hexagon:2611.01111v2

PHASE CONES FOR GAMMA-WEIGHTED ATOMS OF ADDITIVE DIRICHLET SERIES

LOUIS P. SLOTHOUBER

Abstract. In the continuation of an additive Dirichlet series, several pairs of zeros can contribute to the same candidate singularity. The numerators are products of Gamma values with complex coefficients, so the sum of the numerators, which we call the atom, can vanish. We prove a uniform non-cancellation theorem for these sums. Stirling's formula uniformly on fixed vertical strips is a classical theorem, taken as an explicit premise. The pairs have the form $b + \mathrm{i}(u + \tau)$ and $c - \mathrm{i}u$, with bounded τ, constant $b + c$, and real parts in $[1 - B, B]$, where $1/2 \le B < 1$. The coefficients lie in a common sector of half-angle $\delta < \pi(1 - B)$. Every unit bin of heights is finite, and the total coefficient mass in every unit bin with nonnegative base satisfies a uniform polynomial growth bound in that base. The weighted Gamma sum converges absolutely and is nonzero if the least height is sufficiently large and a coefficient there has modulus bounded below. The height threshold depends only on the bounds in the hypotheses. After a common rotation every nonzero head term, at a height close to the least height, has positive projection on the real axis. Exponential decay bounds the sum of the remaining terms. The angle restriction is sharp. We recover the integer-weight cone of the author's earlier paper on the Goldbach generating series. We also explain the two ways a nonzero atom is used. A nonzero atom gives an interior pole where the candidate points are locally finite and both the correction and Γ are regular at the point. Gamma regularity is automatic for sums with positive real part. When the remaining pair sums lie on the boundary line and their numerators are absolutely summable, a nonzero atom gives a nonzero normal limit of the corrected series. The sector theorem and its two specializations have been checked in Lean 4 with Stirling on strips as an explicit premise.

1. Introduction

1.1. **The obstruction and the strategy.** Let Λ be the von Mangoldt function, so $\Lambda(p^j) = \log p$ for primes p and integers $j \ge 1$, and $\Lambda(n) = 0$ otherwise. An additive Dirichlet series is a sum $\sum a(n_1) \cdots a(n_r)(n_1 + \cdots + n_r)^{-s}$ over ordered tuples of positive integers. For $r = 2$ and $a = \Lambda$ this is the Goldbach generating series

$$\Phi_2(s) = \sum_{m,n \ge 1} \frac{\Lambda(m)\Lambda(n)}{(m+n)^s}.$$

Put $F(x) = \sum_{n \ge 1} \Lambda(n) \mathrm{e}^{-nx}$ for $x > 0$. We call this kind of exponential series a heat sum. For $\Re s > 2$, the Mellin transform of $F(x)^2$ is $\Gamma(s)\Phi_2(s)$. To continue the transform, one uses the explicit formula for F near $x = 0$. Write $\rho = \beta + \mathrm{i}\gamma$ for each nontrivial zero of the Riemann zeta function ζ, counted once, and $m(\rho)$ for its multiplicity. The

Date: November 26, 2026.

2020 *Mathematics Subject Classification.* 11M41, 11M06, 11P32.

Key words and phrases. natural boundary, additive Dirichlet series, Goldbach series, Gamma function.

Email: lpslot@me.com. ORCID: https://orcid.org/0009-0005-4535-3306.

formula has a main term x^{-1}, a sum $-\sum_\rho a_\rho x^{-\rho}$ with $a_\rho = m(\rho)\Gamma(\rho)$, and a remainder. The coefficients a_ρ are absolutely summable. Squaring the zero sum and integrating over $(0,1)$ gives the pair sum

$$\sum_{\rho,\rho'} \frac{a_\rho a_{\rho'}}{s-\rho-\rho'}.$$

The rest of the square gives the Goldbach correction stated in Section 2. Under the Riemann Hypothesis (RH), which places every nontrivial zeta zero on $\Re z = 1/2$, the pair sums lie on $\Re s = 1$ and are dense there [8, Lemma 4.1].

Two different pairs can have the same sum. At a point s_0 the numerator for $1/(s-s_0)$ is the total of $a_\rho a_{\rho'}$ over all ordered pairs with $\rho+\rho' = s_0$. We call this total the *atom* at s_0. The two orders of a pair of distinct zeros give equal terms. A coincidence between different unordered pairs can bring together Gamma products with different phases, so cancellation has to be ruled out. A zero atom removes the contribution of the pairs with that sum. Other sums can still accumulate at the same point.

The role of a nonzero atom depends on the location of the sum. Where the candidate points are locally finite and the correction is regular, a nonzero atom gives a pole. A sequence of such poles approaching the boundary from the right prevents a meromorphic extension across its limit. With all the pair sums on the boundary line, we use the limit along the normal instead. At $1+\mathrm{i}\tau$, multiplication by the normal distance η gives each term the factor $\eta/(\eta+\mathrm{i}(\tau-\gamma-\gamma'))$. This factor is bounded in modulus by one and tends to the indicator of $\gamma+\gamma' = \tau$. Dominated convergence gives the atom as the normal limit of the pair sum. For Φ_2 after subtraction of the correction, the limit is the atom divided by $\Gamma(1+\mathrm{i}\tau)$. A meromorphic extension at that point would have a pole if the atom were nonzero. A dense set of such atoms prevents an extension through any point of the line.

For either argument we can choose the points where we test the atom. In a bounded window of ordinates there are only finitely many sums with a pair having a bounded ordinate in modulus. Excluding them leaves pairs of two zeros of large, opposite ordinates. Write such a pair as $b+\mathrm{i}(u+\tau)$ and $c-\mathrm{i}u$, with $u > 0$. Stirling's formula gives the Gamma product the approximate modulus $2\pi \mathrm{e}^{-\pi u-\pi\tau/2}u^{b+c-1}$ and phase $\tau\log u+\pi(b-c)/2$. The two phases of size $u\log u$ have cancelled. What remains varies slowly with u and has a bounded offset from $b-c$. If u_0 is the least height and ω is the common direction of the coefficient sector, we multiply every weighted term by $\overline{\omega}\,\mathrm{e}^{-\mathrm{i}\tau\log u_0}$. At heights at most $u_0+\sqrt{u_0}$, every nonzero term then lies in a common sector in the right half-plane and has positive projection on the real axis, provided the coefficient sector leaves enough room for the offset. The terms at larger heights decay exponentially and together contribute less than the guaranteed projection from a term at the least height.

The author's earlier paper on the Goldbach generating series [22], which we call the Goldbach paper, uses this rotation, square-root window and tail estimate for integer weights [22, Lemmas 5.2–5.3 and Theorem 5.4]. Here the coefficients may be complex. Their phases may vary in a fixed sector, and the polynomial bound on their mass permits larger weights than the squared-logarithm bound used for multiplicities.

1.2. **Results.** *The sector cone* (Theorem 3.1). By Stirling on strips we mean Stirling's formula uniformly on every fixed real-part strip as the absolute imaginary part tends to infinity (Section 2). It is a classical theorem [20, §5.11], taken as an explicit premise because Mathlib lacks it. Assume it, and fix $1/2 \le B < 1$, $Y > 0$, $a > 0$, a real K, a

real $A \geq 0$ and $0 \leq \delta < \pi(1-B)$. There is a threshold $U \geq Y+1$ that depends only on these six parameters and works for every shift $|\tau| \leq Y$ and every family of the following kind. The two real parts b, c of each pair lie in $[1-B, B]$ and their sum is the same for every pair. The coefficients lie in a sector of half-angle δ.

The least height is attained and is at least U, and some term at that height has a coefficient of modulus at least a. Every unit bin $[v, v+1)$ with real v is finite, and for $v \geq 0$ its coefficient mass is at most $K(v+Y+4)^A$.

For such a family the sum of the Gamma products $\Gamma(b+\mathrm{i}(u+\tau))\Gamma(c-\mathrm{i}u)$ multiplied by their coefficients converges absolutely and is nonzero. Corollary 3.2 is the case of squared-logarithm bins, and Lemma 3.3, its case of integer weights, recovers the phase cone of [22, Theorem 5.4].

Section 2 gives the Mellin calculation, the two ways atoms obstruct continuation and the comparison with the prime-pair series. A remark there shows that, under H_∞, the line $\Re s = 1$ is a natural boundary of the difference between the Goldbach series and the prime-pair series, so in general the boundary of one does not follow from that of the other by subtraction. When only finitely many zeros lie off the critical line, the same remark deduces the boundary of the prime-pair series from the continuation of the Goldbach series. The cone and its specializations are proved in Section 3, which also has examples at the limits of the hypotheses. The proof of the sector theorem in Section 3 can be read directly after this introduction, since it uses only Stirling on strips and the Gamma product $w_{b,c}$, which Section 3 defines. In Section 4 we ask whether a gap between the lowest two pair heights can be used for Liouville coefficients. The formal statements are in Appendix A.

Two companion papers by the author apply these results. *The natural boundary of the filtered prime-pair Dirichlet series* [25] proves under H_∞ that the filtered series T_0 of Section 2.7 has the natural boundary $\Re s = 1$. Its pair weights are complex, and the sector theorem with polynomial bins rules out their cancellation, with an ordinate window and a height threshold fixed in advance. Where infinitely many zeros lie off the critical line it does so at sums to the right of the line all of whose representations are high. Where only finitely many do, the proof absorbs all pairs involving off-line zeros into the correction and excludes the candidate sums that have a low critical-line representation, so every critical-line representation of a surviving sum is high and contributes to its atom. That paper also accounts for the other contributions at each selected point. *Natural boundaries and real-point limits for class-one Goldbach series* [24] proves, under the Generalized Riemann Hypothesis at modulus q, the natural boundary $\Re s = r-1$ of the r-fold von Mangoldt series in which every summand is 1 modulo q, for $q, r \geq 2$. There the character average gives integer pair weights once the common factor $\varphi(q)^{-2}$ is removed, and Lemma 3.3 applies.

1.3. **Prior work.** *The ordinary Goldbach series.* Egami and Matsumoto [8] gave a pair-sum expansion for Φ_2, established its continuation to $\Re s > 1$ under RH and conjectured the natural boundary [8, Theorems 2.1 and 2.2]. Their pair term in (4.3) has numerator $\Gamma(s+1-\rho_1)\Gamma(\rho_1)$ and denominator $(s-\rho_1-\rho_2)\rho_2\Gamma(s)$. Following Fujii's discussion of pair sums [11, p. 50], cited in [8, §2], they consider two conditions. Their condition (A) says that two pairs of ordinates with the same nonzero sum are the same pair up to order. Their condition (B) requires a lower bound $\exp(-\alpha\sum_{j=1}^{4}|\gamma_j|)$ for some $0 < \alpha < \pi/2$ on the distance between the sums of two different unordered pairs (γ_1, γ_2) and (γ_3, γ_4)

when the first sum is nonzero [8, pp. 7 and 13]. It implies (A), and under RH and (B) they prove the boundary. Let Φ_r be the unrestricted r-fold additive Dirichlet series with $a = \Lambda$. Bhowmik and Schlage-Puchta [5, Theorem 2(1)] state that under RH the boundary of Φ_r at $\Re s = r - 1$ for every $r \geq 2$ follows from that of Φ_2 at $\Re s = 1$. Their circle-method proof uses a bound $\ll \min(x^3, \alpha^{-3})$ for $T_3(\alpha) = \sum_{|n|\leq x}(x - |n|)^2 e(n\alpha)$, where $e(\theta) = \mathrm{e}^{2\pi \mathrm{i}\theta}$ and x is a large integer. The bound fails at $\alpha = 1/2$, where $T_3 = x$ [22, Remark 8.3.2]. Proposition 7.4 of [22] gives an independent proof of the reduction. Bhowmik and Schlage-Puchta separately proved the binary boundary under RH and (A) [5, Theorem 2(2)].

