The tensor rank of the permanent

Yilin Liu

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Abstract

We prove that the ordinary tensor rank of the $n \times n$ permanent, with one factor for each row, is 2^{n-1} over every field of characteristic different from two. Every shortest decomposition for $n \geq 3$ is Glynn's sign formula up to diagonal changes of variables and rescaling of the factors in each summand. In characteristic two the rank is strictly larger for $n \geq 3$. More generally, the rank of the multilinear map computing all square-free coefficients of a product of k linear forms in n variables is $\sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2j}$ in characteristic different from two. The proof evaluates arbitrary rank-one summands on subsets of the input slots. A quadratic relation acts injectively on one evaluation space and maps it into the kernel of the next multiplication map. Iteration gives the lower bound. Character interpolation attains it and extends the formula to uniform higher truncations.

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1 The permanent and square-free multiplication

Let $V = F^n$. The ordinary rank of an n-linear form is the smallest number of products of n linear forms, one in each argument, whose sum is the given form. For an $n \times n$ matrix $X = (X_{ij})$, the permanent is the n-linear form on its rows

$$\mathrm{per}_n(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)}.$$

The linear forms in a summand may differ from row to row.

Theorem 1.1. *For every field F and integer $n \geq 1$, $\mathrm{R}_F(\mathrm{per}_n) \geq 2^{n-1}$. Equality holds when $\mathrm{char}\, F \neq 2$. For $n \geq 3$ every shortest decomposition over such a field is a diagonal rescaling of Glynn's formula, and in characteristic two the inequality is strict.*

Glynn's formula gives the matching upper bound in characteristic different from two [5]. Han, Ju and Kim [7, Question 5.8] asked whether this formula is shortest for every $n \geq 5$. Han and Song [8, Theorem A] proved the $n = 5$ case in characteristic zero; the ordinary-rank problem for $n \geq 6$ is recorded in Han [6, §4.2]. Theorem 1.1 settles it over every field of characteristic different from two. Earlier lower bounds use matrix substitution [9, Theorem 14] and recursive Koszul flattenings [7].

The symmetric Waring rank of $x_1 \cdots x_n$ follows from the monomial formula of Carlini, Catalisano and Geramita [3]; Han [6] proves its border Waring rank in characteristic zero. Buczyńska, Buczyński and Teitler [2] classify the symmetric Waring decompositions of monomials over $\mathbb{C}$. We prove that every shortest ordinary decomposition of the permanent is already termwise symmetric, and classify the minimizing Segre schemes of complete-intersection tensors.

Consider the square-free algebra $A = F[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ with its usual grading. For $1 \leq k \leq n$, let $M_{n,k} : V^{\otimes k} \to A_k$ multiply k degree-one inputs. Its outputs are the coefficients of the square-free monomials of degree k. In the subset bases, its tensor has the ordered-partition support of sizes $(n-k, 1, \ldots, 1)$, using the complement of a monomial to label the output. At $k = n$ its single coefficient is per_n.

Theorem 1.2. *Let $b(n,k) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2j}$. Over every field, $\mathrm{R}_F(M_{n,k}) \geq b(n,k)$. If $\mathrm{char}\, F \neq 2$, then $\mathrm{R}_F(M_{n,k}) = b(n,k)$. Replacing the commutative square-free product by the exterior product preserves the lower bound in every characteristic.*

In particular, the same argument gives $\mathrm{R}_F(\det_n) \geq 2^{n-1}$ in every characteristic. Stronger determinant lower bounds over the complex numbers follow from recursive Koszul flattenings [7]. In characteristic two the determinant and permanent tensors coincide; already for $n = 3$ their ordinary rank is five [10].

2 Evaluation on subsets of the inputs

We state the argument for a finite-dimensional associative graded algebra A generated by $V = A_1$. We assume either that degree-one elements commute or that interchanging them changes the product by a minus sign. Write $h_j = \dim A_j$ and $A_0 = F$. Fix k and assume that each nonzero element of A_j, $0 \le j \le k$, has a nonzero product with some element of A_{k-j}. We use the maps $M_s : V^{\otimes s} \to A_s$ given by ordered multiplication.

