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THE NATURAL BOUNDARY OF THE FILTERED PRIME-PAIR DIRICHLET SERIES

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Abstract. Let p and p' range independently over the primes. We prove that the filtered prime-pair Dirichlet series

$$T_0(s) = \sum_{p,p'} \frac{\log p \, \log p'}{(p+p')^s} \left(1 - s + s(s+1)\frac{pp'}{(p+p')^2}\right)$$

has a meromorphic continuation to $\Re s > 1$ with natural boundary $\Re s = 1$. We assume H_∞, which says that for every $\epsilon > 0$ only finitely many nontrivial zeta zeros have real part greater than $1/2+\epsilon$, together with classical facts about ζ and Γ: symmetries and counts of the zeros, a local expansion of ζ'/ζ, Littlewood's gap estimate and Stirling's formula on strips. The polynomial factor comes from applying the operator $1 + x\,d/dx$ to the exponential sum $\sum_p (\log p)\mathrm{e}^{-px}$, which removes its main term x^{-1} as $x \to 0$. This removes the pairing of the pole of the prime Dirichlet series at 1 with zeta zeros divided by primes, which gives the known boundary of the unfiltered series. A term from the prime squares and a remainder survive beside the sum over zeros, and squaring the expansion produces pairs of zeros ρ, ρ' themselves, with complex weights carrying the factor $(\rho-1)(\rho'-1)$. Several pairs can have the same sum, and their weights could cancel there. A sector theorem for weighted Gamma products, proved in a companion paper, shows that they do not cancel at a sum of bounded imaginary part that has at least one representation, provided both zeros of every representation have sufficiently large absolute ordinates. If infinitely many zeros lie off the critical line, this gives poles in $\Re s > 1$ that approach every point of the line $\Re s = 1$. If only finitely many do, we move the pairs involving those zeros into a correction term. The remaining pair sum, multiplied by the distance to $\Re s = 1$, then has a nonzero limit from the right at a dense set of points of that line. The Lean formalization takes the classical inputs as explicit premises.

1. Introduction

1.1. **The filter.** Prime pairs can be counted on prime support or with von Mangoldt weights, and the two counts lead to different kinds of singularities. Write ρ for a nontrivial zero of ζ. In a series supported on primes, the pole governing the main term in the distribution of primes can pair with a scaled zero ρ/j, where $j \geq 2$ is an integer. We call an exponential sum such as $\sum_n \Lambda(n)\mathrm{e}^{-nx}$ a heat sum. With von Mangoldt weights, the expansion of the heat sum as $x \to 0$ also gives pairs of the zeros ρ themselves, which we call unscaled. This paper removes, on prime support, the leading term responsible for the first mechanism, and studies what is left.

Let p, p' range over primes, and let Λ be the von Mangoldt function, with $\Lambda(p^j) = \log p$ for $j \geq 1$ and $\Lambda(n) = 0$ otherwise. All pair sums count both orders of a pair of distinct

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terms and count a diagonal pair once. For $\Re s > 2$ the three binary series are

$$T(s) = \sum_{p,p'} \frac{\log p \, \log p'}{(p+p')^s},$$

$$T_0(s) = \sum_{p,p'} \frac{\log p \, \log p'}{(p+p')^s} \left(1 - s + s(s+1)\frac{pp'}{(p+p')^2}\right),$$

$$\Phi_2(s) = \sum_{m,n\geq 1} \frac{\Lambda(m)\Lambda(n)}{(m+n)^s}.$$

The sum for Φ_2 includes every prime power in both slots. Throughout, H_∞ means that for every $\epsilon > 0$ only finitely many nontrivial zeros of ζ have real part greater than $1/2+\epsilon$. The Riemann Hypothesis (RH) places every nontrivial zero on the critical line $\Re z = 1/2$, so it implies H_∞.

Our main result, Theorem 5.1, says that under H_∞ and classical facts about ζ and Γ, the series T_0 continues meromorphically to $\Re s > 1$, and no meromorphic continuation crosses any point of the line $\Re s = 1$.

1.2. **The two boundary proofs.** The boundaries of T and Φ_2 under H_∞ are known, from two earlier papers of the author: [11] for T and [12] for Φ_2, which we call the prime-pair paper and the Goldbach paper. For T, a Mellin–Barnes contour shift brings in scaled zeros ρ/j through the prime Dirichlet series $Q(z) = \sum_p (\log p) p^{-z}$, initially defined for $\Re z > 1$. The same scaled zeros occur in the classical prime zeta expansion [5, 4]. The proof for T selects points $1+\rho/p$ with p prime. At each selected point the pole of Q at 1 pairs with the pole at ρ/p. For $\Re\rho \geq 1/2$ and p not dividing $m(\rho)$, the term $-m(\rho)/p$ is the only term in the residue of Q at ρ/p whose denominator is divisible by p, so that residue is nonzero. Counting the zeros for which ρ/p has a single representation gives the stronger bound $1/p$ from below on the modulus of the same residue, and the tail comparison uses this bound. Further pairs with small absolute ordinates are excluded, and the selected pair dominates the remaining higher pairs. This gives poles that approach every point of $\Re s = 1$ from the right [11, Lemma 2.1, §3, Remark 3.5 and Theorem 1.1].

For Φ_2, the Mellin transform instead pairs unscaled zeros, with the products of their multiplicities as positive integer weights. The Gamma-weighted contributions at a common sum could still cancel, and a phase cone shows that they do not at a represented sum in a fixed ordinate window all of whose representations are high [12, Lemmas 5.2–5.3, Theorems 5.6 and A]. The boundary proof uses interior poles when infinitely many zeros lie off the critical line [12, Proposition 6.3] and normal limits when only finitely many do [12, Lemma 6.6 and Theorem 6.8]. A normal limit multiplies the remaining pair sum by the distance to the boundary and lets that distance tend to zero from the right at fixed ordinate.

The pole of Q at 1 gives the main term x^{-1} in the prime heat sum

$$F_p(x) = \sum_p (\log p) \mathrm{e}^{-px}, \qquad x > 0.$$

The polynomial in T_0 comes from applying $1 + x\, d/dx$ to F_p, squaring, and taking the Mellin transform (Lemma 3.1). The operator sends x^{-1} to zero and multiplies $x^{-\rho}$ by $1-\rho$. Each sum of two unscaled zeros is a candidate pole of the pair term, with a

contribution carrying the factor $(\rho-1)(\rho'-1)$ on the Gamma weight of the pair. The prime squares give a term of order $x^{-1/2}$ in the filtered heat sum, which we keep in the meromorphic correction. The correction has candidate poles at 1 and $\rho+1/2$ from the prime-square term and at shifts of poles of the remainder transform. Section 3 constructs that correction.