Rüppel used a separation hypothesis in her study of Goldbach generating series restricted to residue classes [21]. Matsumoto's survey gives the binary boundary for Φ_2 under RH and linear independence over $\mathbb{Q}$ of the positive ordinates [19, Theorem 5.4]. Bhowmik and Halupczok survey the asymptotics of Goldbach representations [2]. Tanigawa and Zhai state under RH that a class of series including Φ_2 and the corresponding higher-fold series Φ_r cannot be continued meromorphically beyond a half-plane [26, Corollary 1.4]. Egami and Matsumoto write that the proof of the theorem on which that corollary rests seems incomplete without a condition like their (B), which also controls near coincidences [8, pp. 5 and 7]. In our reading, the issue is the possible cancellation of numerators at an exact coincidence. Bhowmik and Schlage-Puchta address the distinction between candidate and actual singularities for Euler products [6]. Their criteria do not apply to the additive Gamma sums here.

Under RH, Bhowmik, Ramaré and Schlage-Puchta pass the normal limit termwise through the Goldbach pair sum, in the normalization of Egami and Matsumoto [4, p. 297]; we use the bounded kernel and dominated convergence (Section 2). Under RH, after subtracting the correction, the normal limit is the atom divided by $\Gamma(1 + \mathrm{i}\tau)$ [22, Corollary B(iii)]. The Goldbach paper removed the ordinate-sum assumptions for Φ_2 and Φ_r under the hypothesis H_∞, which requires that only finitely many nontrivial zeta zeros lie to the right of $1/2 + \epsilon$ for every $\epsilon > 0$ [22, Theorem A]. It combines the pair-sum decomposition, the exclusion of low representations, the normal limit and the integer-weight cone [22, Proposition 3.7, Lemmas 6.1 and 6.6, and Theorem 5.4].

Zaharescu and Zaki study natural boundaries for a family of Dirichlet series with the congruence condition $mn \equiv b \pmod q$ and weight $H(\alpha \log(m+n))$. For $q = 1$, the series is the double von Mangoldt sum $\sum_{m,n\geq 1} H(\alpha \log(m+n))\Lambda(m)\Lambda(n)(m+n)^{-s}$ [28]. The function-field analogue of Egami and Matsumoto distinguishes cases according to the characteristics of the fields [9]. For series with random signs, Bhowmik and Matsumoto relate the natural boundary to the abscissa of convergence [3]. The Liouville convolution returns to pairs of zeta zeros but with different coefficients [7, Corollary 2]. We discuss those coefficients in Section 4.

Our treatment of the heat sums follows the explicit-formula work of Hardy and Littlewood and the approach of Linnik and Languasco and Zaccagnini [12, 16, 15]. The remainder bound printed in [15, Lemma 4.1] omits the constant $-\log(2\pi)$. Languasco adds an $O(1)$ term for it, and on the positive real axis the constant is absorbed into $O(1 + \sqrt{x})$ [14, Lemma 5.1 and §6]. The contour shift below isolates the constant $-\log(2\pi)$ explicitly, as in [22, Lemma 3.5].

2. Mellin transforms, atoms and standing inputs

The Mellin transform identifies the sum of two exponents as a candidate pole.

2.1. **From a heat sum to a pair sum.** For a heat sum $F(x) = \sum_{n\geq 1} a_n \mathrm{e}^{-nx}$ with $|a_n| \leq \log n$, the integral of $F(x)^r x^{s-1}$ over $(0,\infty)$ is $\Gamma(s)$ times the corresponding additive Dirichlet series for $\Re s > r$: the total absolute weight at sum N is at most $N^{r-1}(\log N)^r$, and with the Gamma integral this bound justifies termwise integration. In the applications an explicit formula writes part of a heat sum as $\sum_\rho A_\rho x^{-\rho}$ with $\sum_\rho |A_\rho| < \infty$. For $\Re s > 2\sup_\rho \Re\rho$, and in particular for $\Re s > 2$, the square of this sum integrates termwise over $(0,1)$ to

$$\sum_{\rho,\rho'} \frac{A_\rho A_{\rho'}}{s-\rho-\rho'},$$

with the majorant $(\sum_\rho |A_\rho|)^2/(\Re s - 2\sup_\rho \Re\rho)$. This double series is the *pair sum*. A *collision* at s_0 is the set of ordered pairs (ρ,ρ') with $\rho+\rho' = s_0$, and the sum of the numerators over a collision is the *atom* at s_0. A pair of distinct zeros occurs in both orders, and a diagonal pair (ρ,ρ) occurs once.

The square has other terms besides the pair sum. Products with the main term of the heat sum are completed to integrals over $(0,\infty)$ and give explicit functions, such as $\Gamma(s-1)$ times a logarithmic derivative, that are meromorphic across the boundary line. Products with the remainder of the explicit formula give integrals that converge beyond the line when the remainder is small enough against the other factor. For a finite set of zero terms with large real part, one can also use shifted Mellin transforms of the remainder, provided those transforms are meromorphic far enough to the left. The comparison in Section 2.7 uses this treatment for the zeros to the right of a chosen cutoff. All of these terms, divided by $\Gamma(s)$, make up the *correction*. The continuation is then the pair sum divided by $\Gamma(s)$, plus the correction. The boundary arguments use corrections meromorphic across the line. They subtract the correction or avoid its poles.

For Φ_2, let $P(s)$ denote the pair sum with $A_\rho = a_\rho$ and put $\psi = -\zeta'/\zeta$. At $s_0 = \rho_1+\rho_2$, the Gamma recurrence reduces the coefficient of $1/(s-s_0)$ for the ordered summand indexed by (ρ_1,ρ_2) in Egami and Matsumoto's (4.3) to $\Gamma(\rho_1)\Gamma(\rho_2)/\Gamma(s_0)$. At a collision, the full coefficient sums the contributions of all matching ordered pairs. The identity used here, with numerators independent of s in $P(s)$, is the regularized identity of [22, Proposition 3.7(c), equation (13), with (9) and (12)]. It gives

$$\Gamma(s)\Phi_2(s) = P(s) + 2\Gamma(s-1)\psi(s-1) - \frac{1}{s-2} + H(s),$$

where H is the combination $A - 2c_0 L$ in the notation of that paper and is holomorphic for $\Re s > B$, with $B = \sup_\rho \Re\rho$. The identity holds first for $\Re s > 2$ and continues meromorphically to $\Re s > 1$ under H_∞ [22, Proposition 4.2(b)]. Dividing the terms after $P(s)$ by $\Gamma(s)$ gives the correction for Φ_2.

2.2. **What a nonzero atom gives.** *Interior poles.* Where the collisions are locally finite, a nonzero atom at s_0 gives a simple pole of the continuation if the correction is regular at s_0 and Γ is regular there. The residue is the atom divided by $\Gamma(s_0)$. If such poles accumulate at a point of the line, no function meromorphic on a disc about that point can agree with the continuation on the right, because the poles of a meromorphic function do not accumulate inside its domain. This is the argument when infinitely many zeros lie off the critical line. Not every pair sum that involves such a zero lies to the

right of the line, since the zeros are symmetric about the critical line. The proof selects pairs whose sums do, and under H_∞ these sums approach the line.

Normal limits. When all the pairs that remain have their sums on the boundary line, multiplication by the normal distance η gives each term the bounded kernel $\eta/(\eta + \mathrm{i}v)$, where v is the difference between the ordinate of the point and that of the pair sum. The kernel tends to one at $v = 0$ and to zero elsewhere, even when the nonzero v accumulate at zero. For absolutely summable numerators the normal limit of the pair sum is the atom, by dominated convergence [22, Lemma 6.6]. Where a meromorphic extension exists, a finite nonzero limit of this kind forces a simple pole by the Laurent expansion, so nonzero limits at a dense set of points rule out extension. This is the argument for the Goldbach series when only finitely many zeros lie off the critical line and their terms have been included in the correction [22, Lemma 6.5 and Theorem A]. A zero limit can come from cancellation within a collision or from an empty sum: the pair ordinates form a countable set, and the sum is empty at every ordinate outside it. Nor does a zero limit give any information about nearby collisions.

Choosing the points. To obtain points at which the cone applies, first fix a bounded window of sums. A representation with one bounded ordinate then has two bounded ordinates. Local finiteness makes the set of such sums finite, so a smaller open interval avoids every low representation at once. We call a representation *high*, for a given threshold, when both of its ordinates exceed the threshold in modulus, and *low* otherwise. At a point all of whose representations are high, every representation has ordinates of opposite signs as soon as the threshold exceeds the bound for the window, and this is the configuration the cone treats.

For Φ_2 under RH the limit of η times the corrected series at $1 + \mathrm{i}\tau + \eta$ is the atom at $1 + \mathrm{i}\tau$ divided by $\Gamma(1 + \mathrm{i}\tau)$ [22, Corollary B(iii)]. It suffices here to subtract $2\psi(s - 1)/(s - 1) - 1/(\Gamma(s)(s - 2))$, since $H(s)/\Gamma(s)$ is holomorphic at the line under RH. The same limit also gives a coefficient at the real boundary point. The pairs with real sum consist of conjugate zeros, and each contributes $|a_\rho|^2$. At $\tau = 0$, the linear term of the Mellin expansion is singular as well. Without subtraction the normal limit of $\eta\Phi_2(1+\eta)$ includes the extra term $2\psi(0) = -2\log(2\pi)$ [22, Remark 8.3.1].

2.3. **Choosing pair sums near the boundary.** For the Goldbach series, under H_∞ and the zeta symmetries below, two zeros with sufficiently large absolute ordinates have a sum with real part less than $1 + \varepsilon$ for any fixed $\varepsilon > 0$. On a compact subset of $\Re s > 1$, these pairs therefore stay a positive distance from the compact set. When one zero has bounded ordinate, a partner with sufficiently large ordinate also gives a sum away from the compact set. Only finitely many pairs remain. Absolute summability gives locally uniform convergence away from the resulting locally finite set of candidate poles [22, Proposition 4.2(a)].

