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THE DUAL SMALE MEAN VALUE CONJECTURE FOR DEGREE TEN: BREAKING THE INTERIOR-ZERO BARRIER $P(\zeta_1) = 0$ VIA SECOND-SMALLEST INVERSION GWS, LEMNISCATE SPREAD, AND LEAN 4 / OUTWARD-ROUNDED CERTIFICATION

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ABSTRACT. Let $P(z) = z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C}[z]$ be a complex polynomial of degree $n \geq 2$ with $a_n \neq 0$. The Dual Smale Mean Value Conjecture (Dubinin–Sugawa and Ng, 2009) asserts that

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{n},$$

with equality if and only if $P(z) = \frac{1-(1-cz)^n}{nc}$ for some $c \in \mathbb{C} \setminus \{0\}$. Previously established unconditionally for $n \leq 7$ (Hinkkanen–Kayumov–Khammatova, 2025), for odd polynomials of all degrees (Tang, 2025), and for degrees $n = 8$ and $n = 9$ (Besevic, 2026), the conjecture faces a sharp qualitative phase transition at degree $n = 10$: normalizing a minimal-modulus critical point to $\zeta_1 = 1$, the diagonal polynomial $G_{10}(w) = P(1)\big|_{(w,\ldots,w)}$ vanishes *strictly inside* the open unit disk $\mathbb{D}$ for the first time, at $w_0 \approx 0.123268 \pm 0.881372\,i$ ($|w_0| \approx 0.889950 < 1$). Consequently, $P(\zeta_1)$ vanishes identically on a real-codimension-2 hypersurface inside $\mathbb{D}^8$, single-variable Grace–Walsh–Szego (GWS) near-1 wedge exclusion fails ($10|H_{10}(1, v)| = 10|G_9(v)|$ drops to $0.336 < 1$), and $P(1)$ alone can no longer separate even the *second* smallest inverse critical point $w_{(2)}$ from the pole $w = 0$ of $P(\zeta_j)/\zeta_j$.

In this paper we break the $P(\zeta_1) = 0$ interior-zero barrier and prove the Dual Smale Mean Value Conjecture unconditionally for all complex polynomials of degree $n = 10$, together with strict uniqueness of the extremal family. Our proof combines five new mechanisms: (i) a *Second-Smallest Inversion GWS Theorem* applied to $Q_2 = P(\zeta_2)/\zeta_2$ with bounded ratio $\rho = w_{(1)}/w_{(2)} \in \overline{\mathbb{D}}$, certifying $|w_{(2)}| > 0.110$ globally in 4D, paired with our sharp *Geometric-Mean Lemniscate Spread Theorem* ($2 \cdot 4^9 |w_{(1)}|^8 < \prod_{k=2}^{8} |w_{(k)}|$) to eliminate $|w_{(1)}| \leq 0.095$ unconditionally; (ii) an analytical Stage 0 quantitative local ball for the two-variable form $H_{10}^{(2)}(a, b, v)$ at radius $\delta = 1/10$ via an exact $O(\delta^2 X)$ triple-product estimate and sum-of-squares certificate, enabling 6D polar exclusion of any *pair* of coordinates in $W_{\text{near1}} = [0.726, 1] \times [-24^\circ, +24^\circ]$; (iii) *Max-Modulus Q_j GWS* `SafeQBoxes` collapsing the 82° $P(1) = 0$ interior-zero region directly in 4D via off-center closed-disk GWS; (iv) exact-integer outward-rounded 16D multiset branch-and-bound featuring a 112-term generalized McCormick degree-2 bilinear envelope, optimal supporting-hyperplane directions, and signal-to-noise Q_j selection; and (v) a two-layer audited artifact consisting of a Lean 4.36.0 formalization (2,035 lines, 27 theorems, zero `sorry`, zero `native_decide`, standard axioms only) and a self-contained IEEE-754 outward-rounded Rust certifier (2,494 lines, exact integer coefficients $[1260, -420, 210, -126, 84, -60, 45, -35, 28]/252$) that certifies 100% of $\overline{\mathbb{D}}^8$ in 976.80 seconds (16 minutes 17 seconds) on 10 CPU cores across 374,552,971 leaves.

1. INTRODUCTION

In 1981, Smale [10] posed the mean value conjecture asking whether every complex polynomial $P(z) = z + a_2 z^2 + \cdots + a_n z^n$ admits a critical point ζ with $|P(\zeta)/\zeta| \leq 1 - 1/n$ (or $\leq K$ for a universal constant). In 2026, the primal conjecture was disproved with sharp global supremum $K^* = 4$ [1, 6].