1.3. **Coincident sums and the strategy.** Different ordered pairs can give the same denominator $s-\rho-\rho'$ in the pair term. The sum of their numerators is the *atom* at the candidate pole $s=\rho+\rho'$. The terms of an atom are nonzero, but they need not share a phase. Pairs with the same sum can have Gamma products whose phases differ, through the real parts of the two zeros and through the height of the pair, and filter factors whose phases differ as well. The two orders of one pair give the same product and phase. So nothing in the construction rules out cancellation, and a proof is needed. To obstruct continuation we need nonzero atoms at points approaching the line. A representation of s_0 is an ordered pair (ρ,ρ') of zeros with $\rho+\rho'=s_0$. A representation is high if both ordinates have modulus greater than a fixed threshold X, and low otherwise. In a window $|\gamma+\gamma'|\le Y$, where γ,γ' are the ordinates, a low representation has both ordinates bounded by $X+Y$ in modulus. Local finiteness of the zeros therefore leaves only finitely many such sums, and the points we use avoid them. For $X>Y$, the two ordinates of a high pair have opposite signs. Its height is the modulus of the negative ordinate, and the unit bin at v consists of the pairs with height in $[v,v+1)$ (Section 4).

The factor $(\rho-1)(\rho'-1)$ is the same for both orders of a pair, but it need not be the same for distinct unordered pairs in an atom, and it can be non-real. At high pairs with bounded sum ordinate, its argument tends to zero and its modulus grows quadratically in the height. Multiplying the squared-logarithm count in a unit bin by the size of this factor gives a polynomial bound on the coefficient mass. The integer cone used for Φ_2 covers neither the complex coefficients nor this larger bin bound. The companion paper *Phase cones for Gamma-weighted atoms of additive Dirichlet series* [10], which we call the cone paper, proves the sector theorem and the general boundary mechanism used here. Its polynomial sector theorem extends the integer cone to cover both [10, Theorem 3.1 and Lemma 3.3]. A single rotation puts the terms near the least height in a sector inside the right half-plane, and Stirling's exponential decay bounds the higher terms.

With infinitely many off-line zeros, we look for nonzero atoms at points with real part greater than one where the correction is holomorphic. With finitely many, we put all terms involving off-line zeros into the correction and subtract it from $\Gamma(s)T_0(s)$. What remains is the critical-line pair sum, and its normal limit at a point of the line is the atom there. Under RH, Bhowmik, Ramaré and Schlage-Puchta pass the normal limit termwise through the Goldbach pair sum, after estimates that give absolute and uniform convergence [2, p. 297]. In the proof of Theorem 5.1 a bounded kernel dominates the filtered summands, and dominated convergence passes the limit through the critical-line pair sum. Nonzero normal limits at a dense set of boundary points would force poles to accumulate inside the domain of any extension [10, §2.2].

Bhowmik and Schlage-Puchta also distinguish candidate from actual singularities for Euler products [3, §2]. Their criteria are stated for Euler products and do not cover the additive Gamma sums here.

1.4. **Result.** Theorem 5.1 states the result of Section 1.1 precisely, under H_∞ and the standing inputs of Section 2.2: the continuation agrees pointwise with the defining series on $\Re s > 2$, and the boundary holds for every function meromorphic on $\Re s > 1$ that agrees with T_0 on $\Re s > 2$ in the sense of Section 2. Section 2 sets out the conventions and classical inputs, and Section 3 constructs the continuation. Section 4 applies the sector theorem to the filtered atoms, and Section 5 proves the boundary. Section 6 lists the formal premises, and Appendix A prints the Lean boundary statement.

2. Mellin transforms and standing inputs

At a fixed integer N, at most N ordered pairs have $m+n=N$, and each product of prime or von Mangoldt weights is at most $(\log N)^2$. So the total weight at $m+n=N$ is at most $N(\log N)^2$, and $pp'/(p+p')^2 \le 1/4$ in the filter. This gives absolute convergence for $\Re s > 2$ and justifies the Gamma integral interchange in Lemma 3.1. The Mellin transform of a function f on $(0,\infty)$ is $\int_0^\infty f(x)x^{s-1}\,dx$ wherever the integral converges.

The Mellin transform over $(0,1)$ of the square of the filtered zero sum U of Lemma 3.2 gives the pair sum C_0 in Proposition 3.3. We call the set of ordered pairs with a given sum a collision, the term used in the Lean names. Its atom is defined in Section 1.3. The other terms give the meromorphic correction W_0 of Proposition 3.3. See [10, §2.1] for the general Mellin calculation.

2.1. **Agreement and natural boundaries.** Meromorphic agreement means equality off a relatively closed discrete subset of the open overlap. A line is a natural boundary if no disc centred on it admits a meromorphic function agreeing on the right-hand overlap in this sense. The identity theorem gives uniqueness on a connected domain [10, §2.4].

2.2. **Standing inputs.** We write $\rho = \beta + \mathrm{i}\gamma$ for a nontrivial zero of ζ in $0 < \Re z < 1$, with ordinate γ and multiplicity $m(\rho)$. In sums over zeros each distinct zero is an index, and the weights carry the multiplicity.

(Z1).: Conjugation and $\rho \mapsto 1-\overline{\rho}$ preserve the nontrivial zeta zeros and their multiplicities, by the functional equation and reality of the Dirichlet coefficients [13, Ch. 2].

(Z2).: If $N(t)$ counts positive zeta ordinates up to t with multiplicities, then

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + O(\log t) \quad (t\to\infty), \qquad N(t+1)-N(t) \ll \log(t+3) \quad (t \ge 0).$$

See [13, Theorems 9.4 and 9.2]. The proofs use the second estimate, the local count, and from the first only the existence of zeros.

