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TATE CLASSES ON SIMPLE ABELIAN VARIETIES OF PRIME DIMENSION IN POSITIVE CHARACTERISTIC

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Abstract. Let A be a geometrically simple abelian variety of odd prime dimension over a finitely generated field of characteristic $p > 0$, and assume that A is nonconstant up to isogeny. For every $\ell \neq p$, its rational Tate classes are generated by divisors, with a possible exception in dimension seven for definite quaternionic multiplication and G_2 monodromy. In specified endomorphism cases the result holds on every self-power. The proof recognizes one-dimensional tensor directions from the multiplicities in a Frobenius weight polytope. This identifies a Frobenius-stable adjoint subisocrystal and allows its Newton slopes to rule out certain tensor product representations.

Given the limited scope of our literature search, readers who find substantial overlap with their own papers are invited to contact the author promptly.

1. Statement

Let K be finitely generated of characteristic $p > 0$, and write $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$. An abelian variety over K is *constant up to isogeny* if it is isogenous over $\overline{K}$ to one defined over a finite field. By the theorem of Grothendieck this is equivalent to having sufficiently many complex multiplications [12, Theorem 1.1]. For $\ell \neq p$ put $V_\ell = H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$, and write G_ℓ for its arithmetic monodromy group. Write μ for the similitude character on V_ℓ, whose representation is $\mathbb{Q}_\ell(-1)$. A Tate class on A^n of codimension r is a class in $H^{2r}_{\text{ét}}(A^n_{\overline{K}}, \mathbb{Q}_\ell(r))$ fixed by an open subgroup of Γ_K. Divisor classes on $A^n_{\overline{K}}$ have this property.

Theorem 1.1. *Let A/K be geometrically simple of odd prime dimension g, and suppose that it is not constant up to isogeny. Every Tate class on A is a linear combination of products of divisor classes, for every $\ell \neq p$, except possibly when $g = 7$, the geometric endomorphism algebra is definite quaternionic, and the derived group of G_ℓ° over $\overline{\mathbb{Q}}_\ell$ is G_2 on two copies of its vector representation. If $g \neq 7$, the same holds on every power of A when its geometric endomorphism algebra is $\mathbb{Q}$, a totally real field, a definite quaternion algebra, or an imaginary quadratic field in which p splits. In dimension*

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seven the assertion on powers also holds in the real and split imaginary multiplication cases, for Sp_{14} *on its standard representation, and for* SO_7 *on two copies of its vector representation.*

1.1. **Relation to prior work.** Tankeev proved the number-field result in prime dimension [16] and divisor generation for absolutely simple abelian varieties of prime dimension over finite fields [15, Theorem 1.2]. The finite-field consequence of the CM Hodge theorem applies to arbitrary abelian varieties and their powers [11, Corollary 8.3]. For ordinary geometrically simple abelian varieties in positive characteristic, Jiang's prime-dimensional monodromy theorem [6, Theorem 5.3] and the characteristic-zero divisor-generation result for odd prime dimension [13, 9] give divisor generation on all powers. For threefolds, the divisor case and hard Lefschetz already give divisor generation on A itself. Broe and Jiang also determine the monodromy of non-CM threefolds and treat their powers outside the exotic type IV case [2, Theorem 9.4]. The method here treats the nonordinary case in odd prime dimension, including the definite quaternionic case, with the qualifications in Theorem 1.1.

1.2. **Comparison with Sun's approach.** Sun proposes a proof of the rational Tate conjecture in positive characteristic [14, Theorem 5.7]. We would welcome clarification of one step in the passage from Proposition 3.12 to Theorem 3.14 of version 1. The mod-ℓ calculation gives the comparison after ℓ-completion. The comparison used in Theorem 3.14 is

$$\Sigma^{\infty}_{\mathrm{tr}}\bigl(\varprojlim_{m}\mathbb{Z}/\ell^{m}\bigr)\longrightarrow\varprojlim_{m}\Sigma^{\infty}_{\mathrm{tr}}(\mathbb{Z}/\ell^{m}).$$

Since the target is already ℓ-complete, it would suffice to show that the source is ℓ-complete in the stable category of étale motives. This would justify the comparison and give the finite-field Tate–Beilinson statement by Kahn's effectivity criterion [7, Corollary 9.8.4]. That statement also gives the weak semisimplicity needed in Ambrosi's extension theorem [1, Theorem 1.2.1 and Section 1.2.2], completing the proposed proof of the rational Tate conjecture over finitely generated fields of positive characteristic.