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The *Dual Smale Mean Value Conjecture*, formulated independently by Dubinin–Sugawa [5] and Ng [9], asks for the sharp lower bound on the maximum over all critical points. Our main result resolves the first degree past the interior-zero threshold $P(\zeta_1) = 0$:

Theorem 1.1 (Main Theorem — Dual Smale Mean Value Conjecture for $n = 10$)**.** *Let $P(z) = z + a_2 z^2 + \cdots + a_{10} z^{10} \in \mathbb{C}[z]$ be any complex polynomial of degree 10 with $a_{10} \neq 0$. Then*

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{10}. \tag{1}$$

Moreover, equality holds in (1) if and only if $P(z) = \frac{1-(1-cz)^{10}}{10c}$ for some $c \in \mathbb{C} \setminus \{0\}$.

Prior to this work, the dual conjecture was proved for $n \leq 7$ by Hinkkanen, Kayumov, and Khammatova [7], for odd polynomials of all degrees by Tang [11], unconditionally for all complex polynomials of degree $n = 8$ [2] and degree $n = 9$ [3], and globally for all $n \geq 2$ up to the unconditional bound $1/(n^2 \cos^2(\pi/(2n))) > 1/n^2$ and the sharp geometric-mean lemniscate spread regime $r_{\max} \geq 4 \cdot 2^{1/(n-1)} G(P)$ by the author [4] (improving earlier general lower bounds of Dubinin–Sugawa [5] and Jatar–Ng [8]).

1.1. **The Degree-10 Phase Transition: $P(\zeta_1) = 0$ Inside $\mathbb{D}$.** Normalize P by scaling $z \mapsto \zeta_1 z$ so that a critical point of minimal modulus lies at $\zeta_1 = 1$. Writing the remaining 8 critical points as $\zeta_k = 1/w_k$ $(k = 2, \ldots, 9)$, every w_k lies in the closed unit disk $\overline{\mathbb{D}} = \{w \in \mathbb{C} : |w| \leq 1\}$, and

$$P'(z) = (1 - z) \prod_{k=2}^{9} (1 - w_k z).$$

Integrating from 0 to 1 yields the 8-variable symmetric multiaffine form

$$P(1) = \int_0^1 (1 - u) \prod_{k=2}^{9} (1 - u w_k)\, du = \sum_{m=0}^{8} \frac{(-1)^m}{(m+1)(m+2)}\, e_m(w_2, \ldots, w_9), \tag{2}$$

with diagonal polynomial $G_{10}(w) = P(1)\big|_{(w,\ldots,w)}$, single-variable collapse $H_{10}(a, v) = P(1)\big|_{(a,v,\ldots,v)}$, and two-variable collapse $H_{10}^{(2)}(a, b, v) = P(1)\big|_{(a,b,v,\ldots,v)}$.

Degree $n = 10$ is the historic threshold where three obstructions hit simultaneously:

(1) *Interior zeros of $G_{10}(w)$ in $\mathbb{D}$.* Whereas $n|G_n(w)| \geq 1$ for $n \leq 7$ and G_8, G_9 remain zero-free on $\overline{\mathbb{D}}$ ($\min_{\overline{\mathbb{D}}} 9|G_9| \approx 0.302 > 0$), the degree-8 polynomial $G_{10}(w)$ has a conjugate pair of roots *strictly inside* $\mathbb{D}$:

$$w_0 \approx 0.123268 \pm 0.881372\, i, \qquad |w_0| \approx 0.889950 < 1, \qquad G_{10}(w_0) = 0.$$

At $w = (w_0, \ldots, w_0)$, $P(\zeta_1)/\zeta_1 = P(1) = 0$ vanishes identically, so the conjecture survives strictly via the secondary critical points, where $10|P(\zeta_j)/\zeta_j| \approx 1.100784 > 1$.

(2) *Double pole-separation failure ($k = 1$ and $k = 2$).* Both $10|H_{10}(0, v)| = 10|G_9(v)|$ (minimum ≈ 0.336) and $10|H_{10}^{(2)}(0, 0, v)| = 10|G_8(v)|$ (minimum ≈ 0.866) dip below 1 on $\overline{\mathbb{D}}$. Thus $P(1)$ alone gives *no* positive lower bound on $|w_{(1)}|$ or $|w_{(2)}|$ (only on $|w_{(3)}|, \ldots, |w_{(8)}|$), so the degree-9 Inversion GWS method on Q_1 collapses because $w_{(2)}$ can also approach 0.

(3) *Single-variable near-1 wedge failure.* Because $10|H_{10}(1, v)| = 10|G_9(v)|$ drops to $0.336 < 1$ on $\overline{\mathbb{D}}$, no single variable w_k near 1 can be excluded by $P(1)$ alone. Only a *pair* (a, b) of variables near 1 is excluded via $H_{10}^{(2)}(a, b, v)$, forcing a dedicated Stage 2 covering all configurations with exactly one variable inside the near-1 wedge W_{near1}.

2. Exact Integer Identities and Off-Center Grace–Walsh–Szego

Clearing denominators in (2) via $\operatorname{lcm}(2, 6, 12, 20, 30, 42, 56, 72, 90) = 2520 = 252 \cdot 10$ eliminates every non-integer rational coefficient from both the formal proofs and the interval arithmetic engine.

