The border Waring rank of powers of a square-free monomial

Yilin Liu

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Abstract

For all $n, d \geq 2$, the geometric border Waring rank of $(x_1 \cdots x_n)^{d-1}$ in characteristic zero is d^{n-1}. Flag-orbit flattenings and digit projections certify the lower bound. In the auxiliary characteristic, Frobenius makes the tensor invariant under the special linear group, and the flattening carries a highest flag vector to the next one. A nonzero integral minor transfers this bound to characteristic zero at prime-power heights. Splitting the multiplicities into digits multiplies these linear rank bounds and gives the formula at every height. A Schur-complement step propagates sharp certificates to unequal exponents in two directions. Normalized matching determines the border Waring rank when two exponents are sufficiently large compared with all the others. A strict growth argument also gives the conjectured value for every four-variable monomial with a linear factor.

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For $\boldsymbol{a} = (a_1, \ldots, a_n)$ with $a_i > 0$, let $N = \sum_i a_i$ and let $\mathcal{D}_{\boldsymbol{a}} \in (F^n)^{\otimes N}$ be the sum of the distinct words containing a_i copies of the ith basis vector. We take geometric border Waring rank in the closure of spans of Veronese points $v^{\otimes N}$ over the algebraic closure. In characteristic zero $\mathcal{D}_{\boldsymbol{a}}$ corresponds, up to a nonzero scalar, to $x_1^{a_1} \cdots x_n^{a_n}$. The word definition also applies in positive characteristic.

Landsberg and Teitler [6, Theorem 11.2] gave the monomial upper bound $\prod_i (a_i + 1)/(1 + \max_i a_i)$ in characteristic zero. Equality for all monomials is a longstanding conjecture. Oeding proposed a family of Young flattenings for this question [7]. Han [4] proved equality for the square-free monomial through a higher Koszul flattening and a congruence modulo two.

Theorem 1. *For $n, d \geq 2$ over a field of characteristic zero,*

$$\underline{\mathrm{WR}}\big((x_1 \cdots x_n)^{d-1}\big) = d^{n-1}.$$

The same value holds when a fixed set of at most $2(d-1)$ slots can use independent factors in every summand, with a common factor on the remaining slots.

We first construct a linear rank bound at prime-power heights and then multiply these bounds by a digit projection.

Theorem 2. *Let $q = p^e$ be a prime power, $m = q - 1$, and $n \geq 2$. In characteristic zero and in characteristic p,*

$$\underline{\mathrm{WR}}(\mathcal{D}_{(m,\ldots,m)}) = q^{n-1}.$$

The same is true if a fixed set of at most $2m$ slots can use independent linear factors in each summand, with a common factor in the other slots.

The flag orbit. For $V = F^n$ define

$$W = \bigotimes_{j=0}^{n-1} \big(\bigwedge^{j} V\big)^{\otimes m}, \qquad W' = \bigotimes_{j=0}^{n-1} \big(\bigwedge^{j+1} V\big)^{\otimes m}.$$

The $N = nm$ factors include m scalar factors at $j = 0$. For $T \in V^{\otimes N}$ let $L_T : W \to W'$ be the linear map wedging the letter of T into its corresponding exterior factor, extended linearly from words. Put $h = \bigotimes_{j=0}^{n-1} (e_1 \wedge \cdots \wedge e_j)^{\otimes m}$ and $h' = \bigotimes_{j=0}^{n-1} (e_1 \wedge \cdots \wedge e_{j+1})^{\otimes m}$. Let C be the integer matrix whose columns are the coefficient vectors of the vector-valued polynomial $g \mapsto gh$ in the entries of a generic $n \times n$ matrix g. Over any infinite field its column space is the linear span of

the GL_n-orbit of h: a linear functional annihilates all the evaluations exactly when it annihilates all coefficients. We use the integral linear flattening $T \mapsto L_T C$.