Lemma 2.1. *Suppose $d \ge 2$ and the annihilator $(\ker M_d)^{\perp} \subset (V^*)^{\otimes d}$ contains no nonzero pure tensor over an algebraic closure. In a rank-R decomposition of M_k, the evaluation spaces E_J defined below satisfy*

$$\dim E_J \ge h_{|J|}, \qquad \dim E_J \ge h_{|J|} + \dim E_{J\setminus I} \quad (I \subseteq J \subseteq [k],\ |I| = d).$$

Consequently $\mathrm{R}(M_k) \ge \sum_{j=0}^{\lfloor k/d \rfloor} h_{k-jd}$.

Proof. Extend the field to its algebraic closure for the lower bound. A rank-R decomposition supplies $u_s \in A_k$ and nonzero $p_{is} \in V^*$ such that

$$v_1 \cdots v_k = \sum_{s=1}^{R} u_s \prod_{i=1}^{k} p_{is}(v_i).$$

For $J \subseteq [k]$, let $E_J \subseteq F^R$ be the image of $\bigotimes_{i\in J} V$ under evaluation at the R tuples $(p_{is})_{i\in J}$. The empty evaluation space is the constant line. Multiplication on the slots in J, taken in their original order, factors through a surjection $E_J \to A_{|J|}$. Indeed if a tensor on J evaluates to zero, the displayed algorithm vanishes after filling the other slots arbitrarily. Moving those slots together incurs only a fixed sign. Their products span $A_{k-|J|}$, so the extension hypothesis forces the original product to vanish. Surjectivity follows because V generates A.

Fix $I \subseteq J$ of size d. Each point $(p_{is})_{i\in I}$ has a nonzero evaluation on some element of $\ker M_d$ by hypothesis. A finite union of proper linear hyperplanes over an infinite field cannot cover $\ker M_d$. Choose $H \in \ker M_d$ whose evaluations are nonzero at all R points. Multiplication by its evaluation vector, coordinate by coordinate, embeds $E_{J\setminus I}$ into E_J. This entire image maps to zero in $A_{|J|}$ because the product of H is zero. Rank–nullity gives the second inequality. Iterate it, using $\dim E_J \ge h_{|J|}$ on the remaining slots and $\dim E_{[k]} \le R$. □

Proof of Theorem 1.2. Each degree-j piece has the $\binom{n}{j}$ square-free monomials as a basis. For a nonzero coefficient on S, multiplication by any $k - j$ distinct variables outside S retains that coefficient on the union, up to sign in the exterior case. This checks the extension hypothesis.

For the commutative product, $\ker M_2$ contains $e_i \otimes e_i$ and $e_i \otimes e_j - e_j \otimes e_i$. A pure tensor $p \otimes q$ annihilating these relations would satisfy $p_i q_i = 0$ for all i, and its two factors would be proportional by the exchange relations. A nonzero proportional pair has a nonzero diagonal entry. For the exterior product the exchanges are $e_i \otimes e_j + e_j \otimes e_i$. Choose i with $p_i \ne 0$. The diagonal forces $q_i = 0$ and the exchanges then force every $q_j = 0$. Lemma 2.1 with $d = 2$ gives both lower bounds.

For the upper bound, evaluate an ordinary product of k linear forms on sign vectors $\epsilon \in \{1, -1\}^n$. Its character expansion uses only $\chi_S(\epsilon) = \prod_{i\in S} \epsilon_i$ with $|S| = k, k-2, \ldots$. These $b(n,k)$ characters are independent by orthogonality when $2 \ne 0$ in F. Select $b(n,k)$ evaluation points giving an invertible character matrix. For $|S| = k$ the character coefficient is exactly the square-free coefficient indexed by S. Each evaluation is one product of k linear forms. □

The rank equality in Theorem 1.1 follows by taking $k = n$ and adding the binomial coefficients of one parity.