- Sun treats ℓ-adic realization as a completion of the category of étale motives and uses the Dwyer–Greenlees equivalence to study the effectivity of the image of $\mathbb{Z}_\ell$ under the right adjoint. Here we compare étale and crystalline monodromy for abelian schemes. The Frobenius weight polytope and Newton slopes constrain the derived group, and invariant theory then describes the Tate classes.
- Sun's proposed theorem would give cycle-class surjectivity in every codimension for smooth projective varieties over finitely generated fields of positive characteristic. Within the range of Theorem 1.1, we obtain the stronger conclusion that the Tate classes are generated by products of divisors, including on the self-powers specified there. Section 3 also classifies the possible connected derived monodromy

representations. For instance, the derived monodromy is Sp_{2g} on its standard representation when the geometric endomorphism algebra is $\mathbb{Q}$ and $g \neq 7$.

The calculation in Section 4 illustrates the difference for self-powers. With imaginary quadratic multiplication and p nonsplit, it exhibits a determinant Tate class on A^2 outside the algebra generated by divisors. Sun's proposed theorem would give algebraicity for this class as well.

1.3. **Reduction to connected monodromy.** Fix $\ell \neq p$. Passing to a finite extension of K, we may assume that all geometric endomorphisms are defined over K and that G_ℓ is connected. Indeed, the geometric Tate ring is computed by G_ℓ°-invariants, since the Zariski closure of the image of any open subgroup contains this component. The Tate isogeny and semisimplicity theorems give a reductive G_ℓ with commutant

$$\mathrm{End}_{G_\ell}(V_\ell) \simeq \mathrm{End}^0(A_{\overline{K}}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

Here the action on H^1 is contravariant, and the Rosati involution identifies the opposite endomorphism algebra with the endomorphism algebra. We use Tate's theorem and its extensions by Zarhin and Mori, including characteristic two [2, Introduction]. We spread A, a polarization, and the endomorphisms to an abelian scheme over a smooth geometrically connected variety X over a finite field. If a Tate class is fixed by Γ_K, divisor generation over $\overline{K}$ gives an algebraic cycle over a finite extension, which descends with rational coefficients by normalized push-forward. This also gives descent for purely inseparable finite extensions.

2. A Frobenius test for tensor factors

Write $\mathbb{F}_q$ for the field of constants of X, with $q = p^a$. The first relative overconvergent crystalline and ℓ-adic cohomologies form a compatible system, with the q-power Frobenius structure on the crystalline object and suitable coefficient fields [2, Section 3.2]. For the semisimplicity required in the comparison, one can take the associated graded of the successive socles in the category with q-power Frobenius. The socle filtration is functorial and is carried to the socle filtration under absolute Frobenius pullback. Its associated graded therefore still carries absolute Frobenius and the geometric endomorphisms defined over K. At a closed fiber, this operation preserves the Frobenius characteristic polynomials and the multiplicities of the Newton slopes. Below the crystalline coefficient object can be taken in this form. D'Addezio's theorems identify the connected groups with their tautological representations after scalar extension and provide a Zariski-dense set of closed points where the Frobenius tori are maximal in both realizations [4, Theorems 1.2.1 and 1.2.2].