For a matrix g, write $g_{[j]}$ for its first j columns. Test the output of L_T against a second flag orbit. The resulting kernel on a word $T = \bigotimes_{j,s} v_{j,s}$ is

$$K_T(g,k) = \prod_{j=0}^{n-1} \prod_{s=1}^{m} \det(v_{j,s}, g_{[j]}, k_{[n-1-j]}).$$

Extend linearly in T. We regard the coefficients in the monomials of g and k as a matrix. This composes $L_T C$ with a second integral matrix on the output, so its rank is at most that of $L_T C$.

Lemma 3. *In characteristic zero and in characteristic p, the matrix of K_T has rank at most $q^{\binom{n-1}{2}}$ on one pure term with arbitrary factors at the $2m$ endpoint slots $j = 0, n-1$ and a common vector in the other slots.*

Proof. First use a common nonzero vector v at all slots and work in characteristic zero. By equivariance take $v = e_1$. The orbit span of h is the irreducible Cartan component $S_\lambda V$ in the exterior tensor product, with highest weight $\lambda = m(n-1, n-2, \ldots, 1, 0)$ [3]. The map L_T kills every weight space in which e_1 occurs: each basis wedge tensor there already has e_1 in a nonempty column. The remaining weights have total multiplicity $\dim S_\lambda(F^{n-1})$, by the semistandard tableau character using entries $2, \ldots, n$. On $n-1$ letters the row lengths of λ are $m(n-1), m(n-2), \ldots, m$, so the Weyl dimension formula gives $q^{\binom{n-1}{2}}$. The integral coefficient matrix for $T = e_1^{\otimes N}$ has at most this rank modulo p, and equivariance handles every common nonzero vector there as well.

An endpoint $j = 0$ contributes to K_T a factor depending only on k, and $j = n-1$ contributes one depending only on g. Multiplication by a nonzero polynomial in just one set of variables preserves coefficient-matrix rank, since that multiplication is an injective linear map. Replacing all the endpoint vectors arbitrarily consequently preserves the rank, or yields the zero matrix. □

Proof of Theorem 2. In characteristic p, the ideal $(x_1^q, \ldots, x_n^q)$ is invariant under linear changes of variables by Frobenius. Its quotient has a one-dimensional top piece in degree N with character $\det^m$, as seen on diagonal matrices. Its top multiplication form is $T = \mathcal{D}_{(m,\ldots,m)}$ on dual input variables, so T is SL_n-invariant. Directly, $L_T h = \pm h'$: all m columns of height $n-1$ must receive e_n, exhausting that letter, and descending the column heights forces the entire word. Hence $L_T(gh) = \pm gh'$ for $g \in \mathrm{SL}_n$.

Let U be the finite lower unitriangular group over $\mathbb{F}_q$. For $g, k \in U$ the preceding identity gives

$$K_T(g,k) = \pm f(g^{-1}k), \qquad f(u) = \prod_{j=1}^{n-1} \det\left(u_{\{n-j+1,\ldots,n\},[j]}\right)^{q-1}.$$

Set $\nu = \binom{n}{2}$. As a polynomial in the ν entries strictly below the diagonal, f has total degree at most $\nu(q-1)$. The coefficient of $\prod_{a>b} u_{ab}^{q-1}$ is one. Indeed, on a generic matrix use lexicographic order on the entries with earlier columns first and earlier rows first within a column. The main diagonal leads in each displayed minor. These diagonals together cover each strictly lower entry exactly once. Specializing diagonal entries to 1 and upper entries to 0 introduces no further contribution to this monomial of full total degree. Summing a variable over $\mathbb{F}_q$ kills its constant power and all positive powers not divisible by $q-1$. The degree bound therefore leaves only this one monomial, giving $\sum_{u \in U} f(u) = (-1)^\nu$.