With infinitely many zeros off the line, choose zeros $\rho_n = \beta_n + \mathrm{i}\gamma_n$ with $\beta_n > 1/2$ and $\gamma_n \to +\infty$. To approach $1 + \mathrm{i}y$, the gap estimate below supplies a second zero with ordinate $\gamma_n + y + o(1)$. Reflection in the critical line lets its real part be at least $1/2$. The sum of this zero and $\overline{\rho_n}$ approaches $1 + \mathrm{i}y$ from the right by H_∞. The sequence has infinitely many distinct points, so the finite exclusions of low sums and correction poles near the limit leave an accumulating sequence [22, Lemma 6.1 and Theorem A].

When there are only finitely many zeros off the line, the terms with at least one such zero form a finite combination of shifted single-zero sums and finite pair sums. These

are meromorphic across the line, since the individual zeros are locally finite and their weights are absolutely summable. For the remaining pairs on the line, choose a bounded open interval, discard the finitely many low sums, and choose an open subinterval (l, h) left over. A sufficiently large positive ordinate γ is on the critical line. The gap estimate at $\gamma + (l + h)/2$ gives another critical ordinate γ' with $\gamma' - \gamma \in (l, h)$. The pair formed with the conjugate of the first zero gives the chosen sum. This produces a represented high sum in any interval, so the cone gives a dense set of nonzero normal coefficients [22, Lemma 6.7 and Theorem 6.8].

2.4. **Agreement and natural boundaries.** Meromorphic agreement means agreement away from a relatively closed discrete exceptional set in the open overlap. A line is a natural boundary here if no disc centred on it admits a meromorphic function agreeing on the right-hand overlap in this sense. A Mellin identity throughout the initial half-plane gives pointwise agreement there. The identity theorem identifies any competing continuation on the same connected domain that agrees on the initial half-plane in the meromorphic sense. Adding a function G meromorphic on a disc does not change extendibility across it: a meromorphic f agrees with F on the right-hand overlap exactly when $f+G$ agrees with $F+G$, so F has a meromorphic extension across the disc exactly when $F + G$ does.

2.5. **Standing inputs.** Unless specified otherwise, zeros of ζ are nontrivial zeros $\rho = \beta + \mathrm{i}\gamma$ in $0 < \beta < 1$, each counted once, with multiplicity $m(\rho)$. The critical line is $\beta = 1/2$. RH implies H_∞.

(Z1).: Conjugation and $\rho \mapsto 1 - \overline{\rho}$ preserve the nontrivial zeta zeros and their multiplicities, by the functional equation and reality of the Dirichlet coefficients [27, Ch. 2].

(Z2).: If $N(t)$ counts positive zeta ordinates up to t with multiplicities, then

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + O(\log t) \quad (t \to \infty), \qquad N(t+1) - N(t) \ll \log(t+3) \quad (t \geq 0).$$

See [27, Theorems 9.4 and 9.2]. The proofs use the second estimate, the local count, and from the first only the existence of zeros.

(Z3).: For $-1 \leq \Re z \leq 2$ and $|\Im z| \geq 2$, away from zeros,

$$\frac{\zeta'}{\zeta}(z) = \sum_{|\gamma - \Im z| \leq 1} \frac{m(\rho)}{z - \rho} + O(\log|\Im z|).$$

The large-height estimate is [27, Theorem 9.6(A)], and conjugation gives the negative heights. For $2 \leq |\Im z| \leq t_0$ the bound holds with a larger constant, because the difference of the two sides is bounded on that compact part of the strip, being continuous away from the zeros and the cutoffs $|\gamma - \Im z| = 1$ and bounded near each of them. The estimate is not claimed for $|\Im z| < 2$, where the principal part of ζ'/ζ at $z = 1$ would have to be subtracted as well. Where the proofs below need bounded heights, they use compact pieces of vertical lines that contain no zero and do not pass through $z = 1$.

Littlewood's gap estimate.: For every sufficiently large positive t some zeta ordinate differs from t by less than $A_{\mathrm{L}}/\log\log\log t$, with an absolute constant A_{L} [17][27, Theorem 9.12].

Stirling on strips.: On every fixed real-part strip, uniformly at sufficiently large absolute imaginary part, $\Gamma(z) = \sqrt{2\pi}\exp((z-\frac{1}{2})\operatorname{Log} z - z)(1+O(|\Im z|^{-1}))$, with the principal logarithm [20, §5.11].

Under H_∞ and (Z1), the real parts tend uniformly to $1/2$ at large absolute ordinate. The finite set outside any fixed strip about $1/2$ accounts for all other real parts, so $B = \sup_\rho \beta < 1$ and symmetry puts the zeros in $[1-B, B]$. The full family of zeta zeros uses this supremum in the cone. A family consisting only of critical-line zeros uses $B = 1/2$. In the abstract theorem B is a strip parameter. The remark in Section 2.7 uses that ζ has no zero in $(0,1)$, which follows from the positive alternating Dirichlet sum there. For Γ we use the recurrence, the reflection formula and the absence of zeros [20, §§5.2, 5.5], and the bound $|\Gamma(z)| \le \Gamma(\Re z)$ for $\Re z > 0$, immediate from Euler's integral. With the recurrence it gives $|\Gamma(z)| \ll_k |\Im z|^{-k}$ on every strip $0 < d \le \Re z \le D$, for every k.

The labels (Z1)–(Z3) follow those of the Lean hypotheses used in the applications in the companion papers. In [22], (Z1) denotes the local count, which is the second estimate of our (Z2), and (Z2) denotes Littlewood's gap estimate. The symmetries are derived there. The label (Z3) has the same meaning in both papers.

2.6. **Contour shifts for the heat sum.** Mellin inversion represents $F(x)$ by the integral of $\Gamma(z)\psi(z)x^{-z}/(2\pi\mathrm{i})$ on $\Re z = 2$, where $\psi = -\zeta'/\zeta$. On that line the Euler series for ψ converges absolutely, and Stirling bounds the Gamma factor. To shift the integral to $\Re z = -1/2$, choose heights in $[N, N+1]$ separated by at least $c_0/\log(N+3)$ from the zero ordinates and their reflections in $[N-1, N+2]$. The local count in (Z2) makes the total length of the excluded intervals less than one for a sufficiently small fixed $c_0 > 0$. Then (Z3) gives the bound $O(\log^2(N+3))$ for ψ on both horizontal segments. The Gamma factor decays exponentially there. On the new vertical line the horizontal separation from the zeros gives $O(\log(2+|\Im z|))$ for ψ at large heights, and continuity gives a bound on the compact part of the line.

The pole at $z = 1$ supplies x^{-1}, the nontrivial zeros give $-\sum_\rho a_\rho x^{-\rho}$, and the Gamma pole at $z = 0$ gives $\psi(0) = -\log(2\pi)$. The shifted integral is $O(x^{1/2})$ for $0 < x \le 1$. The local count and Stirling give absolute summability of $|a_\rho|$. In particular, $F(x) - x^{-1} = -\sum_\rho a_\rho x^{-\rho} + O(1)$ on this interval. The contour shift in one variable gives each zero its own residue. Squaring the zero sum brings together contributions at a shared sum [22, Lemma 3.5 and Proposition 3.7].

2.7. **The prime-pair series and the Goldbach series.** Let p, p' range over primes, and for $\Re s > 2$ put

$$T(s) = \sum_{p,p'} \frac{\log p \, \log p'}{(p+p')^s}.$$

The heat sum for T is $F_p(x) = \sum_p (\log p)\mathrm{e}^{-px}$. The prime-pair proof brings in scaled zeros from $Q(z) = \sum_p (\log p)p^{-z}$ and selects poles at $1+\rho/p$. The Goldbach proof instead uses unscaled pairs of zeros. The filtered prime-pair series is

$$T_0(s) = \sum_{p,p'} \frac{\log p \, \log p'}{(p+p')^s}\left(1 - s + s(s+1)\frac{pp'}{(p+p')^2}\right) \qquad (\Re s > 2),$$

and $\Gamma(s)T_0(s)$ is the Mellin transform of $\big((1+x\,d/dx)F_p(x)\big)^2$. The filtered example in Section 3 verifies the cone hypotheses for the unscaled pairs in its expansion at a represented sum $s_0=\rho_1+\rho_2$ with only high representations. The boundaries of T and of Φ_2 both lie on $\Re s=1$ under H_∞ [23, Theorem 1.1][22, Theorem A].

Remark (How T and Φ_2 are related). Let $\psi=-\zeta'/\zeta$, and write μ for the Möbius function. For $\Re z>1$ put

$$C(z)=\sum_p\sum_{j\ge2}(\log p)p^{-jz}=\psi(z)-Q(z)=-\sum_{k\ge2}\mu(k)\psi(kz).$$

The prime-power sum converges for $\Re z>1/2$. The equalities hold for $\Re z>1$, and the last form continues C meromorphically to $\Re z>0$, with at most simple poles at the points $1/k$ and ρ/k for squarefree $k\ge2$. This expansion and the scaled zeros it produces are classical for the prime zeta function [13, 10]; for Q see [23, Lemma 2.1].

Assume H_∞, let $B<1$ be the supremum of the real parts of the zeros (Section 2), and let $F_{\ge2}=F-F_p$ be the heat sum over proper prime powers. For $\Re s>2$,

$$\Gamma(s)\big(\Phi_2(s)-T(s)\big)=\int_0^\infty\big(2FF_{\ge2}-F_{\ge2}^2\big)x^{s-1}\,dx.$$

The part $2x^{-1}F_{\ge2}$ of the integrand integrates to $2\Gamma(s-1)C(s-1)$. For the rest, fix c with $\max\{B/2,1/3\}<c<1/2$ and $b\in(1/2,1-c)$. Shifting the Mellin inversion integral of $\Gamma(z)C(z)$ to $\Re z=c$ crosses only the pole at $1/2$ and gives $F_{\ge2}(x)=dx^{-1/2}+O(x^{-c})$ on $(0,1]$, with $d=\frac12\Gamma(\frac12)$.

The terms with $k\ge3$ in the series for $C(z)$ are uniformly absolutely summable on $\Re z\ge c$, since $3c>1$. For $k=2$, the real part of $2z$ is at least $2c-B>0$ to the right of every nontrivial zero. Thus (Z2)–(Z3) give $\psi(2z)=O(\log(2+|\Im z|))$ at large heights on the part of the strip with $2\Re z\le2$. Absolute convergence bounds $\psi(2z)$ uniformly for $2\Re z\ge2$. Stirling makes the horizontal integrals tend to zero and makes the integral on $\Re z=c$ absolutely convergent with bound $O(x^{-c})$.

Let E be the finite set of zeros with real part greater than b, and insert the explicit formula $F(x)-x^{-1}=-\sum_\rho a_\rho x^{-\rho}+O(1)$. The products of the terms with $\rho\in E$ and $F_{\ge2}$, completed to integrals over $(0,\infty)$, the products of the other zero terms with $dx^{-1/2}$, and the square of $dx^{-1/2}$ have the transforms

$$-2\sum_{\rho\in E}a_\rho\Gamma(s-\rho)C(s-\rho),\qquad -2d\sum_{\rho\notin E}\frac{a_\rho}{s-\rho-1/2},\qquad -\frac{d^2}{s-1},$$

which are meromorphic on $\Re s>B$. The middle sum has only shifted individual zeros as possible poles, and its coefficients are absolutely summable. What remains is an entire contribution from $x\ge1$ and an integrand that is $O(x^{-b-c})$ on $(0,1)$, with transform holomorphic on $\Re s>b+c$. Hence, with $\sigma_*=\max\{B,b+c\}<1$,

$$\Phi_2(s)-T(s)=\frac{2C(s-1)}{s-1}+R(s),\qquad R\text{ meromorphic on }\Re s>\sigma_*,$$

first for $\Re s>2$. The right side is meromorphic on $\Re s>1$, so a continuation of either series to $\Re s>1$ gives one of the other, and the identity persists.