3 Uniform truncations

Let $A = F[x_1, \dots, x_n]/(x_i^d : 1 \le i \le n)$, $d \ge 2$, and set $h_j = [z^j](1 + z + \cdots + z^{d-1})^n$. For $1 \le k \le n(d-1)$ denote multiplication of k degree-one inputs by $M_{n,k}^{(d)}$.

Theorem 3.1. *Over every field,* $\mathrm{R}_F(M_{n,k}^{(d)}) \ge \sum_{0 \le j \le k,\ j \equiv k \pmod d} h_j$. *Equality holds if F contains d distinct dth roots of unity. In particular, the tensor summing all words in which each of n symbols occurs $d-1$ times has ordinary rank d^{n-1} under this field hypothesis.*

Proof. A monomial with all exponents below d and total degree j can be extended within these bounds to degree k. Multiplying by the extending monomial preserves that coefficient without interference from other terms. The kernel of M_d contains exchanges of any two inputs and all $e_i^{\otimes d}$. A nonzero pure functional annihilating the exchanges would be fixed by each transposition, so all its factors would be proportional. An $e_i^{\otimes d}$ then evaluates nonzero for a coordinate where the common factor is nonzero. Lemma 2.1 gives the lower bound.

Evaluate on the grid of dth roots of unity. The possible characters of a product of k linear forms have exponent vectors $\alpha \in \{0, \dots, d-1\}^n$ with $|\alpha| \le k$ and $|\alpha| \equiv k \pmod d$. These characters are independent by the one-variable Vandermonde determinant and tensor product. Interpolate on as many grid points as there are such characters. Those with $|\alpha| = k$ recover exactly the desired output coefficients. In top degree every possible residue vector of the prescribed total residue occurs; the first $n-1$ residues determine the last one. This gives d^{n-1}. □

4 Rigidity of shortest decompositions

Put $N = n(d-1)$ and let $D_{n,d}$ denote the top tensor in Theorem 3.1. Every argument of this form is concise: the flattening on a single input has rank n, since for each symbol a complementary word picks out its coefficient. Write $\mu_d(F) = \{z \in F : z^d = 1\}$.

Theorem 4.1. *Suppose $n, d \ge 2$ and $(n,d) \ne (2,2)$. Equality* $\mathrm{R}_F(D_{n,d}) = d^{n-1}$ *holds exactly when $|\mu_d(F)| = d$. In that case every decomposition of length d^{n-1} is termwise symmetric. Its projective factor vectors, common to all N slots, form one full grid*

$$\big\{[a_1 : a_2\epsilon_2 : \cdots : a_n\epsilon_n] : \epsilon_i \in \mu_d(F)\big\}, \qquad a_1 \cdots a_n \ne 0.$$

For each choice of this grid the weights are unique. For the permanent this is precisely the family of diagonally rescaled Glynn decompositions.

Proof. Suppose there is a decomposition of length $R = d^{n-1}$. Fix d slots I and enlarge them to a set K of size $k = d\lfloor N/d \rfloor$. The remaining slots test A_k by the perfect top pairing. Thus the evaluation spaces of the chosen decomposition on subsets of K satisfy Lemma 2.1: equivalently one may merge the complementary slots into an output and project it onto the natural copy of A_k. Since $N - k < d$, the sum $h_k + h_{k-d} + \cdots + h_0$ includes every coefficient of the Hilbert polynomial in its residue class and equals d^{n-1}. Each inequality in the recurrence, peeling d slots at a time until I remains, must therefore be equality. In particular $\dim E_I = h_d + 1$.