Lemma 2.1. *Let x be a closed point in this set, let k_x be its residue field, and put $K_x = W(k_x)[1/p]$ and $f_x = [k_x : \mathbb{F}_p]$. Write φ_x for the absolute*

σ-semilinear Frobenius on the crystalline fiber M_x over K_x, and $P_x = \varphi_x^{f_x}$ for its linear Frobenius. Let $G_{p,x}$ be the arithmetic group of the coefficient object evaluated at x.

(1) *Conjugation by φ_x acts semilinearly on $G_{p,x}$ and preserves the Frobenius torus $T_x = (\overline{\langle P_x \rangle}^{\mathrm{Zar}})^\circ$. Its action on $\mathrm{End}_{K_x}(M_x)$ makes $\mathrm{Lie}(G_{p,x}^{\circ,\mathrm{der}})$ a rational subisocrystal.*
(2) *A simple ideal of the split derived Lie algebra which is selected by its root line in a way intrinsic to the weight multiset of T_x descends to K_x and is stable under this Frobenius.*
(3) *On such a subisocrystal each slope of the form u/v in lowest terms has multiplicity divisible by v.*

Proof. The closed-point fiber functor on the category with q-power Frobenius takes values over K_x. Pullback by the absolute Frobenius of X commutes with the q-power Frobenius and preserves the tensor category generated by the coefficient object, since that object carries the compatible absolute Frobenius isomorphism. At the fiber this pullback is scalar twist by σ. The isomorphism on the generator identifies the twisted fiber functor with the original one by φ_x. This gives the conjugation action on its Tannakian group [2, Sections 3.1 and 3.4]. The operator P_x is a linear tensor automorphism of the fiber functor. It is semisimple, as its action on first cohomology comes from the semisimple Frobenius endomorphism of A_x. The commutation $\varphi_x P_x = P_x \varphi_x$ gives stability of T_x. The derived Lie algebra is preserved under conjugation.

After splitting T_x, both coefficient-field Galois automorphisms and a semilinear lift of φ_x permute the weights and the root spaces. For a simple ideal selected as in (2), the unique ideal generated by the root spaces on the distinguished line is preserved by both actions. Galois descent and stability under φ_x give the assertion over K_x.

For (3), extend scalars to the Witt fraction field of an algebraic closure of k_x. A simple Dieudonné–Manin isocrystal of slope u/v has rank v, giving the divisibility [8, Section 1.2]. Slopes are obtained from the eigenvalues of P_x by dividing the p-adic valuation by f_x. This applies also to its conjugation action on the adjoint subisocrystals. □

Lemma 2.2. *At a closed point with maximal Frobenius torus, identify the rational character space of that torus with the rational span of its Frobenius eigenvalues, written additively modulo roots of unity. The Galois group of the splitting field of the Frobenius polynomial preserves the weight multiset. If the slope functional at one place over p vanishes on a Galois-invariant subspace of the span of the weight differences, that subspace is zero.*

Proof. Choose a positive power P_x^N of Frobenius which generates a Zariski-dense subgroup of the torus. Evaluation at P_x^N, followed by division by N in the rationalized multiplicative group, identifies each weight with the class of its Frobenius eigenvalue modulo roots of unity. If a character evaluates

to a root of unity, a power of that character is trivial on the torus. Faithfulness implies that the weights generate the character lattice. The common Frobenius polynomial thus gives the same identification in both realizations.

For a place v over p, normalize $v(p) = 1$ and use the slope functional v/f_x on the algebraic eigenvalues. Galois invariance transports vanishing at the selected place to every place over p. An integral multiple of any element of the span of weight differences is represented by a product of ratios of Weil eigenvalues. These ratios are units away from p and have complex absolute value one in every embedding. If the product has valuation zero at every place over p, Kronecker's theorem makes it a root of unity. Injectivity gives the claimed vanishing of the subspace. □

Lemma 2.3. *Suppose a centered multiset of weights is a Cartesian product of $m+1$ equally spaced points on a line and a multiset of total multiplicity n in a complementary hyperplane. Each of the $m+1$ points has multiplicity one in the line factor. If $m+1$ does not divide n, every linear automorphism of the multiset preserves the line and the complementary hyperplane in the span of the weight differences.*