Under H_∞ the line $\Re s=1$ is a natural boundary of the difference Φ_2-T itself. Fix a real $y\ne0$. For large primes p choose a zero ρ_p with $\Re\rho_p\ge1/2$ and $\Im\rho_p/p\to y$, by Littlewood's gap estimate and the symmetries of the zeros. The residue of C at ρ_p/p

contains the term $-m(\rho_p)/p$, and $0 < m(\rho_p) < p$ for large p by the local zero count. Any other representation ρ'/k of the same point has $k < 2p$ and $k \neq p$, so p does not divide k, its term $\mu(k)m(\rho')/k$ has a denominator prime to p, and the terms cannot cancel. This is the prime-denominator argument from Titchmarsh's proof of (9.5.7) [27, §9.5, p. 216], also used in [23, §3 and Remark 3.5]. Hence the points $1 + \rho_p/p$ are poles of the shifted term. They approach $1 + \mathrm{i}y$ from the right, and R, being meromorphic across the line, can cancel only finitely many of them near the limit point. At $y = 0$ one uses the points $1 + 1/p$ for primes p. There C has the residue $1/p$ from the term $k = p$, and no point ρ/k is real because ζ has no zero in $(0, 1)$. So the points $1 + 1/p$ are poles of the shifted term, and they accumulate at $s = 1$.

The identity does not transfer a natural boundary by subtraction, in either direction. The difference is itself singular at every point of the line, and for $S(s) = 2C(s-1)/(s-1)$ the abstract pairs $(T, \Phi) = (0, S)$ and $(-S, 0)$ satisfy the same identity with the boundary on opposite members.

When only finitely many zeros lie off the critical line, in particular under RH, the poles of the continuation of Φ_2 in $\Re s > 1$ have only finitely many real parts, so a strip to the right of the line is free of them. Near any boundary point the poles of the shifted term then survive in $T = \Phi_2 - S - R$, and in that case the boundary theorem for T of [23] follows from the continuation of Φ_2. With infinitely many zeros off the line this deduction would need control of cancellation between the poles of Φ_2 and those of the shifted term. The two papers prove their boundaries by different means, and neither proof invokes the other theorem: [23] works at the candidates $1 + \rho/p$, where one pair dominates, and [22] applies the phase cone to unscaled pairs. The filtered prime-pair series discussed after Lemma 3.3 also uses unscaled pairs.

3. The phase cone

In the Goldbach example the coefficients of the Gamma products are products of multiplicities. Other series supply extra factors, which can change the phases or the size of the coefficients. The estimate is stated for an indexed family of Gamma products with coefficients.

3.1. **Why the phases align.** Fix a point $s_0 = \alpha + \mathrm{i}\tau$ with $|\tau| \le Y$, and consider pairs of zeros that land on it and whose ordinates exceed Y in modulus. Two ordinates of the same sign, both larger than $|\tau|$ in modulus, cannot add up to τ. So every such pair has one positive and one negative ordinate and can be written $b + \mathrm{i}(u + \tau)$, $c - \mathrm{i}u$ with $u > 0$. We call u the *height* of the pair. Its Gamma product is

$$w_{b,c}(\tau, u) = \Gamma(b + \mathrm{i}(u + \tau))\Gamma(c - \mathrm{i}u).$$

By Stirling's formula $\Gamma(b+\mathrm{i}t)$ has, for large $t > 0$, modulus about $\sqrt{2\pi}\, t^{b-1/2}\mathrm{e}^{-\pi t/2}$ and argument about $t \log t - t + \pi(b - 1/2)/2$. The leading phase $t \log t - t$ has derivative $\log t$, so the argument of a single Gamma value varies rapidly with t. In the product the second factor is $\Gamma(c - \mathrm{i}u) = \overline{\Gamma(c + \mathrm{i}u)}$, whose argument is about $-(u \log u - u) - \pi(c - 1/2)/2$. The two leading phases differ only through the shift τ, and $(u + \tau)\log(u + \tau) - (u + \tau) - u \log u + u = \tau \log u + O(1/u)$. Hence

$$w_{b,c}(\tau, u) \approx 2\pi \mathrm{e}^{-\pi u - \pi\tau/2} u^{b+c-1} \exp\bigl(\mathrm{i}[\tau \log u + \pi(b - c)/2]\bigr),$$

which is (1) below with its error term. The modulus decays like $\mathrm{e}^{-\pi u}$, so higher pairs are exponentially smaller; this is what bounds the tail. The power u^{b+c-1} has the same

exponent for all pairs with sum s_0, since $b + c = \alpha$, so the moduli at two heights can be compared through the difference of the heights. The argument is $\tau \log u$, whose derivative is only τ/u, plus an offset $\pi(b - c)/2$ that depends only on the real parts. Over a short range of heights the phases therefore differ by little more than their offsets. The offset vanishes when both zeros are on the critical line. When the real parts lie in $[1 - B, B]$ it is at most $\pi(B - 1/2)$ in modulus, a bound that is attained only for $\{b, c\} = \{B, 1 - B\}$.

Suppose the family has a least height u_0, as it does for the zero pairs considered below. Let w_0 be the Gamma product of a pair at that height, and multiply every product by $\mathrm{e}^{-\mathrm{i}\tau \log u_0}$. A pair at a height u in the window $[u_0, u_0 + \sqrt{u_0}]$ then has argument $\tau \log(u/u_0) + \pi(b - c)/2 + O(1/u_0)$, and the first term is at most $Y/\sqrt{u_0}$ in modulus. So all the products in the window, which we call the *head*, point into a sector about the positive real axis of half-angle slightly more than $\pi(B - 1/2)$ (Figure 1). Because $B < 1$ this sector is strictly inside the right half-plane. Complex numbers in such a sector cannot cancel: each has real part at least a fixed positive multiple λ of its modulus. With positive coefficients d_i the head therefore has real part at least $\lambda |d_{i_0} w_0|$, where d_{i_0} is the coefficient of the lowest term. This is useful only with a lower bound $|d_{i_0}| \geq a > 0$, because the tail will be measured against $a|w_0|$.

If the coefficients are complex but lie within an angle δ of a common direction ω, one rotates by $\overline{\omega}$ as well. The sector widens to half-angle $\pi(B - 1/2) + \delta$, and it stays inside a half-plane as long as $\delta < \pi(1 - B)$.

Above the window only the moduli matter. By (1), at height $u_0 + n$ the ratio to $|w_0|$ is bounded by a constant times $(1 + n)\mathrm{e}^{-\pi n}$ because the exponent $b + c - 1 \leq 1$ is common. A bin starting at $u_0 + n$ can meet the tail only when $n + 1 > \sqrt{u_0}$. Then $u_0 < (n + 1)^2$, so the polynomial bound on the mass of that bin is at most a constant times $(n + 2)^{2A}$. The whole tail is bounded by a constant times $|w_0| \sum_{n+1>\sqrt{u_0}} (n + 2)^{2A+1} \mathrm{e}^{-\pi n}$. This tends to zero relative to $|w_0|$ as u_0 grows, so the tail can be made smaller than half the lower bound $\lambda a |w_0|$ for the head.

The square-root width gives both properties needed here. Its ratio to u_0 tends to zero, keeping the phase drift small, while a tail bin has index large enough to bound the polynomial in $u_0 + n$ by a polynomial in n alone. In a window of width W, the drift is at most YW/u_0, so it tends to zero for any width $W = o(u_0/Y)$. For polynomial bins a width $W = C \log u_0$ suffices whenever $C > A/\pi$. Summing $(n + 2)^{A+1} \mathrm{e}^{-\pi n}$ over $n + 1 > W$ in the bin estimate below bounds the tail divided by $|w_0|$ by

$$O\big(u_0^{A - \pi C} (1 + \log u_0)^{A+1}\big).$$

This tends to zero. For squared-logarithm bins a sufficiently large multiple of $\log \log u_0$ suffices. In that case the exponential factor at the start of the tail outweighs the squared-logarithm bin bound. The square-root width works in both cases without choosing a constant for the width.

3.2. **What the hypotheses mean for pairs of zeros.** Theorem 3.1 is stated for an indexed family (b_i, c_i, u_i, d_i). In the applications the family is the set of high ordered pairs from the zero terms under consideration with sum $s_0 = \alpha + \mathrm{i}\tau$. The point is chosen to exclude low representations from those zero terms. An application to a Dirichlet series must also account for every other contribution at s_0.

The index. An index is an ordered pair of zeros with sum s_0, both of ordinate greater than $|\tau|$ in modulus. It is written with its member of positive ordinate as $b + \mathrm{i}(u + \tau)$

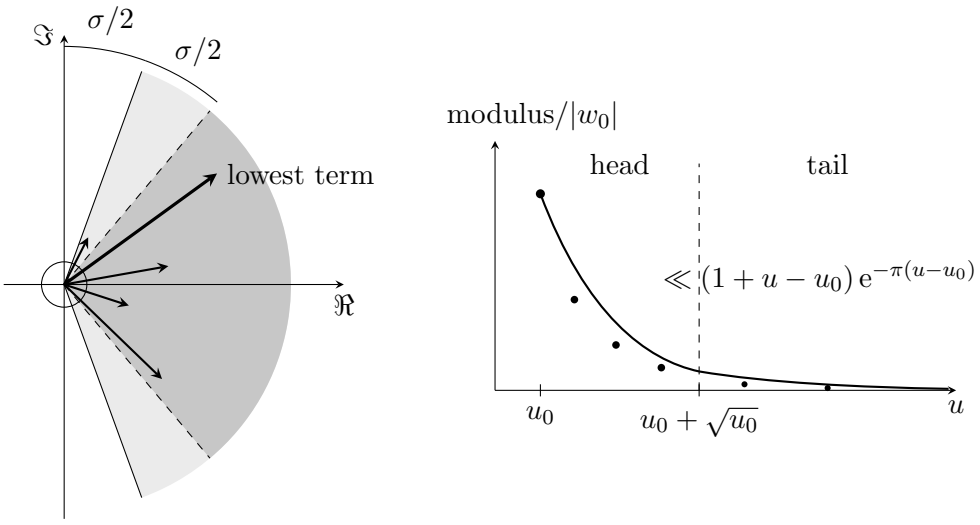

FIGURE 1. The head and the tail at a fixed ordinate τ, in the notation of Theorem 3.1: the coefficients d_i lie within an angle δ of the direction ω, the least height u_0 is attained at the index i_0, and w_0 is the Gamma product at that index. Left: the terms $d_i w_{b_i,c_i}(\tau, u_i)$ of the head after multiplication by $\overline{\omega}\,\mathrm{e}^{-\mathrm{i}\tau\log u_0}$; the long arrow is the rotated lowest term $d_{i_0}w_0$. The dark sector, of half-angle $\pi(B-1/2)+\delta$, contains the offsets $\pi(b-c)/2$ together with the coefficient phases. The light sector, of half-angle $\pi/2-\sigma/2$ with $\sigma=\pi(1-B)-\delta$, contains every term of the head once the drift and the Stirling error have used half of the margin σ. The other half of the margin gives each term a real part of at least $\sin(\sigma/2)$ times its modulus. The small circle bounds the whole tail; its radius is less than half of the guaranteed real part of the lowest term. Right: a sketch of the moduli of the Gamma products, without coefficients, against the height u. Pairs at one height may have different real parts, but the bound shown holds for all of them. Terms up to $u_0+\sqrt{u_0}$ are compared by phase, terms above it by size alone.

and its member of negative ordinate as $c-\mathrm{i}u$. The two zeros are distinct, the two orders of the pair are separate indices with the same data (b,c,u), and two different pairs may share a height. So the theorem is stated for an indexed family and not for a set of triples.