The restrictions to $\ker M_d$ of the R evaluation functionals on I are consequently all proportional. Indeed their evaluations span the one-dimensional kernel of $E_I \to A_d$, and the basepoint-free condition of Lemma 2.1 makes each restriction nonzero. This conclusion descends to F by linear algebra. For $a, b \in I$, interchange those input factors. The alternating relations belong to $\ker M_d$, so the covectors

$$(p_{as} \wedge p_{bs}) \otimes \bigotimes_{c \in I \setminus \{a,b\}} p_{cs} \qquad (1 \le s \le R)$$

are either all zero or all nonzero and proportional. In the latter case, if $d \ge 3$, any remaining factor p_{cs} spans a line independent of s, contradicting conciseness. If $d = 2$, proportional nonzero decomposable bivectors put all p_{as} in one fixed two-dimensional space, again contradicting

conciseness since $n \geq 3$. Thus p_{as} and p_{bs} are proportional for every s. Varying I and the pair identifies all projective factors of the sth term with one point $[p_s]$.

The further relations $e_i^{\otimes d} \in \ker M_d$ now show that the vectors $(p_{s,1}^d, \dots, p_{s,n}^d)$ are all proportional. Each coordinate of the common projective vector is nonzero by conciseness. Normalize the first coordinate of every p_s to the same nonzero value, and use one of them as $(a_1, \dots, a_n)$. Every point then belongs to the displayed grid. Distinct terms have distinct projective points, since terms at one point could be combined. The grid has $|\mu_d(F)|^{n-1}$ points, which is at most R. This proves the field condition and exhausts the grid. The converse follows from Theorem 3.1.

To recover the weights, use any exponents $0 \leq u_2, \dots, u_n < d$ in the last $n-1$ coordinates and put $u_1 = N - \sum_{i>1} u_i \geq 0$. Evaluation of the tensor on the corresponding basis word tests the grid character $\prod_{i>1} \epsilon_i^{u_i}$ up to a fixed nonzero factor. These are all characters, so the weights are unique. Explicitly the weight at ϵ is

$$\frac{\prod_{i>1} \epsilon_i}{d^{n-1} \left(\prod_i a_i\right)^{d-1}}.$$

The case $d = 2$ completes Theorem 1.1. □

5 Complete intersections and their shortest schemes

Let $S = \mathrm{Sym}(V)$, $\dim V = n \geq 2$, and let $f_1, \dots, f_n$ be a homogeneous regular sequence of degrees $d_i \geq 2$. Write $A = S/(f_1, \dots, f_n)$, $N = \sum_i (d_i - 1)$, $d = \max_i d_i$, $H = \{i : d_i = d\}$ and $C = \prod_i d_i/d$. The algebra has perfect pairings $A_s \times A_{N-s} \to A_N$, with $\dim A_N = 1$. Choose an identification $\tau : A_N \to F$ and let $T(v_1, \dots, v_N) = \tau(v_1 \cdots v_N)$. Every input is concise, since A has no linear relations and its top pairing is perfect. A finite scheme in $(\mathbb{P}V^*)^N$ is apolar to T when its Segre span contains $[T]$. The least length of such a scheme is the Segre cactus rank. Ranestad and Schreyer [11] established the corresponding symmetric cactus-rank bound from the generator degrees of the apolar ideal. Gesmundo, Oneto and Ventura [4, Theorem 3.12] state a corresponding cactus-rank identity for partially symmetric monomials. We classify every minimizing Segre scheme for complete-intersection top forms.

Theorem 5.1. *The Segre cactus rank of T is C. Except for $(n, d) = (2, 2)$, every apolar scheme of length C is the diagonal copy of a fiber of*

$$X = V(f_i : i \notin H) \subseteq \mathbb{P}V^* \longrightarrow \mathbb{P}^{|H|-1}, \qquad x \longmapsto [f_h(x)]_{h \in H}.$$

Every fiber has length C and is apolar. Outside the stated exception, $\mathrm{R}_F(T) = C$ precisely when a fiber consists of C distinct F-rational points, and every ordinary decomposition of this length is termwise symmetric.

Proof. Work first over an algebraic closure. For a finite apolar scheme Z, trivialize each factor's hyperplane bundle on Z and form the evaluation spaces E_J in $D = \Gamma(Z, \mathcal{O}_Z)$. The proof of Lemma 2.1 gives $E_J \twoheadrightarrow A_{|J|}$: a section vanishing on Z annihilates T after multiplication by all sections on its complementary slots, and the top pairing is perfect.

