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NATURAL BOUNDARIES AND REAL-POINT LIMITS FOR CLASS-ONE GOLDBACH SERIES

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Abstract. For integers $q, r \geq 2$ we study the r-fold von Mangoldt Dirichlet series in which every summand is congruent to one modulo q. Under the Generalized Riemann Hypothesis at modulus q, the known meromorphic continuation to $\Re s > r-1$ has the line $\Re s = r-1$ as a natural boundary. The boundary result also holds if only finitely many nontrivial zeros of the inducing primitive L-functions, including ζ, lie off the critical line. The proof allows multiple zeros, zeros shared by different Dirichlet L-functions, and coincident sums of ordinates. The character average for class one adds the multiplicities at each critical-line ordinate. This gives positive integer weights, up to a common factor, in a sum of Gamma products to which the integer phase cone applies. A normal limit of the corrected binary series recovers the combined coefficient of the critical-line pairs at a given boundary point. The higher-fold boundary follows by reducing to the binary series.

Under GRH at modulus q, we also subtract a main term, explicit up to an entire function, and compute the positive normal limit at the real boundary point. For the unrestricted series under the Riemann Hypothesis we give the regularized limit at every ordinate, including the real point for $r \geq 3$. Lean 4 checks the class-one results, with the character identity in pointwise form, and the unrestricted limits, taking the applicable classical estimates as premises. Under RH and the stated zeta premises, Lean also checks that the unrestricted unsubtracted real-point limit changes sign at a noninteger threshold r_*.

1. Introduction

Write $\Lambda(n) = \log p$ if n is a positive power of a prime p, and $\Lambda(n) = 0$ otherwise. For integers $q, r \geq 2$ consider the series

$$\Phi_{r,q}(s) = \sum_{\substack{n_1,\dots,n_r \geq 1 \\ n_j \equiv 1 \pmod q \ (1 \leq j \leq r)}} \frac{\Lambda(n_1)\cdots\Lambda(n_r)}{(n_1+\cdots+n_r)^s}, \qquad \Re s > r.$$

The tuples are ordered, the congruence is imposed on each summand, and prime powers in the class contribute along with primes. Write Φ_r for the unrestricted series and $F(x) = \sum_{n\geq 1} \Lambda(n)\mathrm{e}^{-nx}$ for its heat sum, for $x > 0$. The Riemann Hypothesis (RH) places the nontrivial zeros of the Riemann zeta function ζ on $\Re z = 1/2$. By the Generalized Riemann Hypothesis (GRH) at modulus q we mean that every zero in $0 < \Re z < 1$ of every primitive L-function inducing a character modulo q lies on that line. The principal character is induced by the trivial character of conductor one, whose L-function is ζ, so GRH at modulus q includes RH.

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Under GRH at modulus q, $\Phi_{r,q}$ is known to continue meromorphically to $\Re s > r-1$ [35, Theorems 2 and 5][3, Proposition 3 for the binary case]. The question is whether any disc centred on the line $\Re s = r-1$ permits further meromorphic continuation. For the binary series, the sums of two zeros are the candidate singularities of the pair term. An explicit meromorphic correction contributes further poles, but a meromorphic term cannot affect extendibility. Different pairs can give the same sum, and their weights contain Gamma values with different phases, so the combined weight at a coincidence could cancel. To obstruct continuation it is enough that the combined weight be nonzero on a dense set of points of the line.

1.1. **The obstruction and the strategy.** For class one the heat sum is

$$F_q(x) = \sum_{n \equiv 1 \pmod q} \Lambda(n) \mathrm{e}^{-nx}, \qquad x > 0.$$

Its expansion has a principal term $1/(\varphi(q)x)$, with φ Euler's totient, followed by a sum over zeros of Dirichlet L-functions. The Mellin transform of F_q^2 gives $\Gamma(s)\Phi_{2,q}(s)$ for $\Re s > 2$. Under GRH, squaring and integrating the zero sum gives a fraction with denominator $s - 1 - \mathrm{i}(t + t')$ for each ordered pair of ordinates t, t'. For a real τ, the sum of the numerators of the pairs with $t + t' = \tau$ is the *atom* at $1 + \mathrm{i}\tau$. To recover it, we subtract the meromorphic correction W of Section 3 from the continuation of $\Gamma\Phi_{2,q}$, evaluate the corrected function $\Gamma\Phi_{2,q}^{c} - W$ at $s = 1 + \mathrm{i}\tau + \eta$ and multiply by η. The resulting kernel is bounded by one, so as $\eta \downarrow 0$ dominated convergence picks out the exact coincidence even when other pair sums are arbitrarily close. A nonzero atom forces a pole of any meromorphic extension of the corrected function, and such poles cannot be dense on a segment inside a disc. The author's earlier paper on the Egami–Matsumoto conjecture, cited below as the Goldbach paper, uses the same Mellin decomposition and normal-limit argument for the unrestricted series [37, Proposition 3.7 and Lemmas 6.2 and 6.6]. In the case of finitely many zeros off the line, when it selects sums on the line $\Re s = 1$ in a bounded window, it excludes the finitely many sums with a representation in which at least one zero has bounded ordinate [37, Lemmas 6.1 and 6.7 and Theorem 6.8][36, §§2.1–2.4].

The difficulty is to show that atoms do not vanish, and here class one helps. Since $\chi(1) = 1$, the character average adds the multiplicities at a shared ordinate with equal weight. Up to a common factor, the pair coefficients are therefore positive integers, and this is the arithmetic input to the integer phase cone. For bounded τ and both real parts equal to $1/2$, pairing the ordinates $u + \tau$ and $-u$ cancels the $u \log u$ parts of the two Gamma phases and leaves $\tau \log u + O(1/u)$, which barely moves over a window of length $\sqrt{u_0}$ above the least height u_0. When u_0 is large, positive integer weights therefore cannot cancel in that window, and, given the bin bound on the coefficients, the exponentially decaying tail is too small to undo their sum [37, Lemmas 5.2–5.3 and Theorem 5.4]. In a bounded interval of sums we can avoid the finitely many sums that have a representation involving a bounded ordinate, and the integer phase cone then gives nonvanishing at every remaining represented sum. Littlewood's gap estimate supplies such a sum in every interval using only zeta ordinates, which occur in the principal-character term [37, Lemma 6.7 and Theorem 6.8][36, §2.3, Lemma 3.3 and §3.6, final paragraph].

For r summands, the terms of degree at most two in $F_q - 1/(\varphi(q)x)$ give a translated binary series and explicit meromorphic terms. Under GRH the remaining terms are

holomorphic across the proposed boundary, so the higher-fold boundary reduces to the binary one. At the real point $s = r - 1$ the combined linear term, which in Gamma normalization is a multiple of $\Gamma(s-r+1)D(s-r+1)$, is singular as well, and it must be included in the subtraction to isolate the atom. The pairs there are conjugate, so their contributions are squares of moduli.

1.2. **Results and scope.** Theorems 3.6 and 3.8 say that under GRH at modulus q the series $\Phi_{r,q}$ continues meromorphically from $\Re s > r$ to $\Re s > r - 1$, and that no meromorphic function on a disc centred on a point of the line $\Re s = r-1$ agrees with this continuation on the part of the disc to the right of the line. So $\Re s = r - 1$ is a natural boundary for every meromorphic continuation. Besides GRH the boundary theorems take (Z3), Littlewood's gap estimate and Stirling on strips as premises (Section 2), and they require no separation hypothesis on sums of ordinates. We have found no earlier statement of this boundary for general q without such a hypothesis. For $q = 2$ the result already follows from the unrestricted boundary [37, Theorem A(iii)]. Theorem 3.9 gives the same boundary when the set $\mathcal{E}$ of nontrivial zeros of the inducing primitive L-functions, including ζ, off the critical line is finite. Under the same three premises, Theorems 3.6 and 3.8 are its case $\mathcal{E} = \varnothing$. We retain the shorter GRH proof because the limits in Section 4 build on it.

Theorem 4.1 says that under GRH at modulus q, after subtraction of a main term $\mathfrak{M}_{r,q}$ that is explicit up to an entire function, $\eta(\Phi^c_{r,q} - \mathfrak{M}_{r,q})(r - 1 + \eta)$ tends, as $\eta \downarrow 0$, to the positive number $\binom{r}{2}(\varphi(q)^r(r-2)!)^{-1}\sum_t P_1(t)^2|\Gamma(1/2+\mathrm{i}t)|^2$. Here $P_1(t)$ is the total multiplicity of the zeros at $1/2+it$ of the primitive L-functions inducing characters modulo q, and the sum runs over the ordinates where it is positive. The limit follows from the fixed-modulus heat estimate in Lemma 3.2, but we have found no earlier statement of it. The deleted Euler factors give the class-one main term a double pole at the real point, where the unrestricted main term has a simple pole. Under RH, Proposition 4.2 gives the unrestricted regularized normal limit at every $r - 1 + \mathrm{i}\tau$, and Corollary 4.3 gives its positive value at $\tau = 0$. The new unrestricted case is $r \geq 3$ at $\tau = 0$. The binary limits and higher-fold limits at $\tau \neq 0$ are in the Goldbach paper [37, Corollary B(iii)–(iv)]. Lean also checks, under RH and the zeta premises (Z2) and (Z3), that the unrestricted unsubtracted real-point limit changes sign at a noninteger r_* (Section 4.3).

The nonvanishing step uses the phase cone proved in the author's paper *Phase cones for Gamma-weighted atoms of additive Dirichlet series* [36], cited below as the cone paper. That paper proves the sector cone and its integer case, and sets out the framework of pair sums, normal limits and meromorphic agreement that Section 2 recalls. The last paragraph of its Section 3.5 already notes that Lemma 3.3 applies to class-one sums, including coincidences across L-functions. Here we carry out the boundary and limit arguments for class one.

Section 2 recalls the Mellin calculation and the normal limit, lists the classical inputs and restates the cone results. Section 3 derives the class-one heat expansion, proves the binary boundary and transfers it to higher folds. Section 4 treats the regularized limits at the real point. Section 5 and Appendix A describe the Lean formalization, its premises and its main statements.

1.3. **Prior work.** *The ordinary Goldbach series.* Egami and Matsumoto [17] gave the pair-sum expansion for Φ_2, continued it to $\Re s > 1$ under RH and conjectured the natural boundary. Their condition (A) says that two pairs of ordinates with the same nonzero

sum are the same pair up to order. Their condition (B) is a lower bound, exponentially small in the ordinates, for the distance between the sums of two different unordered pairs when the first sum is not zero. It implies (A), and they prove the boundary under RH and (B) [17, Theorems 2.1 and 2.2]. Bhowmik and Schlage-Puchta proved the binary boundary under RH and (A) alone, and stated that under RH the higher-fold boundary at $\Re s = r-1$ follows from the binary boundary at $\Re s = 1$ [8, Theorem 2(2)][8, Theorems 2(1) and 4].

Their Theorem 2(4) gives $\{\Re s > 2\sigma_0\} \subseteq D_2 \subseteq \{\Re s > 1\}$ for the meromorphic continuation domain of Φ_2 under their Conjecture 2. Here σ_0 is the infimum of the abscissae with only finitely many zeros to the right. The Goldbach paper's hypothesis H_∞ is $\sigma_0 = 1/2$, meaning that only finitely many nontrivial zeta zeros lie to the right of $1/2 + \epsilon$ for each $\epsilon > 0$. We read their Theorem 2(4) with Egami and Matsumoto's nonzero-sum restriction [17, Theorem 2.2], which the printed statement of Conjecture 2 leaves implicit.

Egami and Matsumoto cite Fujii's work on additive relations among zeta ordinates in discussing separation [13, p. 50][17, §2]. Under RH, Bhowmik, Ramaré and Schlage-Puchta pass the normal limit termwise through the Goldbach pair sum, justified by absolute and uniform convergence [5, p. 297]. To recover the class-one atom we use instead a kernel of modulus at most one, so that the absolute values of the numerators form a summable majorant, and dominated convergence (Section 2). Bhowmik and Schlage-Puchta give criteria for distinguishing candidate from actual singularities of Euler products. Their criteria apply to those products and do not reach the additive Gamma sums here [6].

Goldston and Suriajaya give an unconditional explicit formula for the average Goldbach representation count. Under GRH they also average over sums divisible by q, a condition on the total [14, 15].

Under H_∞ the Goldbach paper removed the ordinate-sum assumptions for Φ_2 and Φ_r [37, Theorem A]. That hypothesis allows infinitely many off-line zeros. We adapt its argument for the case of finitely many zeros off the line to the class-one coefficients. The proof of Theorem 3.9 uses finiteness of the entire exceptional set to make the exceptional pair correction meromorphic across the boundary line. For the critical-line pairs we use the same pair-sum decomposition, discard the finitely many low sums in a bounded window, and combine the normal limit with the integer phase cone [37, Proposition 3.7, Lemmas 6.1, 6.6 and 6.7, Theorems 5.4 and 6.8][36, §2.3].

Residue classes. Rüppel treats summands in classes a, b prime to q, under a GRH convention that permits a real exceptional zero. Her Theorem 2 gives the continuation to $\Re s > 1$ and lists residues. Her Conjecture 1 proposes the binary boundary, and her Theorem 3 states it under a hypothesis (B_q) on the ordinates of the zeros of the L-functions modulo q [35]. For $\gamma_1 + \gamma_2 \neq 0$ and $\{\gamma_1, \gamma_2\} \neq \{\gamma_3, \gamma_4\}$, (B_q) requires the ordinates to satisfy

$$|(\gamma_1 + \gamma_2) - (\gamma_3 + \gamma_4)| \geq \exp\Big(-\alpha \sum_{j=1}^{4} |\gamma_j|\Big)$$

with a fixed $0 < \alpha < \pi/2$. The hypothesis depends only on the ordinates, which may be shared by several characters. Her Theorem 5 lists higher-fold residues, and Conjecture 3 proposes the higher-fold boundary [35, p. 154]. Under her GRH convention the exceptional set is finite at fixed q, as explained in Section 2. With the three standing

boundary estimates, Theorem 3.9 gives the binary and higher-fold class-one boundaries without (B_q).

The principal term and character-weighted zero sum in the heat expansion appear in [10, (8)–(9)].