The augmentation ideal of the group algebra of a finite p-group in characteristic p is nilpotent. Indeed, choose a central element z of order p and induct on the group order: quotienting by $(z-1)$ gives the group algebra of the quotient group, and $(z-1)^p = 0$. A group-algebra element with nonzero augmentation is consequently a unit. Applied to $\sum_u f(u)u$, this shows that the convolution matrix $(f(g^{-1}k))_{g,k \in U}$ is invertible. Its order is q^ν. Since evaluation cannot increase

coefficient-matrix rank, an integral minor of K_T of this size is nonzero modulo p, and is also nonzero in characteristic zero.

The rank of a linear flattening on a limit of sums of r allowed pure terms is at most r times its maximum rank on one term. Taking the ratio gives q^{n-1}. For the upper bound, use the projective chart $[z_1 : \cdots : z_{n-1} : 1]$ with coordinate ring $F[z_1, \ldots, z_{n-1}]/(z_1^q, \ldots, z_{n-1}^q)$. In degree N, extracting the coefficient of $\prod_{i<n} z_i^{q-1}$ from the evaluations gives T. The scheme has length q^{n-1} and deforms to a reduced grid by replacing each z_i^q with a product of q distinct linear factors tending to zero. Explicitly the top coefficient functional on the monomial basis of the flat grid is a linear combination of its point evaluations on the split fibers, and specializes to the stated coefficient functional. This proves the geometric upper bound over every field. □

Splitting the multiplicities. A linear rank bound for a tensor $T \in V^{\otimes N}$ is the ratio $\operatorname{rank} P(T)/s$ for a linear map P to matrices with $\operatorname{rank} P(v^{\otimes N}) \le s$ for all v. Such a ratio bounds geometric border Waring rank from below. In the following lemma, an entry with multiplicity zero can be suppressed.

Lemma 4. *Fix $a \ge 2$ and let $\alpha_i = ab_i + c_i$ with $0 \le c_i < a$. The product of any linear rank bounds for $\mathcal{D}_{\boldsymbol{c}}$ and $\mathcal{D}_{\boldsymbol{b}}$ is a linear rank bound for $\mathcal{D}_{\boldsymbol{\alpha}}$.*

Proof. Keep $\sum_i c_i$ singleton slots and split the rest into $\sum_i b_i$ blocks of a slots. On each block use the linear map $e_i^{\otimes a} \mapsto e_i$ that kills all nonconstant basis words. In a surviving word the singleton multiplicities C_i satisfy $C_i \equiv c_i \pmod a$, hence $C_i \ge c_i$. Their sums agree, forcing equality for every i. The block multiplicities are then b_i. Every resulting pair of words lifts uniquely, so the image is $\mathcal{D}_{\boldsymbol{c}} \otimes \mathcal{D}_{\boldsymbol{b}}$. Meanwhile $v^{\otimes N}$ maps to

$$v^{\otimes \sum c_i} \otimes (v_1^a, \ldots, v_n^a)^{\otimes \sum b_i}.$$

Apply the Kronecker product of the two matrix maps. Both the target ranks and the single-term rank bounds multiply. □

Proof of Theorem 1. Write $d = ab$. For $\alpha_i = d - 1$ the two digit vectors are constant, with entries $a - 1$ and $b - 1$. Starting with the prime-power linear bounds of Theorem 2, Lemma 4 yields d^{n-1} by induction on the prime factors of d. For the partial symmetry, place at most $2(a - 1)$ exceptional slots among the singletons and the remaining ones in at most $2(b - 1)$ blocks, using $2(d-1) = 2(a-1) + 2a(b-1)$. All other blocks receive the same vector $(v_i^a)_i$, so both single-term bounds persist. The dehomogenized grid used above gives the upper bound. □

Corollary 5. *In characteristic zero partition $n - 1$ variables into nonempty groups G_j, assigning every variable in G_j an arbitrary common positive exponent $m_j = q_j - 1$. Give the last variable exponent $B \ge \sum_j m_j$. Then the border Waring rank of this monomial is $\prod_j q_j^{|G_j|}$. If all q_j are powers of the same prime p, the result also holds for the divided-power tensor in characteristic p.*