For a d-set I of slots, the kernel $K = \ker M_d$ has no common zero. Exchanges force proportional factors at any common zero. At this projective point one generator f_i is nonzero, and multiplying it by a linear form nonzero there raises its degree to d while preserving nonvanishing. Lifting this polynomial to the d slots gives a relation nonzero at the putative point. Choose $g \in K$ nonvanishing at every point of the finite support of Z. Its evaluation is a unit in D, even if Z is nonreduced. Multiplication by that unit gives the injection in Lemma 2.1. The Hilbert polynomial $\sum_j (\dim A_j) z^j = \prod_i (1 + \cdots + z^{d_i - 1})$ has total coefficient mass C in each residue class modulo d. Peeling d slots at a time from $d\lfloor N/d \rfloor$ chosen slots yields length $Z \geq C$.

For $[c_h]$ in the target, choose $j \in H$ with $c_j \neq 0$. The fiber is defined by f_i for $i \notin H$ and $c_j f_h - c_h f_j$ for $h \in H \setminus \{j\}$. These forms are a regular sequence since adjoining f_j gives a

homogeneous system of parameters. Their ideal is saturated by Cohen–Macaulayness of the one-dimensional quotient. The fiber thus has length C, and its diagonal copy is apolar because its ideal is contained in $(f_1, \dots, f_n)$. This also proves finiteness of the projective morphism.

If Z has length C, equality in the recurrence puts the image of K on every d-set of slots in a single line Fu generated by a unit. The exchange argument in Theorem 4.1 now works on Z itself. To see this for nonreduced Z, write $z_{i,a}$ for the ath coordinate at slot i. Multiply the minors $z_{i,a}z_{j,b} - z_{i,b}z_{j,a}$ by unit linear sections on the other $d-2$ slots. Their products lie in Fu. If a minor is nonzero, it is a unit. For $d \geq 3$, varying a linear section at a remaining slot then puts its entire n-dimensional evaluation space in one line. For $d = 2$, all minors are fixed scalar multiples of a unit minor. In that Plücker chart the two rows span a fixed two-plane, contradicting conciseness for $n \geq 3$. Thus all minors vanish on Z, so Z lies scheme-theoretically on the small diagonal.

Use a common trivialization there. For $d_i < d$, multiply f_i by $d - d_i$ unit linear sections. If its restriction is nonzero, their product is a unit in Fu; varying one linear section would again put its evaluation space in one line. Hence all such f_i vanish. Each f_h of degree d restricts to $c_h u$. The generators have no common zero, so some $c_h \neq 0$. The scheme lies in the corresponding fiber and equality of lengths identifies it. The line spanned by the evaluations of the long generators is defined over the field of definition of Z. Their linear relations therefore recover the parameter over that field. A split fiber gives an ordinary decomposition over F by linear algebra, and an ordinary decomposition attaining C gives such a fiber. □

Over an algebraically closed field, a general regular sequence of any prescribed degrees attains C in ordinary rank. For example fix $j \in H$, put $f_j = x_j^d$, and choose $f_i = \prod_{a=1}^{d_i}(x_i - \lambda_{ia}x_j)$ with distinct λ_{ia} for each $i \neq j$. The fiber selected by f_j is a reduced grid. Transversality of the other $n-1$ forms and nonvanishing of f_j on their intersection are open. If all degrees are d, every regular sequence over an algebraically closed field of characteristic prime to d attains d^{n-1} in ordinary rank: the finite morphism from $\mathbb{P}^{n-1}$ has degree d^{n-1} prime to the characteristic, hence is generically separable with reduced general fiber.