The strip and the common sum. The condition $b_i, c_i \in [1-B, B]$ says that the zeros lie in a strip symmetric about the critical line. For the zeros of ζ under H_∞ it holds with $B=\sup\beta<1$, and for critical-line zeros with $B=1/2$. The sum b_i+c_i equals $\Re s_0$ for every pair, so it is constant over the family. The proof uses this in one place. The moduli carry the power u^{b+c-1}, and because the exponent $b+c-1$ is common to the family, the ratio of two moduli is bounded in terms of the difference of the heights.

The coefficient and the sector. The coefficient d_i is the arithmetic factor that multiplies the Gamma product. For the ordinary Goldbach series it is $m(\rho)m(\rho')$, a positive integer. For the filtered prime-pair series T_0 it also includes the factors $1-\rho$ and $1-\rho'$, as explained below. For sums in which both summands are 1 modulo q, the coefficient carries the factor $\varphi(q)^{-2}$, where φ is Euler's totient. Once this factor is removed, the coefficient is a

product of two sums of multiplicities over the characters, one at each complex zero. It is again a positive integer, even when L-functions share a complex zero. In the application below the critical-line assumption gives the zeros equal real parts, so a shared ordinate identifies a shared complex zero. For other residue classes the multiplicities are combined with character values, and the coefficient can be non-real or zero. The sector condition says that the coefficients of all the pairs with the given sum lie within an angle δ of one common direction ω. The direction may depend on the point; only the spread matters. The bound $\delta < \pi(1-B)$ comes from the offsets. The strip allows an offset as large as $\pi(B-1/2)$, and $\pi(B-1/2)+\delta$ must stay below a right angle. When all the zeros of the family are on the critical line one may take $B = 1/2$; the offsets vanish, and any closed sector smaller than a half-plane is admitted.

The minimum. The condition asks for a lowest pair, at a height u_0 above the threshold, whose coefficient has modulus at least a. The head and the tail are both compared with $a|w_0|$, where w_0 is the Gamma product of the lowest pair. For multiplicities $a = 1$ is automatic. The least height is attained for a nonempty high family since below any represented height there are only finitely many unit bins, each of which contains finitely many indices. For coefficients with no known lower bound, a lowest term with a very small coefficient can be cancelled by a higher one (see the minimum-coefficient example below).

The bins. Every unit bin $[v, v+1)$ with real v must be finite. For $v \geq 0$ the condition bounds the total coefficient mass in the bin by $K(v+Y+4)^A$. For ordered pairs weighted by products of multiplicities, the bin mass is at most twice the product of two local multiplicity counts, the factor two allowing for both orders. The member of negative ordinate lies in the window $(-v-1, -v]$ and the other member in $[v+\tau, v+\tau+1)$, and each window holds $O(\log(v+Y+4))$ zeros with multiplicity. This gives squared-logarithm bins for multiplicity weights. For T_0 the filter increases each weight by a factor of order u^2. The same count of pairs still applies. The exponential decay in the tail controls the resulting polynomial bin mass.

None of the hypotheses requires a representation to be unique.

3.3. **The sector theorem.** Recall that $w_{b,c}(\tau, u) = \Gamma(b + \mathrm{i}(u+\tau))\Gamma(c - \mathrm{i}u)$.

Theorem 3.1 (Sector cone with polynomial bins)**.** *Assume Stirling on strips. Fix* $1/2 \leq B < 1$, $Y > 0$, $a > 0$, *a real* K, *a real* $A \geq 0$ *and* $0 \leq \delta < \pi(1-B)$. *There is* $U \geq Y+1$, *depending only on* B, Y, K, a, δ, A, *such that for each real* τ *with* $|\tau| \leq Y$ *the following holds. Let* (b_i, c_i, u_i, d_i) *be a family, over an arbitrary index set, of real numbers* b_i, c_i, u_i *and complex numbers* d_i, *with* $b_i, c_i \in [1-B, B]$ *and* $b_i + c_i$ *constant. Suppose that*

1. *the heights attain a minimum* $u_0 = u_{i_0} \geq U$, *and* $|d_{i_0}| \geq a$;
2. *for some fixed* ω *with* $|\omega| = 1$, *every coefficient is* $d_i = \omega|d_i|\mathrm{e}^{\mathrm{i}\phi_i}$ *with* $|\phi_i| \leq \delta$;
3. *every unit bin* $I(v) = \{i : v \leq u_i < v+1\}$, v *real, is finite, and for* $v \geq 0$

$$\sum_{i \in I(v)} |d_i| \leq K(v+Y+4)^A.$$

Then $\sum_i d_i w_{b_i,c_i}(\tau, u_i)$ *converges absolutely and is nonzero.*

Proof. We use Stirling on strips. For $b \in [1-B, B]$ and large $t > 0$, $\mathrm{Log}(b+\mathrm{i}t) = \log t + \mathrm{i}\pi/2 - \mathrm{i}b/t + O(t^{-2})$ uniformly. At $z = b+\mathrm{i}t$ the Stirling exponent $(z-\frac{1}{2})\,\mathrm{Log}\, z - z$

is therefore

$$\begin{aligned}&\big(b-\tfrac12+\mathrm{i}t\big)\Big(\log t+\frac{\mathrm{i}\pi}{2}-\frac{\mathrm{i}b}{t}\Big)-(b+\mathrm{i}t)+O(t^{-1})\\&\qquad=\big(b-\tfrac12\big)\log t-\frac{\pi t}{2}+\mathrm{i}\Big[t\log t-t+\frac{\pi(b-\frac12)}{2}\Big]+O(t^{-1}),\end{aligned}$$

where the product of $\mathrm{i}t$ and $-\mathrm{i}b/t$ gives b, which cancels the $-b$. Thus

$$\Gamma(b+\mathrm{i}t)=\sqrt{2\pi}\,t^{b-1/2}\mathrm{e}^{-\pi t/2}\exp\big(\mathrm{i}[t\log t-t+\pi(b-1/2)/2]\big)(1+O(t^{-1})).$$

Apply this with $t=u+\tau$ and use conjugation for the factor at $c-\mathrm{i}u$. Since $(u+\tau)\log(1+\tau/u)-\tau=O_Y(u^{-1})$ and $(1+\tau/u)^{b-1/2}=1+O_Y(u^{-1})$, we obtain

$$w_{b,c}(\tau,u)=2\pi\mathrm{e}^{-\pi u-\pi\tau/2}u^{b+c-1}\exp\big(\mathrm{i}[\tau\log u+\pi(b-c)/2]\big)(1+O_{B,Y}(u^{-1})). \tag{1}$$

This relative estimate is uniform for $|\tau|\le Y$ and sufficiently large u.

Write $\sigma=\pi(1-B)-\delta>0$. Rotate by $\overline{\omega}\mathrm{e}^{-\mathrm{i}\tau\log u_0}$. The range $u_0\le u_i\le u_0+\sqrt{u_0}$ meets finitely many unit bins, each of them finite, so the head is a finite sum. In this range the drift satisfies $|\tau\log(u_i/u_0)|\le Y/\sqrt{u_0}$. The offset is bounded by $\pi(B-1/2)$, and the coefficient adds at most δ. The relative error in (1) contributes an angle tending uniformly to zero, since $|\arg(1+e)|\le 2|e|$ for $|e|\le 1/2$. Choose $U_c=U_c(B,Y,\delta)\ge Y+1$ so that for $u_0\ge U_c$ the two errors total at most $\sigma/2$. The rotated real part of each term in this range is therefore at least $\lambda|d_i w_{b_i,c_i}(\tau,u_i)|$ with $\lambda=\sin(\sigma/2)$. The term i_0 is among them, so the sum of these real parts is at least $\lambda|d_{i_0}w_0|\ge\lambda a|w_0|$, where $w_0=w_{b_{i_0},c_{i_0}}(\tau,u_0)$.

For $u\ge u_0$ the leading terms in (1) of $w_{b,c}(\tau,u)$ and of w_0 have the ratio $(u/u_0)^{b+c-1}\mathrm{e}^{-\pi(u-u_0)}$ in modulus, with the same exponent $b+c-1\le 1$ in both. Since $u_0\ge 1$, $(u/u_0)^{b+c-1}\le 1+u-u_0$. Choose $U_d=U_d(B,Y)\ge 1$ so that for $u_0\ge U_d$ the relative errors in the two products have modulus at most $1/2$. Then, with a constant $D>0$ depending only on B,Y,

$$|w_{b,c}(\tau,u)/w_0|\le D(1+u-u_0)\mathrm{e}^{-\pi(u-u_0)}.$$

The ratio is defined because Gamma has no zeros and the real parts are positive.

Write $K_+=\max\{K,0\}$. Partition the indices into the finite bins based at u_0+n, $n=0,1,\dots$. In the nth bin the kernel ratio is at most $D(n+2)\mathrm{e}^{-\pi n}$. We have $u_0+n+Y+4\le(u_0+Y+4)(n+2)$. Thus the sum of the absolute contributions divided by $|w_0|$ is bounded by

$$K_+D(u_0+Y+4)^A\sum_{n\ge0}(n+2)^{A+1}\mathrm{e}^{-\pi n},$$

which converges by the ratio test. If a bin meets the tail $u>u_0+\sqrt{u_0}$ then $n+1>\sqrt{u_0}$ and $u_0<(n+1)^2$. Consequently

$$u_0+n+Y+4\le(Y+6)(n+2)^2,$$

so the mass in that bin is at most $K_+(Y+6)^A(n+2)^{2A}$. Choose an integer $N\ge0$ so large that

$$K_+D(Y+6)^A\sum_{n\ge N}(n+2)^{2A+1}\mathrm{e}^{-\pi n}<\lambda a/2.$$

Again the series converges by the ratio test. Set $U=\max\{U_c,U_d,(N+1)^2\}$. If $u_0\ge U$, every bin meeting the tail has $n+1>\sqrt{u_0}\ge N+1$, so its index is at least N. The tail

has modulus less than $\lambda a|w_0|/2$, and the full rotated sum has positive real part by the head bound. □

Remark (Nonvacuity of the bin bound). The bin at $v = u_0$ has mass at least a, so when $A = 0$ the hypotheses require $K \geq a$, and when $A > 0$ they require $K > 0$. Conversely these conditions suffice. Take a single term with $b = c = 1/2$ and positive coefficient a at an arbitrarily large height u_0. For $A = 0$ every bin containing the term has mass $a \leq K$. For $A > 0$ every such bin has base $v > u_0 - 1$, so its bound is at least $K(u_0 + Y + 3)^A$, which exceeds a for all sufficiently large u_0.