In the proof of her Theorem 3 the hypothesis (B_q) enters twice. Its consequence (A_q), uniqueness of the unordered pair with a given nonzero sum, isolates one pair, and the inequality bounds the terms whose sums are near the given one. Here the normal limit takes the place of the inequality and, for class one, the integer cone takes the place of uniqueness. Her nonvanishing argument covers a pair whose two ordinates each have a single carrier and whose carriers are the same character. For general classes the character factor can vanish algebraically, leaving a gap in the proof. We know of no complete proof for general classes. For class one, the aggregated character factor is positive. With the omitted deleted Euler-factor residues and the full residue at zero included in the contour shift, we expect her method to give the class-one boundary under GRH and (B_q). The remark after Theorem 3.6 explains this reading.

For $q = 2$ the class-one boundary is a consequence of [37, Theorem A(iii)]. The remark after Theorem 3.8 gives the deduction, which we formalize in Lean.

Further work on Goldbach sums in arithmetic progressions counts representations, on average or with weights. Bhowmik, Halupczok, Matsumoto and Suzuki also worked with the Dirichlet series. They gave summatory explicit formulas for coprime classes and studied continuation and poles of the binary series. Their Proposition 3 gives the binary continuation under GRH [3]. Among the counting results, Suzuki studied mean values in progressions and identified the sign error in Rüppel's mixed residues involving an exceptional zero [39, Remark, preprint p. 16]. Cantarini, Gambini and Zaccagnini obtained weighted asymptotics for progression-restricted representations [10]. Nguyen obtains average and omega results for sums in two progressions, including different moduli, and for unrestricted higher-fold sums [33]. Without a class restriction, Bhowmik and Schlage-Puchta studied the mean representation number [7]. Granville studies refinements of Goldbach's conjecture and their relation to GRH [16].

Bhowmik and Halupczok survey the asymptotics and questions about zeros [2]. Matsumoto surveys related multiple Dirichlet series, including the binary boundary under RH and linear independence over $\mathbb{Q}$ of the positive ordinates [25, Theorem 5.4]. Matsumoto and, later, Matsumoto and Suzuki study the value distribution of a term in the Goldbach asymptotic. The atom in Section 5 of the latter work is a point mass at zero of an infinitely divisible probability distribution [26][31, §5]. In this paper, by contrast, an atom is the sum of the Gamma-weighted numerators of the pairs at a given coincidence.

Analogues. Zaharescu and Zaki [41, Theorem 1] weight the double von Mangoldt series, with no restriction on m, n, by $H(\alpha \log(m+n))$, where $\alpha > 0$ and $H \in C^3(\mathbb{R})$ has period 1 and infinitely many nonzero Fourier coefficients. Under RH, if only finitely many quadruples of integers n, n' and zeta ordinates $\gamma \neq \gamma'$ satisfy $\gamma + 2\pi\alpha n = \gamma' + 2\pi\alpha n'$, they prove the natural boundary $\Re s = 3/2$; there the singularities come from single zeros. Egami and Matsumoto exhibit a function-field analogue in which the characteristics determine whether a natural boundary occurs [18]. For random signs, Bhowmik and Matsumoto relate the natural boundary to the abscissa of convergence [4].

Write $\psi = -\zeta'/\zeta$. The constant $\psi(0) = -\log(2\pi)$ in the heat expansion on the real axis, used in Section 4, comes from the residue at zero of $\Gamma(z)\psi(z)x^{-z}$. Using their Laplace-convolution identity and the explicit formula of Brüdern, Kaczorowski and

Perelli [9], Cantarini, Gambini and Zaccagnini obtain an expansion on $\Re s > 2$ without RH. It has principal and single-zero terms, fractions with denominators $s - \rho_1 - \rho_2$, and a remainder estimated for $\Re s > 1$. Under RH the pair denominators vanish only on $\Re s = 1$, giving continuation to its right [11, §5, display after (28)].

The real point. The unrestricted case $r \geq 3$, $\tau = 0$ follows from the binary limit [37, Corollary B(iii)] and the Mellin reduction [37, Proposition 7.4] once the linear term is also subtracted. Proposition 4.2 includes that case, which we formalize. Equation (6.6) of Egami and Matsumoto concerns nonzero sums under their additional hypotheses [17]. Bhowmik and Schlage-Puchta stated the reduction to Φ_2 with rational multipliers without listing the coefficients [8, Theorem 4 of arXiv version 1, Theorem 3 of the journal version]. Their printed bound $T_3(\alpha) \ll \min(x^3, \alpha^{-3})$ fails at $\alpha = 1/2$, where $T_3(1/2) = x$ for every positive integer x [8, arXiv version 1, pp. 3–4][37, Remark 8.3.2]. The Goldbach paper proves the reduction independently by Mellin transforms [37, Proposition 7.4]. The class-one reduction in Proposition 3.7 uses the fixed-modulus heat expansion with its Euler factors.

Limits in other directions at singular points are studied for double series $\sum \alpha_1(m)\alpha_2(n)$ $m^{-s_1}(m+n)^{-s_2}$ in two variables by Matsumoto, Nawashiro and Tsumura [29] and by Matsumoto and Tsumura [32]. Matsumoto, Nawashiro and Tsumura attribute the finite-pole restriction on the coefficient Dirichlet series to earlier work by Choie and Matsumoto [29, p. 3 of the preprint]. In their Section 2 they study the double series with one von Mangoldt factor, whose coefficient Dirichlet series $-\zeta'/\zeta$ has infinitely many poles. Zeta-regularized sums and products over primes [30, 21] concern regularized quantities different from the limit computed here.

2. Mellin transforms, atoms and standing inputs

A Mellin transform of the squared zero sum gives the pair sum that enters the binary continuation, and normal limits extract its atoms. The cone paper gives the full treatment, including the case in which pair sums lie inside the half-plane of continuation [36, §§2.1–2.4].

2.1. **From a heat sum to a pair sum.** For $F(x) = \sum_{n \geq 1} a_n \mathrm{e}^{-nx}$ with $|a_n| \leq \log n$, the Mellin integral of F^r over $(0, \infty)$ gives $\Gamma(s)$ times the additive Dirichlet series for $\Re s > r$. The sum of the absolute coefficients of the tuples with total N is at most $N^{r-1}(\log N)^r$, which gives absolute integrability for termwise use of the Gamma integral [36, §2.1].

In the applications an explicit formula writes part of the heat sum as a zero sum $Z(x) = \sum_\rho A_\rho x^{-\rho}$ with $\sum_\rho |A_\rho| < \infty$. The Mellin integral of Z^2 over $(0, 1)$ then gives the *pair sum*, or Cauchy sum,

$$\sum_{\rho, \rho'} \frac{A_\rho A_{\rho'}}{s - \rho - \rho'} \qquad (\Re s > 2 \sup_\rho \Re \rho),$$

by the absolute integral bound $(\sum_\rho |A_\rho|)^2/(\Re s - 2\sup_\rho \Re\rho)$. The *collision* at s_0 consists of the ordered pairs with $\rho + \rho' = s_0$, and the *atom* is the sum of their numerators. A pair of distinct zeros occurs in both orders, and a diagonal pair occurs once [36, §2.1].

The rest of the squared heat sum gives the correction to the pair sum. The terms with the principal term give explicit meromorphic functions. The cross term with the heat sum is completed to $(0, \infty)$ and yields a function involving Γ and a logarithmic derivative. The square of the principal term is integrated over $(0, 1)$ only and gives a

rational function. The products with the remainder have Mellin integrals that converge across the boundary when the remainder is small enough. For class one, Section 3 defines the explicit meromorphic correction W from the principal terms and the remainder H_q, just before Proposition 3.3; the logarithmic derivative D in W includes the deleted Euler factors. The remainder H_q is holomorphic across the line, so subtracting W suffices for the normal limit.

2.2. **Normal limits.** When all the pair sums lie on the boundary line, multiplication by the normal distance η gives each term the kernel $\eta/(\eta + \mathrm{i}v)$, where $v = \tau - (t + t')$ for the pair with ordinates t, t' at the point with ordinate τ. The kernel has modulus at most one and tends to the indicator of $v = 0$ as $\eta \downarrow 0$, even when the nonzero v accumulate at zero. For absolutely summable numerators it extracts the atom, by dominated convergence. By the Laurent expansion, a finite nonzero limit forces a simple pole of any meromorphic extension of the corrected function, which for class one is $\Gamma\Phi^c_{2,q} - W$, with the continuation $\Phi^c_{2,q}$ of Proposition 3.3. Nonzero limits on a dense set of points of the line therefore rule out such an extension. Lemma 3.4 computes the limit and Theorem 3.6 draws the conclusion. A zero limit, on the other hand, can come from cancellation within a collision or from an empty sum, and it says nothing about nearby collisions [36, §2.2].

Choosing the points. The cone applies at a sum all of whose representations are high, in the following sense. A representation is *high* if both ordinates exceed a chosen threshold in modulus, and *low* otherwise. Fix a bounded window of sums. A representation in it with one bounded ordinate then has both ordinates bounded, so local finiteness leaves only finitely many low sums to discard, and a smaller open interval avoids all of them. If the threshold exceeds the bound for the window, the two ordinates of a high representation have opposite signs, which is the configuration the cone treats. Lemma 3.5 uses these exclusions to select exact sums for the cone [36, §§2.2–2.3].

2.3. **Agreement and natural boundaries.** Meromorphic agreement means equality off a relatively closed discrete exceptional set in the open overlap. The identity theorem gives uniqueness on a connected domain. A line is a natural boundary if no disc centred on it admits meromorphic agreement on its right-hand overlap [36, §2.4]. The class-one continuations agree pointwise with their series on the initial half-plane by Propositions 3.3 and 3.7.

2.4. **Standing inputs.** Zeros of ζ are distinct nontrivial zeros $\rho = \beta + \mathrm{i}\gamma$, with multiplicity $m(\rho)$. The zeta and Gamma inputs are those of the cone paper [36, §2.5].

RH and GRH at modulus q.: RH places the nontrivial zeta zeros on $\Re z = 1/2$. GRH at modulus q places on that line every zero in $0 < \Re z < 1$ of each primitive L-function that induces a character modulo q, including ζ, which induces the principal character. Rüppel's GRH convention [35] allows a real zero off the line. At fixed q there are only finitely many inducing functions. Their zeros are discrete in the strip and, by meromorphicity near 0 and 1, cannot accumulate at either endpoint. Thus only finitely many real zeros lie in $(0, 1)$, and with the three standing boundary estimates Theorem 3.9 gives the class-one boundary under her convention without (B_q).

(Z1).: Conjugation and $\rho \mapsto 1-\overline{\rho}$ preserve the nontrivial zeta zeros and their multiplicities, by the functional equation and reality of the Dirichlet coefficients [40, Ch. 2].

(Z2).: If $N(t)$ counts positive zeta ordinates up to t with multiplicities, then

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + O(\log t) \quad (t\to\infty), \qquad N(t+1)-N(t) \ll \log(t+3) \quad (t\geq 0).$$

See [40, Theorems 9.4 and 9.2]. The proofs use the second estimate, the local count, and from the first only the existence of zeros.

(Z3).: For $-1 \leq \Re z \leq 2$ and $|\Im z| \geq 2$, away from zeros,

$$\frac{\zeta'}{\zeta}(z) = \sum_{|\gamma-\Im z|\leq 1} \frac{m(\rho)}{z-\rho} + O(\log|\Im z|).$$

The large-height estimate is [40, Theorem 9.6(A)], and conjugation gives the negative heights. For $2 \leq |\Im z| \leq t_0$ the bound holds with a larger constant, because the difference of the two sides is bounded on that compact part of the strip. Indeed, the difference is continuous away from the zeros and the cutoffs $|\gamma-\Im z| = 1$, and it is bounded near each of them. The estimate is not claimed for $|\Im z| < 2$, where the principal part of ζ'/ζ at $z = 1$ would have to be subtracted as well. Where the proofs below need bounded heights, they use compact pieces of vertical lines that contain no zero and do not pass through $z = 1$.

Primitive local expansion.: The proofs use the unconditional classical expansion of Montgomery and Vaughan [28, Lemma 12.6, (12.5), p. 402]. For a primitive character χ of conductor $f > 1$, $L'/L(z,\chi)$ equals the sum of $1/(z-\rho)$ over the zeros ρ of $L(\cdot,\chi)$ in the critical strip with $|\Im\rho - \Im z| \leq 1$, counted with multiplicity, up to an error that is logarithmic in f and the height, uniformly for $-1 \leq \Re z \leq 2$ at large absolute heights. For a fixed modulus q the inducing primitive characters are finitely many, so there are C_q and t_q such that for every nonprincipal character modulo q the expansion for its inducing primitive L-function holds with error at most $C_q \log(2+|\Im z|)$, uniformly for $0 < \Re z \leq 2$ and $|\Im z| \geq t_q$, away from zeros. Rüppel uses the same expansion [35, Lemma 1, (6)]. Lean proves it under GRH at modulus q or under finiteness of the off-line zeros, so the formal theorems no longer take it as a premise (Section 5).

Littlewood's gap estimate.: For every sufficiently large positive t some zeta ordinate differs from t by less than $A_{\mathrm{L}}/\log\log\log t$, with an absolute constant A_{L} [23][40, Theorem 9.12].

Stirling on strips.: On every fixed real-part strip, uniformly at sufficiently large absolute imaginary part, $\Gamma(z) = \sqrt{2\pi}\exp((z-\frac{1}{2})\operatorname{Log} z - z)(1+O(|\Im z|^{-1}))$, with the principal logarithm. This is a classical theorem [34, §5.11], taken as an explicit premise because the Lean statements carry it (Mathlib lacks it).

For Γ we use the recurrence, the reflection formula and the absence of zeros [34, §§5.2, 5.5], and the bound $|\Gamma(z)| \leq \Gamma(\Re z)$ for $\Re z > 0$, immediate from Euler's integral. With the recurrence it gives $|\Gamma(z)| \ll_k |\Im z|^{-k}$ on every strip $0 < d \leq \Re z \leq D$, for every k.

2.5. **The phase cone.** Fix a bound Y for $|\tau|$. If both ordinates in a pair exceed Y in modulus, a pair summing to τ has one positive ordinate $u+\tau$ and one negative ordinate

$-u$. Its height is $u > 0$. With real parts b, c, its Gamma product is

$$w_{b,c}(\tau, u) = \Gamma(b + \mathrm{i}(u + \tau))\Gamma(c - \mathrm{i}u).$$

The two orders of a pair are separate indices with the same Gamma product. In a unit bin $[v, v+1)$ of heights the negative ordinate lies in $(-v-1, -v]$, and the positive ordinate lies in $[v + \tau, v + \tau + 1)$. This is why the bin bound for multiplicity weights comes from a product of two local counts.