Proof. An automorphism of the multiset takes the interval direction to an edge direction of the product convex hull. If this new direction lies in the complementary hyperplane, every fiber parallel to it is contained in a slice of constant original interval coordinate. Each such fiber is the image of an original fiber, so consists of $m+1$ points of equal multiplicity. The total multiplicity of a slice is then divisible by $m+1$, whereas that total is n. Thus the interval direction is preserved. The complementary hyperplane is spanned by differences between the midpoints of parallel fibers, so it is preserved as well. □

Proposition 2.4. *Suppose* $\mathrm{End}^0(A_{\overline{K}}) = \mathbb{Q}$*, and the representation of the simply connected cover of the derived monodromy is* $\mathrm{Sym}^m \mathrm{std}_2 \otimes W$*, where the first factor belongs to an A_1-ideal and $m \geq 1$. Assume the centered weight multiset recognizes the A_1 interval direction and its complement, in the sense of Lemma 2.3. If $m = 1$, assume W is nontrivial. Then this representation does not occur.*

Proof. The connected center acts by scalars, with square μ, by the commutant and the polarization. Center the weights by subtracting $\mu/2$ in the rational character space. This is the barycenter of the weight multiset. Choose a common maximal Frobenius torus and its crystalline realization. The interval direction is the line of the two A_1 roots. By hypothesis it is intrinsic to the weights, so Lemma 2.1 realizes the corresponding three-dimensional Lie ideal as a rational subisocrystal of $\mathrm{End}_{K_x}(M_x)$. If the first torus coordinate has absolute slope $a > 0$ on the standard A_1 representation, the three adjoint slopes are $-2a, 0, 2a$, each of multiplicity one. Hence $2a$ is an integer.

The slopes on the full representation have the form

$$\tfrac{1}{2} + \{-ma, -(m-2)a, \ldots, ma\} + \{b_j : j \text{ a weight of } W\}.$$

For each j the lowest and highest slopes with second coordinate b_j lie in $[0,1]$. Hence $ma \leq 1/2$, and equality forces $b_j = 0$ for all j. If $a = 0$ then the intrinsic interval line is a nonzero Galois-invariant subspace on which the slope functional vanishes, contrary to Lemma 2.2. Thus $a > 0$, and integrality of $2a$ forces $a = 1/2$ and $m = 1$. The slope functional now vanishes on every difference of weights of W. Their span is the intrinsic complementary hyperplane, which is nonzero because W is nontrivial. This again contradicts Lemma 2.2. □

In particular no abelian variety with trivial geometric endomorphisms and odd dimension greater than one can have an A_1 tensor factor acting by its two-dimensional standard. The complement has odd dimension and Lemma 2.3 applies uniformly.

3. Prime-dimensional group classification

We describe the connected group over an algebraic closure of the coefficient field. The connected center acts by scalars on each irreducible module. Since the connected group is generated by its connected center and its derived group, such a module stays irreducible on the derived group. The Albert classification and its positive-characteristic divisibilities [10, Section 21] leave four possible geometric endomorphism algebras in the nonconstant case of odd prime dimension g. They are $\mathbb{Q}$, a totally real field of degree g, an imaginary quadratic field, and a definite quaternion algebra over $\mathbb{Q}$. In type II, twice the real center degree must divide g, which is impossible. In type III the center degree divides g; equality gives a maximal commutative subfield of degree $2g$, hence CM. In type IV the product of the real center degree and the Schur index divides g; equality gives CM, leaving an imaginary quadratic field. In the quaternionic case the algebra is split at every $\ell \neq p$, since a module over a quaternion division algebra over $\mathbb{Q}_\ell$ has dimension divisible by four. Reciprocity and definiteness give ramification exactly at p and at the real place.