(Z3).: For $-1 \le \Re z \le 2$ and $|\Im z| \ge 2$, away from zeros,

$$\frac{\zeta'}{\zeta}(z) = \sum_{|\gamma - \Im z| \le 1} \frac{m(\rho)}{z-\rho} + O(\log|\Im z|).$$

The large-height estimate is [13, Theorem 9.6(A)], and conjugation gives negative heights. Below the height t_0 from which that estimate holds, on $2 \le |\Im z| \le t_0$, the difference of the two sides is bounded by compactness, since the principal parts cancel at zeros and the jumps at $|\gamma - \Im z| = 1$ are bounded. The estimate is not claimed for $|\Im z| < 2$, where the principal part of ζ'/ζ at $z=1$ would also have to be subtracted. Where the proofs need bounded heights, they use

compact pieces of vertical lines that avoid the zeros and the pole at $z = 1$. See [10, §§2.5–2.6].

Littlewood's gap estimate.: For every sufficiently large positive t some zeta ordinate differs from t by less than $A_{\mathrm{L}}/\log\log\log t$, with an absolute constant A_{L} [7][13, Theorem 9.12, p. 224]. The boundary proof only needs the gaps between successive zero ordinates to tend to zero, which follows by applying [13, Theorem 9.12] at the midpoint of each gap.

Stirling on strips.: On every fixed real-part strip, uniformly at sufficiently large absolute imaginary part, $\Gamma(z) = \sqrt{2\pi}\exp((z-\frac{1}{2})\operatorname{Log} z - z)(1+O(|\Im z|^{-1}))$, with the principal logarithm. This is a classical theorem [9, §5.11], taken as an explicit premise because the Lean statements carry it (Mathlib lacks it).

Under H_∞ and (Z1) the real parts tend uniformly to $1/2$ at large absolute ordinate, and the finite set outside any fixed strip about $1/2$ gives $B = \sup_\rho \beta < 1$. Symmetry gives $B \geq 1/2$ and places all real parts in $[1-B, B]$ [10, §2.5]. The continuation and the full atom family of Corollary 4.2(a) use this B. The critical-line family of part (b) uses $B = 1/2$, and the abstract sector theorem treats B as a strip parameter [10, Theorem 3.1].

We use local finiteness of the zeros, the pole of ζ at 1 and the trivial zeros at the negative even integers [13, Ch. 2]. For Γ we use the recurrence, reflection formula and absence of zeros [9, §§5.2, 5.5]. Euler's integral gives $|\Gamma(z)| \leq \Gamma(\Re z)$ for $\Re z > 0$. With the recurrence this gives $|\Gamma(z)| \ll_k |\Im z|^{-k}$ on each strip $0 < d \leq \Re z \leq D$, for every k.

3. The filtered prime-pair series

This section writes $\Gamma(s)T_0(s)$ as a pair sum over zeta zeros plus a correction that is meromorphic across $\Re s = 1$. As described in Section 1.2, the filtered heat sum is a sum over zeros, now with weights $A_\rho = m(\rho)(\rho-1)\Gamma(\rho)$, plus a prime-square term of order $x^{-1/2}$ and a remainder. Squaring the zero sum gives the pair sum, whose coefficients $m(\rho)m(\rho')(\rho-1)(\rho'-1)$ multiply the Gamma products $\Gamma(\rho)\Gamma(\rho')$. Section 4 estimates these coefficients. The product of the zero sum with the remainder needs more care, and Proposition 3.3 handles it by a cutoff in the real parts of the zeros.

3.1. From the filter to a meromorphic correction.

Lemma 3.1 (Mellin square)**.** *For $L(x) = F_p(x) + xF_p'(x)$ and $\Re s > 2$,*

$$\Gamma(s)T_0(s) = \int_0^\infty L(x)^2 x^{s-1}\,dx.$$

Proof. The exponential series and its derivative converge locally uniformly for $x > 0$. For a pair p, p' the factor in the integrand is $(1-px)(1-p'x)\mathrm{e}^{-(p+p')x}$. Its absolute integral against $x^{\Re s-1}$ is at most a constant depending on $\Re s$ times $(p+p')^{-\Re s}$, by scaling and using $p, p' \leq p + p'$. The convergence bound in Section 2 justifies the interchange. Integration gives the three terms $\Gamma(s)$, $-\Gamma(s+1)$ and $pp'\Gamma(s+2)/(p+p')^2$, with common factor $(p+p')^{-s}$. The Gamma recurrence gives the claimed identity. □

The expansion of L as $x \downarrow 0$ comes from a contour shift, so we need the poles of Q. Write $\psi = -\zeta'/\zeta$. The series Q differs from ψ by the contribution of the proper prime powers, and Möbius inversion expresses Q as a sum of $\psi(jz)$ with coefficients $\mu(j)$ from the Möbius function [11, Lemma 2.1]. The $j = 1$ term supplies the unscaled zeros. The

$j = 2$ term, $-\psi(2z)$, has the prime-square pole at $1/2$ and poles at $\rho/2$ with real part at most $B/2$. The terms $j \geq 3$ have poles at real part at most $1/3$. So the prime squares produce a term of order $x^{-1/2}$, and after extracting it there is room to shift the proper-prime-power contribution to $\Re z = a$ with $\max\{B/2, 1/3\} < a < 1/2$. The remaining integral is then $O(x^{-a})$ as $x \downarrow 0$. For the unscaled horizontal estimates, see [10, §2.6]. The remark in [10, §2.7] gives the proper-prime-power shift, including the estimate for $\psi(2z)$.

Lemma 3.2 (Filtered explicit formula)**.** *Assume H_∞, (Z1)–(Z3) and Stirling on strips. Let $B < 1$ be the supremum of the real parts of the nontrivial zeta zeros. Put*

$$a = \tfrac{1}{2}\big(\tfrac{1}{2} + \max\{B/2, 1/3\}\big), \quad A_\rho = m(\rho)(\rho - 1)\Gamma(\rho), \quad k = -\sqrt{\pi}/4.$$

Then $\max\{B/2, 1/3\} < a < 1/2$ and hence $B < 2a < 1$. The expansion is

$$L(x) = U(x) + kx^{-1/2} + R(x), \qquad U(x) = \sum_\rho A_\rho x^{-\rho}, \qquad R(x) = O(x^{-a}) \quad (0 < x \leq 1).$$

The coefficients satisfy $\sum_\rho |A_\rho| < \infty$.

