hexagon:2611.01105v1

Turán's Hermite strip problem (1967) is resolved: the sharp half-width is 0.90366…, and other miscellanea

Dave Goel

First version: November 17, 2026. This version (5.40): November 25, 2026.

Abstract

Turán asked whether a single strip $|\operatorname{Im} z| \le c$ contains a zero of every polynomial $1 + 2z + aH_n(z) + bH_m(z)$, where H_j is the jth Hermite polynomial (normalized so that $H_1(z) = 2z$), $2 \le n < m$, and a, b are arbitrary complex numbers. The question appeared in Hayman's 1967 collection of research problems. For trinomials ($b = 0$), Makai and Turán proved that such a strip exists, and Schmeisser found its least half-width, 0.52891952…, in 1971. For the full family we prove that such a strip exists and find its least half-width, $0.903669747226\ldots = (\sqrt{3}/2)\cosh(\log(1+\sqrt{2})/3)$.[1] Only two of these polynomials have no zero in the open strip of this half-width: a cubic with a triple zero, and the cubic obtained by conjugating its coefficients a and b. If the index pair $(2, 3)$ is excluded, the least half-width is 0.87004537354988…, attained by exactly two conjugate quintics. A compactness argument, without computation, shows that some disk $|z| < R$, and so a strip, contains a zero of every such polynomial; this alone answers Turán's question, but it gives no value of R. The proof of the exact constant combines analytic bounds with exact arithmetic certificates for finitely many cases; the analytic bounds are proved in the paper, and the certificates and the programs that check them are given in a supplement [1]. The numbered results outside Section 5 and Appendix A.5 are verified in Lean.[2]

2020 Mathematics Subject Classification: Primary 30C15; Secondary 33C45, 65G30.
Keywords: zeros of polynomials, Hermite polynomials, Hermite expansions, Turán's problem, computer-assisted proof.

This is the expanded version of the focused paper [2]. It adds Section 5, Appendix A.5 and Appendix B, on the least half-width 0.87004537354988… *when the index pair* $(2, 3)$ *is excluded.*

1 Introduction

Let H_j denote the Hermite polynomials, normalized by $H_0 = 1$, $H_1 = 2z$ and $H_{j+1} = 2zH_j - 2jH_{j-1}$. Turán asked whether the Hermite expansion of a polynomial has a theory corresponding to the Landau–Fejér–Montel theory of polynomials, in which the lowest coefficients alone locate some of the zeros, whatever the higher terms are; see [7]. Landau proved the first results of this kind, among them that every polynomial $1 + z + az^n + bz^m$ with $1 < n < m$ has a zero of modulus less than an absolute constant, whatever a, b, n and m are [3, Ch. IV, §1]. The first theorem for Hermite expansions, due to Makai and Turán [7, Theorem I], states that there is an absolute constant A such that every trinomial $1 + 2z + aH_n(z)$, with $n \ge 2$ and $a \in \mathbb{C}$, has a zero in the strip $|\operatorname{Im} z| \le A$.

[1] This is an expanded version of the following focused version [2]: https://doi.org/10.5281/zenodo.24547227

[2] Proved in Lean 4 from Lean's three standard axioms and checks in [1], and checked with `lake comparator`: https://github.com/david44hello/paper_turan-1967-hermite-strip/tree/v5.40/lean

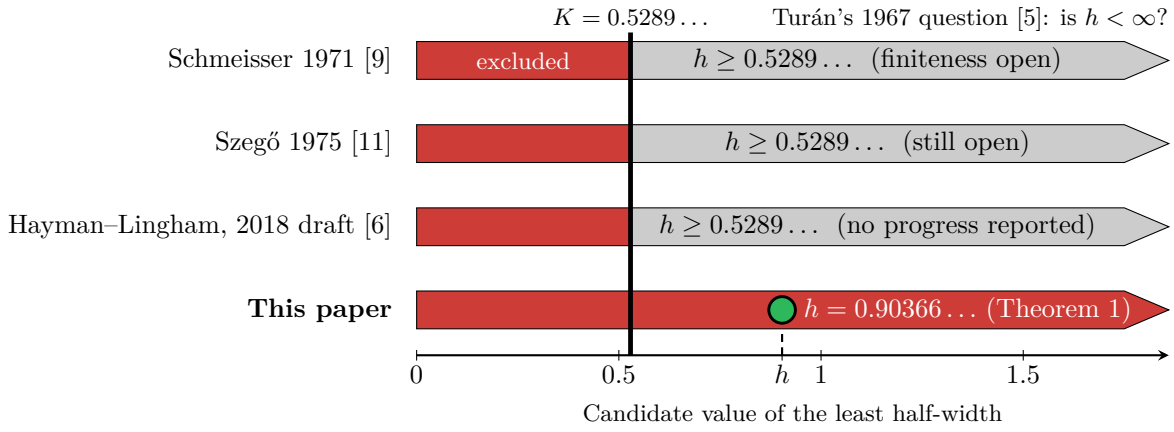

Figure 1: What is known about the least strip half-width h of the quadrinomial family $F_{n,m}(z) = 1 + 2z + aH_n(z) + bH_m(z)$ ($2 \leq n < m$, $a, b \in \mathbb{C}$), to scale. In each row the red parts are excluded as the least half-width, and the grey part is where the least half-width may still lie; the pointed bars continue without bound. The black vertical line marks Schmeisser's 1971 trinomial lower bound $K = 0.52891952\ldots$ (the case $b = 0$); no finite upper bound was known. Szegő [11] recorded the quadrinomial problem as open in 1975, and the 2018 draft of Hayman and Lingham [6] still reported "No progress on this problem has been reported to us." Theorem 1 shrinks the unbounded grey region to the single point $h = 0.903669747226\ldots$ (green dot), so every closed strip $|\operatorname{Im} z| \leq C$ with $C \geq h$ contains a zero of every $F_{n,m}$.

Schmeisser [9, pp. 147–148] found the least value of A, $K = 0.52891952\ldots$, and showed that it is attained only by two conjugate cubic trinomials.

Turán asked for the next case. For integers $2 \leq n < m$ and $a, b \in \mathbb{C}$ put

$$F_{n,m}(z) = 1 + 2z + aH_n(z) + bH_m(z).$$

Is there an absolute constant c such that every $F_{n,m}$ has at least one zero in the strip $|\operatorname{Im} z| \leq c$? According to the tables of [6], the question first appeared in Hayman's 1967 collection of research problems [5]; it is Problem 4.6 of the fiftieth anniversary edition [6]. Turán stated it again, as Problem LXX of [12, p. 62]. In 1975 Szegő recorded Schmeisser's theorem and added that "the corresponding problems for quadrinomials and more general equations remain open" [11, p. 158]. The 2018 draft of the anniversary edition states: "No progress on this problem has been reported to us" [6, Update 4.6]. We know of no later work on it.

We answer the question and determine the exact constant. Sums with more Hermite terms are not treated here; see the Discussion.

Theorem 1. *Every $F_{n,m}$ has a zero in the closed strip $|\operatorname{Im} z| \le h$, where*

$$h = \frac{\sqrt{3}}{2}\cosh\left(\frac{\log(1+\sqrt{2})}{3}\right) = 0.903669747226\ldots.$$

The constant h is sharp. Exactly two of the polynomials $F_{n,m}$ have no zero in the open strip $|\operatorname{Im} z| < h$, namely the cubic

$$F_{2,3}(z) = 8b\,(z + \tfrac{1}{2} - W)^3, \qquad b = (12W^2 - 12W + 9)^{-1}, \quad a = (3 - 6W)b,$$

and the cubic with the conjugate coefficients $\bar{a}, \bar{b}$. Here $W = u + ih$, and u is the positive solution of $32u^3 + 6u = 1$, so that $h^2 = 3/4 + 3u^2$; equivalently, W is the zero of $4w^3 + 3w + 1$ in the upper half-plane.

Thus the answer to Turán's question is yes. This answer alone, without the value of the constant, follows from Proposition 3, which uses no computation. As with trinomials, the extremal polynomials are cubics. All three zeros of an extremal cubic lie at one point, at distance h from the real axis.

The family $F_{n,m}$	**Before this paper**	**This paper**
Does one strip contain a zero of every $F_{n,m}$?	Open: Turán's question [6, Problem 4.6].	Yes, without computation (Proposition 3).
The least half-width	At least $K = 0.52891952\ldots$, because $b = 0$ is allowed and Schmeisser's cubic trinomials attain K [9].	Exactly $h = 0.903669747226\ldots$
The polynomials that attain it	Not known.	Exactly two conjugate cubics, each with a triple zero.

For a nonconstant polynomial F let

$$\mathcal{H}(F) = \min_{F(\zeta)=0} |\operatorname{Im}\zeta|$$

be the distance from the real axis to the nearest zero of F. We call $|\operatorname{Im}\zeta|$ the *height* of a zero ζ, and $\mathcal{H}(F)$ the height of F. Theorem 1 says that the largest value of $\mathcal{H}(F_{n,m})$ is h and that it is attained only for the index pair $(2,3)$. For the uniqueness statement we prove that every other pair stays below $9/10$.

Proposition 2. *For every index pair $(n,m) \neq (2,3)$ and all $a, b \in \mathbb{C}$ we have $\mathcal{H}(F_{n,m}) < 9/10$. Moreover, $\mathcal{H}(F_{n,m}) < 87/100$ for every pair outside $\{(2,3),(3,5)\}$.*

Section 5 (Theorem 9) shows that the largest height among the pairs other than $(2,3)$ is the real algebraic number $h_0 = 0.87004537354988\ldots$ of degree 114, and it is attained by exactly two conjugate quintics with the index pair $(3,5)$. Theorem 1 does not use this.

Plan of the proofs, and the part played by computation. Section 2 proves, by compactness and without computation, that one disk $|z| < R$ contains a zero of every $F_{n,m}$ (Proposition 3). This already answers the existence part of Turán's question, but it gives no value of R. For the pair $(2,3)$ the polynomial has degree at most three, and Section 3 proves by hand that $\mathcal{H}(F_{2,3}) \le h$, with equality only for the two cubics of Theorem 1. Section 4 proves Proposition 2 in three ranges. For $m \ge 450$ we use analytic estimates; one of them (Lemma 17) uses a finite certificate for the trinomials with $4 \le n \le 79$. For $7 \le m \le 449$, exact certificates cover all 100,118 pairs. For the nine

pairs with $m \le 6$ other than $(2,3)$, five have written proofs (four with steps in exact arithmetic), and four have certificates. Since $9/10 < h$, Theorem 1 follows.

Thus Proposition 3 and the case $(2,3)$ of Theorem 1 are proved by hand. Proposition 2, and through it the rest of Theorem 1, also uses exact computations, recorded in the supplement [1]. It holds the certificates, their input data and the programs that check them; its file READ_FIRST lists, for each computed statement, the records that prove it. Section 4 states what each kind of certificate asserts and why it suffices. The analytic estimates are proved in this paper; the longer proofs are in the Appendix.

2 Existence: one disk contains a zero of every $F_{n,m}$

Proposition 3. *There is a radius R such that every $F_{n,m}$ has a zero in the disk $|z| < R$. In particular every $F_{n,m}$ has a zero in the strip $|\operatorname{Im} z| < R$.*

Proof. Since $1 + 2z = H_0 + H_1$, every polynomial of the family is a sum of at most four Hermite terms cH_k with $c \neq 0$. We call these terms its *slots*; two of them are H_0 and H_1. Throughout the proof we pass to subsequences without further comment.

Suppose that no radius R has the stated property. Then there are polynomials f_j of the family such that f_j has no zero in $|z| < R_j$, where $R_j \to \infty$. We may assume that the number of slots of f_j does not depend on j, and that the index $k_{v,j}$ of each slot v is either independent of j or tends to infinity with j.

We use the operator $T = \partial_z^2 - 2z\partial_z$, for which $TH_k = -2kH_k$. Write $\lambda_{v,j} = 2k_{v,j}$. For $\lambda > 0$ the operator $1 + T/\lambda$ multiplies H_k by $1 - 2k/\lambda$: it removes the Hermite term with $2k = \lambda$ and multiplies every other Hermite term by a nonzero number. We call a product of such operators a *filter*.