Only the choice of N uses K, a or A. The threshold is fixed before τ and the family, so a single threshold works at every point of a bounded window. With that threshold, only finitely many points of the window have a low representation for the zero families described above.

Corollary 3.2 (Squared-logarithm bins)**.** *Assume Stirling on strips. Fix $1/2 \leq B < 1$, $Y > 0$, $a > 0$, a real K and $0 \leq \delta < \pi(1 - B)$. There is $U \geq Y + 1$, depending only on B, Y, K, a, δ, such that for every real τ with $|\tau| \leq Y$, the conclusion of Theorem 3.1 holds with the same strip, common sum, minimum, sector and finite-bin hypotheses and with*

$$\sum_{i \in I(v)} |d_i| \leq K \log^2(v + Y + 4) \qquad (v \geq 0)$$

as the bin bound.

Proof. For $x \geq 1$, $0 \leq \log x = 2 \log \sqrt{x} \leq 2(\sqrt{x} - 1) \leq 2\sqrt{x}$, so $\log^2 x \leq 4x$. Apply Theorem 3.1 with $A = 1$ and bin constant $4 \max\{K, 0\}$. □

3.4. **Integer weights.** A positive integer has phase zero and modulus at least one. For nonnegative integer coefficients that are positive at the least height, the sector condition therefore holds with $\omega = 1$ and $\delta = 0$, and the lower bound holds with $a = 1$.

Lemma 3.3 (Integer weights)**.** *Assume Stirling on strips. Fix $1/2 \leq B < 1$, $Y > 0$ and a real K. There is $U \geq Y + 1$, depending only on B, Y, K, such that for each real τ with $|\tau| \leq Y$ the following holds. Let (b_i, c_i, u_i) be as in Theorem 3.1, with nonnegative integer coefficients N_i. Suppose that the heights attain a minimum $u_0 = u_{i_0} \geq U$ with $N_{i_0} \geq 1$, that every unit bin $I(v)$ is finite, and that $\sum_{i \in I(v)} N_i \leq K \log^2(v + Y + 4)$ for $v \geq 0$. Then $\sum_i N_i w_{b_i,c_i}(\tau, u_i)$ converges absolutely and is nonzero.*

Proof. Take $a = 1$, $\omega = 1$ and $\delta = 0$ in Corollary 3.2. □

This is the phase cone of [22, Theorem 5.4]. In the Goldbach paper the coefficients are the products $m(\rho)m(\rho')$ of zeta multiplicities. The cone is applied there at interior collisions and, when only finitely many zeros lie off the critical line, to the critical-line pair sum after the off-line contributions have been removed [22, Theorem A and §§5–6].

3.5. **Limits of the hypotheses.** The sharpness proof and the cancellation and minimum-coefficient examples in this subsection are informal and not formalized in Lean. They lie outside the three numbered Lean statements.

Remark (The angle at the endpoint). For no $B \in [1/2, 1)$ can the condition $\delta < \pi(1 - B)$ be relaxed to $\delta \leq \pi(1 - B)$.

At $B = 1/2$ take two indices at the same height u, with $b = c = 1/2$ and coefficients $+\mathrm{i}$ and $-\mathrm{i}$. With $\omega = 1$ their phases are $\pm\pi/2$, so the sector condition holds with $\delta = \pi/2 = \pi(1-B)$. The minimum coefficient has modulus one, and every bin has mass at most $2 \le \log^2(v+5)$, so the minimum and bin conditions hold with $a = K = Y = 1$. The two Gamma products are equal, and the sum is $\mathrm{i}w - \mathrm{i}w = 0$ for every u and every $|\tau| \le 1$.

For general B put $B' = 1 - B$, $\delta = \pi B'$, $\theta = \pi(B - \frac{1}{2})$, $\omega = 1$ and $a = K = 1$. Fix $Y > 0$ and $\tau \in [-Y, 0)$, and for $u \ge Y + 1$ take the two indices

$$(b_0, c_0, u_0, d_0) = (B, B', u, \mathrm{e}^{\mathrm{i}\delta}), \qquad (b_1, c_1, u_1, d_1) = (B', B, u+1, -d_0 w_0/w_1),$$

where $w_0 = w_{B,B'}(\tau, u)$ and $w_1 = w_{B',B}(\tau, u+1)$. Then $d_0 w_0 + d_1 w_1 = 0$ by construction. The real parts lie in $[1-B, B]$ and $b_i + c_i = 1$, the minimum is attained at the index 0 with $|d_0| = 1$, and d_0 lies on the edge of the closed sector $|\arg d| \le \delta$. We show that for large u the coefficient d_1 lies in the same closed sector and the bin condition holds.

Let $R(t) = \Gamma(B + \mathrm{i}t)/\Gamma(B' + \mathrm{i}t)$. For $t > 0$ the reflection formula [20, §5.5] gives

$$R(t) = \frac{\pi}{\sin(\pi(B+\mathrm{i}t))\,|\Gamma(B'+\mathrm{i}t)|^2}, \qquad \sin(\pi(B+\mathrm{i}t)) = \tfrac{1}{2}\mathrm{e}^{\pi t - \mathrm{i}\theta}\bigl(1 - \mathrm{e}^{2\pi\mathrm{i}B - 2\pi t}\bigr),$$

so $R(t)$ has the argument $\theta - A(t)$ with $A(t) = \arg(1 - \mathrm{e}^{2\pi\mathrm{i}B - 2\pi t}) = O(\mathrm{e}^{-2\pi t})$. Since $w_{B,B'}(\tau,u)/w_{B',B}(\tau,u) = R(u+\tau)/\overline{R(u)}$, this ratio has the argument $2\theta - A(u+\tau) - A(u)$. The function $x \mapsto w_{B',B}(\tau, x)$ has no zeros, and a continuous choice of its argument has the derivative

$$D(x) = \Re\frac{\Gamma'}{\Gamma}(B' + \mathrm{i}(x+\tau)) - \Re\frac{\Gamma'}{\Gamma}(B + \mathrm{i}x) = \frac{\tau}{x} + O_Y(x^{-2}),$$

by the expansion $\Gamma'/\Gamma(z) = \log z - 1/(2z) + O(|z|^{-2})$ [20, §5.11]. Hence w_0/w_1 has the argument $2\theta + \varepsilon$, where

$$\varepsilon = -A(u+\tau) - A(u) - \int_u^{u+1} D(x)\,dx = \frac{|\tau|}{u} + O_Y(u^{-2}).$$

Now $2\theta + \pi = 2\pi - 2\delta$, so $d_1 = -d_0 w_0/w_1$ has the argument $\delta + (-2\delta + \varepsilon) = -\delta + \varepsilon$ modulo 2π. Because $\tau < 0$ we have $0 < \varepsilon < 2\delta$ for all sufficiently large u, and then d_1 lies in the closed sector.

The heights differ by one, so each unit bin contains at most one index. A bin that contains the index 0 has mass $1 \le \log^2(v + Y + 4)$. A bin that contains the index 1 has $v > u$ and mass $|d_1| = |w_0|/|w_1|$, which tends to e^{π} by (1), since $b_i + c_i = 1$; this is less than $\log^2(u + Y + 4)$ for large u. Thus both examples meet the squared-logarithm bound and hence the polynomial bound with $A = 1$ and bin constant $4K$. They give zero sums at the endpoint with arbitrarily large least height.

Remark (Abstract cancellations). Two abstract spectra on the critical line show how an atom can vanish. The first models the classes $1, 2$ modulo 5. Let χ be the character modulo 5 with $\chi(2) = \mathrm{i}$, and fix $u > 2$. The spectrum has four ordinates: $u + 1/2$ and u, labelled by χ, and $-u$ and $-u - 1/2$, labelled by $\overline{\chi}$, so that conjugate ordinates carry conjugate labels. An ordered pair whose members have labels (χ_1, χ_2) gets the coefficient $\overline{\chi_1(1)\chi_2(2)}$. The sum $\tau = 1/2$ has exactly two ordered representations, $(u + 1/2, -u)$ and $(-u, u + 1/2)$. Their coefficients are i and $-\mathrm{i}$, so no sector of half-angle less than $\pi/2$ contains both. Their Gamma products are equal and the two terms cancel.

The second has the positive ordinates $u, u+1, v, v+1$ with $v = u + 1/2$ and u large, and their negatives. Each ordinate t carries a multiplier, and the coefficient of a pair is the product of the multipliers of its members. The sum $\tau = 1$ is represented by the two unordered pairs $\{u+1, -u\}$ and $\{v+1, -v\}$ and by no other: a sum of two ordinates of the same sign is not 1, and among the differences of two of the positive ordinates only $(u+1) - u$ and $(v+1) - v$ equal 1. Write $w_u = w_{1/2,1/2}(1, u)$ and $w_v = w_{1/2,1/2}(1, v)$ for their Gamma products. The model assigns the multiplier $-w_u/w_v$ at $v+1$, its conjugate at $-v-1$, and the multiplier one elsewhere. Each unordered pair occurs in two orders, and the atom is $2(w_u - (w_u/w_v)w_v) = 0$. Here $1 \le |w_u/w_v| \le \mathrm{e}^{\pi/2}$, and (1) gives $w_u/w_v \to \mathrm{e}^{\pi/2}$, so the two coefficients have bounded moduli. With $B = B' = 1/2$ and $\tau = 1$, the phase derivative above is $1/x + O(x^{-2})$. Integrating over $[u, u+1/2]$ gives the angular separation $\pi - 1/(2u) + O(u^{-2})$ between 1 and $-w_u/w_v$. For each sufficiently large u, a sector of half-angle less than $\pi/2$ contains both coefficients, but the smallest such half-angle tends to $\pi/2$, so no fixed admissible half-angle works along the family.

In both models the coefficients obey two-sided size bounds and the conjugation symmetry of actual zeros. So size bounds and conjugation do not replace the sector condition.

Remark (The coefficient at the least height). In a two-term model take $B = 1/2$, $\tau = 1$ and $w(u) = w_{1/2,1/2}(1, u)$. By (1), $w(u)$ has a continuous phase $\theta(u) = \log u + O(1/u)$ for large u. Since this phase tends to infinity, for each sufficiently large u there is $v > u$ with $\theta(v) - \theta(u) = \pi$. This equation gives $v/u \to \mathrm{e}^{\pi}$, so $v > u + \sqrt{u}$ for large u. Give the term at v coefficient one and the term at u coefficient $|w(v)/w(u)|$. The phases are opposite, so the terms cancel exactly. The reflection formula gives $|w(u)| = \pi/\sqrt{\cosh(\pi u)\cosh(\pi(u+1))}$, so the coefficient at the least height u is exponentially small in $v - u$ and tends to zero. Both coefficients are positive and at most one, and each unit bin contains at most one term. The sector half-angle is zero and the bin bound is constant. Without the lower bound at the minimum, cancellation thus occurs at arbitrarily large minimum heights.