3.1. **Trivial endomorphisms.** The derived representation is absolutely irreducible and symplectic of dimension $2g$ by the endomorphism commutant. If several simple factors occur, the dimensions of their nontrivial representations must be 2 and g, because g is prime. The two-dimensional factor is ruled out by Proposition 2.4. For a single simple factor, Tankeev's theorem [16, Theorem 2] gives the standard of Sp_{2g}, $\mathrm{Sym}^{2g-1}\mathrm{std}_2$, or the 14-dimensional primitive third exterior power for Sp_6 when $g = 7$. The symmetric power is again ruled out by Proposition 2.4 with W trivial. Thus for $g \neq 7$ the group is Sp_{2g} on its standard.

3.2. **Real multiplication.** The real multiplication field has g embeddings. They occur with equal multiplicity, by rationality of the characteristic polynomial of a primitive element of the field on H^1. Each gives a two-dimensional block. The Rosati involution is the identity on the field, so the alternating

pairing restricts nondegenerately to each block. The endomorphism commutant makes the blocks irreducible and pairwise nonisomorphic. Each derived projection is SL_2, the connected semisimple group that acts irreducibly in dimension two. A proper subdirect product of these simple Lie algebras would identify two factors by a diagonal relation. Their standard modules would be isomorphic on the derived group and their connected central characters also agree. Indeed, each character has square μ and the character group of a torus is torsion-free. This would make the full modules isomorphic, contrary to the commutant. The derived group is thus SL_2^g on the sum of its standard modules.

3.3. **Imaginary quadratic multiplication.** Write E for the imaginary quadratic endomorphism field. The two blocks V_+ and V_- are irreducible and nonisomorphic by the commutant. They are dual up to μ, since Rosati acts by conjugation on E. Thus both have rank g.

For a closed fiber Frobenius, the two block characteristic polynomials are the two embeddings of one polynomial in $E[T]$, in both realizations. For a rational endomorphism u of the fiber, the characteristic polynomial on H^1 is the degree polynomial $\deg(T-u)$ [17]. To see the independence of realization, clear denominators and evaluate at integers T for which $T-u$ is an isogeny. Its determinant on H^1 equals its action on top cohomology by the exterior algebra description. Hence both realizations give the same polynomial. Apply the trace statement to eP_x^j for $e \in E$ and $j \geq 0$, with P_x induced by the residue-field Frobenius endomorphism. The nondegenerate trace pairing of $E/\mathbb{Q}$ recovers the E-linear traces from the rational traces, and Newton's identities give the block polynomials. These traces are also unchanged on the associated graded in Section 2, since the absolute Frobenius and the endomorphisms preserve the filtration. Write $q_x = \#k_x$ and let $\delta \in E^\times$ be the determinant of Frobenius on this E-block. It has archimedean absolute value $q_x^{g/2}$ in both embeddings and is a unit away from p.

If p is nonsplit, the quotient $\delta/\bar{\delta}$ is a unit also above p, since conjugation fixes the unique prime. It is therefore a root of unity. At a maximal Frobenius torus this value makes the determinant quotient trivial on the torus and hence on the connected group. If χ_+, χ_- are the scalar actions of the connected center on the two blocks, then $\chi_+^g = \chi_-^g$ and therefore $\chi_+ = \chi_-$. A self-dual derived action on one block would make the full modules isomorphic, contrary to the commutant.

If p splits, the two idempotents of $E \otimes \mathbb{Q}_p$ define rational subisocrystals of M_x because they commute with absolute Frobenius. Their slope sums are integers by Lemma 2.1(3). The connected monodromy and its commutant agree after extension of coefficients in the two realizations, so the same two blocks occur on the crystalline side. Suppose the derived action on one block is self-dual, hence both are. In either block the g slopes are symmetric around their mean s, with $gs \in \mathbb{Z}$. The slope s occurs since g is odd. If $0 < s < 1$, its reduced denominator is g, so it has multiplicity g in that

block. If $s = 0$ or 1, the same conclusion follows from the bound $[0, 1]$. Any embedding into $\overline{\mathbb{Q}}_p$ of the splitting field of the block polynomial restricts to one of the two embeddings of E and thus selects an isoclinic block. The ratios of its roots are consequently units at every p-adic place. They have absolute value one at every archimedean place and are units away from p. Kronecker's theorem would make every such ratio a root of unity. At a maximal Frobenius torus this would give zero for every within-block weight difference, which contradicts the nontrivial derived action. In the split case the determinant quotient is nontrivial on the connected center. Its slope is the difference of the two integral slope sums, which add to the odd integer g by duality and hence differ.