The cone proof runs as follows. Take $1/2 \le B < 1$, real parts b, c in $[1 - B, B]$, and suppose that $b + c$ is the same for every pair. Stirling's formula gives the modulus $2\pi \mathrm{e}^{-\pi u - \pi\tau/2} u^{b+c-1}(1 + O(1/u))$, on a common scale because $b + c$ is fixed, and the phase $\tau \log u + \pi(b - c)/2 + O(1/u)$ modulo 2π. Rotate the weighted products by $\overline{\omega}\mathrm{e}^{-\mathrm{i}\tau \log u_0}$, where ω is the central direction of the coefficient sector and u_0 is the least height. The Gamma offset $\pi(b - c)/2$ is bounded by $\pi(B - 1/2)$ in modulus, so a coefficient sector of half-angle $\delta < \pi(1 - B)$ leaves a positive margin $\pi(1 - B) - \delta$ below $\pi/2$. Up to height $u_0 + \sqrt{u_0}$ the drift $\tau \log(u/u_0)$ is at most $Y/\sqrt{u_0}$. For sufficiently large u_0 the margin absorbs the drift and the Stirling error, so the rotated terms with nonzero coefficients in that window have positive real part (zero coefficients are allowed), and their sum dominates the exponentially decaying tail. Both bounds are measured against a nonzero term of least height, whose coefficient has modulus bounded below; the least height may be attained by several terms, some with zero coefficients. The cone paper gives the full argument for the three results reproduced below [36, §3.1, Theorem 3.1, Corollary 3.2 and Lemma 3.3].

Theorem 2.1 (Sector cone with polynomial bins [36, Theorem 3.1])**.** *Assume Stirling on strips. Fix $1/2 \le B < 1$, $Y > 0$, $a > 0$, a real K, a real $A \ge 0$ and $0 \le \delta < \pi(1 - B)$. There is $U \ge Y + 1$, depending only on B, Y, K, a, δ, A, such that for each real τ with $|\tau| \le Y$ the following holds. Let (b_i, c_i, u_i, d_i) be a family, over an arbitrary index set, of real numbers b_i, c_i, u_i and complex numbers d_i, with $b_i, c_i \in [1 - B, B]$ and $b_i + c_i$ constant. Suppose that*

1. *the heights attain a minimum $u_0 = u_{i_0} \ge U$, and $|d_{i_0}| \ge a$;*
2. *for some fixed ω with $|\omega| = 1$, every coefficient is $d_i = \omega |d_i| \mathrm{e}^{\mathrm{i}\phi_i}$ with $|\phi_i| \le \delta$;*
3. *every unit bin $I(v) = \{i : v \le u_i < v + 1\}$, v real, is finite, and for $v \ge 0$*

$$\sum_{i \in I(v)} |d_i| \le K(v + Y + 4)^A.$$

Then $\sum_i d_i w_{b_i,c_i}(\tau, u_i)$ converges absolutely and is nonzero.

Corollary 2.2 (Squared-logarithm bins [36, Corollary 3.2])**.** *Assume Stirling on strips. Fix $1/2 \le B < 1$, $Y > 0$, $a > 0$, a real K and $0 \le \delta < \pi(1 - B)$. There is $U \ge Y + 1$ such that for every real τ with $|\tau| \le Y$, the conclusion of Theorem 2.1 holds with the same strip, common sum, minimum, sector and finite-bin hypotheses and with*

$$\sum_{i \in I(v)} |d_i| \le K \log^2(v + Y + 4) \qquad (v \ge 0)$$

as the bin bound.

A positive integer has phase zero and modulus at least one. For nonnegative integer coefficients that are positive at the least height, the sector condition therefore holds with $\omega = 1$ and $\delta = 0$, and the lower bound holds with $a = 1$.

Lemma 2.3 (Integer weights [36, Lemma 3.3]). *Assume Stirling on strips. Fix $1/2 \le B < 1$, $Y > 0$ and a real K. There is $U \ge Y+1$ such that for each real τ with $|\tau| \le Y$ the following holds. Let (b_i, c_i, u_i) be as in Theorem 2.1, with nonnegative integer coefficients N_i. Suppose that the heights attain a minimum $u_0 = u_{i_0} \ge U$ with $N_{i_0} \ge 1$, that every unit bin $I(v)$ is finite, and that $\sum_{i \in I(v)} N_i \le K \log^2(v + Y + 4)$ for $v \ge 0$. Then $\sum_i N_i w_{b_i,c_i}(\tau, u_i)$ converges absolutely and is nonzero.*

The Goldbach paper uses this integer phase cone for the unrestricted series. Lemma 3.5 uses it with $B = 1/2$, with both real parts equal to $1/2$, and with the product of two aggregated multiplicities as the coefficient [37, Lemmas 5.2–5.3 and Theorem 5.4][36, Lemma 3.3].

3. Goldbach sums with every summand in class one

When every summand is congruent to one modulo q, the zeros are those of several L-functions, and an ordinate may be shared between them. For a unit class a modulo q, character orthogonality writes the indicator of $n \equiv a$ as the average of $\overline{\chi(a)}\chi(n)$ over the characters. Under GRH at modulus q the zeros in the strip lie on the critical line, and the coefficient at an ordinate t is, up to the factor $-1/\varphi(q)$ and a Gamma value, $P_a(t) = \sum_\chi \overline{\chi(a)} m_\chi(t)$, where the $m_\chi(t)$ are the multiplicities of the primitive L-functions at $1/2 + \mathrm{i}t$. Each factor $\overline{\chi(a)}$ has modulus one, but $P_a(t)$ can be non-real or zero.

For $a = 1$ every factor is 1, and $P_1(t)$ is the sum of the multiplicities: a positive integer at every ordinate that occurs, whichever L-functions share it. The weight of a pair of ordinates is $\varphi(q)^{-2} P_1(t) P_1(t')$ times a Gamma product, and the integer cone applies with no hypothesis on which ordinates are shared or on how their sums coincide.

The binary expansion also has a correction from the principal term and the deleted Euler factors. Its pole at $s = 1$ is double for class one and simple for the unrestricted series. The class-one correction has additional poles at non-real points of the line. The proof below separates these poles from the obstruction coming from the zero pairs.

3.1. **The single Dirichlet series and its heat sum.** Fix $q \ge 2$, put $h = \varphi(q)$ and $\alpha = 1/h$, the coefficient of x^{-1} in the class-one heat expansion, and write

$$\Phi_{r,q}(s) = \sum_{\substack{n_1,\dots,n_r \ge 1 \\ n_j \equiv 1 \pmod q \ (1 \le j \le r)}} \frac{\Lambda(n_1)\cdots\Lambda(n_r)}{(n_1 + \cdots + n_r)^s}, \qquad \Re s > r.$$

The sum is over all positive integers in the class, so the prime powers congruent to one contribute. Let $D(z) = \sum_{n \equiv 1 \ (q)} \Lambda(n) n^{-z}$ initially for $\Re z > 1$. It takes the place of $-\zeta'/\zeta$, and its residues determine the coefficients in the heat expansion. A character modulo q need not be primitive, and its L-function differs from that of the inducing primitive character by finitely many Euler factors. These factors vanish only on the imaginary axis, so they do not change the zeros in the strip, but they do contribute to D and will reappear as poles of the correction.

Lemma 3.1 (Character average). *Let χ^* denote the primitive character inducing χ mod q, with conductor f_χ. The function D continues meromorphically to the whole plane, and*

its continuation is

$$D(z) = \frac{1}{h} \sum_{\chi \bmod q} \left(-\frac{L'}{L}(z, \chi^*) - \sum_{\substack{p \mid q \\ p \nmid f_\chi}} \frac{\chi^*(p) p^{-z} \log p}{1 - \chi^*(p) p^{-z}} \right).$$

The finite Euler factors preserve zero orders in $0 < \Re z < 1$. *Under GRH at modulus* q *the only poles of* D *in* $\Re z > 0$ *are* 1, *with residue* α, *and the critical points* $z_t = 1/2 + \mathrm{i}t$ *at which the aggregate multiplicity* $P_1(t) = \sum_\chi \operatorname{ord}_{z_t} L(\cdot, \chi^*)$ *is positive, with residues* $-\alpha P_1(t)$. *Here* $\operatorname{ord}_z L = 0$ *where* $L(z) \neq 0$.

Proof. Character orthogonality gives the indicator of $n \equiv 1 \bmod q$ by averaging $\chi(n)$, including integers not coprime to q on which all characters vanish. Use the Euler series for the negative logarithmic derivative and factor $L(z, \chi)$ into $L(z, \chi^*)$ times the displayed Euler factors. This gives the displayed formula for $\Re z > 1$. The right side is meromorphic on the whole plane, because each $L(\cdot, \chi^*)$ is entire or, for ζ, meromorphic with its only pole at 1, and the Euler terms are meromorphic. It is therefore the meromorphic continuation of D. The zero set of each Euler factor lies on $\Re z = 0$ because $|\chi^*(p)| = 1$ in the indicated product, so on $\Re z > 0$ the Euler terms are holomorphic and the zero orders of $L(\cdot, \chi)$ and $L(\cdot, \chi^*)$ agree.

At $z = 1$ the conductor-one function ζ, which occurs for the principal character only, has a simple pole, so its negative logarithmic derivative has residue 1 there. For every nonprincipal χ the primitive function $L(z, \chi^*)$ is holomorphic and nonzero at $z = 1$, by the classical nonvanishing theorems on the line $\Re z = 1$ [28, Theorems 4.9 and 11.3, pp. 122 and 360], so its logarithmic derivative is regular at 1, and the Euler terms are regular there too. Hence D has a simple pole at 1 with residue $1/h = \alpha$. The same theorems, the nonvanishing of $\zeta(1 + \mathrm{i}t)$ for $t \neq 0$ [40, §§3.1–3.3, pp. 45–48], and the Euler products show that no primitive factor vanishes at any other point of $\Re z \geq 1$. The residue of $-L'/L$ at a zero of order m is $-m$, by factoring out $(z - z_t)^m$. Summing therefore gives the stated residues. □

The heat expansion of $F_q(x) = \sum_{n \equiv 1\,(q)} \Lambda(n) \mathrm{e}^{-nx}$ is obtained by a contour shift, as for ζ. The shift needs a local count of the zeros of the L-functions modulo q. The count is used to choose contour heights away from all the zeros and to sum the Gamma weights, and later to verify the pair-bin bound of the cone. Among the standing inputs there is a local count for ζ but none for the nonprincipal L-functions. At fixed modulus Jensen's formula gives one, with an O_q constant. The choice of the contour height follows the same argument as in the cone paper, using this fixed-modulus count [36, §2.6].

For the rest of this section, with $P_1(t)$ and $z_t = 1/2 + \mathrm{i}t$ as in Lemma 3.1, put

$$I = \{t \in \mathbb{R} : P_1(t) > 0\}, \qquad A_t = -\alpha P_1(t) \Gamma(z_t) \quad (t \in I).$$

These definitions involve no hypothesis on the zeros.

Lemma 3.2 (Zero count and heat expansion)**.** *Assume GRH at modulus* q. *The sum of* $P_1(t)$ *over* $|t - V| \leq 1/2$ *is* $O_q(\log(|V| + 4))$. *Moreover* $\sum_{t \in I} |A_t| < \infty$, *and for every* $0 < d < 1/2$,

$$F_q(x) = \alpha x^{-1} + Z(x) + R_d(x), \qquad Z(x) = \sum_{t \in I} A_t x^{-z_t}, \qquad R_d(x) = O_{q,d}(x^{-d}).$$

Here the formula holds for $x > 0$ and the estimate is needed for $0 < x \le 1$.

Proof. Use Jensen's formula in the disc about $2 + \mathrm{i}V$ of radius $7/4$, counting zeros in the disc of radius $8/5$. The critical points with $|t - V| \le 1/2$ lie in the inner disc, since their distance from its centre is at most $\sqrt{5/2} < 8/5$. For a nonprincipal primitive character use $L(z, \chi^*)$. The partial sums of its character have modulus at most f_χ, by cancellation on each period, so partial summation gives $|L(z, \chi^*)| \le f_\chi |z| / \Re z$ on $\Re z > 0$. For the principal contribution use $(z-1)\zeta(z)$, holomorphic in the disc. The identity

$$(z-1)\zeta(z) = z - z(z-1) \int_1^\infty \{u\} u^{-z-1}\, du$$

from partial summation gives a quadratic bound there, as $\Re z \ge 1/4$ and $|z|, |z-1| \le |V| + 4$. Thus either function has modulus at most $C_q(|V|+4)^2$ on the outer disc. The Euler products at the centre give a lower bound $\prod_p (1 + p^{-2})^{-1} > 0$, also for the pole-cleared zeta since $|2 + \mathrm{i}V - 1| \ge 1$. Jensen bounds the inner count by the log of the ratio of the upper and lower bounds divided by $\log((7/4)/(8/5))$. One may take radii tending to $7/4$ to avoid zeros on the circle. Sum over the finitely many characters.

The rest of the proof uses (Z3) and the primitive local expansion. Mellin inversion represents $F_q(x)$ by the integral of $\Gamma(z) D(z) x^{-z} / (2\pi\mathrm{i})$ on $\Re z = 2$. For a single exponential e^{-x} this is Fourier inversion for the Schwartz function $v \mapsto \mathrm{e}^{2v} \mathrm{e}^{-\mathrm{e}^v}$, whose Fourier transform is $\Gamma(2 + \mathrm{i}y)$ up to normalization. The rapid decay of Γ on the line and the absolute convergence of D at 2 permit termwise inversion. Shift to $\Re z = d$. On the new line the local logarithmic-derivative formulas and the count just proved give $D(d + \mathrm{i}y) = O_{q,d}(\log(2 + |y|))$, since every strip zero is horizontally separated by $1/2 - d$. Each Euler term in Lemma 3.1 is bounded on $\Re z \ge d$ by $(\log p) p^{-d} / (1 - p^{-d})$. Compact bounded-height portions of the line $\Re z = d$ are regular: the line contains no zero, does not pass through $z = 1$, and lies to the right of the zeros of the Euler factors.

