 4.1 associate a satisfying assignment to that family, contrary to the finite obstruction. Thus the characteristic-two Grassmannian fiber is empty. Lemma 2.1 excludes border rank fifteen in characteristics two and zero. The sixteen-point upper bound proves equality. The same argument gives equality outside a finite set of odd characteristics. □

A A dual check of the finite obstruction

The linear-map clauses can also be reconstructed from rows. This gives an independent check on the use of the characteristic-two orbit bases. For a contraction map, insert the deleted letter at the specified position of a target representative word. Its resulting source component determines the corresponding matrix row, with zero for a discarded component. For the coefficient of t^ℓ in $e_a \mapsto e_a + te_b$, start from a target representative word and change ℓ of its occurrences of b back to a. Record the parity of the number of resulting words in each retained source component. These parities form the matrix row. In particular this computation does not require a choice of one preimage for each orbit.

For a proposed source pivot i, restrict each row of M to the coordinates $i, i+1, \dots$ and read the rows from first to last. Record each row whose addition increases the rank of the row span. Stop as soon as the first coordinate functional e_i^* belongs to this span. If it never belongs, the coset of possible images includes zero and this test gives no implication. Otherwise, exactly the recorded row indices can be the initial index of Mx among the choices with $x_i = 1$ and $x_j = 0$ for $j < i$. Indeed the preceding rows vanish on a space on which e_i^* is still nonzero. A new independent row can be nonzero there together with e_i^*, and a dependent row vanishes there. Once e_i^* is in the span, all subsequent first indices are excluded. This is row elimination on the dual of the affine coset in Lemma 4.1.

The reverse construction and this row test independently validate all distinct clauses involving only initial-coordinate variables, 20 510 in case A and 21 322 in case X. The unary-counter clauses

can be checked against their four truth-table implications. Separate replay of each annotated proof then certifies the empty clause from these axioms.

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