Proof. Write $B = \sum_j b_j$ with $b_j \ge m_j$, and form disjoint slot blocks of sizes $|G_j|m_j + b_j$. Restrict each slot in block j to the coordinates in G_j and the last variable. Every small variable can occur in only one block, so the restricted tensor is exactly the ordinary tensor product of the multiplicity tensors $(m_j, \ldots, m_j, b_j)$. Contract $b_j - m_j$ slots of each block against the last coordinate and apply its prime-power or composite flattening. On the monomial the matrix ranks multiply, as do the single-term upper bounds on $v^{\otimes N}$, yielding the asserted lower bound. Dehomogenizing the last coordinate in the same grid construction gives the upper bound. □

Corollary 6. *Suppose $a_1 + 1 \mid a_2 + 1 \mid \cdots \mid a_{n-1} + 1$ and $a_n \ge a_{n-1}$. In characteristic zero the border Waring rank of $x_1^{a_1} \cdots x_n^{a_n}$ equals $\prod_{i<n}(a_i + 1)$.*

Proof. Contract $a_n - a_{n-1}$ copies of the last coordinate and put $a = a_1 + 1$. All surviving widths $a_i + 1$ are divisible by a. The low digit in Lemma 4 has width a at each variable and contributes a^{n-1}. In the high digit the widths are $(a_i + 1)/a$; discard the coordinates of width one and iterate the divisibility chain. Multiplying the bounds gives $\prod_{i<n}(a_i + 1)$. The last-coordinate grid attains it. □

Two-coordinate propagation.

Lemma 7. *Let $\boldsymbol{\alpha} = (\boldsymbol{\beta}, a, b)$, where $1 \le a \le b$ and every entry of $\boldsymbol{\beta}$ is at most b. Put $P = \prod_i(\beta_i + 1)$. If a linear matrix map L has rank at most s on pure powers and rank $sP(a+1)$ on $\mathcal{D}_{\boldsymbol{\alpha}}$, then $\mathcal{D}_{(\boldsymbol{\beta},a+1,b+1)}$ has a linear rank bound $P(a+2)$. The original L gives the bound Pa for $\mathcal{D}_{(\boldsymbol{\beta},a-1,b+1)}$.*

Proof. Restrict L to a nonsingular minor of order $sP(a+1)$ at the given tensor. Denote this restriction by L' and its invertible value by A_0. On two slots of the larger tensor keep only the coordinates for a, b and take their 2×2 matrix flattening. Apply L' on the other slots. A pure power gives a Kronecker product of matrices of ranks at most one and s. On the target, swapping the two block columns gives

$$B = \begin{pmatrix} A_0 & A_- \\ A_+ & A_0 \end{pmatrix}, \qquad A_\mp = L'(\mathcal{D}_{(\boldsymbol{\beta},a\mp1,b\pm1)}).$$

The grid upper bound gives $\operatorname{rank} A_- \le sPa$ and $\operatorname{rank} B \le sP(a+2)$, since $b+1$ is maximal in both targets. The Schur complement of A_0 gives the reverse chain

$$\operatorname{rank} B \ge 2 \operatorname{rank} A_0 - \operatorname{rank} A_- \ge sP(a+2).$$

Thus both inequalities are equalities. The uncropped value of L on the minus tensor has rank at least that of A_- and satisfies the same upper bound sPa. □

Corollary 8. *In characteristic zero, if positive integers $a_1, \dots, a_{n-1}, b$ satisfy $b \ge m + \sum_{i<n} |a_i - m|$ for some positive integer m, then*

$$\underline{\mathrm{WR}}(x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^b) = \prod_{i<n}(a_i + 1),$$

and a linear rank bound attains this value.