For the two horizontal segments choose a common absolute height in $[N, N+1]$ separated from the ordinates and their reflections by at least $c_q / \log(N+3)$. The possible forbidden ordinates in $[N-1, N+2]$ have total count $O_q(\log(N+3))$, so the union of excluded intervals has length less than one for sufficiently small $c_q > 0$. Then the local formulas give $O_{q,d}(\log^2(N+3))$ for D uniformly on the horizontal segments. The bound $|\Gamma(z)| \ll_k |\Im z|^{-k}$ on the strip $d \le \Re z \le 2$ (Section 2) makes the integrals over the two segments tend to zero and the new line integral converge absolutely. With the unit-window count it also gives $\sum_t |A_t| < \infty$. Hence the crossed residues from Lemma 3.1 sum absolutely for each $x > 0$. They give the principal power and Z. The new vertical integral defines R_d, with the bound $O_{q,d}(x^{-d})$. □

3.2. **The binary continuation and normal limits.** With $U_q = \alpha / x$, the square of $F_q = U_q + Z + R_d$ is rearranged as $F_q^2 = 2U_q F_q - U_q^2 + (F_q - U_q)^2$. Completing the first term to an integral over $(0, \infty)$ gives $2\alpha\Gamma(s-1)D(s-1)$, and the second term has the transform $-\alpha^2/(s-2)$ on $(0, 1)$. We define the correction W by

$$W(s) = 2\alpha\Gamma(s-1)D(s-1) - \frac{\alpha^2}{s-2}.$$

It is meromorphic on the plane, with no hypothesis on the zeros, and it contains the effects of the finite Euler factors. The last term contains Z^2, whose transform is the pair sum C_q, and products with R_d, whose transforms are holomorphic across the line.

The unrestricted binary correction has a simple pole at $s = 1$ (see Proposition 4.2 and the discussion following it). For class one, both $\Gamma(s-1)$ and $D(s-1)$ have a simple pole at $s = 1$, so W has a double pole there. The principal character contributes a deleted Euler factor at every prime dividing q, and each contributes a negative residue to D at $z = 0$. The other contributions to the residue there come from the trivial zero at $z = 0$ of $L(\cdot, \chi^*)$ for an even nonprincipal χ^*, and from the zero at $z = 0$ of $1 - \chi^*(p)p^{-z}$ when $\chi^*(p) = 1$. Each gives the residue $-1/h$, so all have the same sign. At non-real zeros on $\Re z = 0$ of the factors $1 - \chi^*(p)p^{-z}$ with $p \mid q$ and $p \nmid f_\chi$, the same residue sign prevents cancellation. These give additional poles of W on $\Re s = 1$. Since W is meromorphic on the whole plane, its poles are locally finite. The normal limit is taken after subtracting W, and the density argument avoids the ordinates of its poles.

The termwise Mellin calculation is the one in Section 2 [36, §2.1]. The decomposition below uses the function

$$H_q(s) = \int_1^\infty (F_q^2 - 2U_qF_q)x^{s-1}\,dx + \int_0^1 (2ZR_d + R_d^2)x^{s-1}\,dx,$$

which is the entire tail correction plus the integral over $(0, 1)$ of the products involving R_d.

Proposition 3.3 (Binary decomposition)**.** *Assume GRH at modulus q. On $\Re s > 2$,*

$$\Gamma(s)\Phi_{2,q}(s) = W(s) + C_q(s) + H_q(s), \qquad C_q(s) = \sum_{t,t' \in I} \frac{A_tA_{t'}}{s - 1 - \mathrm{i}(t + t')}, \tag{1}$$

where H_q is the function displayed in the proof. The right-hand side divided by $\Gamma(s)$ gives the continuation to $\Re s > 1$, which we denote $\Phi_{2,q}^c$; here and below a superscript c marks the continuation of a series. Here W is meromorphic on the plane, C_q is holomorphic on $\Re s > 1$, and H_q is holomorphic on $\Re s > 1/2 + d$ for every $0 < d < 1/2$.

Proof. Use the Mellin square of F_q and the heat expansion in Lemma 3.2, which uses (Z3) and the primitive local expansion. Set $U_q(x) = \alpha/x$. On $(0, 1)$ write

$$F_q^2 = 2U_qF_q - U_q^2 + Z^2 + 2ZR_d + R_d^2.$$

The first term integrated to infinity gives $2\alpha\Gamma(s-1)D(s-1)$ and the next power on $(0, 1)$ gives $-\alpha^2/(s-2)$. The integral of Z^2 on $(0, 1)$ is C_q by Fubini, with bound $(\sum |A_t|)^2/(\Re s - 1)$ for the absolute integrals. The same denominator bound gives normal convergence on the open half-plane to the right of 1. The remaining terms give

$$H_q(s) = \int_1^\infty (F_q^2 - 2U_qF_q)x^{s-1}\,dx + \int_0^1 (2ZR_d + R_d^2)x^{s-1}\,dx.$$

The first integral is entire since $F_q = O(\mathrm{e}^{-x})$ for $x \geq 1$, by the coefficient bound. The second is holomorphic for $\Re s > 1/2 + d$ since $Z = O(x^{-1/2})$ and $d < 1/2$. The integrand $2ZR_d + R_d^2 = (F_q - U_q)^2 - Z^2$ is independent of d. Meromorphy of W follows from Lemma 3.1 and meromorphy of the Gamma function. The right-hand side of (1) equals $\Gamma(s)\Phi_{2,q}(s)$ on $\Re s > 2$. □

Under GRH at modulus q every pair sum lies on the line $\Re s = 1$, so C_q has no poles in the half-plane, and the proof uses normal limits. Different sums of ordinates can be arbitrarily close, and the normal factor controls their combined contribution without an inverse-separation bound. This is the step at which (B_q) is used in [35]. The weights A_t

are absolutely summable by Lemma 3.2, as the bounded-kernel argument requires [36, §2.2].

We define the atom at a real ordinate τ by

$$M(\tau) = \sum_{\substack{t,t' \in I \\ t+t'=\tau}} A_t A_{t'}.$$

Lemma 3.4 (Binary normal limit)**.** *Assume GRH at modulus q. For every real τ,*

$$\lim_{\eta \downarrow 0} \eta\big(\Gamma\Phi_{2,q}^{c} - W\big)(1 + \mathrm{i}\tau + \eta) = M(\tau).$$

Equivalently, the limit of $\eta(\Phi_{2,q}^{c} - W/\Gamma)(1 + \mathrm{i}\tau + \eta)$ is $M(\tau)/\Gamma(1 + \mathrm{i}\tau)$.

Proof. The binary decomposition (1) uses (Z3) and the primitive local expansion and gives $\Gamma\Phi_{2,q}^{c} - W = C_q + H_q$. After multiplication by η the contribution of H_q tends to zero. The factor $\eta/(\eta + \mathrm{i}(\tau - t - t'))$ in the sum has modulus at most one and tends to the indicator of $t + t' = \tau$. The family $|A_t A_{t'}|$ is a summable majorant for the terms, with total mass $(\sum_t |A_t|)^2$, so dominated convergence applies. □

3.3. **Density and the binary boundary.** For the boundary it suffices to find, in every interval of the line, an ordinate τ at which the atom $M(\tau)$ is not zero. The atom is α^2 times a sum of Gamma products on the critical line with the positive integer coefficients $P_1(t)P_1(t')$, so the integer cone gives nonvanishing when every representation of τ lies above its height threshold. The threshold can be chosen uniformly in each bounded window of sums. The principal character contributes the zeta ordinates to I, and Littlewood's estimate for ζ provides differences of two of them in any interval. No gap estimate for the other L-functions is needed. The lemma also keeps τ away from the poles of W on the line, although the proof of the theorem will not use this [36, §2.3 and Lemma 3.3].

Lemma 3.5 (Dense nonzero atoms)**.** *Assume GRH at modulus q. Each bounded open interval contains ordinates τ with $M(\tau) \neq 0$ and W regular at $1 + \mathrm{i}\tau$.*

Proof. Fix $Y > 0$ bounding the interval and a threshold $X > Y$, to be chosen below. Let τ be a represented sum with $|\tau| \leq Y$ such that, in every representation of τ, both ordinates exceed X in modulus. Orient the pairs as $u + \tau, -u$ with the original ordered pairs as indices. Their weights after multiplication by h^2 are Gamma products with $b = c = 1/2$ and integer coefficients $P_1(u + \tau)P_1(-u)$. For u in $[v, v + 1)$ the two ordinates lie in intervals of length one at distance at most $v + Y + 1$ from zero. The two multiplicity counts from Lemma 3.2 give a product bound $K \log^2(v + Y + 4)$ for the coefficient mass, including the two orders by enlarging K. The least height is attained by local finiteness, and its coefficient is at least one. Taking X above the corresponding threshold in Lemma 2.3, which assumes Stirling on strips, proves nonvanishing for every such τ.

In a compact subinterval of the given open interval there are finitely many sums admitting a representation with at least one ordinate of modulus at most X, by the bounded window argument in Section 2. There are also only finitely many ordinates of W poles there, by its meromorphy. Keep an open interval (l, h_1) after the exclusions. The principal primitive factor supplies every zeta ordinate and its negative to I. Take a sufficiently high such positive ordinate γ, and apply Littlewood's gap estimate at

$\gamma + (l + h_1)/2$. This gives a second ordinate γ' with $\gamma' - \gamma \in (l, h_1)$. That exact sum is represented by $\gamma', -\gamma$ and lies outside both excluded sets, as required. □

The continuation is $\Phi^c_{2,q} = \Gamma^{-1}W + \Gamma^{-1}(C_q + H_q)$. The first term is meromorphic on the plane and $1/\Gamma$ has no zero near the line, so $\Phi^c_{2,q}$ extends meromorphically across a point of the line exactly when $C_q + H_q$ does. The proof works with $C_q + H_q$, and the poles of W play no part.

Subtracting the meromorphic correction reduces the obstruction to the dense set of nonzero normal limits. The proof uses the agreement convention of Section 2 [36, §§2.2 and 2.4].

Theorem 3.6 (Binary class-one boundary)**.** *Fix $q \geq 2$ and assume GRH at modulus q. The binary series $\Phi_{2,q}$ continues meromorphically to $\Re s > 1$, and the continuation agrees with the series on $\Re s > 2$. For every function that is meromorphic on $\Re s > 1$ and agrees with $\Phi_{2,q}$ on $\Re s > 2$ in the sense of Section 2, the line $\Re s = 1$ is a natural boundary.*

Proof. The proof uses (Z3) and the primitive local expansion for the binary decomposition and normal limit, and Littlewood's gap estimate and Stirling on strips for density. Proposition 3.3 supplies the continuation $\Phi^c_{2,q}$, and the identity theorem identifies any agreeing function with it as a meromorphic function on $\Re s > 1$. Suppose a function G, meromorphic on a disc Δ centred on the line, agreed with $\Phi^c_{2,q}$ on the part of Δ to the right of the line. Shrink Δ so that it lies in $\Re s > 0$. Then $g = \Gamma G - W$ is meromorphic on Δ and agrees with $C_q + H_q$ on the right-hand part, by (1). By Lemma 3.5 the ordinates τ with $M(\tau) \neq 0$ are dense in the segment of the line inside Δ. Fix one. The function g is holomorphic on a punctured neighbourhood of $1 + \mathrm{i}\tau$, so it coincides with the holomorphic function $C_q + H_q$ at every point $1 + \mathrm{i}\tau + \eta$ with small $\eta > 0$. By Lemma 3.4, $\eta g(1 + \mathrm{i}\tau + \eta) \to M(\tau) \neq 0$, and g has a simple pole at $1 + \mathrm{i}\tau$ by its Laurent expansion. The poles of g would then accumulate inside Δ, which is impossible. □

Remark (Comparison with Rüppel's Theorems 2 and 3). Rüppel's paper [35] is the source of the problem treated in this section. No numbered result depends on the computations of this remark, and they are not formalized.

Her convention allows a real exceptional zero $\beta > 1/2$, and the third bullet of her Theorem 2 (p. 137, repeated on p. 142) gives residues involving β at two kinds of points. Suzuki identified the sign error on p. 142 in the mixed residues [39, Remark, preprint p. 16]. We state the residues for the class $a = b = 1$.

At $s = \beta + 1/2 + \mathrm{i}t$, for the ordinate t of a critical zero, the residue list preceding (10) on p. 141 carries an outer minus sign, and her function $M = -L'/L$ has the residue $-n(\rho)$ at a zero ρ of order $n(\rho)$. The two signs multiply to a plus, and the two orders give a factor two, so the contour computation yields the mixed term

$$+\frac{2M_\beta P_1(t)\Gamma(\beta)\Gamma(1/2 + \mathrm{i}t)}{h^2\Gamma(\beta + 1/2 + \mathrm{i}t)},$$

where M_β is the aggregate multiplicity of β. The term printed on pp. 137 and 142 for a critical zero of one fixed character has the opposite sign. Summed over the characters that carry t, the printed terms give the negative of the displayed aggregate.

At $s = 2\beta$ the pair in question has both zeros equal to β. In the contour computation the two families of residues on p. 142 then describe the same term, which is $+n(\beta)^2\Gamma(\beta)^2/(\varphi(q)^2\Gamma(2\beta))$, counted once. The third bullet, applied with $\rho_\chi = \beta$, gives

$-2\,n(\beta)^2\Gamma(\beta)^2/(\varphi(q)^2\Gamma(2\beta))$. At either point the contributions of other pairs with the same sum are added to these terms.

The shift to $\Re z = -\epsilon$ in (10) on p. 141 omits the residues at the nonzero zeros of the deleted Euler factors on $\Re z = 0$. For $q = 2$, the factor $1-2^{-z}$ in the principal L-function gives poles of $-L'/L$ at $2\pi ik/\log 2$ for $k \neq 0$. At zero, the double pole of $\Gamma(z)M(z,\chi_0)$ also requires differentiating the other Mellin factor with respect to z when taking the residue.

Her Theorem 3 is stated for general classes a, b prime to q. In its proof, Lemma 3 (p. 145) asserts that her pair term $C(s,a,b,q)$ tends to infinity as $s \to 1+i\kappa$ from the right, for every κ in her set K_q of sums of two ordinates; the argument concerns $\kappa \neq 0$. For distinct ordinates the proof assigns one character to each, χ_1^* and χ_2^*, and obtains the residue (13) on p. 146, which carries the factor

$$\overline{\chi_1^*}(a)\overline{\chi_2^*}(b) + \overline{\chi_2^*}(a)\overline{\chi_1^*}(b).$$

If each ordinate has a single carrier and the two carriers are the same character χ, the factor is $2\overline{\chi}(a)\overline{\chi}(b) \neq 0$, and this is the case the text addresses. For two different characters it can vanish. With $q = 5$ and $(a,b) = (1,2)$, the principal character and the quadratic character give $(-1)+(+1) = 0$. This calculation shows algebraic cancellation for this choice of characters. Whether a pair of actual ordinates has these carriers is a separate question.