Proposition 2.1 (Exact Integer Identities for Degree 10)**.** *Let* $w = (w_2, \dots, w_9) \in \mathbb{C}^8$ *and* $c = (1260, -420, 210, -126, 84, -60, 45, -35, 28)$*. Then:*

$$2520\, P(1) = \sum_{m=0}^{8} c_m\, e_m(w_2, \dots, w_9), \tag{3}$$

$$\begin{aligned} 2520\, G_{10}(w) &= 1260 - 3360w + 5880w^2 - 7056w^3 + 5880w^4 \\ &\quad - 3360w^5 + 1260w^6 - 280w^7 + 28w^8, \end{aligned} \tag{4}$$

$$2520\, H_{10}^{(2)}(a, b, v) = A(v) + (a + b)B(v) + ab\, C(v), \tag{5}$$

where

$$\begin{aligned} A(v) &= 1260 - 2520v + 3150v^2 - 2520v^3 + 1260v^4 - 360v^5 + 45v^6, \\ B(v) &= -420 + 1260v - 1890v^2 + 1680v^3 - 900v^4 + 270v^5 - 35v^6, \\ C(v) &= 210 - 756v + 1260v^2 - 1200v^3 + 675v^4 - 210v^5 + 28v^6. \end{aligned}$$

Moreover, for any $j \in \{2, \dots, 9\}$ *with* $w_j \neq 0$*, letting* $z_j = 1/w_j$ *and* $w_{-j} \in \mathbb{C}^7$ *denote the remaining* 7 *coordinates,*

$$2520\, \frac{P(\zeta_j)}{\zeta_j} = \sum_{r=0}^{7} e_r(w_{-j}) \left(c_r z_j^r + c_{r+1} z_j^{r+1}\right), \tag{6}$$

and on the diagonal $w_{-j} = (uw_j, \dots, uw_j)$ *(so* $w_k/w_j = u$*),*

$$2520\, Q_{\mathrm{diag}}(z_j, u) = P_0(u) + z_j\, P_1(u), \tag{7}$$

with

$$\begin{aligned} P_0(u) &= 1260 - 2940u + 4410u^2 - 4410u^3 + 2940u^4 - 1260u^5 + 315u^6 - 35u^7, \\ P_1(u) &= -420 + 1470u - 2646u^2 + 2940u^3 - 2100u^4 + 945u^5 - 245u^6 + 28u^7. \end{aligned}$$

Proof. All five identities are formally verified in Lean 4.36.0 in `DualSmaleN10/PolynomialReduction.lean` (`exact_p1_deg10_integer_identity`, `exact_G10_integer_identity`, `exact_H10_2_integer_identity`, `exact_qj_deg10_zpoly_identity`, and `exact_Qdiag_deg10_integer_identity`). □

In addition to Laguerre's polar derivative theorem and the Grace–Walsh–Szego coincidence theorem on origin-centered disks (`grace_walsh_szego_unitDisk` and `grace_walsh_szego_disk`), we formalize the general off-center closed-disk coincidence theorem (`grace_walsh_szego_closedDisk`) by explicit coefficient translation:

$$\mathrm{shiftCoeff}(m, c, c_0)_k = \sum_{j=0}^{m-k} \binom{m-k}{j} c_0^j\, c_{k+j},$$

which backs both Stage 1a (`SafeDisks`) and Stage 1a-Q (`SafeQBoxes`).

3. Breaking the Pole Barrier: Second-Smallest Inversion GWS and Theorem 2

At degree $n = 10$, Stage 1b certifies by 4D outward-rounded polar branch-and-bound on the two-block GWS polynomials $F_k(v_1, v_2) = 2520 \int_0^1 (1-u)(1-uv_1)^k (1-uv_2)^{8-k}\, du$ that whenever $k \in \{3, 4, 5, 6, 7, 8\}$ coordinates lie in $|w| \leq R_k$, we have strictly $10|P(1)| > 1$, for the six certified radii

$$(R_3, R_4, R_5, R_6, R_7, R_8) = (0.300,\ 0.495,\ 0.605,\ 0.678,\ 0.726,\ 0.763). \tag{8}$$

Sorting the 8 moduli as $|w_{(1)}| \leq |w_{(2)}| \leq \cdots \leq |w_{(8)}|$, any potential counterexample therefore satisfies $|w_{(k)}| > R_k$ for all $k = 3, \dots, 8$, leaving ONLY $|w_{(1)}|$ and $|w_{(2)}|$ unseparated from 0.

To separate $w_{(2)}$ first without any assumption on $w_{(1)}$, we evaluate the secondary quotient $Q_2 = P(\zeta_{(2)})/\zeta_{(2)}$ at $b = w_{(2)} \neq 0$. Because $|w_{(1)}| \leq |b|$, the ratio $\rho := w_{(1)}/b$ satisfies $|\rho| \leq 1$ automatically!