The main step. Let f_j be sums of a fixed number $s \le 4$ of slots, among them H_0 and H_1 with arbitrary nonzero coefficients. Assume that for every R the polynomial f_j has no zero in $|z| \le R$ once j is large, and that the largest index tends to infinity. Put

$$\Lambda_j = \max_v \lambda_{v,j}, \qquad \Omega_j = \sqrt{\Lambda_j}, \qquad \gamma_v = \lim_{j\to\infty} \lambda_{v,j}/\Lambda_j \in [0,1],$$

and form the filter

$$\Theta_j = \prod_{\gamma_v > 0} (1 + T/\lambda_{v,j}).$$

It removes the slots with $\gamma_v > 0$. A slot v with $\gamma_v = 0$ is multiplied by $\prod_{\gamma_{v'}>0}(1 - \lambda_{v,j}/\lambda_{v',j})$, which tends to one. *We claim that for every R the polynomial $\Theta_j f_j$ has no zero in $|z| \le R$ once j is large.* The coefficients of the slots of f_j can have arbitrarily different sizes, so the two polynomials cannot be compared term by term. Instead we show that $\Theta_j f_j / f_j \to 1$ uniformly on every circle $|z| = R$; the claim then follows from Rouché's theorem.

Step 1 (a growth bound). Each operator $T + \lambda_{v,j}$ annihilates its own slot, and these operators commute. Hence

$$\prod_v \Big(\frac{T}{\Omega_j^2} + \frac{\lambda_{v,j}}{\Lambda_j}\Big) f_j = 0. \tag{1}$$

With $D = \Omega_j^{-1}\partial_z$ we have $T/\Omega_j^2 = D^2 - (2z/\Omega_j)D$, so (1) is a linear differential equation which is monic of order $2s \le 8$ in D. Expand the product in (1) and move each D to the right of the factors $-2z/\Omega_j$; each move adds the term $D(-2z/\Omega_j) = -2/\Omega_j^2$. So the coefficients are polynomials in z/Ω_j, $1/\Omega_j^2$ and the numbers $\lambda_{v,j}/\Lambda_j \in [0,1]$, and they stay bounded on each fixed disk. Multiply

f_j by a constant so that the largest of the numbers $|f_j^{(\ell)}(0)|/\Omega_j^\ell$, $0 \le \ell < 2s$, equals one. Such a constant exists, because a solution of (1) whose derivatives of order less than $2s$ all vanish at the origin is identically zero. The vector $(f_j, Df_j, \dots, D^{2s-1}f_j)$ satisfies a first-order linear system whose coefficients are $O(\Omega_j)$ on each fixed disk. Integrating along radii (Gronwall's inequality) yields

$$|f_j(z)| \le \exp(C_R \Omega_j) \qquad (|z| \le R),$$

with a constant C_R that depends only on R and s. In the other direction, some $\ell < 2s$ has $|f_j^{(\ell)}(0)| = \Omega_j^\ell \ge 1$ once $\Omega_j \ge 1$, and Cauchy's estimate gives $\sup_{|z|\le 1} |f_j(z)| \ge 1/(2s-1)!$.

Step 2 (a harmonic limit). For large j the function $u_j = \Omega_j^{-1} \log |f_j|$ is harmonic on $|z| < R$, and $u_j \le C_R$ there by Step 1. By the last estimate of Step 1 the supremum of u_j over the unit disk is at least $-\Omega_j^{-1} \log(2s-1)!$, which tends to zero. Harnack's inequality for the positive harmonic functions $C_R - u_j$ now shows that the u_j are bounded on every compact subset of $|z| < R$. Hence a diagonal subsequence converges, locally uniformly in the plane, to a harmonic function u.

Step 3 (the limit is symmetric). We show that $u(\bar z) = u(z)$, although the coefficients of f_j may be complex. The operator

$$\prod_{v' \ne v} \frac{T + \lambda_{v',j}}{\lambda_{v',j} - \lambda_{v,j}}$$

maps f_j to its slot v. The differences $|\lambda_{v',j} - \lambda_{v,j}|$ are at least two. By Cauchy's estimate, the value of this operator applied to f_j at a point z is bounded by the supremum of $|f_j|$ on the disk $|w - z| \le \epsilon$, times a fixed power of $1 + \Lambda_j$ and a constant depending on z and ϵ. Each slot has the same modulus at z and at $\bar z$, because H_k has real coefficients. Summing over the slots, taking logarithms, dividing by Ω_j, letting $j \to \infty$ and using $\log(1 + \Lambda_j)/\Omega_j \to 0$ gives $u(\bar z) \le \sup_{|w-z|\le\epsilon} u(w)$. Now let $\epsilon \to 0$ and interchange z and $\bar z$.

Step 4 (the limit is constant). Put $\Phi = u_x - iu_y$, an entire function. Derivatives of harmonic functions converge with the functions, so $f_j'/(\Omega_j f_j) = (\partial_x - i\partial_y)u_j \to \Phi$ locally uniformly. By induction

$$\frac{f_j^{(k)}}{\Omega_j^k f_j} \to \Phi^k \qquad (k \ge 1),$$

locally uniformly: the derivative of the quotient for k, divided by Ω_j, tends to zero, and it differs from the quotient for $k+1$ by the product of the quotient for k with $f_j'/(\Omega_j f_j)$. Dividing (1) by f_j and letting $j \to \infty$ (the terms with a factor z/Ω_j or $1/\Omega_j^2$ tend to zero, since the quotients $D^k f_j/f_j$ converge) gives

$$\prod_v (\Phi^2 + \gamma_v) = 0.$$

On the real axis Φ is real, by Step 3, and every γ_v is nonnegative. Hence Φ vanishes on the real axis, and therefore everywhere.

Step 5 (proof of the claim). By Step 4, for every fixed $k \ge 1$ the quotient $(T/\Omega_j^2)^k f_j/f_j$ tends to zero locally uniformly. For the slots v with $\gamma_v > 0$ the numbers $\Omega_j^2/\lambda_{v,j}$ stay bounded. Expanding the product Θ_j therefore gives $\Theta_j f_j/f_j \to 1$ uniformly on each circle $|z| = R$, and Rouché's theorem shows that $\Theta_j f_j$, like f_j, has no zero in $|z| \le R$ once j is large. This proves the claim.

End of the proof. The polynomials $\Theta_j f_j$ again satisfy the assumptions of the main step, with fewer slots, unless all their indices are bounded. We apply the main step again in the first case. Since there are at most two slots besides H_0 and H_1, after at most two applications we reach polynomials g_j with the following properties: all indices of g_j are independent of j; for every R the polynomial g_j has no zero in $|z| \le R$ once j is large; and, up to a common constant factor, the coefficients of H_0 and H_1 in g_j tend to one. (If the indices of the f_j were bounded from the start, take $g_j = f_j$.)

Remove the common constant factor, and let $\tilde{g}_j$ be g_j divided by its Hermite coefficient of largest modulus. A subsequence of the $\tilde{g}_j$ converges to a polynomial g. If the Hermite coefficients of g_j in the slots other than H_0 and H_1 stay bounded along a subsequence, then the coefficient of H_1 in g is not zero, as that of H_1 in g_j tends to one. Otherwise the coefficients of H_0 and H_1 in g vanish, and the coefficient of at least one H_k with $k \geq 2$ in the Hermite expansion of g has modulus one. In both cases g is not constant, so it has a zero. By Hurwitz's theorem $\tilde{g}_j$, and so g_j, has a zero near it for all large j, a contradiction. □

3 The cubic case and the constant h

We prove the case $(2,3)$ of Theorem 1: $\mathcal{H}(F_{2,3}) \leq h$, with equality exactly for the two cubics given there. Throughout this section u is the positive solution of $32u^3 + 6u = 1$, and $h = \sqrt{3/4 + 3u^2}$ and $W = u + ih$, as in Theorem 1.

Lemma 4. *The zeros of* $4w^3 + 3w + 1$ *are* $-2u$, W *and* $\overline{W}$. *Moreover* $0 < u < 1/6$, $|W|^2 = 1/(8u)$, *and* $h = (\sqrt{3}/2)\cosh(\frac{1}{3}\log(1+\sqrt{2}))$.

Proof. The number $-2u$ is a zero by the definition of u, and $6u < 1$. The polynomial is increasing on the real axis, so its other two zeros are conjugate, say w_1 and $\bar{w}_1$ with $\operatorname{Im} w_1 > 0$. The three zeros have sum 0, sum of products of pairs $3/4$ and product $-1/4$. Hence $\operatorname{Re} w_1 = u$, $|w_1|^2 = 3/4 + 4u^2$ and $|w_1|^2 = 1/(8u)$. The first two relations give $(\operatorname{Im} w_1)^2 = 3/4 + 3u^2 = h^2$, so $w_1 = W$. Finally write $2u = \sinh t$ with $t > 0$. Then $\sinh 3t = 4\sinh^3 t + 3\sinh t = 1$, so $t = \frac{1}{3}\log(1+\sqrt{2})$, and $h^2 = \frac{3}{4}(1 + \sinh^2 t) = \frac{3}{4}\cosh^2 t$. □

A relation between the zeros. Put $w = z + 1/2$; this does not change imaginary parts. In the variable w,

$$F_{2,3} = 2w + a(4w^2 - 4w - 1) + b(8w^3 - 12w^2 - 6w + 5).$$

If $a = b = 0$ the only zero is real. If $b = 0$ and $a \neq 0$, the two zeros have product $-1/4$, so one of them has modulus at most $1/2$, and $\mathcal{H}(F_{2,3}) \leq 1/2 < h$. Let $b \neq 0$, and divide by $8b$. The resulting polynomial $w^3 + Aw^2 + Bw + C$ has $A + 4C = 1$. In terms of its zeros x, y, v this reads

$$4xyv + x + y + v + 1 = 0. \tag{2}$$

By Lemma 4 the triple $x = y = v = W$ satisfies (2). It comes from a polynomial of our family: comparing the coefficients of $8b(w - W)^3$ with the display above gives $a = (3 - 6W)b$ and $b\,(12W^2 - 12W + 9) = 1$, and then the constant terms agree by Lemma 4. The number $12W^2 - 12W + 9$ is not zero, because both zeros of $12w^2 - 12w + 9$ have real part $1/2$, while $\operatorname{Re} W = u < 1/6$. This is the first cubic of Theorem 1; its only zero is $z = W - 1/2$, of height h. Conjugating a and b gives the second one. No other $F_{2,3}$ has the triple zero $w = W$, or $w = \overline{W}$: a monic cubic is determined by its zeros, and a, b by the comparison of coefficients just made. If $\mathcal{H}(F_{2,3}) \geq h$ and $b \neq 0$, the three zeros x, y, v satisfy the hypotheses of the next proposition; so the case $(2,3)$ of Theorem 1 follows from it.

Proposition 5. *Let* x, y, v *be complex numbers that satisfy* (2) *and have* $|\operatorname{Im} x|, |\operatorname{Im} y|, |\operatorname{Im} v| \geq h$. *Then* $x = y = v = W$ *or* $x = y = v = \overline{W}$.

Reduction. Two of the three numbers lie on the same side of the real axis. Since (2) has real coefficients, we may conjugate all three and rename them, and so assume that $\operatorname{Im} x \geq h$ and $\operatorname{Im} y \geq h$. We must show that $x = y = v = W$. The proof has two steps: first x and y are replaced by one

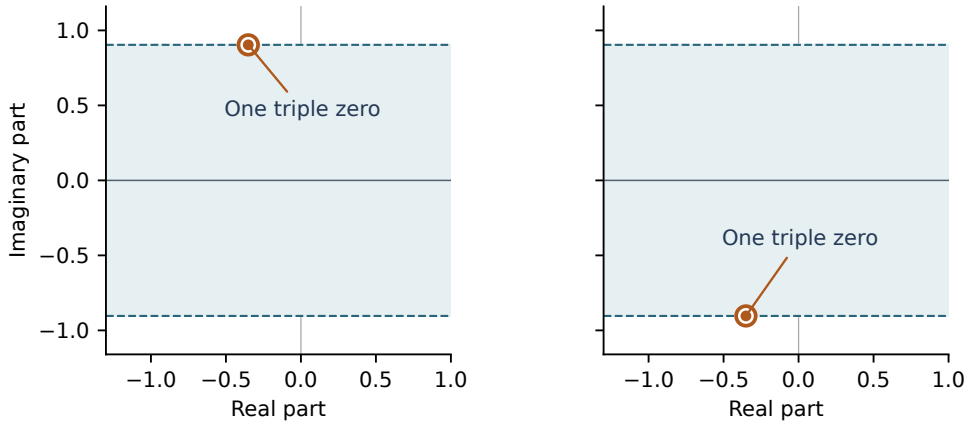

Figure 2: The two extremal cubics of Theorem 1. Each panel shows all three zeros of one of them, at a single point. The dashed lines $|\operatorname{Im} z| = h$ bound the sharp closed strip.

double zero, without leaving the half-plane $\operatorname{Im} w \geq h$; then the third zero is a function of the double zero alone, and it is studied on the boundary of that half-plane.