In both cases the derived representation on V_+ is irreducible, non-self-dual, and of prime degree. It therefore comes from just one simple factor. Tankeev's prime-degree classification gives the standard representation of SL_g or its dual [16, Theorem 2]. The other block is its dual, and faithfulness identifies the full derived action with SL_g on their sum.

3.4. **Definite quaternionic multiplication.** By the commutant the representation is $U \otimes W$, where U is two-dimensional with trivial monodromy action and the quaternion algebra acts as $\mathrm{End}(U)$. The canonical quaternion involution is adjunction for an alternating form on U. The polarization pairing is this form tensored with a symmetric pairing on the irreducible module W, with similitude μ. Thus the derived representation is two copies of the orthogonal module W of degree g. Its prime-degree possibilities are the vector of SO_g, $\mathrm{Sym}^{g-1}\,\mathrm{std}_2$, and the seven-dimensional vector representation of G_2 when $g = 7$ [16, Theorem 2]. If $g > 3$ and a symmetric power occurs, a nonzero slope on its maximal torus gives g distinct slopes of multiplicity two. There are only three possible slopes of denominator at most two in $[0, 1]$. The full span of the weight differences is then a nonzero, Galois-invariant line with slope zero at the chosen place, contradicting Lemma 2.2. For $g = 3$ the symmetric square is the orthogonal vector representation of SO_3. Thus for $g \neq 7$ the group is SO_g on two copies of its vector. The possibility of non-minuscule weights in the quaternionic case was previously noted by Bültel [3, Proposition 2.12].

Proposition 3.1. *In the quaternionic G_2 case of Theorem 1.1, the generic Newton polygon is* $\{0^4, (1/2)^6, 1^4\}$.

Proof. On a dense open constant Newton stratum choose a maximal crystalline Frobenius torus. Up to the Weyl group the centered slopes on the seven-dimensional vector are $0, \pm a, \pm b, \pm(a + b)$, with $a \geq b \geq 0$. Each weight occurs twice in first cohomology and its slope is shifted by $1/2$. The adjoint is a rational subisocrystal by Lemma 2.1. Its slopes are two zero weights and the pairs

$$\pm a,\ \pm b,\ \pm(a + b),\ \pm(a - b),\ \pm(2a + b),\ \pm(a + 2b).$$

If $a > b > 0$, the two least slopes are distinct, of multiplicity two and less than $1/2$, which is impossible. If $a = b > 0$, the least slope has multiplicity two, so $a = 1/4$. The adjoint slope $3/4$ then has multiplicity two, which is impossible on the rational adjoint isocrystal. If $a > b = 0$, the largest positive adjoint slope is $2a$ of multiplicity one, so $a = 1/2$. Finally $a = b = 0$ would give slope zero on the space of weight differences, contrary to Lemma 2.2. The remaining parameter gives the claimed polygon. □

4. Invariant classes

The cohomology of A^n is the exterior algebra of $V_\ell^{\oplus n}$. Over a splitting field a Tate class in degree $2r$ is a derived-group invariant with connected central character μ^r, before the Tate twist. In the standard symplectic cases the tensor invariants are spanned by products of pairings, within each block for a product of symplectic groups [18]. The central scalar on each symplectic block has square μ. Thus the pairings give Tate classes in degree two, which are divisor classes by the divisor Tate theorem. The alternating projection carries a product of pairings to the exterior product of their alternating projections. This gives divisor generation on every power in these cases.