Proof. Start with the uniform certificate at exponent m. Move each of the first $n-1$ exponents one unit at a time to a_i, increasing the last exponent at every step. It stays maximal. Contract any surplus copies of the last coordinate from the desired tensor to the certified one. The grid gives the matching upper bound. □

In particular, a block of any number of equal positive exponents may be followed by two arbitrary larger exponents. In three variables, taking m to be the smaller nonmaximal exponent supplies a sharp linear certificate for every monomial. The ternary border-rank values were obtained by Buczyńska and Buczyński [1, Theorem 6.21 and Example 6.23].

A Unequal exponents

A.1 Two large exponents

Theorem 9. *Let $a_1 \le \cdots \le a_{n-2}$ be positive, $n \ge 3$, $A = \sum_{i \le n-2} a_i$, and $H = \prod_{i \le n-2}(a_i + 1)$. If*

$$B \ge c \ge \max\left\{A, \frac{(n-3)A}{2} + a_1\right\},$$

then $\underline{\mathrm{WR}}(\mathcal{D}_{(a_1,\dots,a_{n-2},c,B)}) = H(c+1)$ in every characteristic. In four variables the hypothesis is $B \ge c \ge a_1 + a_2$.

We use the normalized matching property of products of chains [5]. If P is a product of coordinate intervals and $K \subseteq P$ is a downset, the proportion $|K_t|/|P_t|$ at total degree t is nonincreasing. Here intervals can be unbounded, by truncation. One proof couples uniform vectors on successive levels in coordinatewise increasing order. Induct on the number of coordinates. For last interval $[0, b]$, the sum of the earlier coordinates has weights u_s on $[t-b, t]$, where u_s are their level sizes. Shifting this window makes the sum stochastically increase. The last coordinate also increases stochastically: its weights are $u_{t-\ell}$, and the ratio $u_{t+1-\ell}/u_{t-\ell}$ increases in ℓ by log-concavity of convolutions of interval indicators. The interval endpoints obey the same order. The common quantile coupling therefore increases the earlier sum by zero or one. Conditional induction gives the desired vectors.

Proof of Theorem 9. Contraction in the last coordinate reduces to $B = c$, and the coefficient grid gives the stated upper bound. Write $\alpha = (a_1, \dots, a_{n-2}, c, c)$. If the border rank were at most $r = H(c+1) - 1$, border apolarity [2, Theorem 1.1] would give an ideal inside $(y_i^{\alpha_i+1})_{i=1}^n$ with quotient Hilbert function $\min(r, \binom{n-1+t}{n-1})$. Taking a monomial initial ideal preserves both properties.

All monomials in the box $0 \le u_i \le \alpha_i$ are standard. Group the remaining standard monomials by the first i where $u_i > \alpha_i$. Subtracting $\alpha_i + 1$ in this coordinate gives a downset in

$$P_i = \prod_{j<i} [0, \alpha_j] \times \mathbb{N}^{n-i+1}.$$

Let $e_i(d)$ count this part in original degree $d = A + c$. For $i \le n-2$, fix its first $i-1$ coordinates v. Among the remaining $n-i+1$ free coordinates the ratio of consecutive level sizes is

$$1 + \frac{n-i}{d - a_i - |v|} \le 1 + \frac{n-i}{c + \sum_{i<j\le n-2} a_j} \le 1 + \frac{2}{A}.$$

For the last inequality the required thresholds $C_i = (n-i)A/2 - \sum_{i<j\le n-2} a_j$ satisfy $C_i - C_{i+1} = A/2 - a_{i+1}$. They form a convex sequence with both endpoints at most c. Summing these bounds controls the growth of the rank layers of P_i; normalized matching then gives $e_i(d+1) \le (1 + 2/A) e_i(d)$.

For $i = n-1, n$ the shifted levels are $A-1, A$, where the earlier bound c on either of the two large coordinates is inactive. A small-box vector v has $A - |v|$ and $A + 1 - |v|$ completions in the two coordinates. The level counts are $S = HA/2$ and $S + H$, since $v \mapsto a - v$ pairs their weights. Thus the same growth bound applies to both last parts.