Lemma 6. *Let x, y, v satisfy (2), with $\operatorname{Im} x \geq h$ and $\operatorname{Im} y \geq h$. Then $v \neq 0$, and with $\tau = -1/(4v)$ there is a number w such that*

$$(w-\tau)^2 = (x-\tau)(y-\tau), \qquad \operatorname{Im} w \geq h,$$

and the triple w, w, v satisfies (2). If $\operatorname{Im}\tau < h$ and $\operatorname{Im} w = h$, then $x = y = w$.

Proof. If $v = 0$, then (2) gives $x + y = -1$, which is impossible because $\operatorname{Im}(x+y) \geq 2h$. For fixed $v \neq 0$, relation (2), divided by $4v$, says that $(x-\tau)(y-\tau) = \tau^2 - (v+1)/(4v)$, a number depending only on v. So w, w, v satisfies (2) for both solutions $w = \tau \pm \sigma$ of $(w-\tau)^2 = (x-\tau)(y-\tau)$. If $\operatorname{Im}\tau \geq h$, one of the two has $\operatorname{Im} w \geq h$. Let $\operatorname{Im}\tau < h$, and put $U = x - \tau$, $V = y - \tau$, so that $\sigma^2 = UV$ and $\operatorname{Im} U, \operatorname{Im} V \geq h - \operatorname{Im}\tau > 0$. The Cauchy–Schwarz inequality in the real plane gives $|U||V| \geq \operatorname{Re} U \operatorname{Re} V + \operatorname{Im} U \operatorname{Im} V$, hence

$$(\operatorname{Im}\sigma)^2 = \frac{|U||V| - \operatorname{Re}(UV)}{2} \;\geq\; \operatorname{Im} U \operatorname{Im} V \;\geq\; (h - \operatorname{Im}\tau)^2.$$

Choose the sign of σ so that $\operatorname{Im}\sigma > 0$. Then $\operatorname{Im} w = \operatorname{Im}\tau + \operatorname{Im}\sigma \geq h$.

Suppose in addition that $\operatorname{Im} w = h$. Then both inequalities in the last display are equalities. The second shows that $\operatorname{Im} U = \operatorname{Im} V = h - \operatorname{Im}\tau$. The first shows that U and V, as vectors in the real plane, point in the same direction. Hence $U = V$, that is, $x = y$. Then w is x or its reflection $2\tau - x$, and the reflection has imaginary part $2\operatorname{Im}\tau - \operatorname{Im} x < h$. So $w = x$. □

Lemma 7. *The function $f(w) = -(2w+1)/(4w^2+1)$ is analytic in the half-plane $\operatorname{Im} w \geq h$ and satisfies $|\operatorname{Im} f(w)| \leq h$ there. Equality holds only at $w = W$, where $f(W) = W$.*

Proof. The poles $\pm i/2$ of f lie outside the half-plane, since $h > 1/2$. On its boundary write $w = s + ih$ with s real. A direct expansion gives

$$|4w^2+1|^2 \,(h \mp \operatorname{Im} f(w)) = h\, q_{\mp}(s),$$

where

$$q_-(s) = 16(s^2+h^2)^2 - 16h^2 - 8s + 3, \qquad q_+(s) = 16(s^2+h^2)^2 + 16s^2 + 8s - 1.$$

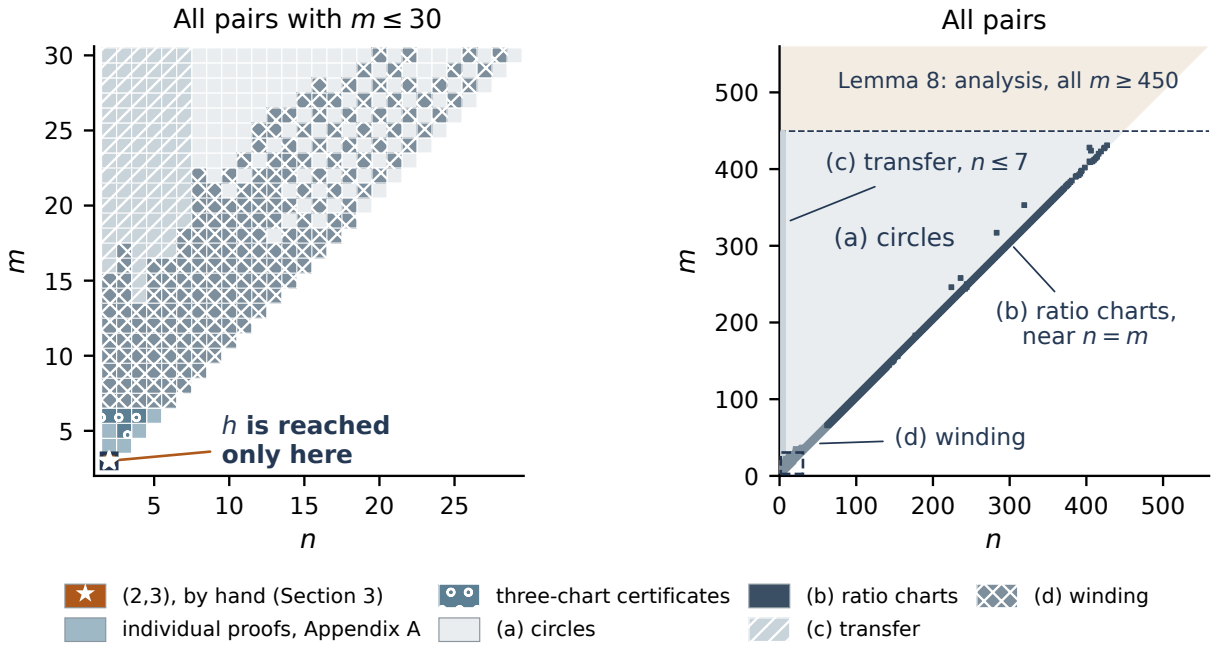

Figure 3: The index pairs (n, m), $2 \le n < m$, and the method that bounds the height for each. The pair $(2, 3)$, marked by a star, is the only pair whose height reaches h (Section 3). Every other pair has height less than $9/10$ (Proposition 2); for $m \ge 450$ by Lemma 8. In the legend, (a)–(d) are the four kinds of certificate of Section 4 (under "The indices $7 \le m \le 449$"), and the three-chart certificates are described under "The indices $m \le 6$". Left: the pairs with $m \le 30$. Right: all pairs.

The polynomial q_- is strictly convex. Its derivative $64s(s^2 + h^2) - 8$ vanishes at $s = u$, because $u^2+h^2 = |W|^2 = 1/(8u)$ by Lemma 4. Since $4W^3+3W+1 = 0$ we have $f(W) = W$, so $\operatorname{Im} f(W) = h$ and $q_-(u) = 0$. Thus $q_-(s) > 0$ for $s \ne u$. Also $q_+(s) \ge 16h^4 - 2 > 0$, because $16s^2 + 8s - 1 \ge -2$ and $h^4 > 1/8$. So on the boundary line $-h < \operatorname{Im} f \le h$, with equality on the right only at $w = W$.

The harmonic function $\operatorname{Im} f$ tends to zero at infinity. Applying the maximum principle to $\pm \operatorname{Im} f$ on large half-disks gives $|\operatorname{Im} f| \le h$ in the half-plane, and by the strong maximum principle the values $\pm h$ are not taken at an interior point. □

Proof of Proposition 5. Let $\operatorname{Im} x, \operatorname{Im} y \ge h$ and $|\operatorname{Im} v| \ge h$, as arranged above, and let w and τ be as in Lemma 6. Since w, w, v satisfies (2) and $4w^2 + 1 \ne 0$, solving (2) for v gives $v = f(w)$. Lemma 7 gives $|\operatorname{Im} v| \le h$. Hence $|\operatorname{Im} v| = h$, and by the same lemma $w = W$ and $v = W$. Now $\tau = -1/(4W)$ has imaginary part $h/(4|W|^2) = 2uh < h$, and $\operatorname{Im} w = h$. The last statement of Lemma 6 gives $x = y = w = W$. □

4 Excluding all other index pairs

We prove Proposition 2 in three parts: $m \ge 450$, $7 \le m \le 449$, and $m \le 6$; Figure 3 shows which method bounds each pair. We write F with $\psi_j = H_j/\sqrt{2^j j!}$ in place of H_j; since ψ_j is a positive multiple of H_j, this only rescales a and b.

The indices $m \ge 450$.

Lemma 8. *If $m \ge 450$, then $\mathcal{H}(F_{n,m}) < 87/100$ for every $n < m$ and all a, b.*

Proof. Suppose that $m \ge 450$ and that $F = 1 + 2z + a\psi_n + b\psi_m$ has no zero in the open strip $|\operatorname{Im} z| < 0.87$. We derive a contradiction, in two cases.

Case $n \le m/2$. On a small circle about a real zero of ψ_m far from the origin, both ψ_n and $1+2z$ are much smaller than ψ_m; so b must be very small, or F would have a zero inside that circle (Steps 1–3). But at a zero c of the trinomial $1 + 2z + a\psi_n$ near the origin we have $F(c) = b\,\psi_m(c)$, and this number cannot be so small, because all zeros of F stay away from c (Step 4).

Step 1: a lower bound for ψ_m on a circle. Put $\phi_m(z) = e^{-z^2/2}\psi_m(z)$, so that $\phi_m'' + (2m + 1 - z^2)\phi_m = 0$. On the real interval $[\sqrt{9m/5}, \sqrt{9m/5} + 3\pi/\sqrt{m}]$ the coefficient $2m + 1 - t^2$ exceeds $0.14m$, and Sturm's comparison theorem supplies a zero x of ϕ_m, that is, of ψ_m, in this interval. On $[0, x]$ the function

$$\phi_m(t)^2 + \frac{\phi_m'(t)^2}{2m + 1 - t^2}$$

is nondecreasing, since its derivative is $2t\,\phi_m'(t)^2/(2m + 1 - t^2)^2$, and its value at $t = 0$ is at least $1/(2m + 2)$, because $\psi_m(0)^2 = \binom{m}{m/2}2^{-m} \ge 1/(m + 1)$ for even m and $\psi_m'(0) = \sqrt{2m}\,\psi_{m-1}(0)$ for odd m. At $t = x$, where $\phi_m(x) = 0$, this gives $\phi_m'(x)^2 \ge (2m + 1 - x^2)/(2m + 2) > 0.14m/(2m + 2)$, so $|\phi_m'(x)| > 0.26$ (as $m \ge 450$). On $[x - 1/\sqrt{m}, x + 1/\sqrt{m}]$ the coefficient $2m + 1 - t^2$ is at most $m/4$, and Sturm's comparison theorem, applied with the solution of $y'' + (m/4)y = 0$ that has the same value and slope at x, gives $|\phi_m(x \pm 1/\sqrt{m})| \ge |\phi_m'(x)| \sin(1/2)/(\sqrt{m}/2) > 1/(5\sqrt{m})$. Finally, if p is a polynomial with only real zeros, then the minimum of $|e^{-z^2/2}p(z)|$ on a circle centered on the real axis is attained at one of the two real points of the circle: as functions of the cosine of the polar angle, the logarithm of each factor $|z - s|$ with s real is concave, and so is the logarithm of $|e^{-z^2/2}|$. Therefore

$$|\phi_m(z)| > \frac{1}{5\sqrt{m}} \qquad \text{on the circle } \Gamma : \ |z - x| = 1/\sqrt{m}.$$

Step 2: ψ_n and $1 + 2z$ are small on Γ. For $z = X + iY$, Mehler's formula with parameter $1/5$ gives (Lemma 16)

$$|\psi_n(z)|e^{-X^2/2} \le (1 - 1/25)^{-1/4} \exp\{-X^2/3 + Y^2/4 + (n/2)\log 5\}.$$

On Γ we have $X^2 - Y^2 \ge 1.8m - 2.7$ and $|Y| \le 1/\sqrt{m}$, and $|\psi_m| = e^{(X^2-Y^2)/2}|\phi_m|$. Together with Step 1, $n \le m/2$ and $0.6 - \frac{1}{4}\log 5 > 0.197$, this gives

$$|\psi_n| < 20\sqrt{m}\,e^{-0.197m}\,|\psi_m|, \qquad |1 + 2z| < 20\sqrt{m}\,e^{-0.197m}\,|\psi_m| \qquad \text{on } \Gamma.$$

For the second inequality note that $|1 + 2z| \le 2.8\sqrt{m}$ on Γ, so that $|1 + 2z| < 55m\,e^{-0.9m}|\psi_m|$ there, which is stronger.