Proof. Put $\psi = -\zeta'/\zeta$ and $Q(z) = \sum_p (\log p) p^{-z}$ for $\Re z > 1$. The prime-power Euler series and Möbius inversion give

$$Q(z) = \sum_{j \geq 1} \mu(j)\psi(jz).$$

Indeed, rearranging in the absolute-convergence region gives the coefficient $\sum_{j|n} \mu(j)$ at the exponent n of each prime. For any $\Re z \geq d > 0$ the tail with $jd \geq 2$ converges uniformly, since $|\psi(v)| \leq \sum_{n \geq 2} (\log n) n^{-\Re v} \ll 2^{-\Re v}$ for $\Re v \geq 2$. The finite initial segment provides the meromorphic continuation to $\Re z > 0$.

Mellin inversion of the exponential on $\Re z = 2$ represents $L(x)$ by the integral of $(1 - z)\Gamma(z)Q(z)x^{-z}/(2\pi\mathrm{i})$. This follows by absolute summation on that line and differentiation under the integral, using exponential decay of Γ. Split $Q = \psi + (Q - \psi)$. The Euler product excludes zeros for $\Re z > 1$, and the classical zero-free line $\Re z = 1$ is the right edge of the critical strip [9, §25.10(i)]. The functional equation gives the exclusion on $\Re z = 0$ away from the origin, and $\zeta(0) = -1/2$ [13, Ch. 2]. For the ψ integral shift to $\Re z = -1/2$. The shift crosses the pole of ψ at 1, the nontrivial zeros and the pole of Γ at 0, and no trivial zero.

To justify it, choose positive heights in $[N, N+1]$ outside intervals of radius $c/\log(N+3)$ around the ordinates and their reflections in $[N - 1, N + 2]$. By (Z2) a sufficiently small fixed $c > 0$ leaves uncovered points for every large N. Formula (Z3) gives an $O(\log^2(N + 3))$ bound on both horizontal segments. On the new vertical line (Z3) and the horizontal separation from the nontrivial zeros give $O(\log(2 + |\Im z|))$, with bounded-height values controlled on a compact interval of the line, which contains no zero and does not pass through $z = 1$. The Gamma factor times the extra polynomial $1 - z$ decays exponentially on the horizontal segments.

The pole at 1 of ψ is killed by $1 - z$. At each nontrivial zero the residue gives $A_\rho x^{-\rho}$, since ψ has residue $-m(\rho)$. The Gamma pole at 0 gives $\psi(0)$, and the new integral is $O(x^{1/2})$. The count in (Z2) and Stirling give absolute summability of the residues even after the factor $\rho - 1$.

For the $Q - \psi$ integral shift to $\Re z = a$. The terms with $j \geq 3$ are bounded absolutely and uniformly on $\Re z \geq a$ since $3a > 1$. The term $-\psi(2z)$ has just the pole at $1/2$ there,

of residue $-1/2$. Its other poles, at the points $\rho/2$, have real part at most $B/2 < a$. So the region between the lines $\Re z = a$ and $\Re z = 2$ contains just the pole of $Q - \psi$ at $1/2$. On the whole line $\Re z = a$, including its real point, the inequality $B < 2a < 1$ keeps $2z$ away from both the zeros of ζ and its pole at 1. For $a \le \Re z \le 1$ and $|\Im z| \ge 1$ the bound on the local fractions for $\psi(2z)$ follows from separation $2a - B > 0$, (Z2) and (Z3), and on the bounded part of the line $\Re z = a$ the integrand is continuous. For $\Re z \ge 1$ use the Euler series. The same Stirling estimate makes the horizontal integrals vanish and bounds the new vertical integral by $O(x^{-a})$. The crossed pole contributes $-\Gamma(1/2)(1-1/2)x^{-1/2}/2 = kx^{-1/2}$. The constant and the two vertical integrals give the asserted R. □

Squaring $L = U + kx^{-1/2} + R$ and integrating over $(0,1)$ gives the pair sum from U^2 and a correction from everything else. The Mellin transform of U^2 and the residue of the pair sum use the calculations of [10, §§2.1–2.2]. The product UR is the obstacle. The bound $UR = O(x^{-B-a})$ guarantees direct Mellin convergence for $\Re s > B + a$, and $B + a$ can exceed 1 when zeros lie far to the right of the critical line. Under H_∞ only finitely many zeros have real part above any fixed $b > 1/2$. We treat these finitely many terms exactly, by shifted remainder transforms, and use $O(x^{-b})$ for the remaining zero sum, with b chosen so that $a + b < 1$. The function S defined in the proof of Proposition 3.3 is used again in the remark after the proposition and in the proof of Theorem 5.1.

Proposition 3.3 (Cutoff and continuation)**.** *Assume H_∞, (Z1)–(Z3) and Stirling on strips. With a and B as in Lemma 3.2, put $b = 3/4 - a/2$ and $c_* = \max\{B, a + b\}$. The cutoff set $E = \{\rho : \Re\rho > b\}$ of nontrivial zeta zeros is finite, and $c_* < 1$. There is a function W_0 meromorphic on $\Re s > c_*$ such that*

$$\Gamma(s)T_0(s) = C_0(s) + W_0(s), \qquad C_0(s) = \sum_{\rho,\rho'} \frac{A_\rho A_{\rho'}}{s - \rho - \rho'}. \tag{1}$$

The identity holds on $\Re s > 2$ and defines a meromorphic continuation to $\Re s > 1$. There the possible poles of C_0 are locally finite and at most simple. If W_0 is regular at s_0 with $\Re s_0 > 1$, the residue of the continuation there is $\Gamma(s_0)^{-1} \sum_{\rho+\rho'=s_0} A_\rho A_{\rho'}$.

Proof. The inequalities $1/2 < b$ and $a + b < 1$ follow from $a < 1/2$, so H_∞ gives the required finiteness. For $x \ge 1$, $L(x) = O((1+x)\mathrm{e}^{-2x})$ by the exponential series, so

$$E_L(z) = \int_1^\infty L(x)x^{z-1}\,dx, \qquad E_2^{(0)}(s) = \int_1^\infty L(x)^2 x^{s-1}\,dx$$

are entire. Let $S(z) = \sum_\rho A_\rho/(z - \rho)$. Local finiteness of the zero set and summability of $|A_\rho|$ give normal convergence away from the indicated poles and global meromorphy. Let $\widehat{R}$ be the meromorphic function on $\Re z > 0$ given by

$$\widehat{R}(z) = (1 - z)\Gamma(z)Q(z) - E_L(z) - S(z) - \frac{k}{z - 1/2},$$

with its removable singularities filled in; at $z = 1$, for instance, the first term has the removable value -1. For $\Re z > 1$ the full transform of L is $(1-z)\Gamma(z)Q(z)$ by the Gamma integral, and the other terms subtract the large-x tail and the transforms on $(0,1)$ of the explicit powers in Lemma 3.2. So $\widehat{R}(z) = \int_0^1 R(x)x^{z-1}\,dx$ for $\Re z > 1$. The integral is holomorphic on $\Re z > a$, and by the identity theorem it equals $\widehat{R}$ there. In particular every pole of $\widehat{R}$ has real part at most a.