Theorem 3.1 (Second-Smallest Inversion GWS Theorem for Degree 10)**.** *Let $b \in \mathbb{C} \setminus \{0\}$, $\rho \in \mathbb{C}$ with $|\rho| \le 1$, and $w_{\mathrm{rest}} = (w_{(3)}, \dots, w_{(8)}) \in \mathbb{C}^6$ with $|w_{(k)}| \ge R_3 > 0$ for all $k \in \{3, \dots, 8\}$. Define the exact integer degree-6 polynomials*

$$\begin{aligned} V_0(u) &= 45 - 360u + 1260u^2 - 2520u^3 + 3150u^4 - 2520u^5 + 1260u^6, \\ V_1(u) &= 35 - 270u + 900u^2 - 1680u^3 + 1890u^4 - 1260u^5 + 420u^6, \\ V_2(u) &= 28 - 210u + 675u^2 - 1200u^3 + 1260u^4 - 756u^5 + 210u^6. \end{aligned}$$

Then there exists a single complex point $u \in \mathbb{C}$ with $|u| \le |b|/R_3$ such that

$$\left(\prod_{k=3}^{8} |w_{(k)}|\right)\Big(\big|b\,V_0(u) - V_1(u)\big| - \big|b\,V_1(u) - V_2(u)\big|\Big) \;\le\; |b|^7 \cdot \left|2520\,\frac{P(\zeta_{(2)})}{\zeta_{(2)}}\right|. \tag{9}$$

Proof. Multiplying (6) at $w_j = b$ by $b^7 \prod_{k=3}^{8} w_{(k)}^{-1}$ and applying the reciprocal symmetry identity $e_r(w_{\mathrm{rest}}) \prod_{k=3}^{8} w_{(k)}^{-1} = e_{6-r}(w_{\mathrm{rest}}^{-1})$ (`elemSym_mul_prod_inv`) yields a 6-variable symmetric multiaffine form in $z_k = w_{(k)}^{-1}$ (`inv_q2_multiaffine_repr`). Since every $|z_k| \le 1/R_3$, applying `grace_walsh_szego_disk` collapses z_{rest} to a single point z with $|z| \le 1/R_3$. Setting $u = bz$ ($|u| \le |b|/R_3$) produces the exact diagonal identity (`exact_inv_q2_u_factorization`)

$$b^7\Big(\prod_{k=3}^{8} w_{(k)}^{-1}\Big) \cdot 2520\,\frac{P(\zeta_{(2)})}{\zeta_{(2)}} = \big(b\,V_0(u) - V_1(u)\big) - \rho\big(b\,V_1(u) - V_2(u)\big).$$

Taking moduli and applying the reverse triangle inequality with $|\rho| \le 1$ gives (9). Completely formalized as `inversion_gws_q2_deg10` in `DualSmaleN10/InversionGWS.lean`. □

Stage 1d-1 certifies by 4D outward-rounded centered mean-value-form polar B&B over $|b| \le R_{2,\mathrm{inv}} = 0.110$ and $|u| \le R_{2,\mathrm{inv}}/R_3$ (11,608,152 leaves in 25.10 s) that

$$\big|b\,V_0(u) - V_1(u)\big| - \big|b\,V_1(u) - V_2(u)\big| > \frac{252\,R_{2,\mathrm{inv}}^7}{\prod_{k=3}^{8} R_k},$$

which by (9) forces $10|P(\zeta_{(2)})/\zeta_{(2)}| > 1$ whenever $0 < |w_{(2)}| \le 0.110$ (and $w_{(2)} = 0$ contradicts $a_{10} \ne 0$). Hence unconditionally $|w_{(2)}| > 0.110$.

Once $|w_{(2)}|, \dots, |w_{(8)}|$ are all bounded below, the smallest inverse critical point $w_{(1)}$ is eliminated in *zero* floating-point search steps by our all-degree Geometric-Mean Lemniscate Spread Theorem [4, Theorem 2]:

Corollary 3.2 (Global Elimination of $|w_{(1)}| \le 0.095$ via Lemniscate Spread)**.** *Let $P(z) = z + \cdots + a_{10}z^{10}$ ($a_{10} \ne 0$) with minimal-modulus critical point $\zeta_1 = 1$ and sorted inverse critical moduli $|w_{(1)}| \le |w_{(2)}| \le \cdots \le |w_{(8)}| \le 1$ satisfying $|w_{(2)}| \ge 0.110$ and $|w_{(k)}| \ge R_k$ ($k = 3, \dots, 8$) from (8). If $|w_{(1)}| \le 0.095$, then strictly*

$$\max_{P'(\zeta)=0} \left|\frac{P(\zeta)}{\zeta}\right| > \frac{1}{10}.$$

Proof. Because $r_{\max} = \max_k |\zeta_k| = 1/|w_{(1)}|$ and $\prod_{k=1}^{9} |\zeta_k| = \prod_{k=1}^{8} |w_{(k)}|^{-1}$, Theorem 2 of [4] asserts that

$$\max_{P'(\zeta)=0} \left|\frac{P(\zeta)}{\zeta}\right| \ge \frac{r_{\max}^9}{2 \cdot 10 \cdot 4^9 \prod_{k=1}^{9} |\zeta_k|} = \frac{1}{10} \cdot \frac{\prod_{k=2}^{8} |w_{(k)}|}{2 \cdot 4^9\,|w_{(1)}|^8}.$$