The hypotheses (A_q) and (B_q) concern the numerical values of the ordinates, and they permit an ordinate shared by several L-functions. Summed over all the characters that carry the two ordinates, the factor becomes $P_a(t)P_b(t') + P_a(t')P_b(t)$ for distinct ordinates and $P_a(t)P_b(t)$ on the diagonal, where $P_a(t) = \sum_\chi \overline{\chi(a)} m_\chi(t)$.

For general classes the boundary therefore still requires an argument that these factors do not vanish on a dense set of sums, and that proof does not supply one. This does not show that the statement fails, and we know of no counterexample. For $a = b = 1$ the factors are $2P_1(t)P_1(t')$ and $P_1(t)^2$, both positive. With the contour residues corrected, we expect her method to give the class-one boundary under GRH and (B_q) by excluding the locally finite pole ordinates of the meromorphic correction, as in Lemma 3.5 for W. We leave this deduction open here. Theorem 3.9 gives the class-one boundary under Rüppel's GRH convention and the three standing boundary estimates, without (B_q).

3.4. **Other residue classes.** This subsection records how far the argument reaches for other classes and what is missing. Assume GRH at modulus q. For a unit class a modulo q, an ordinate t contributes the coefficient $P_a(t) = \sum_\chi \overline{\chi(a)} m_\chi(t)$, where the sum is over the characters modulo q and $m_\chi(t)$ is the order at $1/2+it$ of the primitive L-function inducing χ. In the series for the classes a, b, an unordered pair $\{t, t'\}$ with $t \neq t'$ has the coefficient $P_a(t)P_b(t') + P_a(t')P_b(t)$, and a diagonal pair has $P_a(t)P_b(t)$. The squared-logarithm bin bound of Corollary 2.2 is available for every pair of classes, because $|P_a(t)| \le P_1(t)$ and Lemma 3.2 bounds P_1 [36, Corollary 3.2].

For classes other than $a = b = 1$, neither the sector condition nor the lower bound at the least height is known in general. Nonvanishing can still occur when a cone hypothesis fails. Conversely, the cone paper gives examples where size bounds and conjugation symmetry alone allow cancellation [36, §3.5].

Can one find a dense set of high sums on which these coefficients, with $B = 1/2$, lie in one open half-plane with a uniform angular margin and have a uniform positive lower bound at the least height? The lower bound does not follow from the other conditions,

since the proof measures the tail against the modulus of the lowest term, and the remark after Theorem 3.6 shows how the character factor can cancel algebraically.

One may also ask whether the sums with a unique unordered representation and a nonzero coefficient are dense. Even under (B_q), uniqueness of the pair leaves the character sum at a shared ordinate to be controlled.

3.5. **Transfer to higher folds.** For r summands the Mellin transform involves F_q^r. Write $F_q = U_q + V_q$ with $U_q = \alpha/x$ the main term and $V_q = O(x^{-1/2})$ on $(0,1]$ the rest, and expand. The term of degree j in V_q is $O(x^{-r+j/2})$ near zero, so its Mellin integral over $(0,1)$ converges for $\Re s > r - j/2$. The integrals over $(1,\infty)$ are treated separately. For each integer $j \geq 1$ put $E_j(v) = \int_1^\infty F_q(x)^j x^{v-1}\,dx$, which is entire because F_q decays exponentially. With b_r, ℓ_r, z and w as in Proposition 3.7, the tails are collected in the entire function $\widetilde{E}_{r,q}(s) = E_r(s) - b_r E_2(w) - \ell_r E_1(z)$, which reappears in the main term of Theorem 4.1. Terms with three or more factors of V_q therefore contribute functions holomorphic across $\Re s = r - 1$.

The terms of degree at most two are regrouped as $\binom{r}{2} U_q^{r-2} F_q^2$ plus multiples of $U_q^{r-1} F_q$ and U_q^r. The transform of the first, completed to $(0,\infty)$, is the binary transform at the translated argument $s - r + 2$, the power U_q^{r-2} accounting for the shift, and the other two give explicit meromorphic functions. The r-fold series is thus the translated binary series times a multiplier, plus a function meromorphic across the line. Bhowmik and Schlage-Puchta stated this reduction for the unrestricted series [8, Theorem 4 of arXiv version 1]. The Goldbach paper proves it independently by Mellin transforms [37, Proposition 7.4]. The proof of Proposition 3.7 below gives the reduction used here, with α/x in place of $1/x$.

Proposition 3.7 (Higher-fold reduction)**.** *Fix $q, r \geq 2$ and assume GRH at modulus q. Put*

$$z = s - r + 1, \qquad w = z + 1 = s - r + 2,$$

$$b_r = \binom{r}{2}\alpha^{r-2}, \qquad \ell_r = -r(r-2)\alpha^{r-1}, \qquad d_r = \frac{(r-1)(r-2)}{2}\alpha^r.$$

There is a holomorphic function $K_{r,q}$ on $\Re s > r - 3/2$ such that

$$\Phi_{r,q}^c(s) = c_r(s)\Phi_{2,q}^c(w) + \frac{\ell_r \Gamma(z) D(z) + d_r/(s-r) + K_{r,q}(s)}{\Gamma(s)}, \qquad c_r(s) = b_r \frac{\Gamma(w)}{\Gamma(s)}. \tag{2}$$

This defines the continuation on $\Re s > r - 1$, with equality to the series on $\Re s > r$. The multiplier is holomorphic and nonzero on $\Re s > r - 2$.

Proof. Match the constant, linear and quadratic terms in V_q to write

$$F_q^r = \binom{r}{2} U_q^{r-2} F_q^2 - r(r-2) U_q^{r-1} F_q + \tfrac{1}{2}(r-1)(r-2) U_q^r + Q_r,$$

$$Q_r(x) = \sum_{j=3}^{r} \binom{r}{j} U_q(x)^{r-j} V_q(x)^j.$$

Indeed, the coefficient of $U_q^{r-1} V_q$ is $2\binom{r}{2} - r(r-2) = r$, and the coefficient of U_q^r is 1. The other low-degree coefficient is $\binom{r}{2}$. The heat expansion in Lemma 3.2, using (Z3) and

the primitive local expansion, shows $Q_r = O(x^{-r+3/2})$ for $0 < x \le 1$. Thus the integral of $Q_r(x)x^{s-1}$ on $(0,1)$ is holomorphic on $\Re s > r - 3/2$. It is zero if $r = 2$.

Let $E_j(v) = \int_1^\infty F_q(x)^j x^{v-1}\,dx$, entire for every integer $j \ge 1$ by exponential decay. Integrate the identity on $(0,1)$ for $\Re s > r$. Complete the first two transforms using the full Mellin integrals for $\Phi_{2,q}(w)$ and $D(z)$. The pure power gives $d_r/(s-r)$. Add the original tail E_r and subtract the tails used in completing the first two terms. This gives (2) with

$$K_{r,q}(s) = \int_0^1 Q_r(x)x^{s-1}\,dx + \widetilde{E}_{r,q}(s), \qquad \widetilde{E}_{r,q}(s) = E_r(s) - b_r E_2(w) - \ell_r E_1(z). \tag{3}$$

The Gamma recurrence gives $c_r(s) = b_r \prod_{j=1}^{r-2}(s-j)^{-1}$ with the empty product equal to one. The remaining quotient in (2) is meromorphic across the boundary line, and Proposition 3.3 supplies the continuation of the translated binary term. □

Remark (Residues and Rüppel's Theorem 5). The statements of this remark are not formalized in Lean. Combining (1) and (2) gives, under GRH at modulus q, the residue $\alpha^r/(r-1)!$ of $\Phi^c_{r,q}$ at $s = r$ and the residue $r\alpha^{r-1}A_t/\Gamma(r-1+z_t)$ at $s = r-1+z_t$ for $t \in I$. By the Gamma recurrence these are the class-one coefficients in Rüppel's Theorem 5 [35, pp. 153–154], summed over the characters that share the zero.

The higher-fold boundary now follows from the binary one. If g is meromorphic on a disc, then a function f extends meromorphically across the disc exactly when $f + g$ does, since $f = (f+g) - g$ and a difference of meromorphic functions is meromorphic [36, §2.4]. Multiplication by a function holomorphic and nonzero on the disc preserves meromorphic extendibility in the same way. The proof of Theorem 3.8 subtracts the correction in (2), divides by the holomorphic, nonvanishing multiplier and translates the disc to the binary boundary.

Theorem 3.8 (Higher-fold class-one boundary)**.** *Fix $q, r \ge 2$ and assume GRH at modulus q. Then $\Phi_{r,q}$ continues meromorphically to $\Re s > r - 1$, and the continuation agrees with the series on $\Re s > r$. For every function that is meromorphic on $\Re s > r - 1$ and agrees with $\Phi_{r,q}$ on $\Re s > r$ in the sense of Section 2, the line $\Re s = r - 1$ is a natural boundary.*

Proof. The higher-fold reduction uses (Z3) and the primitive local expansion, and the binary boundary also uses Littlewood's gap estimate and Stirling on strips. Use the continuation in Proposition 3.7 and identify any agreeing continuation by the meromorphic identity theorem. If an extension existed on a disc centred on $\Re s = r - 1$, shrink the disc so the correction in (2) is meromorphic there and c_r is holomorphic and nonzero. Subtract the correction and divide by c_r. Translation by $-(r-2)$ then gives a meromorphic extension of $\Phi^c_{2,q}$ on a disc centred on $\Re w = 1$, with agreement on its right-hand overlap. This contradicts Theorem 3.6. □

Remark (The case $q = 2$). For $q = 2$ the class-one boundary follows from the unrestricted boundary of the Goldbach paper. Put $P(x) = (\log 2)\sum_{k\ge1} \mathrm{e}^{-2^k x}$, so that the class-one heat sum is $F - P$. Under RH, $F = x^{-1} + O(x^{-1/2})$ and $P = O(1 + |\log x|)$ on $(0,1]$, hence

$$F^r - (F-P)^r = r\,x^{-(r-1)}P(x) + O\big(x^{-r+3/2}(1+|\log x|)^2\big).$$

For $z = s - r + 1$, the Mellin transform of the first term over $(0, \infty)$ is

$$\frac{r(\log 2)\Gamma(z)2^{-z}}{1 - 2^{-z}}.$$

The transform of the remainder is holomorphic on $\Re s > r - 3/2$. After division by $\Gamma(s)$ the difference of the two series is meromorphic across $\Re s = r - 1$, and the boundary for $q = 2$ follows. Lean formalizes this Euler-deletion deduction under RH, (Z2), (Z3) and Littlewood's gap estimate, with no Stirling premise. At $q = 2$, `phiR_natural_boundary_of_proved` together with `grh_two_iff_riemannHypothesis` gives the boundary under RH, (Z3), Littlewood's gap estimate and Stirling on strips, with no (Z2) premise (Section 5).

3.6. **Finitely many off-line zeros.**

Theorem 3.9 (Class-one boundary with finitely many off-line zeros)**.** *Fix integers $q, r \geq 2$. Suppose only finitely many distinct zeros in $0 < \Re z < 1$ of the primitive L-functions inducing the characters modulo q, including ζ, lie off $\Re z = 1/2$. Real exceptional zeros count towards this set. Assume (Z3), Littlewood's gap estimate and Stirling on strips. Then $\Phi_{r,q}$ continues meromorphically to $\Re s > r-1$, with pointwise agreement on $\Re s > r$. Every meromorphic function on $\Re s > r - 1$ agreeing with the series on $\Re s > r$ in the sense of Section 2 has $\Re s = r - 1$ as a natural boundary.*

Theorems 3.6 and 3.8 are the case $\mathcal{E} = \varnothing$ of this theorem, where $\mathcal{E}$ denotes the set of off-line zeros, with the same premises (Z3), Littlewood's gap estimate and Stirling on strips. We keep the shorter GRH proof because the limits in Section 4 build on it. Lean formalizes the theorem for $r = 2$ and for general $r \geq 2$ (Section 5).

Theorem A of the Goldbach paper allows infinitely many off-line zeros under H_∞ [37]. Here the exceptional pair correction must be meromorphic across the line. The proof below obtains this by summing over a finite set of exceptional zeros.

Proof. For each $e \in \mathcal{E}$ put $\widetilde{m}(e) = \sum_\chi \operatorname{ord}_e L(\cdot, \chi^*)$ and $\tilde{a}_e = -\alpha\widetilde{m}(e)\Gamma(e)$. The character average in Lemma 3.1 gives residues α at 1, $-\alpha P_1(t)$ at $z_t = 1/2 + \mathrm{i}t$, and $-\alpha\widetilde{m}(e)$ at $e \in \mathcal{E}$. The deleted Euler factors have their zeros on $\Re z = 0$.

For every real T, the critical-line multiplicity in $|t - T| \leq 1/2$ is $O_q(\log(|T| + 4))$ without GRH. Indeed, the inner Jensen disc in the proof of Lemma 3.2 contains that window, and the bounds on the outer circle and at the centre do not use GRH. Adding the finite total exceptional multiplicity gives the same window bound for all strip zeros, with a constant allowed to depend on $\mathcal{E}$. Let

$$\beta_{\mathcal{E}} = \max\big(\{1/2\} \cup \{\Re e : e \in \mathcal{E}\}\big) < 1, \qquad 0 < d < \min\big(\{1/2\} \cup \{\Re e : e \in \mathcal{E}\}\big).$$

If $\mathcal{E}$ is nonempty, the functional equation pairs each e with $1 - \overline{e}$ in $\mathcal{E}$. The reflection of a zero attaining $\beta_{\mathcal{E}}$ then gives $d < 1 - \beta_{\mathcal{E}}$. For $\mathcal{E} = \varnothing$ this inequality is just $d < 1/2$.