Step 3: b is small. If $|b| > 20\sqrt{m}\,e^{-0.197m}(1 + |a|)$, then $|b\psi_m| > |1 + 2z| + |a\psi_n|$ on Γ by Step 2. By Rouché's theorem F then has as many zeros inside Γ as $b\psi_m$, which has the zero x. But Γ lies in the strip $|\operatorname{Im} z| < 0.87$. Hence

$$|b| \le 20\sqrt{m}\,e^{-0.197m}(1 + |a|). \tag{3}$$

Step 4: b is not small. The trinomial $1 + 2z + a\psi_n$ has a zero c with $|\operatorname{Re} c| \le 2$ and $|\operatorname{Im} c| < 0.76$, by Lemma 17. At this point $F(c) = b\,\psi_m(c)$. Every zero of F is at distance greater than 0.11 from c, because $|\operatorname{Im} c| < 0.76$ and F has no zero in $|\operatorname{Im} z| < 0.87$. Since F has degree at most m, this gives $|F^{(\ell)}(c)/F(c)| \le (100m/11)^\ell$ for every ℓ. Set $E_* = \partial^2/2 - z\partial$ (this is $T/2$, with T as in Section 2)

and $L_j = 1 + E_*/j$. Since $E_*\psi_j = -j\psi_j$, the operator L_j annihilates ψ_j, and it multiplies $2z$ by $1 - 1/j$. Therefore

$$(L_n L_m F)' = 2(1 - 1/n)(1 - 1/m), \qquad L_m F = L_m(1 + 2z) + a\,(1 - n/m)\,\psi_n.$$

The left sides are combinations of derivatives of F of order at most five, with polynomial coefficients. So at c they are bounded by $|F(c)|$ times powers of $100m/11$. In this way the first identity bounds $1/|F(c)|$, and the second identity and its derivative bound $|a| \max(|\psi_n(c)|, |\psi_n'(c)|/\sqrt{2n})$ by a multiple of $|F(c)|$. Since ψ_n and ψ_n' have only real zeros, their moduli at c are at least their moduli at the real point $\operatorname{Re} c$. On the real axis $\psi_n^2 + \psi_n'^2/(2n)$ has the derivative $2t\,\psi_n'^2/n$, so it is smallest at $t = 0$, where it is at least $1/(n+1)$ by the two facts used in Step 1, with n in place of m. Hence the larger of $|\psi_n(c)|$ and $|\psi_n'(c)|/\sqrt{2n}$ is at least $1/\sqrt{2n+2} \ge 1/\sqrt{m+2}$. With explicit constants, these estimates give (Lemma 18)

$$1 + |a| < 110000\, m^{9/2}\, |F(c)|.$$

Finally $|\psi_m(c)| \le e^{8+0.76\sqrt{2m}}$: on the real interval $|t| \le 2$ the function $\psi_m^2 + \psi_m'^2/(2m)$ is at most e^{2t^2}, and the bound is carried from $\operatorname{Re} c$ to c by the first-order system for $(\psi_m, \psi_m'/\sqrt{2m})$ along the vertical segment (Lemma 19). Inserting (3) and the last two bounds in $|F(c)| = |b|\,|\psi_m(c)|$ gives

$$1 < 2200000\, m^5 \exp(8 + 0.76\sqrt{2m} - 0.197m).$$

The logarithm of the right side is below -12 at $m = 450$ and decreases thereafter. This is a contradiction.

Case $m/2 \le n < m$. Put $w_0 = -1/2$, the zero of $1 + 2z$, and

$$U_n = \frac{a\,\psi_n}{F}, \qquad U_m = \frac{b\,\psi_m}{F}.$$

These functions are analytic in the strip $|\operatorname{Im} z| < 0.87$, and $U_n(w_0) + U_m(w_0) = 1$ because $1 + 2w_0 = 0$. We show that $|U_n(w_0)| < 1/2$ and $|U_m(w_0)| < 1/2$, which is absurd. The reason is that U_j ($j = n$ or $j = m$) vanishes at the real zeros of ψ_j, many of which are close to w_0, while it is not too large on two horizontal lines.

Step 1: a bound on the lines $|\operatorname{Im} z| = y_*$, $y_* = 7/11$. *The derivatives.* Let z_0 lie on one of these lines, and put $\Omega = \sqrt{2m + 16(\operatorname{Re} z_0)^2}$. The disk about z_0 of radius $7/\Omega$ has radius at most $7/30$, so it lies in the strip $|\operatorname{Im} z| < 0.87$, where F has no zero. The operator E_* multiplies 1, z, ψ_n and ψ_m by 0, -1, $-n$ and $-m$, so that

$$E_*(E_* + 1)(E_* + n)(E_* + m)F = 0.$$

As in Steps 1 and 2 of the proof of Proposition 3, this differential equation bounds $|F|$ on the disk in terms of the derivatives of F at z_0. Cauchy's estimates bound these derivatives by the maximum of $|F|$ on a smaller concentric disk. Harnack's and Carathéodory's inequalities for the nonnegative harmonic function $C - \log|F|$ (with a suitable constant C) then bound the derivatives of $\log F$ at z_0. On the scale $1/\Omega$ of the disk (the role of $1/\Omega_j$ in Section 2) the constants are absolute, and one obtains (Lemma 21)

$$|F^{(\ell)}(z_0)/F(z_0)| \le (11\Omega)^\ell \qquad (1 \le \ell \le 4).$$

The identity. Put $P_n = L_n(1+2z)$, $P_{nm} = L_n L_m F = 1 + 2(1-1/n)(1-1/m)z$ and $\Xi = P_n/P_{nm}$; the polynomial P_{nm} has no zero off the real axis. Then

$$(n - m)\, b\, \psi_m = (1 - \Xi)(E_* + n)F - \frac{\Xi}{m}\, E_*(E_* + n)F,$$

because $(E_*+n)F = nP_n + (n-m)\,b\,\psi_m$, $E_*P_n = m(P_{nm}-P_n)$ and $E_*\psi_m = -m\psi_m$. Exchanging n and m gives the corresponding identity for $(m-n)\,a\,\psi_n$. After division by F these identities express U_m and U_n through E_*F/F and E_*^2F/F, which contain the quotients $F^{(\ell)}/F$ with $\ell \le 4$ only. For $j \in \{n,m\}$ let j' be the other index, put $P_m = L_m(1+2z)$ and $\Xi_j = P_{j'}/P_{nm}$, so that $\Xi_m = \Xi$. On the two lines $|\Xi_j - 1| < 5/j$. Hence $|U_j(z_0)| \le 4200\,\Omega^4/(j(m-n))$ for $j=n$ and $j=m$ (Lemma 22).

The maximum principle. This bound grows with $|\operatorname{Re} z_0|$. But the polynomial $(1-4iz/\sqrt{2m})^4$ has no zero in the strip $|\operatorname{Im} z| \le y_*$, and at the point z_0 its modulus is at least $(1-4y_*/\sqrt{2m})^4\Omega^4/(4m^2)$. So the quotient of U_j by this polynomial is bounded on the two lines by an absolute constant times $m^2/(j(m-n))$, and it tends to zero as $\operatorname{Re} z \to \pm\infty$ in the strip. Finally, for real s the function $\tanh(\pi(z-s)/(4y_*))$ maps the strip $|\operatorname{Im} z| < y_*$ onto the unit disk; it has modulus one on the two lines and vanishes only at s. We divide also by these functions, for finitely many real zeros s of ψ_j, and apply the maximum principle in the strip. This gives (Lemma 23), for $j=n$ and $j=m$,

$$|U_j(w_0)| \le \frac{25000\,m^2}{j\,(m-n)} \prod_s \Big|\tanh\frac{\pi(w_0-s)}{4y_*}\Big|, \tag{4}$$

where s runs through the real zeros of ψ_j in $[w_0-3, w_0+3]$.

Step 2: the zeros of ψ_j near w_0. On $[w_0-3, w_0+3]$ the coefficient $2j+1-t^2$ is at least $2j-11.25$. By Sturm's theorem consecutive zeros of ψ_j in this interval are at most $\pi/\sqrt{2j-11.25}$ apart. Pairing the zeros on the two sides of w_0 and using the convexity of $\log\coth$ gives (Lemma 24)

$$-\log\prod_s\Big|\tanh\frac{\pi(w_0-s)}{4y_*}\Big| > 0.608\sqrt{2j-11.25} - 0.315.$$

Step 3: comparison. By (4) and Step 2 it is enough to show that

$$0.608\sqrt{2j-11.25} - 0.315 > \log\frac{50000\,m^2}{j\,(m-n)} \qquad (j=n,\ j=m).$$

The case $j=m$ follows from the case $j=n$, because the left side increases with j and the right side decreases. For $j=n$ the difference of the two sides is a concave function of n. At $n=m/2$ the right side is $\log 200000$, and at $n=m-1$ it is at most $\log(50200m)$. At both ends the inequality holds for $m=450$ and persists for larger m, by differentiation. Thus $|U_n(w_0)| < 1/2$ and $|U_m(w_0)| < 1/2$, a contradiction. □

The indices $7 \le m \le 449$. There are $\sum_{m=7}^{449}(m-2) = 100{,}118$ such pairs. For each of them the records [1] contain a *certificate*: finitely many strict inequalities between rational numbers which together prove that, for all complex a and b, the polynomial $F = 1+2z+a\psi_n+b\psi_m$ has a zero in the open strip $|\operatorname{Im} z| < 87/100$. Below we write the inequalities for ψ_j; the records use other real multiples of H_j, such as $128^j H_j$ in (a), which only rescale a and b, and replace irrational numbers, such as $\sqrt{2m}$, by rational bounds in the direction that keeps each inequality valid. The certificates were found by numerical search; only the exact inequalities enter the proof. There are four kinds of certificate, (a)–(d) below; the table gives the number of index pairs of each kind.

Certificate	Index pairs
(a) Circles	96,816
(b) Ratio charts	429
(c) Transfer from the trinomial, one inequality for each pair	2,524
(c) Transfer from the trinomial, square by square	75
(d) Winding rectangles (for $n=2,3$ some boxes use disks)	274
Total	100,118

All four kinds apply Rouché's theorem or the argument principle on circles and rectangles inside the strip. They use three elementary facts.

(i) A dominant coefficient. Let p be a polynomial, or any entire function, and let $p(\xi+\rho v)=\sum_k A_k v^k$. If $|A_\ell|>\sum_{k\neq\ell}|A_k|$ for some $\ell\geq 1$, then p has exactly ℓ zeros in the disk $|z-\xi|<\rho$, by Rouché's theorem. The disk lies in the strip if $|\operatorname{Im}\xi|+\rho<87/100$. When the coefficients of p depend on parameters in a box, it is enough that a lower bound for $|A_\ell|$ on the whole closed box exceeds the sum of upper bounds for the other $|A_k|$.

(ii) A circle about a real zero of ψ_j. If ψ_j has opposite signs at the two real points $\xi\pm\rho$ of the circle $|z-\xi|=\rho$, then it has a zero inside the circle, and on the circle $|\psi_j|$ is at least the smaller of its absolute values at these two points. This follows from the concavity argument in Step 1 of the proof of Lemma 8, because the zeros of ψ_j are real.

(iii) The circle about $-1/2$. On $|z+1/2|=\rho$ we have $|1+2z|=2\rho$.

Suppose that F has no zero in the strip, and consider a circle inside the strip. By Rouché's theorem, if a part P of F has a zero inside the circle, then $|P|\leq|F-P|$ at some point of the circle. Each certificate derives a contradiction from finitely many statements of this form.