The exact rational inequality $2 \cdot 4^9 \cdot (95/1000)^8 < \frac{110}{1000} \cdot \frac{300}{1000} \cdot \frac{495}{1000} \cdot \frac{605}{1000} \cdot \frac{678}{1000} \cdot \frac{726}{1000} \cdot \frac{763}{1000}$ ($3.478 \times 10^{-3} < 3.711 \times 10^{-3}$) is formally verified in Lean 4 as `stage1d2_thm2_deg10`, yielding strict $> 1/10$. □

4. Stage 0 Quantitative Local Ball at Radius $\delta = 1/10$

Because single-variable near-1 wedge exclusion fails at $n = 10$, Stage 1c operates in 6D on the two-variable collapse $H_{10}^{(2)}(a, b, v)$. To make the 6D branch-and-bound tractable, we push the analytical Stage 0 local ball radius around $(1, 1, 1)$ by an order of magnitude compared to degree 9 (from $\delta = 0.01$ to $\delta = 0.10$, reducing 6D leaf counts near $(1, 1, 1)$ by over $10^6\times$).

Theorem 4.1 (Stage 0 Quantitative Local Ball for $H_{10}^{(2)}$ at $\delta = 1/10$)**.** *For all* $(a, b, v) \in \overline{\mathbb{D}}^3 \setminus \{(1, 1, 1)\}$ *satisfying* $\max(|a - 1|, |b - 1|, |v - 1|) \le 1/10$*, we have strictly*

$$10\left|H_{10}^{(2)}(a, b, v)\right| > 1.$$

Proof sketch (fully formalized in `DualSmaleN10/LocalMinimum.lean`*).* Set $d_a = 1 - a = x_a + iy_a$, $d_b = 1 - b = x_b + iy_b$, $d_v = 1 - v = x_v + iy_v$. From $|a|, |b|, |v| \le 1$ we have $x_k \ge |d_k|^2/2 \ge 0$ for each $k \in \{a, b, v\}$. Set $S = |d_a|^2 + |d_b|^2 + 6|d_v|^2 > 0$, $X = x_a + x_b + 6x_v$, and $s = X - S/2 \ge 0$. Expanding $10H_{10}^{(2)}(1 - d_a, 1 - d_b, 1 - d_v) = 1 + L + Q_2 + R_3 + R_4$ by degree in (d_a, d_b, d_v) gives $L = \frac{1}{9}(d_a + d_b + 6d_v)$, $Q_2 = \frac{1}{36}d_a d_b + \frac{1}{6}(d_a + d_b)d_v + \frac{5}{12}d_v^2$, $R_3 = \frac{1}{14}d_a d_b d_v + \frac{5}{28}(d_a + d_b)d_v^2 + \frac{5}{21}d_v^3$, and a 16-term degree-$\ge 4$ tail R_4 with $|R_4| \le \frac{1}{150}|d_v|^2 \le \frac{1}{900}S$ at $\delta = 1/10$.

The quadratic form satisfies the exact sum-of-squares lower bound

$$2\operatorname{Re} L + |L|^2 + 2\operatorname{Re} Q_2 - \left(\frac{5}{324}S + \frac{2}{9}s\right) \ge \frac{15(y_a + y_b - 2y_v)^2 + 20(y_a - y_b)^2}{324} \ge 0.$$

Crucially, instead of losing $O(\delta S)$ on degree-3 terms, every cross term $2\operatorname{Re} L \operatorname{Re} Q_2$, $2\operatorname{Im} L \operatorname{Im} Q_2$, and every triple product real part in $2\operatorname{Re} R_3$ carries at least one factor $x_k \ge 0$ via the elementary identity $|\operatorname{Re}(z_1 z_2 z_3)| \le (\operatorname{Re} z_1 + \operatorname{Re} z_2 + \operatorname{Re} z_3)\delta^2$ (`abs_re_mul3_le`), bounding their combined loss by $\frac{55}{54}\delta^2 X = \frac{55}{5400}(S/2 + s)$. Combining with $|2\operatorname{Re}((L + Q_2)\overline{R_3})| \le \frac{1}{500}S$ yields $|1 + L + Q_2 + R_3| > 1 + \frac{1}{900}S \ge 1 + |R_4|$, hence $10|H_{10}^{(2)}(a, b, v)| > 1$. Verified with zero `sorry` as `stage0_H10_2_local_ball`. □

5. Max-Modulus Q_j SafeQBoxes, McCormick Envelopes, and Global Exhaustion

To overcome the $P(\zeta_1) = 0$ interior zero blob and tame the 16-dimensional branch-and-bound search over $\overline{\mathbb{D}}^8$, we introduce four certified pruning accelerators, each formally proved sound in `DualSmaleN10/IntervalSoundness.lean`:

(1) *Stage 1a-Q: Max-Modulus Q_j GWS* `SafeQBoxes` (`safe_qbox_max_modulus_gws_sound`). Whenever all 8 coordinates lie in a polar sector $S = [r_0, r_1] \times [\theta_0, \theta_1]$ covering part of the 82° $P(1) = 0$ blob, pick $a = w_{j_{\max}}$ of maximum modulus among the 8 points. Every ratio $u_k = w_k/a$ $(k \ne j_{\max})$ lies in $[r_0/r_1, 1] \times [\theta_0 - \theta_a, \theta_1 - \theta_a]$, which is strictly contained in a small closed disk $D(r_c e^{i(\theta_{\text{mid}} - \theta_a)}, R_u)$. Applying off-center GWS (`grace_walsh_szego_closedDisk`) collapses all 7 ratios to a single u in that disk, reducing the 16D box S^8 to a 4D verification of $|P_0(u) + (1/a)P_1(u)| > 252$. Eight overlapping `SafeQBoxes` on $[0.726, 1.0] \times [66°, 94°]$ (and conjugates) cover the entire $P(1) = 0$ region in under 0.32 s alongside 50 GWS safe disks on G_{10} (including the left-half-plane disk $D(-4, 4.11)$ covering 100% of $\overline{\mathbb{D}} \cap \{\operatorname{Re} w \le 0\}$, tested via exact polar-sector-in-disk convexity `polar_sector_in_disk_sound`).

(2) *Generalized McCormick Degree-2 Bilinear Envelope* (`mccormick_bilinear_lb`). For any centered normalized coordinates $x_a, x_b \in [-1, 1]$, signs $s_a, s_b \in \{-1, +1\}$, interval coefficient $c \in c_{\text{iv}}$ with $|c| \le c_{\text{abs}}$, and relaxation weight $0 \le \alpha \le s_a s_b c$,

$$-c_{\text{abs}} + \alpha(s_a x_a + s_b x_b) \le c\, x_a x_b,$$

proved from $(s_a s_b c - \alpha)(1 + s_a s_b x_a x_b) + \alpha(1 - s_a x_a)(1 - s_b x_b) \ge 0$. Applying this relaxation to all $\binom{8}{2} \cdot 4 = 112$ degree-2 bilinear terms in `eval_p1_rigorous` sharpens the linear envelope dramatically on curved non-convex cells.

(3) *Optimal Supporting-Hyperplane Directions and SNR Q_j Selection.* An untrusted 7-step 1D angular search selects a candidate separating unit direction d_{opt} for tilted zonotopes

(rigorously verified to satisfy $|d_{\mathrm{opt}}|^2 \le 1$ with outward rounding), while unique boxes are prioritized for Q_j evaluation by descending signal-to-noise ratio $(q_{\mathrm{mid}} - 1)/(0.04 + \mathrm{rad}/\max(r_0, 0.05))$ instead of raw magnitude, accelerating deep Stage 2–4 subtrees by over 400×.

(4) *Higher-Order Error Attribution Splitting and Dyadic Simplex Combination.* Splitting variables are chosen by attributing higher-order Taylor remainder mass (`hi_k`) alongside linear gradient widths, and whenever no single functional exceeds 252, exact dyadic weights $k_i/256$ ($\sum_i k_i = 256$) certify $\sum_i \frac{k_i}{256} \mathrm{Re}(d_i F_i) > 252$ (`weighted_combo_re_le_max`).

6. Complete Proof and Audited Verification Results

Table 1 reports the complete end-to-end certification log produced by the outward-rounded Rust certifier (`cargo test --release && cargo run --release`) on an Apple Silicon 10-core machine. All 27 formal theorems in Lean 4.36.0 (`lake build DualSmaleN10 && lake env lean check_axioms.lean`) compile cleanly with zero `sorry`, zero `admit`, zero `native_decide`, and depend strictly on the standard Lean axioms `[propext, Classical.choice, Quot.sound]`.

Table 1. Complete outward-rounded IEEE-754 certification summary for degree $n = 10$ ($252 \cdot 10 = 2520$, 10 CPU threads, 0 uncertified nodes).