Use (Z3) for zeta and the primitive local expansion for the other inducing functions. As in Lemma 3.2, shift the Mellin integral for F_q from $\Re z = 2$ to $\Re z = d$. On the new line all strip zeros are a fixed positive horizontal distance away, so the local expansions and window bound give $D(d+\mathrm{i}y) = O(\log(2+|y|))$. The Euler terms are bounded on $\Re z \geq d$. The same window bound lets us choose heights $\pm H_N$, $H_N \in [N, N+1]$, at distance at least $c/\log(N+3)$ from all strip ordinates, by excluding intervals of total length less than one for small enough $c > 0$. On the horizontal segments $D = O(\log^2(N+3))$. Gamma decay makes those integrals tend to zero, gives $R_d = O(x^{-d})$ on $(0, 1]$ for the

shifted integral, and gives $\sum_t |A_t| < \infty$ by the window bound. The residue formula is

$$F_q = U_q + Z + \mathcal{X} + R_d, \qquad Z(x) = \sum_{t\in I} A_t x^{-z_t}, \qquad \mathcal{X}(x) = \sum_{e\in\mathcal{E}} \tilde{a}_e x^{-e}.$$

Thus $Z = O(x^{-1/2})$ and $\mathcal{X} = O(x^{-\beta_\mathcal{E}})$ on $(0,1]$.

For the binary series, use $F_q^2 = 2U_qF_q - U_q^2 + (Z+\mathcal{X}+R_d)^2$ in the Mellin calculation of Proposition 3.3. Completing the $2U_qF_q$ integral to infinity and integrating the pair terms on $(0,1)$ gives, on $\Re s > 2$,

$$\Gamma(s)\Phi_{2,q}(s) = W(s) + C_\mathcal{E}(s) + C_q(s) + H_\mathcal{E}(s),$$
$$C_\mathcal{E}(s) = 2\sum_{e\in\mathcal{E}}\sum_{t\in I} \frac{\tilde{a}_e A_t}{s-e-z_t} + \sum_{e,e'\in\mathcal{E}} \frac{\tilde{a}_e\tilde{a}_{e'}}{s-e-e'}.$$

The termwise integrals converge absolutely there by $\sum_t |A_t| < \infty$, finiteness of $\mathcal{E}$, and $\Re s > 2\beta_\mathcal{E}$. The functions W and C_q are given by the same formulas as in the GRH proof. The tail of $H_\mathcal{E}$ is entire by exponential decay, and its small-x integrand before the Mellin weight is $2(Z+\mathcal{X})R_d + R_d^2 = O(x^{-\beta_\mathcal{E}-d})$. Thus $H_\mathcal{E}$ is holomorphic on $\Re s > \beta_\mathcal{E} + d$, which contains the line.

For each e the candidates $e + z_t$ are locally finite and have real part $\Re e + 1/2 \neq 1$. On any compact set the denominators for large $|t|$ have modulus at least one. Absolute summability gives a normally convergent tail there and hence a meromorphic function on the plane, with at most the indicated poles. The second sum in $C_\mathcal{E}$ is finite and accounts for all exceptional pair candidates on $\Re s = 1$. In particular a Siegel zero $\beta > 1/2$ and its reflected zero $1-\beta$ have mixed candidates with real parts $\beta + 1/2$ and $3/2 - \beta$. The diagonal candidates 2β and $2(1-\beta)$ lie off the line. Their complementary pair contributes $2\tilde{a}_\beta\tilde{a}_{1-\beta}/(s-1)$ to $C_\mathcal{E}$. Principal products with exceptional zeros are in W, and products with R_d are in $H_\mathcal{E}$. Put $W_* = W + C_\mathcal{E}$. This correction is meromorphic on the plane, and C_q is holomorphic on $\Re s > 1$ by the denominator bound $\Re s - 1$. Dividing by $\Gamma(s)$ gives the binary continuation.

The critical-line pair weights are still $\alpha^2 P_1(t)P_1(t')$ times the Gamma products. This uses multiplicity at the full point $1/2 + \mathrm{i}t$, even when an exceptional zero has ordinate t. In a bounded window $|\tau| \le Y$, choose $L > Y$ and consider representations $\tau = t + t'$ with both ordinates larger than L in modulus. Orient each pair as $u + \tau, -u$, keeping both orders as indices. The product of the two unit-window counts bounds the total integer weight in $[v, v+1)$ by $O_q(\log^2(v + Y + 4))$, allowing a factor of two for the two orders. The bins are finite and a nonempty family has a least height. The phase cone of Section 2.5 is applied with its parameter $B = 1/2$, since both real parts are $1/2$. Thus Lemma 2.3 gives $M(\tau) \neq 0$ for represented sums with only such high representations when L exceeds its uniform threshold.

In any closed subinterval of positive length in the window, the sums with a critical ordinate of modulus at most L form a finite set, since the other ordinate then has modulus at most $L + Y$. The poles of W_* there are also finite. Choose an open subinterval (a,b) avoiding these sets. Littlewood's estimate gives arbitrarily large positive zeta ordinates γ. Applied at $\gamma + (a+b)/2$, it supplies a zeta ordinate γ' with $\gamma' - \gamma \in (a,b)$ once γ is sufficiently large. Both heights can be taken above all exceptional ordinates, and conjugation puts $-\gamma$ in I. This gives a critical pair sum with only high representations. The nonzero atoms are therefore dense.

Evaluate the corrected function $\Gamma\Phi^{c}_{2,q} - W_*$ as $C_q + H_{\mathcal{E}}$ on the right. After multiplication by η and evaluation at $1 + \mathrm{i}\tau + \eta$, the $H_{\mathcal{E}}$ term tends to zero as $\eta \downarrow 0$. The kernel $\eta/(\eta + \mathrm{i}(\tau - t - t'))$ tends to the coincidence indicator and is bounded by one, so dominated convergence with the summable majorant $|A_t A_{t'}|$ gives the limit $M(\tau)$. A meromorphic extension G on a disc centred on the line would give a meromorphic function $\Gamma G - W_*$ agreeing with $C_q + H_{\mathcal{E}}$ on the right-hand overlap. By the Laurent expansion a nonzero finite normal limit forces a simple pole, and the dense nonzero atoms would give an accumulation of poles inside the disc. This proves the binary boundary.

For $r \geq 3$ write $V_q = F_q - U_q = \mathcal{X} + V_0$ with $V_0 = Z + R_d = O(x^{-1/2})$ on $(0, 1]$. The algebraic identity from Proposition 3.7 gives

$$F_q^r = \binom{r}{2} U_q^{r-2} F_q^2 - r(r-2) U_q^{r-1} F_q + \tfrac{1}{2}(r-1)(r-2) U_q^r + Q_r,$$

$$Q_r = \sum_{j=3}^{r} \binom{r}{j} U_q^{r-j} (\mathcal{X} + V_0)^j.$$

In each term of Q_r let k be the number of V_0 factors. For $k = 0$, the term is a finite sum of monomials $x^{-\lambda}$ with $\Re\lambda \leq r - j + j\beta_{\mathcal{E}} < r$, whose transforms on $(0, 1)$ are individual multiples of $(s - \lambda)^{-1}$. For $k = 1$, the monomials multiply V_0 and have $\Re\lambda \leq r - j + (j-1)\beta_{\mathcal{E}} < r - 1$. With $u = s - \lambda$ use

$$\int_0^1 V_0(x) x^{u-1}\, dx = \Gamma(u) D(u) - E_1(u) - \frac{\alpha}{u-1} - \sum_{e \in \mathcal{E}} \frac{\tilde{a}_e}{u - e}, \qquad \Re u > 1,$$

where $E_1(u) = \int_1^\infty F_q(x) x^{u-1}\, dx$ is entire. This follows from $V_0 = F_q - U_q - \mathcal{X}$ and continues each one-factor term meromorphically to the plane. These formulas apply to the initial transform on $\Re s > r$. For $k \geq 2$,

$$U_q^{r-j} \mathcal{X}^{j-k} V_0^k = O\big(x^{-r+k/2+(j-k)(1-\beta_{\mathcal{E}})}\big).$$

If $k = 2$ then $j - k \geq 1$, and if $k \geq 3$ then $k/2 \geq 3/2$. Hence these integrals are holomorphic on $\Re s > r - 1 - \delta$, where $\delta = \min(1 - \beta_{\mathcal{E}}, 1/2) > 0$.

Completing the first two Mellin integrals as in Proposition 3.7 gives (2) and (3) with the binary continuation just constructed and with this meromorphic transform of Q_r. The tails in (3) are entire. The $k \leq 1$ terms can put poles on $\Re s = r - 1$. Local finiteness of the poles allows the correction to be subtracted meromorphically across the line. The multiplier c_r is holomorphic and nonzero there. The formula agrees with the series on $\Re s > r$. Subtracting the correction, dividing by c_r and translating by $-(r-2)$ would turn any extension across $\Re s = r - 1$ into a binary extension. The identity theorem identifies every agreeing continuation on the right-hand half-plane, so the binary obstruction proves the claim for every r. □

4. Regularized limits at the real boundary point

Theorem 4.1 and the other limits in this section retain their stated GRH or RH hypotheses, since a real exceptional pair $\beta, 1-\beta$ contributes the extra term $2\tilde{a}_\beta \tilde{a}_{1-\beta}/(s-1)$ at the real point in the Gamma-normalized binary series.

At the real boundary point $s = r - 1$ the normal limit needs no cone. Under RH or GRH the two zeros of each pair with ordinate sum zero are conjugates, so every numerator is the square of a modulus and the atom is positive. The cone would not

apply there in any case. The pairs with ordinate sum zero include low ones, for instance the first zero paired with its conjugate, so the threshold argument fails. It fails at some other points too, such as $\tau = 2\gamma_1$ for the least positive zeta ordinate γ_1, where a diagonal pair is low.

The linear Mellin terms are also singular at the real boundary point. Write $z = s-r+1$. Once the translated binary correction is expanded, its cross term combines with the linear term of the reduction, and in Gamma normalization the combined linear term is $r\alpha^{r-1}\Gamma(z)D(z)$ for class one and $r\Gamma(z)\psi(z)$ for the unrestricted series. The coefficient of the linear term of the reduction alone vanishes at $r = 2$; the combined coefficient does not. The combined term is singular at $z = 0$ because $\Gamma(z)$ has a pole there. For the unrestricted series the limit of $\eta\,\Phi_r^c(r-1+\eta)$ exists and includes a contribution of the linear term, which Section 4.3 computes. For class one the linear term has a double pole at the real point. Subtracting a main term, explicit up to an entire function, isolates the coefficient of the zero-pair square. For the unrestricted series we do this at every real ordinate, under RH. The class-one result in this section treats the real point under GRH at modulus q.

4.1. **The class-one series.** For class one D takes the place of ψ and the weights A_t take the place of the unrestricted zero weights $a_\rho = m(\rho)\Gamma(\rho)$. The main term $\mathfrak{M}_{r,q}$ is $1/\Gamma(s)$ times the sum of the shifted binary correction $b_rW(w)$, the linear and pure-power terms of (2), and the entire function $\widetilde{E}_{r,q}$ of (3). With $\widetilde{E}_{r,q}$ in the main term, the subtraction in Gamma normalization leaves the binary pair term $b_rC_q(w)$, the holomorphic term $b_rH_q(w)$ and the integral over $(0,1)$ in (3). The last two terms are holomorphic near $s = r-1$ and tend to zero after multiplication by η. Since $\widetilde{E}_{r,q}/\Gamma$ is entire, putting it in the main term does not change the limit. We put it there so that $\mathfrak{M}_{r,q}$ is the main term of the formal statement (Appendix A). The main term is therefore explicit up to this entire function. The proof uses the heat estimate in Lemma 3.2 at fixed modulus.

The positive limit in Theorem 4.1 is relative to the stated decomposition and is the limit of $\eta(\Phi_{r,q}^c - \mathfrak{M}_{r,q})(r-1+\eta)$. The class-one main term has a double pole at the real point, where the unrestricted main term has a simple pole. For $q \geq 2$ the principal character has at least one deleted Euler factor, which gives D a pole at $z = 0$ and $\Gamma(z)D(z)$ a double pole there. For $q = 2$, for instance, $D(z) = \psi(z) - (\log 2)2^{-z}/(1-2^{-z}) = -1/z + O(1)$.

Theorem 4.1 (Positive class-one real limit)**.** *Fix $q, r \geq 2$ and assume GRH at modulus q. With $\Phi_{r,q}^c$ as in Proposition 3.7, define*

$$\mathfrak{M}_{r,q}(s) = \frac{r\alpha^{r-1}\Gamma(z)D(z) - (r-1)\alpha^r/(s-r) + \widetilde{E}_{r,q}(s)}{\Gamma(s)}.$$

Then

$$\lim_{\eta\downarrow 0}\eta(\Phi_{r,q}^c - \mathfrak{M}_{r,q})(r-1+\eta) = \frac{\binom{r}{2}}{h^r(r-2)!}\sum_{t\in I}P_1(t)^2|\Gamma(1/2+\mathrm{i}t)|^2 > 0.$$

If $0 \in I$, the ordinate $t = 0$ is counted once.

Proof. The binary decomposition and higher-fold reduction use (Z3) and the primitive local expansion. Using $2\alpha b_r + \ell_r = r\alpha^{r-1}$ and $d_r - \alpha^2 b_r = -(r-1)\alpha^r$, we obtain

$$\Gamma(s)\mathfrak{M}_{r,q}(s) = b_rW(w) + \ell_r\Gamma(z)D(z) + \frac{d_r}{s-r} + \widetilde{E}_{r,q}(s).$$

Thus the subtracted continuation in Gamma normalization is $b_r(C_q(w)+H_q(w))$ plus the small-x integral in (3). Both $b_rH_q(w)$ and the integral are holomorphic near $s=r-1$, so their contributions vanish after multiplication by η as $\eta\downarrow 0$. Lemma 3.4 gives the limit $b_rM(0)/\Gamma(r-1)$ after division by Gamma.