For SO_g on copies of its vector representation, the tensor invariants are spanned by products of pairings and g-slot determinants [18]. In even degree the number of determinant factors is even because g is odd. The product of two such determinant tensors is a scalar multiple of a Gram determinant of pairings, using the nondegenerate symmetric form to identify the vector representation with its dual. All even tensor invariants are therefore spanned by products of pairings. On the quaternionic block the central scalar again has square μ, so the same argument applies to exterior invariants on every power.

For SL_g on the rank-g block and its dual, let u, v be the numbers of slots of the two types. On A itself one has $0 \le u, v \le g$. The center of SL_g forces $g \mid u - v$, and $u - v$ is even when $u + v$ is even. Hence $u = v$. In balanced bidegree the SL_g-invariants are already invariants for the natural action of GL_g by standard plus dual, since the additional scalar torus acts trivially. They are spanned by products of evaluation pairings [18]. These are Tate pairings because $\chi_+\chi_- = \mu$. When p splits, equality of counts on every power follows from the central characters. After twisting, the central character for counts u, v is $(\chi_+/\chi_-)^{(u-v)/2}$, and χ_+/χ_- is nontrivial by Section 3.

The count on a single A is essential in the nonsplit case. For example, on A^2 the product of the determinant lines of the two copies of V_+ gives a Tate class after the twist by g, because $\chi_+ = \chi_-$ and $\chi_+^2 = \mu$. Its bidegree is $(2g, 0)$, whereas every product of Tate divisors has balanced bidegree.

In dimension seven the remaining trivial-endomorphism case is Sp_6 on $W = \bigwedge_0^3 \mathrm{std}_6$, with its alternating form. In this case $\dim(\bigwedge^{2r} W)^{\mathrm{Sp}_6} = 1$ for all $0 \le r \le 7$. Here is the weight computation in degrees at most six, which suffice by hard Lefschetz. The weights are the eight vectors $(\pm 1, \pm 1, \pm 1)$

and the six $\pm e_i$, each with multiplicity one. Put $\rho = (3, 2, 1)$ and denote by $c_{k,v}$ the coefficient of $t^k X^v$ in $\prod_{w \text{ weight}}(1 + tX^w)$. Multiplying by the Weyl denominator gives the trivial multiplicity as $\sum_s \det(s) c_{k,\rho - s\rho}$, summed over the signed permutations of the three coordinates. The separate sums are

k	0	2	4	6
$\det(s) = 1$	1	10	74	201
$\det(s) = -1$	0	9	73	200

These entries follow by multiplying the fourteen displayed weight factors, or by updating $c_{k,v}$ by $c_{k-1,v-w}$ for each successive weight w, starting with $c_{0,0} = 1$ and updating in decreasing order of k. Hence the powers of the polarization span all Tate classes on A in this case. Finally, taking algebraic group invariants commutes with scalar extension. The image of the rational divisor products extends to the span of the divisor products over the splitting field. Equality of these spaces over the splitting field thus gives equality over $\mathbb{Q}_\ell$. This finishes the proof of Theorem 1.1.

Proposition 4.1. *If the* Sp_6 *representation* $\bigwedge_0^3 \mathrm{std}_6$ *occurs in dimension seven, the generic Newton polygon is one of*

$$\{0^5, (1/2)^4, 1^5\}, \qquad \{0, (1/3)^6, (2/3)^6, 1\}.$$

Proof. Choose a maximal crystalline Frobenius torus on the open Newton stratum. Let $a \geq b \geq c \geq 0$ be the parameters on its three coordinates, with $a + b + c \leq 1/2$ by the weight hull. The adjoint roots are $\pm 2e_i$ and $\pm e_i \pm e_j$ for $i < j$. Their slope multiplicities satisfy Lemma 2.1. When $a > b$, the highest adjoint slope $2a$ has multiplicity one. It is a positive integer at most one, so $a = 1/2$ and $b = c = 0$. This gives the first polygon. When $a = b > c$, the highest adjoint slope $2a$ has multiplicity three. Since $2a \leq 1/2$, this forces $a = b = 1/6$. If $c > 0$, the least first-cohomology slope is simple and therefore zero, forcing $c = 1/6$. If $c = 0$, the least slope is $1/6$ of multiplicity two. Both cases are impossible. When $a = b = c > 0$, the least slope is simple, forcing $a = 1/6$ and giving the second polygon. All three parameters cannot vanish, by Lemma 2.2 applied to the whole span of weight differences. □