Stage / Domain	Subtrees	Certified leaves	Max depth	Wall time
Stage 1a (50 GWS disks + 8 `SafeQBoxes`)	—	201,827	28 / 36	0.30 s
Stage 1b (6 two-block $R_3 \dots R_8$)	—	2,874,347	44	5.99 s
Stage 1d (Q_2 Inv. GWS + Thm. 2)	—	11,608,152	36	25.10 s
Stage 1c (6D pair W_{near1} wedge)	—	249,334,867	—	62.50 s
Stage 2 ($n_W = 1$, 112 seeds)	12,875	15,195,760	28	84.84 s
Stage 3 ($n_W = 0$ mixed, 65 seeds)	8,472	59,357,260	33	366.36 s
Stage 4 ($n_W = 0$ same-half, 7 seeds)	9,079	35,980,758	32	431.72 s
Total (100% of $\overline{\mathbb{D}}^8$)	**30,426**	**374,552,971**	**33**	**976.80 s**

Proof of Theorem 1.1. Normalize P so that a critical point of minimal modulus lies at $\zeta_1 = 1$, with remaining inverse critical points $w = (w_2, \dots, w_9) \in \overline{\mathbb{D}}^8$. Assume for contradiction (or equality characterization) that

$$\max\left(10|P(1)|, \max_{2 \le j \le 9,\, w_j \ne 0} 10\left|\frac{P(\zeta_j)}{\zeta_j}\right|\right) \le 1. \tag{10}$$

Step 1 (Modulus thresholds, Stage 1b). Stage 1b certifies strictly $10|P(1)| > 1$ whenever $k \in \{3, \dots, 8\}$ coordinates lie in $|w| \le R_k$ for the radii (8). Sorting $|w_{(1)}| \le \cdots \le |w_{(8)}|$, we must have $|w_{(k)}| > R_k$ for all $k = 3, \dots, 8$. In particular, at most 6 coordinates satisfy $|w_k| \le R_7 = 0.726$.

Step 2 (Global pole separation, Stage 1d). By Theorem 3.1 and the 4D certification of Stage 1d-1, $0 < |w_{(2)}| \le 0.110$ implies $10|P(\zeta_{(2)})/\zeta_{(2)}| > 1$ (and $w_{(2)} = 0$ contradicts $a_{10} \ne 0$). Hence $|w_{(2)}| > 0.110$. By Corollary 3.2 (Theorem 2 of [4] together with `stage1d2_thm2_deg10`), $|w_{(1)}| \le 0.095$ also implies strictly $> 1/10$. Therefore every coordinate satisfies $|w_k| > 0.095$.

Step 3 (Exhaustion of $n_W \in \{0, 1\}$, Stages 2, 3, and 4). Partition the remaining annulus $\overline{\mathbb{D}} \setminus D(0, 0.095)$ disjointly into $W_u = [0.726, 1] \times [0, 24^\circ]$, $W_\ell = \overline{W_u}$, $U_{\mathrm{in}} = [0.095, 0.726] \times [0, \pi]$, $U_{\mathrm{out}} = [0.726, 1] \times [24^\circ, \pi]$, $L_{\mathrm{in}} = \overline{U_{\mathrm{in}}}$, and $L_{\mathrm{out}} = \overline{U_{\mathrm{out}}}$, and let n_W denote the number of coordinates in $W_{\mathrm{near1}} = W_u \cup W_\ell$. By conjugation symmetry and Step 1 ($n_{\mathrm{in}} \le 6$):

- If $n_W = 1$ (WLOG in W_u), the remaining 7 coordinates distribute across $\{U_{\mathrm{in}}, U_{\mathrm{out}}, L_{\mathrm{in}}, L_{\mathrm{out}}\}$ into 112 seeds, all certified strictly > 1 by Stage 2 (15,195,760 leaves: 14,097,568 by $P(1)$, 909,750 by Q_j, 14,171 by simplex combinations).

- If $n_W = 0$ with mixed half-planes ($1 \le n_L \le 4$ by conjugation), all 65 seeds are certified strictly > 1 by Stage 3 (59,357,260 leaves: 52,190,509 by $P(1)$, 5,886,404 by Q_j, 397 by combinations).
- If $n_W = 0$ in the same half-plane ($n_L = 0$, WLOG all 8 in $U_{\text{in}} \cup U_{\text{out}}$), all 7 seeds are certified strictly > 1 by Stage 4 (35,980,758 leaves: 16,892,655 by $P(1)$, 18,297,209 by Q_j, 578,206 by combinations — directly demonstrating the essential role of Q_j across the $P(1) = 0$ interior zero).

Hence (10) forces $n_W \ge 2$.

Step 4 (Pair near-1 wedge and strict uniqueness, Stage 0 + Stage 1c). Since $n_W \ge 2$, pick two coordinates $a, b \in W_{\text{near1}}$. Applying `grace_walsh_szego_unitDisk` to the remaining 6 coordinates $w_{\text{rest}} \in \overline{\mathbb{D}}^6$ produces $v \in \overline{\mathbb{D}}$ such that $P(1) = H_{10}^{(2)}(a, b, v)$. By Theorem 4.1 (Stage 0) together with the 6D outward-rounded polar certification of Stage 1c (249,334,867 nodes in 62.50 s), we have $10|H_{10}^{(2)}(a, b, v)| > 1$ for every $(a, b, v) \in (W_{\text{near1}}^2 \times \overline{\mathbb{D}}) \setminus \{(1, 1, 1)\}$. Thus (10) forces $a = b = 1$, $v = 1$, and $P(1) = 1/10$.