(a) Circles. Each circle that is centered on the real axis and has radius less than $87/100$ gives one linear inequality between $|a|$ and $|b|$. By (iii), a circle about $-1/2$ gives $2\rho\leq|a|M_n+|b|M_m$, where M_j is an upper bound for $|\psi_j|$ on the circle. By (ii), a circle at whose two real points ψ_n has opposite signs gives $|a|\beta\leq M_0+|b|M_m$, where β is the lower bound of (ii) and M_0 is an upper bound for $|1+2z|$; in the same way a circle at whose real points ψ_m has opposite signs gives $|b|\beta\leq M_0+|a|M_n$. Two further types apply (i) with $\ell=1$ on a disk $|z-\xi|\leq\rho$ with real center ξ. There the coefficients δ_n and δ_m of v in the expansions of ψ_n and ψ_m are real; let e_j be the sum of the moduli of the other coefficients of ψ_j. If $\operatorname{Re}(a\bar b)\,\delta_n\delta_m\geq 0$, then $|a\delta_n+b\delta_m|\geq\sigma|a\delta_n|+\tau|b\delta_m|$ for all $\sigma,\tau\geq 0$ with $\sigma^2+\tau^2\leq 1$. If F has no zero in the disk, the term of first order of $a\psi_n+b\psi_m$ cannot dominate the rest of F, and Rouché's theorem gives, as in (i),

$$|a|\,(\sigma|\delta_n|-e_n)+|b|\,(\tau|\delta_m|-e_m)\leq|1+2\xi|+2\rho.$$

The second type is the same inequality for $e^{-z^2/2}F$, which has the same zeros as F; every term of this inequality has the positive factor $e^{-\xi^2/2}$, which the certificates divide out. In this type δ_j is the coefficient of v in the expansion of $\phi_j=e^{-z^2/2}\psi_j$, the right side is a bound for $e^{-z^2/2}(1+2z)$ on the circle, and e_j is replaced by the bound

$$\cosh\bigl(\sqrt{Q_j}\bigr)\,|\phi_j(\xi)|+\Bigl(\frac{\sinh\sqrt{Q_j}}{\sqrt{Q_j}}-1\Bigr)|\delta_j|,\qquad Q_j=\rho^2\bigl(|2j+1-\xi^2|+2|\xi|\rho+\rho^2\bigr),$$

for the rest of ϕ_j on the circle. It follows from the differential equation $\phi_j''+(2j+1-z^2)\phi_j=0$ by iterating the equivalent integral equation along the radii of the disk (Lemma 20). Finally, Giroux's inequality [4], which we quote from [10, Theorem B], bounds the sum of the heights of the zeros of a polynomial by the coefficients of its orthogonal expansion. For F with $b\neq 0$ it states (in the ψ-basis, $1+2z=\psi_0+\sqrt{2}\,\psi_1$) that the heights of the m zeros have a sum of at most $\sqrt{m/2}\,\sqrt{3+|a|^2}/|b|$. If F has no zero in the strip, each of these heights is at least $87/100$, and $\sqrt{3+|a|^2}\leq\sqrt{3}+|a|$, so that

$$\tfrac{87}{100}\sqrt{2m}\,|b|\leq\sqrt{3}+|a|.$$

This holds trivially when $b=0$. A certificate lists a few circles for each of the two cases $\operatorname{Re}(a\bar b)\geq 0$ and $\operatorname{Re}(a\bar b)\leq 0$, such that in each case the inequalities have no common solution with $|a|,|b|\geq 0$; this is decided by eliminating $|b|$ and then $|a|$ from these linear inequalities (Fourier–Motzkin elimination) in exact arithmetic.

(b) Ratio charts. If $a = b = 0$, the zero is $-1/2$. Otherwise multiply ψ_n and ψ_m by fixed positive constants, and divide F by the larger in modulus of its two coefficients with respect to these multiples. This gives

$$\Psi_1 + \chi\Psi_2 + \kappa\,(1+2z), \qquad |\chi| \le 1, \quad \kappa \in \mathbb{C}, \tag{5}$$

where Ψ_1 and Ψ_2 are the two multiples, in one of the two orders. Conjugation allows us to assume that $\operatorname{Im}\chi \ge 0$. The certificate covers this half-disk of χ by squares, and for each square it gives two disks with real centers and radii less than $87/100$. The first disk is centered at $-1/2$. By (iii), if the polynomial has no zero in it, then $|\kappa|$ is at most an explicit bound. The second is a disk in which, for every χ of the square and every κ within this bound, one coefficient of positive order dominates as in (i). Here (i) is applied to the product of the polynomial with $e^{-z^2/2}$, as in the second type of (a). The Taylor coefficients of $e^{-z^2/2}\Psi_1$ and $e^{-z^2/2}\Psi_2$ up to order ten are computed exactly from their differential equation. The sum of the moduli of all the others is bounded by Cauchy's estimate on the disk of twice the radius, on which the same equation bounds $|\phi_j|$. So there is a zero in one of the two disks.

(c) Transfer from the trinomial, for $n \le 7$. The trinomial part of F has a zero in the strip that no small perturbation removes, and the term with ψ_m is shown to be small. A constant multiple of F has the form $F_0 + b_0\psi_m$, where $b_0 \in \mathbb{C}$ and the trinomial part is $F_0 = (1+2z) + \vartheta\Psi$ or $F_0 = \Psi + \vartheta\,(1+2z)$ with $|\vartheta| \le 1$; here Ψ is a fixed real multiple of ψ_n. Conjugating F if necessary, we may assume that $\operatorname{Im}\vartheta \ge 0$. For each n the records contain a subdivision of the rectangle $|\operatorname{Re}\vartheta| \le 1$, $0 \le \operatorname{Im}\vartheta \le 1$ into squares and, for each square, a disk which lies in the strip and in $|z| \le 5/4$, and on which a coefficient of F_0 of positive order dominates as in (i), by a margin greater than $1/64$. So $F_0 + g$ has a zero in this disk for every polynomial g with $|g| < 1/64$ on $|z| \le 5/4$. On the other hand, choose a circle as in (ii) for ψ_m, of radius less than $87/100$. If F has no zero in the strip, then by (ii) the number $|b_0|$ is at most the maximum of $|F_0|$ on this circle divided by the lower bound of (ii). The certificate verifies that this bound gives $|b_0\psi_m| < 1/64$ on $|z| \le 5/4$, a contradiction. One such inequality for each pair suffices when $n = 2, \dots, 7$ and m is at least $24, 26, 29, 30, 33, 34$, respectively. For the 75 pairs with smaller m, down to $16, 18, 14, 17, 17, 19$, respectively, the same argument is carried out on each square separately, with a disk and a circle chosen for that square. The 58 pairs with $n \le 7$ and still smaller m are covered by (d).

(d) Winding rectangles. Bring F to the form (5) again (the normalizing constants differ from those of (b)). By (iii), a circle about $-1/2$ again bounds $|\kappa|$, if F has no zero in the strip. The certificate covers the set of these pairs (χ, κ) by boxes, and for each box it gives a rectangle with vertices $x_1 \pm iy'$ and $x_2 \pm iy'$, where $x_1 < x_2$ are real and $y' \le 55/64$. The boundary of the rectangle is cut into short segments. For each segment an exact inequality shows that, for all (χ, κ) in the box, the values of the polynomial (5) on the segment lie in a disk that does not contain the origin. In particular the disk contains the values of the polynomial of the box's center at the two endpoints, and so the chord joining them. As the disk does not contain the origin, the image of the boundary winds about the origin as often as the closed polygon through these endpoint values. The winding number of this polygon, computed exactly, is positive, and the argument principle gives a zero in the rectangle. For some boxes, which occur only for $n = 2$ and $n = 3$, the certificate instead subdivides the box and applies (i) on a disk.

The classes of the table are disjoint and together contain all 100,118 pairs [1]. Every inequality is checked in exact rational arithmetic by two separately written programs in [1]; its file READ_FIRST names them for each class. In (b), (c) and (d) the squares and boxes are the leaves of dyadic subdivisions of a few fixed closed squares and boxes that cover the region of the parameters, and the programs visit every leaf; so no part of the region is left out.

Equal-value points. Several pairs with $m \le 6$ are treated with real points of the following kind. Let $\nu < \nu'$ be real numbers, different from $-1/2$, at which $H_n/(2z+1)$ takes equal values and $H_m/(2z+1)$ takes equal values. Then $F/(2z+1) = 1 + aH_n/(2z+1) + bH_m/(2z+1)$ takes equal values at ν and ν' for all a, b. So, if F has no real zero, $F(\nu')/F(\nu) = (2\nu'+1)/(2\nu+1)$. For a nonreal point $z = x + iy$ put

$$\theta(z) = \int_\nu^{\nu'} \frac{y\,dt}{(t-x)^2 + y^2}.$$

Its absolute value is the angle that the interval $[\nu, \nu']$ subtends at z, and it has the sign of y. As t runs from ν to ν', the argument of $t - z$ changes by $\theta(z)$. Hence the argument of $F(t)$ changes by $\sum_k \theta(z_k)$, the sum over the zeros of F with multiplicity, and this sum is an odd multiple of π if $-1/2$ lies between ν and ν', and an even multiple otherwise. If the interval has midpoint μ and half-length s, then

$$\cot|\theta(x+iy)| = \frac{(x-\mu)^2 + y^2 - s^2}{2|y|s}. \tag{6}$$

The indices $m \le 6$. For $m \le 6$ the pair $(2,3)$ has been treated in Section 3; the bounds below not given by certificates are proved in Appendices A.3 and A.4. For the other nine pairs, and all a, b, we have: $\mathcal{H}(F) < \sqrt{3}/2$ for $(2,4)$; $\mathcal{H}(F) < 5/7$ for $(3,4)$; $\mathcal{H}(F) < 87/100$ for $(2,5)$; $\mathcal{H}(F) < 29491/32768$ for $(3,5)$; $\mathcal{H}(F) < 17/20$ for $(4,5)$; $\mathcal{H}(F) < 57016/65536$ for $(2,6)$, $(3,6)$ and $(4,6)$; and $\mathcal{H}(F) < 43/50$ for $(5,6)$. For $(2,4)$, Lemma 26 shows that the largest height is attained by the quartics with the zeros $z_*, z_*, z_*, -3z_*$, where z_* is a nonreal zero of $12z^4 + 16z^3 + 12z^2 - 3$; their height is $|\operatorname{Im} z_*| = 0.8286\ldots < \sqrt{3}/2$. For $(3,4)$, $(4,5)$ and $(5,6)$ we use equal-value points. For $(3,4)$, Lemma 27 compares the changes of the argument of F along three intervals $[\nu, \nu']$. For $(4,5)$ and $(5,6)$, if F has no real zero, each pair $\nu < \nu'$ of equal-value points gives $\prod_k (\nu' - z_k)/(\nu - z_k) = (2\nu'+1)/(2\nu+1)$ over the zeros z_k of F, with multiplicity; combining two, respectively five, such pairs fixes the product of $|\mathcal{K}|$ over the zeros of F, for a rational function $\mathcal{K}$ with real zeros and poles. For $(4,5)$ this product exceeds 240^5, while $|\mathcal{K}| < 240$ wherever $|\operatorname{Im} z| \ge 17/20$; for $(5,6)$ it exceeds $(181/100)^6$, while $|\mathcal{K}| < 181/100$ wherever $|\operatorname{Im} z| \ge 43/50$ (Lemmas 28 and 29). Since F has at most five, respectively six, zeros, one of them lies in the strip.

For the three sextic pairs and for $(3,5)$ we use three charts. For a sextic pair, write F in a fixed basis of the span of $1 + 2z$, H_n and H_6, given in [1], and divide by its coefficient of largest modulus in this basis. For $(3,5)$, write F in the basis $P_1 = z + 1/2$, $P_3 = z^3 + 3/4$, $P_5 = z^5 + 15/8$, by $1 + 2z = 2P_1$, $H_3 = 8P_3 - 12P_1$ and $H_5 = 32P_5 - 160P_3 + 120P_1$, and divide by its coefficient of largest modulus in this basis. In each chart the two other coefficients have modulus at most one. The certificates cover each chart by boxes and apply (i) above on a disk for each box. The centers ξ and radii ρ of these disks have denominator 65536, and $|\operatorname{Im}\xi| + \rho$ is at most $57016/65536$ for the sextic pairs and at most $58982/65536 = 29491/32768$ for $(3,5)$ [1].

For $(2,5)$ we use one interval of equal-value points. If $b = 0$, then F has the real zero $-1/2$ or is a trinomial, and $\mathcal{H}(F) \le K$ by Schmeisser's theorem [9]. If $b \ne 0$, let S be the zero of $2S^3 - 5S^2 - 4S + 1$ in $(0, 1/4)$. The real interval with midpoint $X = (S-1)/2$ and half-length $s_* = \sqrt{1+S^2}/2$ contains $-1/2$, and its endpoints are equal-value points for $(2,5)$. If the strip $|\operatorname{Im} z| < 0.87$ contained no zero, then, after conjugation, the signed angles $\theta(z_k)$ would have a sum of at least π, while by (6) each of them is less than $\theta_* = \operatorname{arccot}(5/9)$ in absolute value. Counting these angles, with the identities $\sum z = 0$ and $\sum z^2 = 10$ for the five zeros, shows that exactly three zeros are upper (in the upper half-plane), that their angles have a sum less than $3\theta_*$, and that the two lower angles each exceed $\arctan(1/35)$ in absolute value. Since $\tan(3\theta_* - \pi) = 27/545 < 1/20 < \tan(2\arctan(1/35))$, the signed sum is less than π, a contradiction. Lemma 30 gives the details.