Conjugation permutes the inducing primitive characters via $\chi\mapsto\overline{\chi}$ and sends zeros at $1/2+it$ to zeros of the conjugate functions at $1/2-it$, preserving their orders. Hence $P_1(-t)=P_1(t)$ and $A_{-t}=\overline{A_t}$. Each representation of zero is $(t,-t)$ with first entry $t\in I$, giving $M(0)=\alpha^2\sum_{t\in I}P_1(t)^2|\Gamma(z_t)|^2$. The sum converges by the absolute summability of $\{A_t\}$ and has a strictly positive term from a zeta ordinate in the principal contribution, since ζ has zeros in the critical strip [40, Theorem 9.4, p. 214]. The claimed factor follows from $b_r\alpha^2=\binom{r}{2}h^{-r}$ and $\Gamma(r-1)=(r-2)!$. □

4.2. **The unrestricted series.** Write $\Phi_r(s)=\sum_{n_1,\dots,n_r\geq 1}\Lambda(n_1)\cdots\Lambda(n_r)(n_1+\cdots+n_r)^{-s}$ initially for $\Re s>r$. Let Φ_r^c be its meromorphic continuation to $\Re s>r-1$ under RH, constructed in [37, Corollary B]: for $r=2$ from the Mellin square of F, and for $r\geq 3$ by the reduction in Proposition 7.4 there. Put

$$a_\rho=m(\rho)\Gamma(\rho),\qquad \mu\{\tau\}=\sum_{\Im\rho+\Im\rho'=\tau}a_\rho a_{\rho'},$$

where ρ and ρ' run independently over the distinct zeros, so that a diagonal pair is allowed. The notation $\mu\{\tau\}$ is that of [37]; the braces denote the atom at τ. For the binary series the subtraction is the transform of $2x^{-1}F-x^{-2}$, the cross term of the principal term with the heat sum minus the square of the principal term. For $r\geq 3$ it is combined with the completed linear term of the higher-fold expansion, and the result is the function $\mathfrak{M}_r$ of the proposition.

Proposition 4.2 (Unrestricted normal limit)**.** *Assume RH, and let $r\geq 2$. Set $z=s-r+1$, $\psi=-\zeta'/\zeta$ and*

$$\mathfrak{M}_r(s)=\frac{r\Gamma(z)\psi(z)-(r-1)/(s-r)}{\Gamma(s)}.$$

The singularities of $\Phi_r^c-\mathfrak{M}_r$ in $\Re s>r-1$ are removable, and we write $\Phi_r^c-\mathfrak{M}_r$ for the resulting holomorphic function on that half-plane. For every real τ it satisfies

$$\lim_{\eta\downarrow 0}\eta(\Phi_r^c-\mathfrak{M}_r)(r-1+i\tau+\eta)=\frac{\binom{r}{2}\,\mu\{\tau\}}{\Gamma(r-1+i\tau)}.$$

For $r=2$ this is Corollary B(iii) of [37]. For $r\geq 3$ and $\tau\neq 0$ it follows from Corollary B(iv) there, because $\mathfrak{M}_r$ is regular at $r-1+i\tau$. The case $r\geq 3$, $\tau=0$ is the new one. At $s=r-1$, the main term $\mathfrak{M}_r$ has a simple pole, since $\Gamma(z)$ has residue one at $z=0$ and $\psi(0)=-\log(2\pi)$. For $r=2$ this gives the residue $-2\log(2\pi)$ at $s=1$.

Proof. The proof uses (Z2) and (Z3). Under RH the heat expansion of Hardy and Littlewood [19, (2.214)], in the form of [37, Lemma 3.5], gives

$$F(x)=\sum_{n\geq 1}\Lambda(n)e^{-nx}=x^{-1}-\sum_\rho a_\rho x^{-\rho}+\psi(0)+O(x^{1/2}),\qquad 0<x\leq 1.$$

The Gamma weights are summable by the unit count in (Z2) and the classical Gamma bound. In particular $F(x)-x^{-1}=O(x^{-1/2})$. The same degree-$0,1,2$ binomial identity as in Proposition 3.7, now with $\alpha=1$ and F in place of F_q, yields the RH specialization

of the Mellin reduction. Here $b_r = \binom{r}{2}$, $\ell_r = -r(r-2)$ and $d_r = (r-1)(r-2)/2$ are the coefficients of Proposition 3.7 at $\alpha = 1$, and $w = z+1$. Multiplied through by $\Gamma(s)$, the reduction reads

$$\Gamma(s)\Phi_r^c(s) = b_r\Gamma(w)\Phi_2^c(w) + \ell_r\Gamma(z)\psi(z) + \frac{d_r}{s-r} + K_r(s).$$

The completed linear term uses $\int_0^\infty F(x)x^{z-1}\,dx = \Gamma(z)\psi(z)$ for $\Re z > 1$. The integral on $(0,1)$ of the terms of degree $j \geq 3$ in $F - x^{-1}$ converges locally uniformly for $\Re s > r - 3/2$, since each such term is $O(x^{-r+j/2})$. Adding the entire exponential tails preserves this domain of holomorphy for K_r. Equality first holds in the convergence half-plane and then as a meromorphic identity for the indicated continuation.

By Corollary B(iii) of [37], the function

$$\widetilde{H}(w) = \Phi_2^c(w) - \frac{2\Gamma(w-1)\psi(w-1) - 1/(w-2)}{\Gamma(w)}$$

is holomorphic on $\Re w > 1$, its singularities there being removable. Substituting Φ_2^c from this display into the reduction and using $2b_r + \ell_r = r$ and $d_r - b_r = -(r-1)$ gives

$$\Phi_r^c(s) - \mathfrak{M}_r(s) = b_r\frac{\Gamma(w)}{\Gamma(s)}\widetilde{H}(w) + \frac{K_r(s)}{\Gamma(s)}.$$

The right side is holomorphic on $\Re s > r-1$, and the statement writes $\Phi_r^c - \mathfrak{M}_r$ for this function. By [37, Proposition 3.7], $\Gamma(w)\widetilde{H}(w)$ is the pair sum $\sum_{\rho,\rho'} a_\rho a_{\rho'}/(w-\rho-\rho')$ plus two further terms that under RH are holomorphic across $\Re w = 1$. One is a function holomorphic on $\Re w > 1/2$, and the other is a multiple of $\sum_\rho a_\rho/(w-\rho)$, whose poles have real part $1/2$. The kernel $\eta/(\eta + \mathrm{i}(\tau - \Im\rho - \Im\rho'))$ selects the pairs with $\Im\rho + \Im\rho' = \tau$ by dominated convergence, using $\sum_{\rho,\rho'} |a_\rho a_{\rho'}| < \infty$. Thus $\eta\,\Gamma(w)\widetilde{H}(w)$ tends to $\mu\{\tau\}$ along the normal. The term K_r/Γ contributes zero after multiplication by η, and $1/\Gamma(s)$ is continuous at $r - 1 + \mathrm{i}\tau$, which gives the formula. □

Here the remainder K_r keeps the entire tails, together with the Mellin integral over $(0,1)$ of the terms of degree at least three in $F - x^{-1}$, so unlike $\mathfrak{M}_{r,q}$ the main term $\mathfrak{M}_r$ has no tail term.

The proposition involves no non-cancellation statement. At every τ it gives the limit as the atom times $\binom{r}{2}/\Gamma(r - 1 + \mathrm{i}\tau)$, whether or not the atom vanishes. At $\tau = 0$ the pairs with ordinate sum zero are conjugate pairs under RH, so the atom is a sum of squares of moduli.

Corollary 4.3 (Positive unrestricted real limit)**.** *Assume RH, and let $r \geq 2$. Then*

$$\lim_{\eta \downarrow 0} \eta(\Phi_r^c - \mathfrak{M}_r)(r - 1 + \eta) = \frac{\binom{r}{2}}{(r-2)!}\sum_\rho m(\rho)^2|\Gamma(\rho)|^2 > 0.$$

The sum is over distinct nontrivial zeros, including both members of each conjugate pair.

Proof. Under RH a pair with ordinate sum zero has the form $(\rho, \overline{\rho})$. Conjugation preserves multiplicity and conjugates Gamma, so the numerator for that ordered pair is $m(\rho)^2|\Gamma(\rho)|^2$. Absolute convergence follows from the summability of the pair weights. The sum has a positive term because ζ has zeros, by the first estimate of (Z2), and Γ has none. Apply Proposition 4.2 with $\tau = 0$ and use $\Gamma(r-1) = (r-2)!$. □

Remark (Two evaluations). The formal statements for the subtracted real-point limits use sums with $|\Gamma|^2$ (Section 5). The reflection formula gives $|\Gamma(1/2+\mathrm{i}t)|^2 = \pi/\cosh(\pi t)$ [34, §5.5], so under RH the sum in Corollary 4.3 is $\sum_\rho \pi m(\rho)^2/\cosh(\pi\Im\rho)$. The simple pole of the linear term in $\mathfrak{M}_r$ at $z=0$ comes from $\Gamma(z)$. Its residue at $s=r-1$ is $r\psi(0)/\Gamma(r-1)$, which is the limit of $\eta\,\mathfrak{M}_r(r-1+\eta)$. This is the contribution of the constant $\psi(0)$ in the heat expansion, and the subtraction removes it.

4.3. **The unsubtracted coefficient at the real point.**

Proposition 4.4 (Unsubtracted real limit)**.** *Assume RH and let $r \geq 2$ be an integer. Then*

$$\lim_{\eta\downarrow 0} \eta\,\Phi_r^c(r-1+\eta) = \frac{\binom{r}{2}\mu\{0\} - r\log(2\pi)}{(r-2)!}.$$

Set $r_* = 1+2\log(2\pi)/\mu\{0\}$ and $N = 5639249912025015212$. Then $N < r_* < N+1$. The limit is nonzero for every integer $r \geq 2$, negative for $r \leq N$ and positive for $r \geq N+1$.

Lean formalizes the formula under RH, (Z2) and (Z3) and the gap under RH, (Z2) and the certified low zeros. The release proves the low-zero certificate unconditionally. The final `Tendsto` statement gives the limit and its nonvanishing with the stated signs under RH, (Z2) and (Z3).

Proof. Adding the main term back in Corollary 4.3 gives the formula, since $\eta\,\mathfrak{M}_r(r-1+\eta) \to r\psi(0)/(r-2)!$ and $\psi(0) = -\log(2\pi)$. The formula is a direct consequence of [37, Corollary B(iii) and Proposition 7.4]. Since $\mu\{0\} > 0$, the numerator is negative for $2 \leq r < r_*$ and positive for $r > r_*$.

Certified computation of the gap. The gap is proved by the certified ball-arithmetic computation described below and independently by the Lean theorem `PhiRProof.rStar_gap_RH`. An unconditional argument-principle count in interval arithmetic on the rectangle $1/4 \leq \Re z \leq 2$, $|\Im z| \leq 32$ gives eight zeros of ζ there, counted with multiplicity. Sign changes of Hardy's real-valued zeta function in four disjoint intervals on the critical line identify them as the first four positive ordinates and their negatives, all simple. The four intervals have width 10^{-30} and left endpoints

$$14.134725141734693790457251983562, \quad 21.022039638771554992628479593896,$$
$$25.010857580145688763213790992562, \quad 30.424876125859513210311897530584.$$

The count evaluates ζ by Euler–Maclaurin summation with cutoff 24 and 18 correction terms, and encloses the image of each segment of the contour in an open coordinate half-plane after multiplication by $1 + 0.2718281828459045235\,\mathrm{i}$. Traversing the bottom, right, top and left sides counterclockwise gives quadrant increments 0, 0, -1 and 29, respectively. The winding number is 7, and with the pole at 1 this gives eight zeros. The sign checks use cutoff 128 and 50 correction terms. Hardy's phase $\Im\log\Gamma(1/4+\mathrm{i}t/2) - (t/2)\log\pi$, with a continuous logarithm, varies by less than $20\cdot 10^{-30}$ on each interval.

Under RH, $\mu\{0\} = \sum_\rho \pi m(\rho)^2/\cosh(\pi\gamma)$. For the tail, set $D_0 = \sum_\rho m(\rho)/(1/4+\gamma^2)$. Its value is $2+\gamma_{\mathrm{E}} - \log(4\pi) < 1/20$, where γ_{E} is Euler's constant. Logarithmic differentiation of the Hadamard product of ξ gives $(\xi'/\xi)(1) - (\xi'/\xi)(0) = \sum_\rho m(\rho)/(\rho(1-\rho)) = 2+\gamma_{\mathrm{E}} - \log(4\pi)$, which equals D_0 under RH [12, Ch. 12, (7) and (10)][40, (2.12.5)]. The zeros with $|\gamma| \geq 32$ therefore contribute less than $3.605\cdot 10^{-40}$ to $\mu\{0\}$: the squares of the terms of D_0 sum to at most D_0^2, the function $(1/4+t^2)^2\mathrm{e}^{-\pi t}$ decreases for $t \geq 32$, and $\cosh(\pi t) \geq \mathrm{e}^{\pi|t|}/2$.

The result is $\mu\{0\} \approx 6.518161 \cdot 10^{-19}$ and

$$5639249912025015212.748 < r_* < 5639249912025015212.752.$$

Both bounds lie between N and $N+1$. For $r=3$ the numerator is about -5.514. The script `anc/enclosure.py`, posted with this paper, carries out the computation above in Arb ball arithmetic with python-flint 0.9.0 and certifies each count and inequality in that computation. Its output is `anc/enclosure_out.log`.

The Lean proof reproduces the integer gap using the `zeta` project to certify unconditionally that the nontrivial zeros with $0 < \Im\rho \le 32$ are exactly four, all simple and on the critical line, with one in each of the 10^{-30}-wide brackets above [27]. Under RH, Lean proves convergence and the identity for the D_0 sum. It proves $D_0 < 1/6$ and that the contribution from $|\gamma| > 32$ is less than $4.1 \cdot 10^{-39}$. These bounds suffice for $N < r_* < N+1$ with kernel-checked interval arithmetic. The gap and the sign change of the numerator give the stated signs at the integers. □

5. Formal verification

The formalization extends the Goldbach development. The class-one results and the unrestricted limits were checked with Lean 4.35.0 and Mathlib at commit `56a0058`. The classical premises are propositions about Mathlib's zeta function, Dirichlet L-functions and Gamma function [24][37, Section 10]. The character identity of Lemma 3.1 is formalized in pointwise form (Appendix A).

The development is release `v1.2` of `https://github.com/lpkralv/phase-cone`, Apache-2.0, commit `468b8d234ef08a71d329207fd210d16095928217`, built with the toolchain `leanprover/lean4:v4.35.0`. The folder `class-one` holds this paper's check file and a README that maps each numbered result to its declarations. One build from a clean clone of the release checks the Lean results of all three papers that the repository accompanies. Appendix A gives the premises and prints the main class-one boundary and limit theorems.

The `zeta` commit `4704234df2d97563936adcaf0b89e2f8955389f9` supplies the kernel-checked zero evaluation used here [27]. The release uses this layer throughout, excluding the `native_decide` variant. The `zeta` project pulls in *PrimeNumberTheoremAnd* from the `smmercuri` fork at commit `16a8986` [20]. The proof of the primitive local expansion in `PrimitiveLocalProof` imports `PrimeNumberTheoremAnd.StrongPNT` from that fork and uses its disk estimates `FinalBound` and `ZerosBound`.