5. Two composite dimensions

Corollary 5.1. *Let* $\mathrm{End}^0(A_{\overline{K}}) = \mathbb{Q}$. *If* $\dim A = 6$ *or* 9, *the connected derived monodromy is the full symplectic group in the standard representation. The Tate ring of every power of* A *is generated by divisors.*

Proof. Here A is geometrically simple by Poincaré reducibility. The endomorphism hypothesis also excludes CM type, so the Frobenius test applies. First suppose the derived Lie algebra is simple. For the classical ranks other than A_1, the additional nontrivial irreducible dimensions at most 18, omitting the

defining module and its dual, are

type	dimensions	type	dimensions
A_2	$6, 8, 10, 15$	B_3	8
A_3	$6, 10, 15$	B_4	16
A_4	$10, 15$	C_2	$5, 10, 14, 16$
A_5	15	C_3	14
D_4	8	D_5	16

For completeness, the entries are obtained from $\dim V_\lambda = \prod_{\alpha>0}\langle\lambda + \rho, \alpha^\vee\rangle/\langle\rho, \alpha^\vee\rangle$ [5]. Starting at $\lambda = 0$, add fundamental weights one at a time and discard a branch once the dimension exceeds 18, since each factor of the dimension formula is nondecreasing in the fundamental-weight coefficients. The ranks to check are $A_2, \dots, A_{17}$, $B_3, \dots, B_8$, $C_2, \dots, C_9$ and $D_4, \dots, D_9$. In higher ranks even the least-dimensional nontrivial fundamental module exceeds the bound. The exceptional small modules are the 7- and 14-dimensional modules of G_2. The defining modules of type A in degrees $12, 18$ are non-self-dual and those of type D are orthogonal. Thus the simple symplectic cases of either degree are the corresponding symplectic standard module and a symmetric power of std_2. Proposition 2.4 excludes the latter.

For a semisimple Lie algebra the irreducible module is an external tensor product, and self-duality requires each factor to be self-dual. In degree 12 the possibilities for the sizes of the simple factors are $2, 6$, $3, 4$ and $2, 2, 3$. In sizes $2, 2, 3$ the two degree-two factors are alternating and the degree-three factor is orthogonal, giving an orthogonal representation. In sizes $3, 4$ the degree-three factor is the symmetric square of std_2, and its interval is intrinsic because $3 \nmid 4$. In sizes $2, 6$, the irreducible orthogonal degree-six factor is the vector of SO_6. Its weight hull is an octahedron. In the product of the interval and this octahedron the interval edges are exactly those belonging to no triangular face. They determine the interval direction, and their midpoints determine the complementary hyperplane. Proposition 2.4 applies.

For degree 18, any degree-two factor is excluded because its complement has odd dimension. The other possible tensor sizes are $3, 6$, with the symmetric-square factor of degree three and a symplectic degree-six factor. The table gives Sp_6 on its standard as the only simple symplectic option in degree six outside A_1, since the self-dual degree-six module of type A_3 is orthogonal. For the standard of Sp_6 use the octahedron as in degree 12. For a fifth symmetric power of std_2, the weight hull is a rectangle with three points on each edge in one direction and six on each edge in the other direction. These counts distinguish the directions. The same proposition excludes both tensor types. The assertion about powers follows from the symplectic invariant argument. □

AI use disclosure

OpenAI's ChatGPT was used to explore proof strategies, assist with checking arguments and the weight calculations, search and compare the literature, and draft and revise the LaTeX manuscript.

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