Together the three ranges prove Proposition 2. With Section 3, this completes the proof of Theorem 1.

5 The exceptional quintic

By Proposition 2, every index pair other than $(2,3)$ has height less than $9/10$. This section finds the exact largest height of these pairs.

Theorem 9. *Among the pairs $(n,m) \neq (2,3)$ the largest value of $\mathcal{H}(F_{n,m})$ is*

$$h_0 = 0.87004537354988188745\ldots,$$

a real algebraic number specified exactly below. It is attained by exactly two polynomials, which are conjugate, have Hermite indices $(3,5)$ and have degree five. One of them has a double zero on the upper boundary line of the strip $|\operatorname{Im} z| \leq h_0$ and a simple zero on the lower one; its two remaining zeros have absolute imaginary part strictly greater than h_0.

It is necessary to exclude the whole index pair $(2,3)$ and not only the two extremal cubics, because cubics near an extremal one have nearest zeros at distances arbitrarily close to h, by the continuity of the zeros of a polynomial of fixed degree. Since every pair outside $\{(2,3),(3,5)\}$ has height less than $87/100$ (Proposition 2), and $87/100 < h_0$, it remains to prove the case $(3,5)$: the largest value of $\mathcal{H}(F_{3,5})$ is h_0, attained exactly by two conjugate quintics. If $b = 0$, then $F_{3,5}$ is a trinomial or has a real zero, and $\mathcal{H}(F_{3,5}) \leq K < 0.87$ by Schmeisser's theorem [9]. So let $b \neq 0$. The proof runs as follows. We bring the polynomial to a normal form and define h_0 as the height of one explicit quintic, the candidate. Four real points (Lemma 10) show that a quintic of height at least 0.87 has three zeros on one side of the real axis, two on the other, and all of them in a fixed disk; so a quintic of largest height exists. Such a quintic touches the boundary of its zero-free strip in exactly three zeros, counted with multiplicity, two on one line and one on the other. An exact computation places these three zeros near those of the candidate, and a local estimate shows that there the candidate is the only quintic of height at least h_0.

A normal form. Dividing $F_{3,5}$ by $32b$ gives the monic quintic

$$P(z) = G(z) + CA(z) + EB(z), \qquad G = z^5 - 15z/4, \quad A = z^3 - 3z/2, \quad B = 2z + 1, \tag{7}$$

where $E = 1/(32b)$ and $C = a/(4b) - 5$. Conversely, all complex numbers C and $E \neq 0$ occur, with $b = 1/(32E)$ and $a = 4b(C+5)$. Written out, $P = z^5 + Cz^3 + (2E - 3C/2 - 15/4)z + E$. The coefficients of z^4 and z^2 vanish. Hence the zeros $z_1, \ldots, z_5$ of P satisfy

$$z_1 + \cdots + z_5 = 0, \qquad z_1^3 + \cdots + z_5^3 = 0. \tag{8}$$

In this section a *quintic* is a polynomial (7) with $E \neq 0$, and its height is $\mathcal{H}(P)$. We call the zeros of a quintic with positive imaginary part *upper* and those with negative imaginary part *lower*. If the quintic has height $y > 0$, its zeros on the two lines $|\operatorname{Im} z| = y$ are its *contacts*.

The number h_0. If r is a double zero of P, eliminating E from $P(r) = P'(r) = 0$ gives

$$L = 16r^3 + 12r^2 - 6, \quad N = 15 - 20r^4 - 32r^5, \quad C = N/L, \quad E = 4r^5 + 2Cr^3,$$

and $P(z) = (z-r)^2 Q(r,z)$ with $Q(r,z) = z^3 + 2rz^2 + (3r^2 + C)z + 4r^3 + 2rC$. Here $L(r) \neq 0$: we have $C\,L(r) = N(r)$, and L and N have no common zero, since $(192r^4 + 8r^3 + 186r^2 + 40r - 27)L +$

$(96r^2 + 16r + 86)N = 1452$. So the quintics with a double zero form a family with the one complex parameter r. In this family we look for a quintic whose double zero $r = x + iy$ and one simple zero $\zeta = s - iy$ lie on the two lines at the same height y, and for which this height cannot be increased by moving r. Define (x_0, h_0, s_0) as the unique real solution (x, y, s) of

$$\operatorname{Re} Q(r,\zeta) = \operatorname{Im} Q(r,\zeta) = 0, \qquad \operatorname{Im}\Big(Q_r(r,\zeta)\overline{Q_z(r,\zeta)}\Big) = 0 \tag{9}$$

in the closed cube of radius 10^{-8} in the maximum norm about

$$(-0.707163467293, \quad 0.870045373550, \quad 1.639753235047).$$

Here $C = N/L$ is a function of r, and Q_r includes its derivative. The first two equations say that ζ is a zero of the quintic. The last equation says that the derivative $-Q_r/Q_z$ of the simple zero ζ with respect to the double zero r is real; if it were not real, a small change of r would increase both heights. At the candidate it is positive (proof of Lemma 15). The cube and the equations define h_0 exactly; the digits above are rounded.

That (9) has exactly one solution in the cube is proved by exact arithmetic [1], as follows. We replace Q by the polynomial LQ; since $(LQ)_r = LQ_r$ and $(LQ)_z = LQ_z$ where $Q = 0$, this multiplies the last equation by $|L|^2$. We map the cube affinely onto $[-1,1]^3$ and write the system so obtained as $\Xi = 0$. Summing absolute values of coefficients shows that on $[-1,1]^3$ each component of $\mathrm{id} - \Xi'(0)^{-1}\Xi$ has absolute value less than $1/2$, and so has each absolute row sum of its derivative. So this map is a contraction of $[-1,1]^3$ into itself, and its unique fixed point is the only zero of Ξ there. The same calculation isolates the two remaining zeros of this quintic in disks of radius 10^{-5} centered at $-0.721582 + 0.919236i$ and $0.496156 - 1.789281i$; they are simple and have height greater than h_0. Thus the formulas for C, E, a, b above give a polynomial $F_{3,5}$ of height $h_0 > 0.87$, which we call the *candidate*. The number h_0^2 is algebraic of degree 57: it is the only zero in $[7569/10000, 81/100]$ of an irreducible polynomial of degree 57 with integer coefficients, which is computed by exact elimination in [1]. Appendix B describes the elimination. The norm of h_0^2 is negative [1], so h_0^2 is not a square in the number field that it generates, and h_0 has degree 114.

Four real nodes. The next lemma holds for all quintics at once. It is the source of every bound on their zeros in this section.

Lemma 10. *There are real numbers $\nu_1 < \nu_2 < \nu_3 < \nu_4$, lying respectively in the intervals*

$$(-1.596, -1.595), \quad (-0.258, -0.257), \quad (-0.173, -0.172), \quad (1.587, 1.588),$$

such that every polynomial (7) *satisfies*

$$P(\nu_2)\,B(\nu_1) = P(\nu_1)\,B(\nu_2), \qquad P(\nu_4)\,B(\nu_3) = P(\nu_3)\,B(\nu_4),$$

and $B(\nu_2)/B(\nu_1) < 0$ and $B(\nu_4)/B(\nu_3) > 0$.

Proof. Let $\nu \neq \nu'$ be two points, different from $-1/2$, at which A/B takes the same value and G/B takes the same value. Then $P/B = G/B + C\,A/B + E$ takes the same value at ν and at ν', for all C and E. Write $S = \nu + \nu'$ and $p = \nu\nu'$. Clearing denominators in $A(\nu)/B(\nu) = A(\nu')/B(\nu')$ and dividing by $\nu - \nu'$ gives $p\,(2S - 1) = 3/2 - S^2$. Doing the same for G and substituting p gives

$$10S^4 - 27S^2 + 12S - 3 = 0.$$

This quartic has two real roots, $S = -1.853\ldots$ and $S = 1.414\ldots$, and two nonreal roots, of modulus less than 0.35. For each real root, ν and ν' are the roots of $z^2 - Sz + p$; they are real and lie in the stated intervals. Finally $B(z) = 2z + 1$ changes sign only at $-1/2$, which lies between ν_1 and ν_2 and to the left of ν_3. □

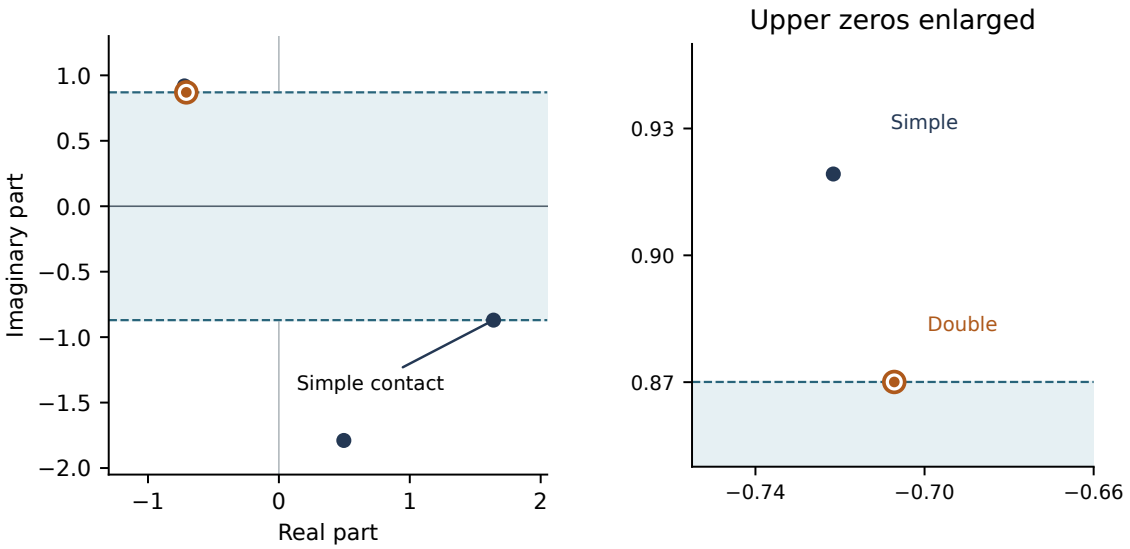

Figure 4: The extremal quintic of Theorem 9. Of its five zeros counted with multiplicity, exactly three lie on the dashed lines $|\operatorname{Im} z| = h_0$. The enlargement separates the double zero from a nearby simple zero strictly above it. Conjugation gives the second extremal quintic.

Put $I = [\nu_1, \nu_2]$ and $J = [\nu_3, \nu_4]$. Let m_I be the midpoint of I and s_I its half-length; then $s_I < 0.67$. For a nonreal point $z = x + iy$ put

$$\theta_I(z) = \int_I \frac{y\,dt}{(t-x)^2 + y^2}.$$

Its absolute value is the angle that I subtends at z, and it is positive above the real axis and negative below. Define θ_J in the same way.

Lemma 11. *Let P be a polynomial (7) without real zeros. Then $\sum_k \theta_I(z_k)$ is an odd multiple of π, and $\sum_k \theta_J(z_k)$ is an even multiple of π.*

Proof. As t runs through I, the argument of $t - z_k$ changes by $\theta_I(z_k)$. So $\sum_k \theta_I(z_k)$ is the change of the argument of $P(t)$ along I. By Lemma 10 the quotient $P(\nu_2)/P(\nu_1)$ is a negative real number, and the change is an odd multiple of π. On J the quotient is positive. □

Lemma 12. *Let P be a quintic of height at least 0.87. Then three of its zeros lie on one side of the real axis and two on the other. All five zeros lie in $|z| < 5.6$, and $|E| \geq 0.87^5$.*

Proof. The imaginary parts of the zeros have sum zero by (8), so the zeros do not all lie on one side. Suppose that four lie above the real axis and one below; the opposite case follows by conjugating the coefficients. The sum $\sum_k \theta_I(z_k)$ is greater than $-\pi$, because its only negative term has absolute value less than π. By Lemma 11 it is at least π. Now $I + 10/7 \subset J$, and for $y \geq 4/5$ and real v

$$5(v^2 + y^2) > (v + 10/7)^2 + y^2,$$

by completing the square. Hence $\theta_J(z) > \theta_I(z)/5$ for every z with $\operatorname{Im} z \geq 4/5$, and the four upper zeros contribute more than $\pi/5$ to $\sum_k \theta_J(z_k)$. The fifth zero has height at least $4 \cdot 0.87$, since the imaginary parts sum to zero, and so $|\theta_J|$ is at most $|J|/(4 \cdot 0.87) < \pi/5$ for it. Thus $\sum_k \theta_J(z_k) > 0$, and by Lemma 11 it is at least 2π. Altogether the four upper zeros have $\sum(\theta_I + \theta_J) \geq 3\pi$. But

$I \cup J \subset [-1.6, 1.6]$, so at a point of height at least 0.87 the two intervals together subtend an angle of at most $2\arctan(1.6/0.87) < 3\pi/4$, so the four upper zeros give less than 3π. This is a contradiction, and the zeros split three to two.