In namespace `ResidueClass`, `logDerivPrimitiveLocal_of_GRH` derives the primitive local expansion `LogDerivPrimitiveLocal q` from `GRH q`. The declarations `phi2_natural_boundary_of_proved` and `phiR_natural_boundary_of_proved` take (Z3), Littlewood's gap estimate and Stirling on strips in addition to GRH. The real-point declaration `phiR_sub_main_real_limit_of_proved` takes (Z3) in addition to GRH.

The declaration `logDerivPrimitiveLocal_of_finite_exceptions` derives the primitive local expansion from finiteness of `OffLineZeros q`, with the height threshold above all exceptional ordinates. Theorem 3.9 is formalized as `phi2_natural_boundary_finite_exceptions` for the binary case and `phiR_natural_boundary_finite_exceptions` for every $r \ge 2$, in the same namespace.

For Proposition 4.4, Lean checks the limit formula in `PhiRProof.phiR_real_limit` and the D_0 identity and tail bound in `PhiRProof.d0_identity` and `PhiRProof.tail_above_32_of_RH`. The proof of the D_0 identity uses the Hadamard factorization of ξ from

PrimeNumberTheoremAnd. The gap follows from `PhiRProof.rStar_gap_RH` with the low-zero certificate `PhiRProof.releaseLowZeros_proved`. The final declaration `PhiRProof.phiR_tendsto_ne_zero` gives nonvanishing and the sign for each integer $r \geq 2$. Both limit declarations take Mathlib's `RiemannHypothesis`, `Paper2.Z2` and `Paper2.Z3`.

The formal statements differ from the text in a few points of form. The natural-boundary predicate uses agreement in the codiscrete filter, which is the meromorphic agreement of Section 2 [36, §§2.4 and 5.2]. In each boundary theorem the existence clause gives pointwise agreement on the initial half-plane, and the universal clause uses meromorphic agreement there. Lean functions are total. An expression such as $\Gamma(z)D(z)$ has a value at every point, including its poles and removable singularities, and that value need not be the limit. The formal statements about continuations therefore assert agreement off a discrete set, or are made at points where the expressions are analytic, while the formulas printed in the text are identities of meromorphic functions. The series $\Phi_{r,q}$ and Φ_r enter the statements only on half-planes where they converge absolutely. The Lean names say *radial* where the text says normal limit, as in the Goldbach paper [37, Lemma 6.6].

5.1. **Axioms.** The reported output of `#print axioms` lists only the three standard axioms of Lean's logic (`propext`, `Classical.choice` and `Quot.sound`) for these `ResidueClass` declarations. `logDerivPrimitiveLocal_of_GRH`, `logDerivPrimitiveLocal_of_finite_exceptions`, `phi2_natural_boundary_of_proved`, `phiR_natural_boundary_of_proved`, `phiR_sub_main_real_limit_of_proved`, `phi2_natural_boundary_finite_exceptions`, `phiR_natural_boundary_finite_exceptions`, `phiR_two_natural_boundary_of_unrestricted`, `phiRc_sub_phiRTwoContinuation_extends` and `grh_two_iff_riemannHypothesis`. The same holds for the two unrestricted subtracted limit statements, `PhiRProof.phiR_real_limit`, `PhiRProof.d0_identity`, `PhiRProof.tail_above_32_of_RH`, `PhiRProof.rStar_gap_RH`, `PhiRProof.releaseLowZeros_proved` and `PhiRProof.phiR_tendsto_ne_zero`. None of these declarations depends on `sorry`, including the declarations whose proofs use *PrimeNumberTheoremAnd* modules that emit `sorry` warnings.

5.2. **The status of RH and GRH.** Alpöge's Lean proofs of RH and of GRH for Dirichlet L-functions are a separate development, built against other versions of Lean and Mathlib, and no result here depends on them. Each result takes the hypothesis on the zeros that it states as a premise [1]. The joint statement of the Lean and Mathlib communities certifies the proof of RH. It confirms that the theorem proved is Mathlib's `RiemannHypothesis` and that the proof passes the kernel and independent checkers [22]. That statement concerns RH only. Sutherland reports independent checks and prints the GRH theorem for primitive characters, which places zeros in $0 < \Re s < 1$ on $\Re s = 1/2$ [38]. No build yet combines that development with ours.

5.3. **Scope of the formal statements.** For $q = 2$, Lean derives the boundary by Euler deletion from the unrestricted one under RH, (Z2), (Z3) and Littlewood's gap estimate in `phiR_two_natural_boundary_of_unrestricted`, with no Stirling premise. Under RH, the difference after subtraction of the explicit Euler term extends holomorphically to $\Re s > r - 3/2$ by `phiRc_sub_phiRTwoContinuation_extends`. At $q = 2$, `phiR_natural_boundary_of_proved` together with `grh_two_iff_riemannHypothesis` gives the boundary under RH, (Z3), Littlewood's gap estimate and Stirling on strips, with no (Z2) premise. These declarations are in namespace `ResidueClass`.

The two comparisons with Rüppel's formulas and the questions in Section 3.4 are outside the formal statements. The pointwise identity $|\Gamma(1/2 + \mathrm{i}t)|^2 = \pi/\cosh(\pi t)$ is available in Lean from *PrimeNumberTheoremAnd* [20] as `gamma_half_vertical_norm_sq`. The subtracted real-point statements use $|\Gamma|^2$, and the unrestricted unsubtracted limit uses the mass in its $\pi/\cosh$ form. The formal statements also cover the integer enclosure for r_* and the resulting signs. The cone paper formalizes the cone results [36, Theorem 3.1, Corollary 3.2 and Lemma 3.3], gives the general extension argument used here [36, §2.4], and describes its formal counterpart [36, §5.4].

Appendix A identifies the classical premises, Section 2 gives their sources, and the Lean kernel checks the implications from those premises.

Acknowledgement and statement on the use of AI

The author is a computer scientist, not a mathematician. He originated the questions treated here, directed an AI research agent through a sequence of questions on natural boundaries of Goldbach series over primes and prime powers, and chose at each step the direction to pursue. He designed the verification methodology: independent adversarial reviews, re-derivation and small-case checks of every result before it was recorded as proved, and a formalization in Lean (Section 5). The mathematics is largely the work of AI models (Anthropic Claude, OpenAI GPT) under his direction. Multiple models reviewed the argument. An OpenAI GPT model drafted the manuscript and an Anthropic Claude model revised it. AI agents wrote the Lean development. The author did not write or check the Lean development line by line; he relies on the Lean kernel and the axiom checks reported in Section 5. No human mathematician has checked the proof line by line. There is no co-reader: queries about the proof will be answered by the author, with the same AI assistance. No AI system is an author. The author takes full responsibility for the content.

The author thanks Anna DeConti, formerly of Osbourn Park High School, who taught him calculus and encouraged his explorations in mathematics.

Appendix A. Formal premises and main Lean statements

This appendix lists the premises of each numbered result and prints the main class-one statements, without proofs or docstrings.

A.1. **Premises by result.** The class-one statements fix a nonzero modulus. The GRH premise is `GRH q`. The finite-exception premise is `(OffLineZeros q).Finite`, where `OffLineZeros q` is the set of exceptional zeros in Theorem 3.9. Lean derives the primitive local expansion under either zero hypothesis. Jensen's formula gives the local count in Lemma 3.2. In Lean, (Z2) and (Z3) have the names `Paper2.Z2` and `Paper2.Z3`, Littlewood's gap estimate is `Paper2.ZLittlewood`, and Stirling on strips is `Phi2HInf.StirlingStrip`. The hypothesis names (Z1)–(Z3) follow the formalization. In the Goldbach paper, (Z1) denotes the local count in our (Z2), (Z2) denotes Littlewood's estimate, the zero symmetries in our (Z1) are derived, and (Z3) is unchanged [37, Appendix A].

Theorem 2.1: The stated polynomial-bin hypotheses and Stirling on strips.
Corollary 2.2: The stated hypotheses, including Stirling on strips.
Lemma 2.3: The stated integer-weight hypotheses and Stirling on strips.
Lemma 3.1: GRH only for the location of the strip poles.

Lemma 3.2: GRH. The expansion adds (Z3), $0 < d < 1/2$ and $x > 0$. The separate bound on the remainder, stated for every real d for the line integral as Lean defines it (Lean's integral is zero when the integrand is not integrable), takes $x > 0$ and requires no zero hypotheses.
Proposition 3.3: GRH and (Z3). The holomorphic correction also takes $0 < d < 1/2$.
Lemma 3.4: GRH. The corrected limit adds (Z3).
Lemma 3.5: GRH, Littlewood's gap estimate and Stirling on strips. The finite-pole exclusion requires no zero hypotheses.
Theorem 3.6: $q \geq 2$, GRH, (Z3), Littlewood and Stirling on strips.
Proposition 3.7: $r \geq 2$, GRH and (Z3).
Theorem 3.8: $q, r \geq 2$, GRH, (Z3), Littlewood and Stirling on strips.
Theorem 3.9: $q, r \geq 2$, finitely many off-line zeros, (Z3), Littlewood and Stirling on strips.
Theorem 4.1: $q, r \geq 2$, GRH and (Z3).
Proposition 4.2: $r \geq 2$, Mathlib's `RiemannHypothesis`, (Z2) and (Z3).
Corollary 4.3: $r \geq 2$, Mathlib's `RiemannHypothesis`, (Z2) and (Z3).
`PhiRProof.phiR_real_limit`**:** Proposition 4.4, limit formula. $r \geq 2$, RH, (Z2) and (Z3).
`PhiRProof.rStar_gap_RH`**:** Proposition 4.4, integer gap. RH, (Z2) and the low-zero certificate `ReleaseLowZeros`.
`PhiRProof.phiR_tendsto_ne_zero`**:** Proposition 4.4, nonvanishing limit and signs. $r \geq 2$, RH, (Z2) and (Z3).

The certificate `ReleaseLowZeros` says that ζ has exactly four nontrivial zeros with $0 < \Im\rho \leq 32$, all simple and on the critical line, with one ordinate in each bracket printed after Proposition 4.4. The release proves it as `PhiRProof.releaseLowZeros_proved` without a zero hypothesis.

The character identity in Lemma 3.1 is formalized pointwise where $\Re z > 0$, $z \neq 1$ and no inducing primitive function vanishes. The unrestricted declarations are `PhiRProof.phiR_sub_main_radial_limit` for Proposition 4.2 and `PhiRProof.phiR_sub_main_real_limit` for Corollary 4.3. The former asserts analyticity of the subtracted function on $\Re s > r - 1$ and, for every real τ, gives the limit as $\eta \downarrow 0$ of η times the subtracted function evaluated at $r - 1 + \mathrm{i}\tau + \eta$.

A.2. **Main statements.** The namespace below is `GoldbachBoundaries.ResidueClass`. The common notation is from the Goldbach development. Here `halfPlane a` is $\{\Re s > a\}$, `HH` is `halfPlane 1`, and `lineAt a` and `lineOne` are their boundary lines. The cone paper prints `IsNaturalBoundary` [37, Appendix A][36, Appendix A.2].

Class one: the boundary theorems (namespace `ResidueClass`).

```
theorem phi2_natural_boundary_of_proved
    (hq : 2 ≤ q) (hGRH : GRH q) (hZ3 : Paper2.Z3)
    (hGap : Paper2.ZLittlewood) (hStir : Phi2HInf.StirlingStrip) :
    (∃ F : ℂ → ℂ, MeromorphicOn F HH ∧ EqOn F (Phi2 q) (halfPlane 2)) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F HH →
      F =ᶠ[codiscreteWithin (halfPlane 2)] Phi2 q →
      IsNaturalBoundary F HH lineOne) :=

theorem phiR_natural_boundary_of_proved
    (hq : 2 ≤ q) (hGRH : GRH q) (hZ3 : Paper2.Z3)
    (hGap : Paper2.ZLittlewood) (hStir : Phi2HInf.StirlingStrip) {r : ℕ} (hr : 2 ≤ r) :
    (∃ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) ∧
      EqOn F (PhiR q r) (halfPlane r)) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) →
      F =ᶠ[codiscreteWithin (halfPlane r)] PhiR q r →
```

```
      IsNaturalBoundary F (halfPlane ((r : ℝ) - 1)) (lineAt ((r : ℝ) - 1))) :=

theorem phiR_natural_boundary_finite_exceptions (hq : 2 ≤ q)
    (hE : (OffLineZeros q).Finite) (hZ3 : Paper2.Z3) (hGap : Paper2.ZLittlewood)
    (hStir : Phi2HInf.StirlingStrip) {r : ℕ} (hr : 2 ≤ r) :
    (∃ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) ∧
      EqOn F (PhiR q r) (halfPlane r)) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) →
      F =ᶠ[codiscreteWithin (halfPlane r)] PhiR q r →
      IsNaturalBoundary F (halfPlane ((r : ℝ) - 1)) (lineAt ((r : ℝ) - 1))) :=
```

Class one: the real point. In the limit statement, `phiRMain q r` is $\mathfrak{M}_{r,q}$ of Theorem 4.1. The set `Ordinates q` is the set of ordinates I, `ordinateWeight q t` is $P_1(t)$, and `criticalPoint t` is z_t. The right-hand neighbourhood filter in `Tendsto` denotes $\eta \downarrow 0$. The prime on the sum marks a Lean infinite sum, which has value zero for a nonsummable family [36, Appendix A.2].

```
theorem phiR_sub_main_real_limit_of_proved
    (hq : 2 ≤ q) (hGRH : GRH q) (hZ3 : Paper2.Z3)
    {r : ℕ} (hr : 2 ≤ r) :
    Tendsto (fun η : ℝ ↦ (η : ℂ) *
      (phiRContinuation q r ((r : ℂ) - 1 + η) -
        phiRMain q r ((r : ℂ) - 1 + η)))
      (𝓝[>] 0)
      (𝓝 (((r.choose 2 : ℝ) / ((q.totient : ℝ) ^ r * ((r - 2).factorial : ℝ)) *
        ∑' t : Ordinates q,
          (ordinateWeight q t : ℝ) ^ 2 *
            ‖Gamma (criticalPoint t)‖ ^ 2 : ℝ) : ℂ)) ∧
    0 < (r.choose 2 : ℝ) / ((q.totient : ℝ) ^ r * ((r - 2).factorial : ℝ)) *
      ∑' t : Ordinates q,
        (ordinateWeight q t : ℝ) ^ 2 * ‖Gamma (criticalPoint t)‖ ^ 2 :=
```

References

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