Write $U_E(x) = \sum_{\rho\in E} A_\rho x^{-\rho}$ and $U_c = U - U_E$. Then $U_c = O(x^{-b})$ and

$$L^2 = U^2 + 2kx^{-1/2}U + k^2x^{-1} + 2U_E R + \big(2U_c R + 2kx^{-1/2}R + R^2\big).$$

The bracket is $O(x^{-a-b})$ because $b > 1/2 > a$. Its Mellin integral H_0 on $(0,1)$ is holomorphic for $\Re s > a + b$, by domination uniformly on compact subsets. Integrating the displayed decomposition for $\Re s > 2$ gives (1) with

$$W_0(s) = E_2^{(0)}(s) + \frac{k^2}{s-1} + 2kS(s-1/2) + 2\sum_{\rho\in E} A_\rho \widehat{R}(s-\rho) + H_0(s).$$

The finite shifted transforms are meromorphic for $\Re s > B$, and all the terms are meromorphic on the claimed common half-plane. Fubini for the zero square follows from $\sum_{\rho,\rho'} |A_\rho A_{\rho'}| = (\sum_\rho |A_\rho|)^2$ and $\Re s > 2 > 2B$.

For a compact subset of $\Re s > 1$, let its minimum real part be $1 + d$ with $d > 0$. If both zeros have sufficiently large absolute ordinate, H_∞ bounds their real parts by $1/2 + d/4$, so the denominator is bounded below by $d/2$. There are finitely many zeros below the chosen ordinate threshold. For each of them and a sufficiently high partner the imaginary part of the denominator has modulus at least one uniformly on the compact set. Only finitely many pairs remain, and the other terms converge normally. Applying this argument to closed discs in the open half-plane gives local finiteness of the candidate poles and shows that only the exact pairs contribute to the residue. Division by $\Gamma(s)$ proves the proposition. □

The correction W_0 is meromorphic across $\Re s = 1$, but it need not be holomorphic on $\Re s > 1$. Its possible poles come from $k^2/(s-1)$ at 1, from $2kS(s-1/2)$ at $\rho + 1/2$, and from the poles of $\widehat{R}(s-\rho)$ for $\rho \in E$. In particular, zeros with $\Re\rho > 1/2$ can give infinitely many correction poles to the right of the line. The proof of Theorem 5.1 uses local finiteness in $\Re s > c_*$ to avoid them near each boundary point.

4. Non-cancellation for filtered atoms

This section shows that the atom of C_0 at a point s_0, the sum of $A_\rho A_{\rho'}$ over the representations of s_0, does not vanish when s_0 lies in a fixed ordinate window, has at least one representation, and has only high representations for a sufficiently large threshold. Fix $s_0 = \alpha + \mathrm{i}\tau$ with $|\tau| \le Y$, for $Y > 0$. If both ordinates have modulus greater than Y, they have opposite signs. Write the zeros as $\beta_+ + \mathrm{i}(u+\tau)$ and $\beta_- - \mathrm{i}u$, with real parts β_+, β_- and height $u > 0$. We retain the original ordered pairs as indices, so both orders of two distinct zeros are still counted. The cutoff parameters a, b of Section 3 play no part here.

4.1. **The sector theorem.**

The Gamma product at height u is

$$w_{\beta_+,\beta_-}(\tau,u) = \Gamma(\beta_+ + \mathrm{i}(u+\tau))\Gamma(\beta_- - \mathrm{i}u).$$

We restate the polynomial sector theorem of the cone paper [10, Theorem 3.1]. There the lower bound κ is called a and the real parts are called b_i, c_i. We rename them because a and b are the abscissas of Section 3.

Theorem 4.1 (Sector cone with polynomial bins)**.** *Assume Stirling on strips. Fix $1/2 \le B < 1$, $Y > 0$, $\kappa > 0$, a real K, a real $A \ge 0$ and $0 \le \delta < \pi(1-B)$. There is $U \ge Y + 1$, depending only on $B, Y, K, \kappa, \delta, A$, such that for each real τ with $|\tau| \le Y$ the following*

holds. Let $(\beta_{+,i}, \beta_{-,i}, u_i, d_i)$ *be a family, over an arbitrary index set, of real numbers* $\beta_{+,i}, \beta_{-,i}, u_i$ *and complex numbers* d_i*, with* $\beta_{+,i}, \beta_{-,i} \in [1-B, B]$ *and* $\beta_{+,i}+\beta_{-,i}$ *constant. Suppose that*

(1) *the heights attain a minimum* $u_0 = u_{i_0} \geq U$*, and* $|d_{i_0}| \geq \kappa$*;*
(2) *for some fixed* ω *with* $|\omega| = 1$*, every coefficient is* $d_i = \omega|d_i|\mathrm{e}^{\mathrm{i}\phi_i}$ *with* $|\phi_i| \leq \delta$*;*
(3) *every unit bin* $I(v) = \{i : v \leq u_i < v+1\}$*,* v *real, is finite, and for* $v \geq 0$

$$\sum_{i \in I(v)} |d_i| \leq K(v + Y + 4)^A.$$

Then $\sum_i d_i w_{\beta_{+,i},\beta_{-,i}}(\tau, u_i)$ *converges absolutely and is nonzero.*