Conjugating if necessary, let three zeros lie above the real axis. At a point $x + iy$ the angle θ subtended by I satisfies

$$\cot\theta = \frac{(x - m_I)^2 + y^2 - s_I^2}{2|y|s_I}.$$

For $|y| \geq 0.87$ this is greater than 13/50, so $|\theta_I(z_k)| < M = \operatorname{arccot}(13/50) < \pi/2$ for all five zeros. The two lower zeros contribute more than $-2M > -\pi$ to $\sum_k \theta_I(z_k)$. By Lemma 11 this sum is at least π, so the three upper angles have sum at least π, and each of them exceeds $\pi - 2M > 0$. The points of the upper half-plane at which I subtends an angle θ form an arc of the circle with center $m_I + is_I \cot\theta$ and radius $s_I \csc\theta$. Hence the three upper zeros lie in a bounded set. Let U be their sum; then $|\operatorname{Im} U| \geq 3 \cdot 0.87 = 2.61$. By (8) the two lower zeros have sum $-U$, and the sum of their cubes is known as well, which gives their product after division by $3U$. So the lower zeros are bounded too. Made explicit, these estimates give $|z_k| < 5.6$ for all k (Lemma 31). Finally $|z_k| \geq 0.87$ for each k, and $E = -z_1 \cdots z_5$. □

By Lemma 12 the quintics of height at least 0.87 form a compact set of polynomials, on which E stays away from zero. It contains the candidate. Hence $\mathcal{H}$ attains a largest value on the quintics, and this value is at least h_0. It remains to show that a quintic of largest height is the candidate or its conjugate.

The contacts of a quintic of largest height. From now on let P be a quintic of largest height, and let y be its height; thus $y \geq h_0 > 0.87$. By Lemma 12 we may assume, after conjugating the coefficients, that three zeros of P are upper and two are lower.

Lemma 13. *Counted with multiplicity, P has exactly two upper contacts and one lower contact.*

Proof. At least three contacts. When the coefficients change by small amounts dC, dE, a simple zero z of P moves, to first order, by $-(A(z)\,dC + B(z)\,dE)/P'(z)$. Suppose first that the only contact is a simple zero. Since B does not vanish there, a suitable dE moves it away from the real axis. Suppose next that the contacts are two distinct simple zeros α, β. If the vectors $(A(\alpha), B(\alpha))$ and $(A(\beta), B(\beta))$ are linearly independent, suitable dC, dE move both contacts away from the real axis. If they are dependent, then A/B takes the same value at α and β, and so does G/B, because $P(\alpha) = P(\beta) = 0$. By the proof of Lemma 10, $S = \alpha + \beta$ is then a root of $10S^4 - 27S^2 + 12S - 3$. If α and β lie on the same line, then $|\operatorname{Im} S| = 2y > 1.74$, but the nonreal roots of the quartic have modulus less than 0.35. If they lie on opposite lines, then S is real, and α, β are two of the real numbers of Lemma 10. Both cases are impossible. Suppose finally that the only contact is a double zero r. Then $L(r) \neq 0$, as shown before (9). So the coefficients $C = N/L$ and E are analytic functions of the double zero near r, and the double zero can be moved away from the real axis. In all three cases the zeros that are not contacts stay farther from the real axis than the contacts, by continuity, so the height increases. This contradicts the choice of P.

Not two lower contacts. The imaginary parts of the zeros sum to zero. Two lower zeros of height y cannot balance three upper zeros of height at least y.

Not three upper contacts. Any two upper contacts of P can be written $x \pm \sqrt{q} + iy$ with x real and $q \geq 0$. Since the three upper angles θ_I have sum at least π, the cotangent formula in the proof of Lemma 12 gives, for every such pair (Lemma 32),

$$(x - m_I)^2 < 3s_I^2 - y^2, \qquad q < 3s_I^2 - y^2 < 0.591. \tag{10}$$

In particular $y^2 < 3s_I^2 < 1.35$, which also follows from $3 \cdot 2\arctan(s_I/y) > \pi$. Suppose now that all three upper zeros are contacts. Let μ be their mean, so that $\operatorname{Im}\mu = y$, let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be their real parts minus $\operatorname{Re}\mu$, and write the two lower zeros as $-3\mu/2 \pm \omega$; this is possible by the first relation of (8). The second relation of (8) then reads

$$\varepsilon_1\varepsilon_2\varepsilon_3 = \tfrac{5}{4}\mu^3 + 3\mu\omega^2 - \mu\,(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2).$$

Its left side is real. Taking imaginary parts, and using (10) for the three pairs of upper zeros and $|\operatorname{Im}\omega| \le y/2$ (both lower zeros have height at least y), gives a bound for $\operatorname{Re}\mu$ and for $\operatorname{Re}\omega$, which puts the real parts $-\frac{3}{2}\operatorname{Re}\mu \pm \operatorname{Re}\omega$ of both lower zeros in $[-3,3]$ (Lemma 33). The imaginary parts of the lower zeros are $-ky$ and $-(3-k)y$ with $1 \le k \le 3/2$. The numbers ε_j are real and have sum zero, so they are the roots of the cubic $t^3 - \frac{1}{2}\Sigma_2\, t - \Pi$, where $\Sigma_2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$ and $\Pi = \varepsilon_1\varepsilon_2\varepsilon_3$. The imaginary and the real part of the last display express Σ_2 and Π through μ and ω, that is, through the two lower zeros. Since the cubic has three real roots, $\Sigma_2^3 \ge 54\,\Pi^2$. Two further real equations hold: the real and the imaginary part of the relation, contained in (7), that the coefficient of z in P is $2E - \frac{3}{2}C - \frac{15}{4}$. So the four real parameters (the real parts of the two lower zeros, k and y^2) satisfy one inequality and two equations. An exact computation [1] shows, by sign bounds on a subdivision of their box $[-3,3]^2 \times [1,3/2] \times [0.7569, 1.35]$, that these three conditions cannot hold together on the closed box, including the case of a repeated ε_j. Two independent programs do this, with 1,540 and 9,868 boxes.

Thus there are at least three contacts, at most one of them lower and at most two upper. □

Reduction to a neighborhood of the candidate. By Lemma 13 the contacts of P are two upper zeros $r \pm \sqrt{q}$ and one lower zero ζ, where

$$r = x + iy, \qquad q \ge 0, \qquad \zeta = s - iy.$$

The case $q = 0$ is a double upper contact.

Lemma 14. *The contacts of P satisfy*

$$|x + 0.707163467293| \le 0.05999, \qquad y \le 0.870045373550 + 0.00999, \qquad q \le 1/50.$$

Proof. Let $e_1 = 2r + \zeta$, $e_2 = r^2 - q + 2r\zeta$ and $e_3 = (r^2 - q)\zeta$ be the elementary symmetric functions of the three contacts; note that $\operatorname{Im} e_1 = y$. By (8) the two remaining zeros have sum $-e_1$ and product $e_2 - e_3/e_1$. Expressing that P has the form (7) gives the complex equation

$$(4e_2 + 8e_3 + 6)(e_1e_2 - e_3) + e_1\{-4e_1e_3 + 6(e_2 - e_1^2) + 15\} = 0. \tag{11}$$

Let $\delta = e_1^2 - 4(e_2 - e_3/e_1)$ be the square of the difference of the two remaining zeros. One of them is upper, of height at least y. The other is lower, and its height is at least $2y$, because the imaginary parts of all five zeros sum to zero. So their imaginary parts differ by at least $3y$, which gives

$$(\operatorname{Im}\delta)^2 \ge 36y^2(\operatorname{Re}\delta + 9y^2). \tag{12}$$

By Lemmas 12 and 32, the point (x, s, q, y^2) lies in the box

$$[-1.70, -0.15] \times [-5.6, 5.6] \times [0, 0.6] \times [0.7569, 1.35].$$

After denominators are cleared, (11) and (12) become polynomial conditions in (x, s, q, y^2). An exact computation [1] subdivides the box into 11,299 cells and writes these polynomials in Bernstein form on each cell. On a cell, a polynomial lies between the least and the greatest of its Bernstein coefficients, since the basis is nonnegative and sums to one; so a polynomial that must vanish has no zero on a closed cell where all its coefficients have one strict sign, and a polynomial that must be nonnegative is negative on a closed cell where all its coefficients are negative. This excludes 11,180 cells. The other 119 cells lie in the stated neighborhood. □

The local peak. The last lemma compares P with the candidate inside the neighborhood of Lemma 14. Write $r_0 = x_0 + ih_0$, and consider quintics with two zeros $r \pm d$ near r_0, where $r = r_0 + \xi + i\eta$ and $d \geq 0$ is real.

Lemma 15. *Let a quintic have two zeros $r \pm d$ as above, with $|\xi| \leq 0.06$, $0 \leq \eta \leq 0.01$ and $d^2 \leq 1/50$. Then it has a lower zero γ with*

$$\operatorname{Im}\gamma + h_0 \;\geq\; 0.005\,\eta + 0.25\,\xi^2 + 0.35\,d^2.$$

In particular its height is at most h_0, with equality only for the candidate.

Proof. For $|\xi| \leq 0.06$, $0 \leq \eta \leq 0.01$ and $q = d^2$ in the disk $|q| \leq 1/50$, the quintic with the zeros $r \pm d$ is determined by r and q. Dividing it by $(z-r)^2 - q$ leaves a cubic which has exactly one zero $\gamma(r,q)$ in the disk of radius $1/10$ about $1.639753235047 - 0.870045373550i$, and γ is analytic in r and q [1]. At the candidate, $\gamma(r_0, 0) = s_0 - ih_0$. The exact calculation on the cube of (9) also gives $\operatorname{Re}\gamma_r(r_0,0) > 0.04$ [1], and the third equation of (9) gives $\operatorname{Im}\gamma_r(r_0,0) = 0$. A second exact calculation [1] divides this region of r into 96 equal rectangles and proves on each that

$$\begin{aligned} &1.9 < \operatorname{Im}\gamma_{rr} < 3.5, && |\operatorname{Re}\gamma_{rr}| < 1.4 && \text{at } q = 0,\\ &0.35 < \operatorname{Im}\gamma_q && && \text{for } |q| \leq 1/50. \end{aligned}$$

Integrate γ_{rr} along the segment from r_0 to r at $q = 0$, and then γ_q along the real segment from 0 to d^2 at fixed r. With $r - r_0 = \xi + i\eta$ this gives

$$\operatorname{Im}\gamma + h_0 \;\geq\; 0.04\,\eta + 0.95\,\xi^2 - 1.4\,|\xi|\eta - 1.75\,\eta^2 + 0.35\,d^2.$$

Completing the square in $|\xi|$ shows that the right side is at least $0.04\,\eta - 2.45\,\eta^2 + 0.25\,\xi^2 + 0.35\,d^2$, and $2.45\,\eta^2 \leq 0.035\,\eta$ for $\eta \leq 0.01$. This is the stated inequality. If its right side is positive, the lower zero γ has height less than h_0. Otherwise $\eta = \xi = d = 0$, and the quintic has the double zero r_0; it is the candidate. □

End of the proof for the pair $(3,5)$. Let P be a quintic of largest height $y \geq h_0$, with three upper zeros. By Lemmas 13 and 14 its upper contacts are $r \pm d$ with $d = \sqrt{q}$ real, $|x + 0.707163467293| \leq 0.05999$, $q \leq 1/50$ and $h_0 \leq y \leq 0.870045373550 + 0.00999$. Since (x_0, h_0) differs from $(-0.707163467293, 0.870045373550)$ by at most 10^{-8} in each coordinate, P satisfies the hypotheses of Lemma 15 with $\eta = y - h_0$. Hence $y \leq h_0$, so $y = h_0$, and P is the candidate. Undoing the conjugation gives the second extremal quintic. With Proposition 2 this proves Theorem 9.