To establish strict uniqueness, consider the 6-variable symmetric multiaffine form $F_6(w_{\text{rest}}) := 2520\, P(1)\big|_{a=b=1} - 252$ on $\overline{\mathbb{D}}^6$, whose diagonal polynomial is $Q(z) := 2520\, H_{10}^{(2)}(1, 1, z) - 252$. By Stage 0 and Stage 1c, $Q(z)$ has $z = 1$ as its unique root in $\overline{\mathbb{D}}$. Moreover, Q has degree 6 with leading coefficient $45 - 2 \cdot 35 + 28 = 3 \neq 0$ and simple derivative $Q'(1) = 252 \cdot (-2/3) = -168 \neq 0$, so $Q(z) = 3(z - 1)\prod_{i=1}^{5}(z - a_i)$ with every $|a_i| > 1$. Suppose some coordinate $w_\ell \in \overline{\mathbb{D}} \setminus \{1\}$ among w_{rest}. Then its polar derivative $D_{w_\ell} Q(z) = 6Q(z) + (w_\ell - z)Q'(z)$ satisfies $D_{w_\ell} Q(1) = (w_\ell - 1)Q'(1) \neq 0$, $D_{w_\ell} Q(w_\ell) = 6Q(w_\ell) \neq 0$, and for every $z \in \overline{\mathbb{D}} \setminus \{1, w_\ell\}$, Laguerre's identity

$$2\,\mathrm{Re}\left((1 - z\overline{w_\ell})\frac{a - w_\ell}{a - z}\right) = (1 - |z|^2)\left|\frac{a - w_\ell}{a - z}\right|^2 + (1 - |w_\ell|^2) + (|a|^2 - 1)\frac{|z - w_\ell|^2}{|a - z|^2}$$

is ≥ 0 at $a = 1$ and strictly > 0 at each exterior root $a_1, \ldots, a_5$. Thus $D_{w_\ell} Q$ is zero-free on all of $\overline{\mathbb{D}}$, and applying Grace–Walsh–Szego to the remaining 5 coordinates yields $F_6(w_{\text{rest}}) \neq 0$, contradicting $P(1) = 1/10$. Therefore $w_2 = \cdots = w_9 = 1$, giving $P(z) = \frac{1-(1-z)^{10}}{10}$ (and $\frac{1-(1-cz)^{10}}{10c}$ upon unscaling). □

7. Reproducibility and Software Artifacts

All Lean 4.36.0 proofs (27 theorems, 2,035 lines) and the outward-rounded Rust certifier (2,494 lines) are publicly available at `https://github.com/ivan-fr/dual-smale-n10` and verified automatically by GitHub Actions CI:

```
# 1. Verify all 27 Lean 4.36.0 theorems and audit axioms
cd lean && lake exe cache get
lake build DualSmaleN10 && lake env lean check_axioms.lean

# 2. Run Rust unit tests and full outward-rounded degree-10 certification
cd verifier && cargo test --release && cargo run --release
```

References

[1] B. Adamczewski and M. J. Colbrook, *Disproof of Smale's mean value conjecture*, arXiv:2610.36158, 2026.

[2] I. Besevic, *The Dual Smale Mean Value Conjecture for Degree Eight: Breaking the Minimal-Modulus Barrier via Exact-Integer Outward-Rounded Verification and Lean 4 Formalization*, Zenodo, 2026, doi:10.5281/zenodo.24528837.

[3] I. Besevic, *The Dual Smale Mean Value Conjecture for Degree Nine: Inversion Grace–Walsh–Szego, Reciprocal Polar Boxes, and Formal Lean 4 / Outward-Rounded Verification*, Zenodo, 2026, doi:10.5281/zenodo.24534765.

[4] I. Besevic, *Global All-Degree Bounds for the Dual Smale Mean Value Conjecture: Unconditional Hermite Radial Projection Beating $1/n^2$ and Sharp Geometric-Mean Lemniscate Spread*, Zenodo, 2026, doi:10.5281/zenodo.24540795.

[5] V. N. Dubinin and T. Sugawa, *Dual mean value problem for complex polynomials*, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), no. 9, 135–137.

[6] A. Goel, *The sharp constant in Smale's mean value problem*, Preprint, 2026.

[7] A. Hinkkanen, I. R. Kayumov, and D. M. Khammatova, *Dual mean value conjecture for polynomials of degree up to 7*, Constr. Approx. (2025), arXiv:2303.17586.
[8] A. Jatar and T. W. Ng, *An improved lower bound for the dual Smale mean value conjecture*, arXiv:2608.27047, 2026.
[9] T. W. Ng, *Smale's mean value conjecture and the dual problem*, in *Complex Analysis and Potential Theory*, World Scientific, 2009.
[10] S. Smale, *The fundamental theorem of algebra and complexity theory*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 1, 1–36.
[11] Q. Tang, *The dual Smale mean value conjecture for odd polynomials*, arXiv:2510.16875, 2025.

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