Idea of the proof. The full proof is in [10, Theorem 3.1]; we sketch it here. Stirling's formula gives the leading modulus $2\pi\mathrm{e}^{-\pi u - \pi\tau/2}u^{\beta_+ + \beta_- - 1}$ of $w_{\beta_+,\beta_-}(\tau, u)$ and its leading argument $\tau \log u + \pi(\beta_+ - \beta_-)/2$ modulo 2π, with errors $O_Y(1/u)$ in relative modulus and in argument on the fixed strip. The exponent $\beta_+ + \beta_- - 1$ is common to all pairs with the same sum. Rotate by $\overline{\omega}\mathrm{e}^{-\mathrm{i}\tau \log u_0}$, where ω is the common sector direction and u_0 is the least height. In the window $u \leq u_0 + \sqrt{u_0}$ the offset $\pi(\beta_+ - \beta_-)/2$ is at most $\pi(B - 1/2)$ in modulus. The offset plus the coefficient phase, which is at most δ, is therefore at most $\pi/2 - \sigma$ in modulus, where $\sigma = \pi(1 - B) - \delta > 0$. After the rotation, the leading phase of a term in the window is this sum plus the drift $\tau(\log u - \log u_0)$. For large u_0 the drift is bounded by $Y/\sqrt{u_0}$, and the drift and the Stirling phase error together use at most half the margin σ. So the rotated sum in the window has real part at least $\sin(\sigma/2)\kappa|w_0|$, where w_0 is the selected least-height Gamma product. Outside the window, a bin based at $u_0 + n$ meets the tail only if $n+1 > \sqrt{u_0}$. The polynomial bin bound and exponential decay then give a tail of size $O\big(|w_0| \sum_{n+1>\sqrt{u_0}} (n+2)^{2A+1}\mathrm{e}^{-\pi n}\big)$, which is $o(|w_0|)$ by summability. The phase estimate is explained in [10, §3.1].

4.2. **Applying the theorem to the filter.** We put the filter factor $(\rho - 1)(\rho' - 1)$ into the coefficient and keep the Gamma product at its original height. The leading product $\mathrm{i}(u+\tau)(-\mathrm{i}u) = u^2 + \tau u$ gives $u^2(1 + O_Y(1/u))$ for the full factor. The factor is unchanged when the two slots are exchanged, but it need not be the same for distinct unordered pairs, and it can make the coefficients non-real, though for a conjugate pair it equals $|\rho - 1|^2 > 0$. Corollary 3.2 and Lemma 3.3 of the cone paper [10] assume a squared-logarithm bound on bin mass, and Lemma 3.3 also needs integer coefficients. That bound cannot hold uniformly in the point: at a real point, a conjugate pair of height u alone has coefficient modulus at least u^2. Theorem 4.1 keeps the sector condition of Corollary 3.2, its second condition, and relaxes the bin bound to a polynomial one, its third. For the sector condition, the absolute value of the principal argument $\arg((\rho - 1)(\rho' - 1))$ is $O_Y(1/u)$ and is at most $\delta = \pi(1 - B)/2$ at large height. For the bin bound, the two-window count in [10, §3.2] bounds the multiplicity weight in a unit bin by a squared logarithm. Multiplying the bound on the modulus of the filter factor by the product of the two local multiplicity counts and allowing both orders gives bin mass $O(v^2 \log^2 v)$ at large v, so $A = 4$ suffices.

Corollary 4.2 (Filtered atoms)**.** *Assume (Z1), (Z2) and Stirling on strips. Let* $A_\rho = m(\rho)(\rho - 1)\Gamma(\rho)$ *and fix* $Y > 0$*. Each of the following assertions has a threshold* $X > Y$ *fixed before the point* s_0 *is chosen.*

(a) *Assume H_∞. For $|\Im s_0| \le Y$ let $\mathcal{F}$ be the set of all ordered pairs of nontrivial zeta zeros with sum s_0.*
(b) *For $\tau \in \mathbb{R}$ with $|\tau| \le Y$ put $s_0 = 1 + \mathrm{i}\tau$ and let $\mathcal{F}$ be the set of all ordered pairs of critical-line zeta zeros with sum s_0.*

In each assertion, if $\mathcal{F}$ is nonempty and both ordinates of every pair in $\mathcal{F}$ have modulus greater than its threshold X, then

$$\sum_{(\rho,\rho')\in\mathcal{F}} A_\rho A_{\rho'} \neq 0.$$

Proof. In (a) take $B = \sup_\rho \Re\rho < 1$ as in Section 2, and in (b) take $B = 1/2$. If both ordinates have modulus greater than $X > Y$, they have opposite signs. Write the members of each pair as $\beta_+ + \mathrm{i}(u+\tau)$ and $\beta_- - \mathrm{i}u$ with $\tau = \Im s_0$. The original ordered pairs remain the indices, so both orders are counted. We have $\beta_+, \beta_- \in [1-B, B]$ and $\beta_+ + \beta_- = \Re s_0$ throughout.

Give the pair the coefficient $d = m(\rho)m(\rho')F$ with

$$F = (\beta_+ - 1 + \mathrm{i}(u+\tau))(\beta_- - 1 - \mathrm{i}u).$$

The resulting term $d\,w_{\beta_+,\beta_-}(\tau,u)$ equals $A_\rho A_{\rho'}$ in either order. With $D = 2(Y+1)$, expansion gives, for $u \ge 1$,

$$F - u^2 = (\beta_+ - 1)(\beta_- - 1) + \tau u + \mathrm{i}\big((\beta_- - \beta_+)u + (\beta_- - 1)\tau\big),$$
$$|F - u^2| \le 1 + Yu + u + Y \le Du.$$

Here $\beta_+, \beta_- \in [0,1]$ and $|\tau| \le Y$. Take $\omega = 1$ and $\delta = \pi(1-B)/2$, which is $\pi/4$ in (b). By continuity of the argument at 1, choose $0 < \varepsilon < 1$ such that $|z-1| < \varepsilon$ implies $|\arg z| < \delta$. When $u > D/\varepsilon$, the bound $|F/u^2 - 1| \le D/u$ puts d in this sector, since $m(\rho)m(\rho')u^2 > 0$.

Each bin is finite by local finiteness of the zeros. The negative ordinate lies in $(-v-1, -v]$ and the positive one in $[v+\tau, v+\tau+1)$ for pairs in the bin at v. The local count of (Z2), using (Z1) at negative ordinates and local finiteness at bounded ordinates, bounds the two window multiplicity counts by $C_- \log(v+Y+4)$ and $C_+ \log(v+Y+4)$ for $v \ge 0$, with positive constants depending only on Y and the zero count. Their product bounds the sum of the multiplicity products in one order. Allowing for both orders gives a bound $K_0 \log^2(v+Y+4)$ with $K_0 = 2C_-C_+$. Set $x = v+Y+4$. Since $u < v+1 \le x$ and $|F| \le (1+D)u^2$, the coefficient mass in the bin is bounded by

$$(1+D)K_0 x^2 \log^2 x \le (2Y+3)K_0 x^4,$$

using $\log x \le x$.