The quotient Hilbert function equals r in degrees $d, d+1$: one may prescribe the first $n-1$ exponents in their box and fill the last degree, giving at least $H(c+1)$ monomials. By reflection the box counts in degrees $d, d+1$ equal its counts in degrees $c, c-1$, namely $H(c+1) - S$ and $Hc - S$. Hence $\sum_i e_i(d) = S - 1$ and $\sum_i e_i(d+1) = S + H - 1$. The latter exceeds $(1+2/A)(S-1)$, a contradiction. □

A.2 Four variables with a linear factor

Theorem 10. *For $1 \le b \le c \le B$ in every characteristic,*

$$\underline{\mathrm{WR}}(\mathcal{D}_{(1,b,c,B)}) = 2(b+1)(c+1).$$

Proof. Theorem 9 covers $b < c$. For $b = c$ contract the surplus $B - c$ and use the grid upper bound. Suppose the rank at $(1, c, c, c)$ is at most $r = 2(c+1)^2 - 1$. Partition the standard monomials outside the box by their first exceeded bound as above. At degree $d = 2c + 1$ and $d + 1$ their counts are respectively

$$e = c(c+2), \qquad e' = (c+2)^2 - 3.$$

Indeed the box counts are those at $c, c-1$, namely $(c+1)^2, c^2$, by reflection.

Write $\Delta_k(s) = s^{\langle k \rangle} - s$ for the increment in Macaulay's growth theorem. On each part it bounds the increment from shifted degree k. It decreases with k and is superadditive in s [1, Lemmas 6.10 and 6.12]. At $e = c(c+2)$ one has $\Delta_c(e) = 2c+1 = e' - e$, with strict superadditivity across every nontrivial split. Here are the details for $c \geq 2$. Put $D_k = \binom{k+2}{2}$ and $L_k(j) = j(k+1) - \binom{j}{2}$. For $s < D_k$ its binomial expansion gives $\Delta_k(s) = \max\{0 \leq j \leq k : L_k(j) \leq s\}$: the first j summands are $\binom{h+1}{h}$ and the remaining nonzero ones are $\binom{h}{h}$. If both parts are below D_c, their increments sum to at most $2c$. Otherwise they are $D_c + s$ and $v > 0$, with $s + v = D_{c-1} - 1 = L_{c-1}(c-1)$. The first increment is $c + 2 + \Delta_{c-1}(s)$, yielding $2c+1$ at $s = D_{c-1} - 1$. Were $\Delta_{c-1}(s) + \Delta_c(v) \geq c - 1$, we could choose p, q within those increments with $p + q = c - 1$ and $q > 0$. Yet

$$s + v \geq L_{c-1}(p) + L_c(q) = L_{c-1}(c-1) + (p+1)q.$$

This proves strictness.

Every shifted degree is at least c, so equality forces all e monomials into one part. The part belonging to x^2 has shifted degree $2c-1$ and $e < D_{2c-1}$, giving increment at most $2c-1$. Thus a large variable Y supplies the sole occupied part. At its residual degree c the only effective bound is $\deg_x \leq 1$: exactly one monomial u among $(c+1)^2$ candidates is missing. In residual degree $c+1$ there are $(c+2)^2$ candidates. The pure powers Z^{c+1}, W^{c+1} of the other large variables are forbidden, since dividing $Y^{c+1}Z^{c+1}$ or $Y^{c+1}W^{c+1}$ by Y gives a monomial of an empty part. Also uY, uZ, uW are all forbidden. At most one coincides with either forbidden pure power. At least four candidates are excluded, contradicting e'. An empty part stays empty by downclosure.

For $c = 1$ one has $e = 3, e' = 6$. Strictness of $\Delta_1(3) = 3$ under splitting again confines all excess to one square Y^2. Its residual level one omits one variable u. Among the ten quadratic candidates, the three other squares and all four products of u with a variable are forbidden, with at most one overlap. At most four candidates remain. □

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