3.6. **Two applications of the hypotheses.** We check the hypotheses of the cone for the two series treated in the companion papers: the filtered prime-pair series T_0 of Section 2.7 and the Goldbach sums whose summands are 1 modulo q. The selected pole $1 + \rho/p$ in the proof for T pairs the pole at 1 of Q with a scaled zero. The pole at 1 supplies the principal term x^{-1} of F_p. The filter in T_0 removes that term. As in the definition of T_0 in Section 2.7, the filter weights the summand of T by

$$1 - s + s(s+1)\frac{pp'}{(p+p')^2}.$$

The operator $1 + x\,d/dx$ of Section 2.7 kills the principal term x^{-1} of F_p and sends $x^{-\rho}$ to $(1-\rho)x^{-\rho}$. The pair weight therefore includes $(1-\rho)(1-\rho')$, in addition to the multiplicities. For this filtered example, choose a represented sum $s_0 = \rho_1 + \rho_2$ of unscaled zeros with only high representations, as in Section 2. The full family consists of all ordered pairs of unscaled zeros with sum s_0. For $\rho = b + \mathrm{i}(u + \tau)$ and $\rho' = c - \mathrm{i}u$ that factor is $u^2 + O_Y(u)$ uniformly in the strip. At large u it has modulus at least one and lies in the sector about the positive real axis of half-angle $\pi(1-B)/2$. The product of the counts in the two ordinate windows bounds the coefficient mass by a constant times $(v + Y + 4)^2 \log^2(v + Y + 4)$, so $A = 4$ in Theorem 3.1 suffices. The family uses $B = \sup_\rho \Re\rho$, or $B = 1/2$ when only critical-line pairs are kept. The filter also sends

$x^{-1/2}$ to $x^{-1/2}/2$ and multiplies $x^{-\rho/k}$ by $1-\rho/k$. At s_0 a residue argument for T_0 must account for all other contributions, including products involving the surviving $x^{-1/2}$ term or scaled zero terms; the companion paper on T_0 does this [25].

For Goldbach sums where both summands are 1 modulo a fixed $q \geq 2$, the indicator of the class is $\varphi(q)^{-1}\sum_{\chi \bmod q}\chi(n)$, the sum over Dirichlet characters modulo q. Each character has $\chi(1) = 1$. When the nontrivial zeros of all the inducing primitive L-functions lie on the critical line, a shared ordinate gives a shared complex zero, and the multiplicities at that zero add with the same sign. Each inducing function of conductor dividing q has total multiplicity $O_q(\log(|t|+3))$ in an ordinate window $[t, t+1)$, by the zero counting estimate of [1, Corollary 1.2] for conductors greater than one and (Z2) for conductor one. Removing the common factor $\varphi(q)^{-2}$ leaves nonnegative integer pair weights and the squared-logarithm bin bound from these local multiplicity counts. Lemma 3.3 applies with $B = 1/2$, including coincidences between pairs drawn from different L-functions.

4. Open question

4.1. **Liouville coefficients.** The Liouville function is $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ counts prime factors with multiplicity. Its additive convolution has Dirichlet series $\sum_{m,n\geq 1}\lambda(m)\lambda(n)(m+n)^{-s}$. Cantarini, Gambini and Zaccagnini [7, Corollary 2] state that under RH and the simplicity of the zeros the Dirichlet series of the Liouville convolution continues to $\Re s > 1$, and that if in addition the imaginary parts of the zeros are linearly independent over $\mathbb{Q}$, the line $\Re s = 1$ is its natural boundary. Independence can be meant only for the positive ordinates, since the ordinates come in pairs $\pm\gamma$. Their expansion over pairs of zeros, Theorem 17, is stated under RH and their hypothesis SZ (Conjecture 7), the bound $\sum_{0<\gamma<T}|\zeta'(\rho)|^{-1} \ll T(\log T)^{1/2}$, which implies simplicity.

Under RH and SZ the coefficient attached to a zero ρ is $\Gamma(\rho)\zeta(2\rho)/\zeta'(\rho)$, from the residue of $\zeta(2s)/\zeta(s)$ at a simple zero, and no uniform sector bound for the factors $\zeta(2\rho)/\zeta'(\rho)$ is known on a dense set of high nonzero sums.

One can ask instead for a gap above the least height. Take an ordinate sum $\tau \neq 0$ that is not itself the ordinate of a zero and all of whose pairs are high. The zeros being simple, the height determines the pair, so the pair of least height u_0 is unique as an unordered pair. Suppose that every other pair has height greater than $u_0 + \sqrt{u_0}$. The Gamma products of the other pairs are then smaller than that of the lowest by a factor $\mathrm{e}^{-\pi\sqrt{u_0}}$ times a power of u_0, and this outweighs polynomial bounds on the coefficients, provided the bounds are two-sided: an upper bound for the coefficients of the higher pairs and a lower bound for the coefficient of the lowest.

Under RH and SZ such bounds are available. Polynomial upper bounds for $|\zeta(2\rho)/\zeta'(\rho)|$ and for its inverse follow from the individual bound contained in SZ, the bounds for ζ and $1/\zeta$ in [7, Lemma 8], and Cauchy's estimate for ζ' on a fixed strip. The local zero count then bounds the coefficient mass in each unit bin polynomially.

The open question is whether the sums τ with the stated gap are dense.

The minimum-coefficient example in Section 3 shows why an upper bound alone cannot replace a lower bound at the least height. A proof along these lines would also have to control the terms of the expansion that come from single zeros and from the principal term. For the normal-limit test, use the heat-sum formula following Theorem 17 in [7, §4]. Its pair terms have the products of the two coefficients given above. The principal and single-zero terms have candidate ordinates 0 and γ, which were excluded in the

choice of τ, and the remainder is $O_\varepsilon(x^{-1/2-\varepsilon}+1)$. It is the whole error term of the heat sum of the convolution. With $\varepsilon < 1/2$ its Mellin integral over $(0,1)$ at $1+\mathrm{i}\tau+\eta$ stays bounded as $\eta \to 0$, so after multiplication by η it contributes nothing to the normal limit. What remains to be shown is the density of the sums τ with the stated gap and the nonvanishing of the atoms at them.

5. Formal verification

Theorem 3.1, Corollary 3.2 and Lemma 3.3 have been checked in Lean 4 with Mathlib [18], at version 4.34.0 of both, as extensions of the Goldbach development [22, Section 10]. Each formal implication takes the stated hypotheses and Stirling on strips as premises. Table 1 identifies the classical premise. Appendix A maps the numbered results to their declarations.

The development is release `v1.0` of `https://github.com/lpkralv/phase-cone`, Apache-2.0, commit `82f618bc4a9491668c43a8dcd948ea933aa6aa75`, built with the toolchain `leanprover/lean4:v4.34.0`. The folder `phase-cone` holds this paper's check file and a README that maps each numbered result to its declarations. One build from a clean clone of the release checks the Lean results of all three papers that the repository accompanies.

5.1. **Formal premise.** The classical premise is a proposition about Mathlib's Gamma function. Its full name is `GoldbachBoundaries.Phi2HInf.StirlingStrip`. The estimates and zero hypotheses used to explain the applications are separate from the sector declarations.

Paper	Formal hypothesis	Used in
Stirling on strips	`Phi2HInf.StirlingStrip`	Theorem 3.1, Corollary 3.2 and Lemma 3.3

Table 1. The classical premise of the cone results. Appendix A prints the definition from [22, Appendix A].

5.2. **Differences in form.** The formal natural-boundary predicate uses eventual agreement in the codiscrete filter on the overlap, which is the meromorphic agreement of Section 2. The cone results do not use the predicate; Appendix A prints it for the companion papers [25, 24]. Lean functions are total, so a formula can have a value at a pole or at a removable singularity. Meromorphic agreement concerns the functions away from a discrete set. A Lean infinite sum has value zero when the family is not summable. The sector declaration asserts summability explicitly as well as nonvanishing.

5.3. **Axioms.** The command `#print axioms` on the sector declarations lists the three standard axioms of Lean's logic, `propext`, `Classical.choice` and `Quot.sound`. The declarations contain no dependency on `sorry`.

5.4. **Scope of the formal statements.** The comparison of T and Φ_2, the sharpness argument, the cancellation and minimum-coefficient examples and the Liouville question lie outside the formal statements of the numbered results. The proofs of those results use none of them. Lean proves the pointwise Gamma identity $|\Gamma(1/2+\mathrm{i}t)|^2 = \pi/\cosh(\pi t)$ used in the examples. Section 2 describes the boundary arguments in general terms.

The development proves the instances used in the applications of the companion papers [25, 24], with supporting lemmas such as `isNaturalBoundary_of_eventuallyEq_add`.

Acknowledgement and statement on the use of AI

The author is a computer scientist, not a mathematician. He originated the questions treated here, directed an AI research agent through a sequence of questions on natural boundaries of Goldbach series over primes and prime powers, and chose at each step the direction to pursue. He designed the verification methodology: independent adversarial reviews, re-derivation and small-case checks of every result before it was recorded as proved, and a formalization in Lean (Section 5). The mathematics is largely the work of AI models (Anthropic Claude, OpenAI GPT) under his direction. Multiple models reviewed the argument. An OpenAI GPT model drafted the manuscript and an Anthropic Claude model revised it. AI agents wrote the Lean development. The author did not write or check the Lean development line by line; he relies on the Lean kernel and the axiom checks reported in Section 5. No human mathematician has checked the proof line by line. There is no co-reader: queries about the proof will be answered by the author, with the same AI assistance. No AI system is an author. The author takes full responsibility for the content.

The author thanks Anna DeConti, formerly of Osbourn Park High School, who taught him calculus and encouraged his explorations in mathematics.

Appendix A. The Lean declarations

The declaration list gives the premises of the three numbered results. The printed statements omit docstrings and proofs.

A.1. **Declarations and premises.** Namespace prefixes in this appendix are relative to `GoldbachBoundaries`. The premise names are those of Table 1.

The cone.

Theorem 3.1.: `PhaseCone.sum_bKer_ne_zero_sector_poly` takes `StirlingStrip` and gives absolute convergence and nonvanishing with a bin bound of degree $A \geq 0$ for any real K.

Corollary 3.2.: `PhaseCone.sum_bKer_ne_zero_sector_of_poly` takes the same premises except for the change to squared-logarithm bins and the omission of A.

Lemma 3.3.: `Phi2HInf.sum_bKer_ne_zero_hinf` is the original integer-weight declaration. The declaration `PhaseCone.sum_bKer_ne_zero_hinf_of_sector` has the same statement and invokes the squared-logarithm theorem `PhaseCone.sum_bKer_ne_zero_sector` in `PhaseCone/Sector.lean`. The corollary above uses the separate declaration `PhaseCone.sum_bKer_ne_zero_sector_of_poly`. Both integer-weight declarations take `StirlingStrip`.