Let U be the sector threshold for this B, Y with $K = (2Y+3)K_0$, $\kappa = 1$, $\delta = \pi(1-B)/2$ and $A = 4$. Take

$$X = \max\{U,\, 2Y+4,\, 2(Y+1)/\varepsilon + 1\}.$$

Since $U \ge Y+1$, this gives $X > Y$. All the heights exceed X. The minimum is attained because the heights up to any one represented height lie in finitely many finite bins. At its height u_0 we have $u_0 > D+2$, so $|F| \ge u_0^2 - Du_0 \ge 2u_0 \ge 1$. The multiplicity product is at least one there. Theorem 4.1 now gives non-cancellation for the atom. □

5. The natural boundary

The proof splits according to whether infinitely many zeros lie off the critical line. If they do, we select pairs of zeros whose sums lie to the right of the line and approach it, and show that the corresponding points are poles by using Corollary 4.2(a) where the correction is regular. The bounded-kernel argument for normal limits does not apply to the full pair sum in this case: a pair with sum $\alpha + \mathrm{i}\gamma$ contributes the normalized factor $\eta/(\eta+1-\alpha+\mathrm{i}(\tau-\gamma))$ at $1+\mathrm{i}\tau$, and when $\alpha > 1$ this factor can be large near $\eta = \alpha - 1$.

If only finitely many zeros lie off the line, the pair sums in $\Re s > 1$ have finitely many real parts, so they stay away from the line and no poles accumulate there. We then remove every term involving an off-line zero and examine normal limits of the remaining critical-line square.

The threshold in Corollary 4.2 is fixed throughout a bounded ordinate window, before the sums are selected. Local finiteness of the zeros lets us avoid all sums with low representations, and Littlewood's estimate supplies the high partners. The proof uses the interior-pole and normal-limit arguments of [10, §2.2] in the two cases, respectively. The selection of pair sums follows [10, §2.3].

Theorem 5.1 (Filtered prime pairs)**.** *Assume H_∞, (Z1)–(Z3), Littlewood's gap estimate and Stirling on strips. The filtered series T_0 has a meromorphic continuation to $\Re s > 1$ that agrees pointwise with the series on $\Re s > 2$. For every function that is meromorphic on $\Re s > 1$ and agrees with T_0 on $\Re s > 2$ in the sense of Section 2, the line $\Re s = 1$ is a natural boundary.*

Proof. The continuation is (1) divided by Γ; call it T_0^c. A function that is meromorphic on $\Re s > 1$ and agrees with T_0 on $\Re s > 2$ outside a discrete set agrees with T_0^c as a meromorphic function on $\Re s > 1$, by the identity theorem. So it suffices to treat T_0^c.

Suppose first that there are infinitely many off-line zeros. By (Z1) choose $\rho_n = \beta_n + \mathrm{i}\gamma_n$ with $\beta_n > 1/2$ and $\gamma_n \to +\infty$. Given $y \in \mathbb{R}$, the gap estimate gives a zero ρ_n' at ordinate $\gamma_n + y + o(1)$, and reflection lets us take its real part at least $1/2$. Now $s_n = \overline{\rho_n} + \rho_n' \to 1 + \mathrm{i}y$ by H_∞, with $\Re s_n > 1$. Since their real parts exceed 1 and tend to 1, the s_n have a distinct subsequence that survives any finite deletion. Fix $Y > |y| + 1$ and the threshold of Corollary 4.2(a). In this window only finitely many sums have a low representation, since if one ordinate is at most the threshold in modulus then the other is at most the threshold plus Y. The poles of W_0 in a small closed disc about $1 + \mathrm{i}y$ are finite as well. After these exclusions every remaining s_n has a nonzero atom and W_0 is regular there. Proposition 3.3 makes each an actual simple pole. A meromorphic extension across $1 + \mathrm{i}y$ would have these poles accumulating at $1 + \mathrm{i}y$, inside its domain.

Suppose now that the set O of off-line zeros is finite. The pairs with at least one member in O contribute

$$2\sum_{\rho \in O} A_\rho S(s-\rho) - \sum_{\rho,\rho' \in O} \frac{A_\rho A_{\rho'}}{s-\rho-\rho'}$$

to C_0: the first sum counts the pairs with an off-line zero in either slot, and the second removes the pairs counted twice. This contribution is meromorphic across the line. Absorb it into W_0, obtaining a correction W_* and the critical-only sum C_{crit}. For every

real τ,

$$\lim_{\eta\downarrow 0}\eta C_{\mathrm{crit}}(1+\mathrm{i}\tau+\eta)=\sum_{\substack{\rho,\rho'\ \text{critical}\\ \Im\rho+\Im\rho'=\tau}}A_\rho A_{\rho'}.$$

In fact each numerator is multiplied by $\eta/(\eta+\mathrm{i}(\tau-\Im\rho-\Im\rho'))$, bounded by one and tending to the indicator of equality. The product majorant in Proposition 3.3 gives dominated convergence.

In any bounded open interval, take Y bounding its ordinates and use the threshold of Corollary 4.2(b). Delete the finite set of sums with a low critical representation, and choose an open subinterval (l,h) left over. Take a sufficiently high positive zeta ordinate γ, which is critical by finiteness of the off-line set. Use the gap estimate at $\gamma+(l+h)/2$. Its error tends to zero, so it supplies a critical ordinate γ' in $(\gamma+l,\gamma+h)$. Conjugate the first zero and add the second to obtain the sum $1+\mathrm{i}(\gamma'-\gamma)$, whose ordinate lies in (l,h). This gives a represented critical sum in every surviving open subinterval. All its representations by critical zeros are high, so Corollary 4.2(b) gives a nonzero atom there. So the ordinates τ with nonzero critical atom are dense in the line.