For the monomial complete intersection with $f_i = x_i^{b_i+1}$ and $b_i > 0$, this gives the Segre cactus rank $\prod_i(b_i+1)/(1+\max b_i)$ for every multiplicity vector. Each fiber is smoothable: on a chart with $c_j \neq 0$ it is cut out by one monic univariate equation per coordinate, which can be deformed independently to equations with distinct roots. If the b_i are unequal, a short coordinate is a nonzero nilpotent in every fiber, so the ordinary rank is strictly greater. A unique largest exponent gives a unique minimizing scheme.

6 Alternating products of two inputs

Let U be the space of alternating $n \times n$ matrices and $D = \binom{n}{2}$. Its slice tensor computes the exterior product $V \times V \to \bigwedge^2 V$, equivalently an alternating matrix–vector product. Fawzi et al. [1, Figure 4] give a general algorithm with $D + n - 1$ products, improving the earlier upper bound of Ye and Lim [12].

Theorem 6.1. *For $n \geq 2$ the ordinary rank of the exterior product of two vectors equals $D+n-1$ over every field. Consequently*

$$\mathrm{R}_F(\det_n) \geq 2^{n-1} + n - 2, \qquad \mathrm{R}_F(\det_n) \geq 2^{n-1} + 2(n-2) \quad (n \geq 4 \text{ even}).$$

The same bounds hold for per_n in characteristic two. In particular $\mathrm{R}_F(\det_4) = 12$ over every field.

Proof. Embed $U = \bigwedge^2 V^*$ in $V^* \otimes V^*$ by $x \wedge y = x \otimes y - y \otimes x$. Let W be a space spanned by rank-one tensors $p_s \otimes q_s$ and containing U, as supplied by a decomposition. Its image under ordinary polynomial multiplication in $\mathrm{Sym}^2 V^*$ is a space Q spanned by $p_s q_s$, with $\dim W = D + \dim Q$. Choose a basis of Q among the products $p_s q_s$, and let L be the common kernel of the first factors

of these basis products. Then $\dim L \geq n - \dim Q$. Every $p_s q_s$ restricts to the zero polynomial on L, so one of its factors vanishes there because $\mathrm{Sym}(L^*)$ is a domain. Hence W vanishes on $L \times L$, forcing $\dim L \leq 1$ since $U \subseteq W$. This proves the lower bound over every field.

For the matching upper bound write x_i for the coordinate forms. Use $A_i = x_i \otimes (x_i - x_n)$ and its transpose for $i < n$, together with $B_{ij} = (x_i - x_j) \otimes (x_i + x_j - x_n)$ for $i < j < n$. Their span contains $A_i - A_i^{\mathsf{T}} = -x_i \wedge x_n$ and $B_{ij} - A_i + A_j = x_i \wedge x_j$. There are $D + n - 1$ displayed generators. Finally apply the exterior version of Lemma 2.1 to the largest even number of inputs of $\det_n$. In its last two slots the evaluation dimension is at least $D + n - 1$, since the evaluations span rank-one matrices whose span contains U. This improves the final term $D + 1$ of the parity sum by $n - 2$.

For even $n \geq 4$, split the inputs into two slots and their $n - 2$ complementary slots. Write the matrix of this flattening as PQ^{T}, with one column in each matrix for each term of a shortest decomposition. Its rank is $D = \binom{n}{2}$ by exterior duality. The two column ranks are the dimensions of the evaluation spaces on the respective sets of slots. On either even set of size $j \geq 2$, iteration of Lemma 2.1 ending in the two-slot bound gives dimension at least $b(n, j) + n - 2$. Since $b(n, 2) + b(n, n-2) - D = 2^{n-1}$, Sylvester's rank inequality gives

$$\mathrm{R}_F(\det_n) \geq 2^{n-1} + 2(n-2) \qquad (n \geq 4 \text{ even}).$$

In particular $\mathrm{R}_F(\det_4) = 12$ over every field. The upper bound in characteristic two is Houston, Goucher and Johnston [9, Corollary 13]; in other characteristics see Han, Ju and Kim [7, Theorem 4.7]. The case of characteristic-two fields larger than $\mathbb{F}_2$ was left open directly after that theorem. □

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