A.2. **The main statements.** In the sector statement, `bKerH b c τ u` is the Gamma product $w_{b,c}(\tau, u)$; its definition is printed below, where `Gamma` is `Complex.Gamma`. The symbol `I` denotes the imaginary unit, and `Icc` and `Ico` denote closed and half-open intervals. The norm bars denote the modulus. The superscript f marks a finite sum, and the prime marks the sum of a family. As in Section 5, the sum of a nonsummable family is zero by convention. The bound variable T_0 in `StirlingStrip` is a height.

The boundary predicate and `NoMeromorphicExtensionAt` are printed for the companion papers, which use them; the cone results do not. In the boundary predicate, `ball` is an open disc. For a function F, a domain U and a line L, `IsNaturalBoundary F U L` says that no meromorphic function on a disc centred on L agrees with F on the part of the disc in U off a relatively closed discrete set there.

Stirling on strips and the boundary predicate (as in [22, Appendix A]).

```
def StirlingStrip : Prop :=
  ∀ x₁ x₂ : ℝ, x₁ ≤ x₂ → ∃ C T₀ : ℝ, 0 ≤ C ∧
    ∀ z : ℂ, x₁ ≤ z.re → z.re ≤ x₂ → T₀ ≤ |z.im| →
      ‖Complex.Gamma z -
          Complex.exp ((z - 1 / 2) * Complex.log z - z +
            (1 / 2) * Complex.log (2 * (Real.pi : ℂ)))‖ ≤
        C / |z.im| *
          ‖Complex.exp ((z - 1 / 2) * Complex.log z - z +
            (1 / 2) * Complex.log (2 * (Real.pi : ℂ)))‖

def NoMeromorphicExtensionAt (F : ℂ → ℂ) (U : Set ℂ) (z₀ : ℂ) : Prop :=
  ∀ r : ℝ, 0 < r → ¬ ∃ f : ℂ → ℂ, MeromorphicOn f (ball z₀ r) ∧
    f =ᶠ[codiscreteWithin (ball z₀ r ∩ U)] F

def IsNaturalBoundary (F : ℂ → ℂ) (U L : Set ℂ) : Prop :=
  ∀ z₀ ∈ L, NoMeromorphicExtensionAt F U z₀
```

The Gamma product (namespace `Phi2HInf`).

```
def bKerH (b c τ u : ℝ) : ℂ :=
  Gamma ((b : ℂ) + (u + τ : ℝ) * I) * Gamma ((c : ℂ) - (u : ℂ) * I)
```

The sector cone (namespace `PhaseCone`).

```
theorem sum_bKer_ne_zero_sector_poly (hStir : StirlingStrip)
    {B Y K a δ A : ℝ}
    (hB : 1 / 2 ≤ B) (hB1 : B < 1) (hY : 0 < Y) (ha : 0 < a)
    (hA : 0 ≤ A) (hδ0 : 0 ≤ δ) (hδ : δ < Real.pi * (1 - B)) :
    ∃ U : ℝ, Y + 1 ≤ U ∧ ∀ τ : ℝ, |τ| ≤ Y →
      ∀ {ι : Type} (b c u : ι → ℝ) (d : ι → ℂ) (i₀ : ι),
      (∀ i, b i ∈ Icc (1 - B) B ∧ c i ∈ Icc (1 - B) B) →
      (∀ i, b i + c i = b i₀ + c i₀) → U ≤ u i₀ → (∀ i, u i₀ ≤ u i) →
      a ≤ ‖d i₀‖ →
      (∃ ω : ℂ, ‖ω‖ = 1 ∧ ∀ i, ∃ φ : ℝ, |φ| ≤ δ ∧
        d i = ω * (‖d i‖ : ℂ) * exp ((φ : ℂ) * I)) →
      (∀ v : ℝ, {i | u i ∈ Ico v (v + 1)}.Finite) →
      (∀ v : ℝ, 0 ≤ v →
        (∑ᶠ i ∈ {i | u i ∈ Ico v (v + 1)}, ‖d i‖) ≤
          K * (v + Y + 4) ^ A) →
      Summable (fun i => ‖d i * bKerH (b i) (c i) τ (u i)‖) ∧
        ∑' i, d i * bKerH (b i) (c i) τ (u i) ≠ 0 :=
```

References

[1] M. A. Bennett, G. Martin, K. O'Bryant and A. Rechnitzer, Counting zeros of Dirichlet L-functions, Math. Comp. 90 (2021), no. 329, 1455–1482, doi:10.1090/mcom/3599.

[2] G. Bhowmik and K. Halupczok, Asymptotics of Goldbach representations, in: Various Aspects of Multiple Zeta Functions (in honor of Professor Kohji Matsumoto's 60th birthday), Adv. Stud. Pure Math. 84, Math. Soc. Japan, 2020, pp. 1–21; doi:10.2969/aspm/08410001; arXiv:1809.06920. Section and display numbers refer to arXiv:1809.06920v2.

[3] G. Bhowmik and K. Matsumoto, Analytic continuation of random Dirichlet series, Proc. Steklov Inst. Math. 282, suppl. 1 (2013), S67–S72; doi:10.1134/S0081543813070043; also Sovrem. Probl. Mat. 17 (2013), 76–81, doi:10.4213/spm44; preprint arXiv:1105.0361 (as "On random multiple Dirichlet series"). Proposition numbers refer to the Sovrem. Probl. Mat. version.

[4] G. Bhowmik, O. Ramaré and J.-C. Schlage-Puchta, Tauberian oscillation theorems and the distribution of Goldbach numbers, J. Théor. Nombres Bordeaux 28 (2016), no. 2, 291–299; doi:10.5802/jtnb.940.

[5] G. Bhowmik and J.-C. Schlage-Puchta, Meromorphic continuation of the Goldbach generating function, Funct. Approx. Comment. Math. 45 (2011), 43–53; doi:10.7169/facm/1317045230; arXiv:1002.4806. Theorem numbers are those of arXiv version 1.

[6] G. Bhowmik and J.-C. Schlage-Puchta, Natural boundaries of Dirichlet series, Funct. Approx. Comment. Math. 37 (2007), no. 1, 17–29; doi:10.7169/facm/1229618738; arXiv:0704.3370.

[7] M. Cantarini, A. Gambini and A. Zaccagnini, On the discrete convolution of the Liouville and Möbius functions, arXiv:2603.10241 (2026).

[8] S. Egami and K. Matsumoto, Convolutions of the von Mangoldt function and related Dirichlet series, in: Number Theory: Sailing on the Sea of Number Theory, Proc. 4th China–Japan Seminar (Weihai), S. Kanemitsu and J.-Y. Liu, eds., Ser. Number Theory Appl. 2, World Scientific, 2007, pp. 1–23; `https://doi.org/10.1142/9789812770134_0001`. Page references are to the authors' version at `https://www.math.nagoya-u.ac.jp/~kohjimat/weihai07.pdf`, numbered 1–23, matching the indexed published page range.

[9] S. Egami and K. Matsumoto, A function-field analogue of the Goldbach counting function and the associated Dirichlet series, Bull. Korean Math. Soc. 61 (2024), no. 1, 135–145; doi:10.4134/BKMS.b230063; arXiv:2302.02549. Theorem and conjecture numbers refer to arXiv:2302.02549v1.

[10] C.-E. Fröberg, On the prime zeta function, BIT 8 (1968), no. 3, 187–202; doi:10.1007/BF01933420.

[11] A. Fujii, An additive theory of the zeros of the Riemann zeta function, Comment. Math. Univ. St. Pauli 45 (1996), no. 1, 49–116; doi:10.14992/00009900.

[12] G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, Acta Math. 41 (1916), 119–196; doi:10.1007/BF02422942.

[13] E. Landau and A. Walfisz, Über die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen definierter Funktionen, Rend. Circ. Mat. Palermo 44 (1920), 82–86; doi:10.1007/BF03014596.

[14] A. Languasco, Applications of some exponential sums on prime powers: a survey, Riv. Mat. Univ. Parma 7 (2016), no. 1, 19–37; arXiv:1606.00860. Lemma, theorem and section numbers refer to arXiv:1606.00860v1.

[15] A. Languasco and A. Zaccagnini, A Cesàro average of Goldbach numbers, Forum Math. 27 (2015), no. 4, 1945–1960; doi:10.1515/forum-2012-0100; arXiv:1206.0251. Lemma and equation numbers are to the journal version unless the arXiv version is named.

[16] Yu. V. Linnik, A new proof of the Goldbach–Vinogradow theorem (in Russian), Mat. Sbornik N.S. 19(61) (1946), 3–8.

[17] J. E. Littlewood, Two notes on the Riemann zeta-function, Proc. Cambridge Philos. Soc. 22 (1924), 234–242; doi:10.1017/S0305004100014158.

[18] The mathlib Community, The Lean mathematical library, in: Proc. 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; doi:10.1145/3372885.3373824.

[19] K. Matsumoto, A survey on the theory of multiple Dirichlet series with arithmetical coefficients as numerators, Lith. Math. J. 61 (2021), no. 3, 391–400; doi:10.1007/s10986-021-09534-9; arXiv:2101.07449. Theorem numbers refer to arXiv:2101.07449v1.

[20] NIST Digital Library of Mathematical Functions, `https://dlmf.nist.gov/`, Release 1.2.8 of 2026-09-15, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl and M. A. McClain, eds.

[21] F. Rüppel, Convolutions of the von Mangoldt function over residue classes, Šiauliai Math. Semin. 7(15) (2012), 135–156.

[22] L. P. Slothouber, The Egami–Matsumoto conjecture: a natural boundary of the Goldbach generating series, hexagon:2611.00834v2 (23 November 2026), `https://hexagonmath.org/pdf/2611.00834v2`; doi:10.5281/zenodo.24483697.

[23] L. P. Slothouber, Natural boundaries of prime-pair Dirichlet series, hexagon:2611.00971v1 (24 November 2026), https://hexagonmath.org/pdf/2611.00971v1; doi:10.5281/zenodo.24490140.
[24] L. P. Slothouber, Natural boundaries and real-point limits for class-one Goldbach series, hexagon:2611.01114v1 (26 November 2026), https://hexagonmath.org/pdf/2611.01114v1; doi:10.5281/zenodo.24551268.
[25] L. P. Slothouber, The natural boundary of the filtered prime-pair Dirichlet series, hexagon:2611.01112v1 (26 November 2026), https://hexagonmath.org/pdf/2611.01112v1; doi:10.5281/zenodo.24551160.
[26] Y. Tanigawa and W. Zhai, Dirichlet series associated with polynomials and applications, J. Number Theory 122 (2007), 466–518; doi:10.1016/j.jnt.2006.05.004.
[27] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.
[28] A. Zaharescu and M. Zaki, Natural boundaries of a family of Dirichlet series, Rocky Mountain J. Math. 43 (2013), no. 2, 661–676; doi:10.1216/RMJ-2013-43-2-661.