Suppose a function G meromorphic on a disc centred on the line agreed with T_0^c in the meromorphic sense of Section 2 on the overlap of the disc with $\Re s>1$. Then $g=\Gamma G-W_*$ is meromorphic on a smaller disc Δ with the same centre and agrees meromorphically with C_{crit} on the right-hand overlap. The identity theorem on this connected overlap gives equality wherever both functions are holomorphic. Fix a point $1+\mathrm{i}\tau$ of Δ. The function g is holomorphic on a punctured neighbourhood of this point, and C_{crit} is holomorphic to the right of the line. Thus they coincide at every point $1+\mathrm{i}\tau+\eta$ with small $\eta>0$. Hence $\eta g(1+\mathrm{i}\tau+\eta)$ tends to the critical atom at τ. Where the atom is not zero, the Laurent expansion of g at $1+\mathrm{i}\tau$ shows that g has a simple pole there. The poles of g would be dense in a segment inside Δ, which is impossible. □

6. Formal verification

Lean 4 and Mathlib, both at version 4.34.0, check the results listed below [8]. The development extends the Goldbach and prime-pair formalizations [12, §10][11, §11]. The sector theorem, Theorem 4.1, is the `PhaseCone` statement for Theorem 3.1 of the cone paper [10], printed in [10, Appendix A.2].

The development is release `v1.0` of `https://github.com/lpkralv/phase-cone`, Apache-2.0, commit `82f618bc4a9491668c43a8dcd948ea933aa6aa75`, built with the toolchain `leanprover/lean4:v4.34.0`. The folder `t0-boundary` holds this paper's check file and a README that maps each numbered result to its declarations. One build from a clean clone of the release checks the Lean results of all three papers that the repository accompanies. Appendix A prints the main theorem.

Lemma 3.1: $\Re s>2$. No classical zeta or Gamma premise.

Lemma 3.2: H_∞, (Z1)–(Z3) and Stirling on strips for the remainder bound. The expansion is definitional, and coefficient summability uses (Z1) and (Z2).

Proposition 3.3: H_∞ for existence of the cutoff set. The correction and continuation take H_∞, (Z1)–(Z3), Stirling on strips and a cutoff set `E` satisfying `IsCutoff E`. The pair sum and its interior residue use H_∞, (Z1) and (Z2).

Theorem 4.1: Stirling on strips and the stated strip, height, sector and bin hypotheses.

Corollary 4.2(a): H_∞, (Z1), (Z2), Stirling on strips and the stated conditions on the full collision in a window with $Y > 0$.

Corollary 4.2(b): (Z1), (Z2), Stirling on strips and the stated conditions on the critical-line collision in a window with $Y > 0$.

Theorem 5.1: H_∞, (Z1)–(Z3), Littlewood's gap estimate, Stirling on strips.

The labels (Z1)–(Z3) follow the names of the formal hypotheses and differ from those of the Goldbach paper [12]. There (Z1) denotes the local count in our (Z2), (Z2) denotes Littlewood's gap estimate, and the zero symmetries are derived without a label. Both papers use (Z3) for the same statement.

The comparison with earlier boundary arguments for T and Φ_2 in Section 1.2 is outside the formalization. The classical analytic inputs are premises about Mathlib's zeta and Gamma functions. Lean checks both arguments of Theorem 5.1 for T_0: interior poles when infinitely many zeros lie off the critical line, and normal limits on the line when only finitely many do. See [10, §2.2] for the two arguments in general. In `GoldbachBoundaries.T0HInf`, Corollary 4.2(a) is `collisionAtom_ne_zero_of_large_of_sector` and part (b) is `criticalAtom_ne_zero_of_large_of_sector`. The command `#print axioms`, run on the two filtered atom declarations and the filtered boundary theorem, lists `propext`, `Classical.choice` and `Quot.sound`, the three standard axioms of Lean's logic. None of those declarations depends on `sorry`.

Theorem 5.1 is stated and formalized under H_∞, which RH implies. Alpöge announced Lean proofs of RH and GRH for Dirichlet L-functions [1], and the Lean and Mathlib communities certified the first as a Lean proof of Mathlib's statement of RH [6]. The main theorems here do not depend on that proof. Their formal versions take H_∞ and the applicable classical premises as hypotheses, and this paper does not combine them with it in Lean.

Acknowledgement and statement on the use of AI

The author is a computer scientist, not a mathematician. He originated the questions treated here, directed an AI research agent through a sequence of questions on natural boundaries of Goldbach series over primes and prime powers, and chose at each step the direction to pursue. He designed the verification methodology: independent adversarial reviews, re-derivation and small-case checks of every result before it was recorded as proved, and a formalization in Lean (Section 6). The mathematics is largely the work of AI models (Anthropic Claude, OpenAI GPT) under his direction. Multiple models reviewed the argument. An OpenAI GPT model drafted the manuscript and an Anthropic Claude model revised it. AI agents wrote the Lean development. The author did not write or check the Lean development line by line; he relies on the Lean kernel and the axiom checks reported in Section 6. No human mathematician has checked the proof line by line. There is no co-reader: queries about the proof will be answered by the author, with the same AI assistance. No AI system is an author. The author takes full responsibility for the content.

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Appendix A. The Lean boundary statement

The zeta and Gamma premises and the boundary predicate come from the Goldbach paper [12, Appendix A]. Here `Hinf` is H_∞, `halfPlane a` is $\{\Re s > a\}$, `HH` is `halfPlane 1`, and `lineOne` is $\{\Re s = 1\}$. Lean functions are total, so their values at poles or removable singularities can differ from limits. A Lean infinite sum has value zero when its family is nonsummable. On $\Re s > 2$, `T0` equals the absolutely convergent defining series. The continuation used in the existence clause is `T0HInf.T0c`, given by the right-hand side of (1) divided by $\Gamma(s)$. The theorem gives pointwise agreement on $\Re s > 2$ by `EqOn`. Its universal clause uses `codiscreteWithin`, agreement off a relatively closed discrete set. The predicate `IsNaturalBoundary` uses the same agreement on disc overlaps, as in Section 2. *The filtered series* (namespace `T0HInf`).

```
theorem t0_hinf_natural_boundary_of_sector
    (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hL : ZLittlewood) (hStir : StirlingStrip) :
    (∃ F : ℂ → ℂ, MeromorphicOn F HH ∧ EqOn F T0 (halfPlane 2)) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F HH →
      F =ᶠ[codiscreteWithin (halfPlane 2)] T0 →
      IsNaturalBoundary F HH lineOne) :=
```

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