Discussion

Before this paper the strip problem was solved only for the trinomials $1 + 2z + aH_n(z)$ [7, Theorem I], [9, pp. 147–148]. Szegő recorded the quadrinomial case as open in 1975 [11, p. 158]. Theorem 1 settles the quadrinomials; as for trinomials, the extremal polynomials are two conjugate cubics. One correction of the record: Turán [12, p. 62] quotes $1/2$ as Schmeisser's value, but $1/2$ is the least constant for $n = 2$ alone, and Schmeisser's constant for all n is $K = 0.52891952\ldots$ [9, pp. 147–148].

We end with three questions.

(1) *More Hermite terms.* We expect that the proof of Proposition 3, with five slots in place of four, gives a disk, and so a strip, that contains a zero of every polynomial $1 + 2z + aH_n(z) + bH_m(z) + cH_k(z)$ with $2 \leq n < m < k$ and $a, b, c \in \mathbb{C}$; this paper does not carry this out

(a general form of the argument is in a note of the supplement [1]). If so, what is the least half-width h_3 of a closed strip $|\operatorname{Im} z| \le h_3$ with this property? These polynomials are the simplest of Szegő's "more general equations" [11, p. 158]. The closest results are Theorem 1 for two Hermite terms and Schmeisser's theorem for one.

(2) *The other index pairs.* Section 5 (Theorem 9) shows that the largest value of $\mathcal{H}(F_{n,m})$ over all pairs other than $(2,3)$ is $h_0 = 0.87004537354988\ldots$, attained only for the pair $(3,5)$. What is the supremum of $\mathcal{H}(F_{n,m})$ over the pairs outside $\{(2,3),(3,5)\}$? Lemma 26 gives the largest height $0.8286\ldots$ for the pair $(2,4)$, and Proposition 2 bounds every height for these pairs by $87/100$.

(3) *A disk.* Proposition 3 shows, by compactness, that some disk $|z| < R$ contains a zero of every $F_{n,m}$, but it gives no value of R. What is the infimum of such R?

Acknowledgments

This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

A Written proofs

This appendix proves the analytic estimates that Section 4 states without proof, and gives the written proofs for the five pairs with $m \le 6$; Appendix A.5 proves the steps of Section 5 for the pair $(3,5)$. Exact computations with explicit rational numbers (root enclosures, signs of explicit polynomials, lists of Rouché disks) are checked by programs in the supplement and marked [1]. A letter introduced inside a proof is local to that proof.

A.1 Estimates for the case $n \le m/2$ of Lemma 8

We prove the estimates that Steps 2 and 4 of the case $n \le m/2$ of the proof of Lemma 8 quote, and, at the end, the bound used by the second type of circle in (a). The notation is that of Section 4.

Lemma 16 (Mehler's bound)**.** *For every integer $n \ge 0$ and every $z = X + iY \in \mathbb{C}$, $|\psi_n(z)|e^{-X^2/2} \le (1-1/25)^{-1/4}\exp\{-X^2/3 + Y^2/4 + (n/2)\log 5\}$.*

Proof. Mehler's formula states that, for real z_1, z_2 and $0 \le r < 1$, $\sum_{k\ge 0} r^k \psi_k(z_1)\psi_k(z_2) = (1-r^2)^{-1/2}\exp\{(2z_1z_2r - (z_1^2+z_2^2)r^2)/(1-r^2)\}$; see [8, (18.18.28)], where $\psi_k(z_1)\psi_k(z_2)$ is written $H_k(z_1)H_k(z_2)/(2^k k!)$. Cauchy's estimate for the generating function $\sum_k H_k(z)t^k/k! = e^{2zt-t^2}$ on $|t| = \sqrt{k}$, and $k! \le k^k$, give $|\psi_k(z)| \le (e/\sqrt{2})^k e^{2|z|\sqrt{k}}$ for $k \ge 1$ and complex z. So for $r = 1/5$ the kth term is at most $(e^2/10)^k e^{2\sqrt{k}(|z_1|+|z_2|)}$, with $e^2/10 < 0.74$; so the series converges uniformly on bounded subsets of $\mathbb{C}^2$, both sides are entire in (z_1,z_2), and as they agree on $\mathbb{R}^2$ they agree on $\mathbb{C}^2$. Take $z_1 = z$, $z_2 = \bar{z}$, $r = 1/5$. As ψ_k has real coefficients, the kth term is $5^{-k}|\psi_k(z)|^2 \ge 0$, and the sum is at least its term with $k = n$. As $z\bar{z} = X^2 + Y^2$ and $z^2 + \bar{z}^2 = 2(X^2 - Y^2)$, the exponent is $\frac{2r}{1+r}X^2 + \frac{2r}{1-r}Y^2 = X^2/3 + Y^2/2$. Hence $5^{-n}|\psi_n(z)|^2 \le (1-1/25)^{-1/2}e^{X^2/3+Y^2/2}$; take square roots and multiply by $e^{-X^2/2}$. □

The arithmetic of Step 2. Let $z = X + iY \in \Gamma$. By Step 1, $\sqrt{9m/5} \le x \le \sqrt{9m/5} + 3\pi/\sqrt{m}$, so $|Y| \le 1/\sqrt{m}$, $X \ge x - 1/\sqrt{m} > 0$ and $X^2 - Y^2 \ge (x - 1/\sqrt{m})^2 - 1/m = x^2 - 2x/\sqrt{m} \ge 1.8m - 2\sqrt{1.8} > 1.8m - 2.7$, the second inequality as $t^2 - 2t/\sqrt{m}$ increases for $t \ge 1/\sqrt{m}$. Lemma 16 gives $|\psi_n| \le (25/24)^{1/4} e^{X^2/6 + Y^2/4 + (n/2)\log 5}$, Step 1 gives $|\psi_m| > e^{(X^2-Y^2)/2}/(5\sqrt{m})$, and $X^2/6 + Y^2/4 - (X^2 - Y^2)/2 = -(X^2 - Y^2)/3 + 5Y^2/12$. With $Y^2 \le 1/m$, $n \le m/2$ and $m \ge 450$, it follows that $|\psi_n/\psi_m| < 5(25/24)^{1/4}\sqrt{m}\, e^{-(0.6 - \frac{1}{4}\log 5)m + 0.9 + 5/(12m)}$, where the exponent is less than $-0.197m + 0.901$, as $0.6 - \frac{1}{4}\log 5 = 0.19764\ldots$; and $5(25/24)^{1/4} e^{0.901} < 12.5$, so $|\psi_n| < 12.5\sqrt{m}\, e^{-0.197m} |\psi_m|$. Next, $|z| \le x + 1/\sqrt{m}$ gives $|1 + 2z| \le (2\sqrt{1.8} + 1/\sqrt{m} + (6\pi + 2)/m)\sqrt{m} < 2.78\sqrt{m}$, so $|1 + 2z|/|\psi_m| < 2.8\sqrt{m} \cdot 5\sqrt{m}\, e^{-(1.8m - 2.7)/2} = 14 e^{1.35} m\, e^{-0.9m} < 55m\, e^{-0.9m}$, and $55m\, e^{-0.9m} < 20\sqrt{m}\, e^{-0.197m}$ because $2.75\sqrt{m} < e^{0.703m}$ for $m \ge 450$.

Lemma 17 (a zero of the trinomial)**.** *For every integer $n \ge 2$ and every $a \in \mathbb{C}$, the trinomial $G = 1 + 2z + a\psi_n$ has a zero c with $|\operatorname{Re} c| \le 2$ and $|\operatorname{Im} c| < 0.76$.*

Proof. Put $w_0 = -1/2$. If $a = 0$, take $c = w_0$. Let $a \ne 0$, and let $\zeta_1, \ldots, \zeta_n$ be the zeros of G. Comparing $G(w_0) = a\psi_n(w_0)$ with the leading coefficient $a2^n/\sqrt{2^n n!}$ of G gives $\prod_k |\zeta_k - w_0| = 2^{-n}|H_n(w_0)|$.

$n = 2$. As $H_2(w_0) = -1$, some zero has $|\zeta - w_0| \le 1/2$, and it lies in the rectangle.

$n = 3$. As $H_3(w_0) = 5$, the product is $5/8$. Suppose that no zero lies in the rectangle. Then every factor is at least 0.76, and a factor with $|\operatorname{Re}\zeta| > 2$ exceeds $3/2$; so, since $1.5 \cdot 0.76^2 = 0.8664 > 5/8$, $|\operatorname{Im}\zeta_k| \ge 0.76$ for $k = 1, 2, 3$. As ψ_3 is odd, $\zeta_1 + \zeta_2 + \zeta_3 = 0$; two imaginary parts have the same sign, so the third has modulus at least 1.52, and the product is at least $0.76^2 \cdot 1.52 = 0.877952 > 5/8$, a contradiction.

$4 \le n \le 79$. Let $N = \lceil\sqrt{2n}\rceil$, and let Ψ be the real multiple of ψ_n with $\Psi(0) = 1$ if n is even and $\Psi'(0) = N$ if n is odd. A constant multiple of G is $N(z - w_0) + \vartheta_0\Psi$. Dividing by ϑ_0 if $|\vartheta_0| > 1$, and passing to conjugate coefficients if the parameter has negative imaginary part (this conjugates the zeros, and the rectangle is symmetric about $\mathbb{R}$), we may take G of the form $N(z - w_0) + \vartheta\Psi$ or $\vartheta N(z - w_0) + \Psi$ with $\vartheta \in [-1, 1] + i[0, 1]$. For each n and each form the records [1] divide $[-1, 0] + i[0, 1]$ and $[0, 1] + i[0, 1]$, by repeated quartering, into closed squares (6,598 in all), and give for each square a center ω with real and imaginary parts in $\frac{1}{16384}\mathbb{Z}$, a radius $\varrho \in \{\frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2\}/N$ and an order $k \in \{1, 2, 3, 4\}$. Exact rational arithmetic [1] verifies: $|\omega| + \varrho < 9/10$; $|\operatorname{Im}\omega| + \varrho < 0.76$ (the largest value is $7753/10240$); and, for every ϑ in the square, the coefficient of order k of the polynomial in powers of $(z - \omega)/\varrho$ exceeds in modulus the sum of the moduli of the others. The coefficients of order at most 12 are bounded by their values at the midpoint of the square, computed with outward rounding, plus their change over the square; the higher ones through $l!\,|\gamma_l| \le 2^l \max(|\gamma_0|, |\gamma_1|)$ for the Taylor coefficients γ_l of $\Psi(\omega + u/N)$ in u, which follows by induction from the recursion $(l + 2)!\gamma_{l+2} = \frac{2\omega}{N}(l + 1)!\gamma_{l+1} + \frac{2(l-n)}{N^2} l!\gamma_l$ of the Hermite equation, whose coefficients have modulus at most 1, as $2|\omega| \le N$, $2n \le N^2$ and $\gamma_l = 0$ for $l > n$. By Rouché's theorem, as in (i) of Section 4, G has k zeros in $|z - \omega| < \varrho$, and by the first two checks they lie in the rectangle.

$n \ge 80$. We show that G has a zero in the disk $|z - w_0| < 0.76$, where $|\operatorname{Re} z| < 1.26$ and $|\operatorname{Im} z| < 0.76$. Suppose not. Then neither has $G^* = 1 + 2z + \bar{a}\psi_n$, since $G^*(z) = \overline{G(\bar{z})}$.

(A) On the circle $|z - w_0| = 0.65$ every zero of G or G^* is at distance at least 0.11, and $|z| \le 1.15$. So $|G'/G| \le n/0.11$ and $|G''/G| \le n(n - 1)/0.11^2$, and, as $L_n G = G + (G''/2 - zG')/n$, $|L_n G/G| \le 1 + 1.15/0.11 + (n - 1)/(2 \cdot 0.11^2) < 43n$. As L_n annihilates ψ_n, $L_n G = 1 + 2(1 - 1/n)z = 2(1 - 1/n)(z - w_0) + 1/n$; on the circle $|1 + 2z| = 1.3$ and $|L_n G| \ge 1.3(1 - 1/n) - 1/n$, so $|(1 + 2z)/L_n G| \le 0.65/(0.65(1 - 1/n) - 1/(2n)) \le 0.65/0.635625 < 1.03$. The same holds for G^*, as $L_n G^* = L_n G$.

(B) As $G^* = (\bar a/a)G + (1-\bar a/a)(1+2z)$ and $|\bar a/a| = 1$, (A) gives $|G^*/G| \le 1 + 2|(1+2z)/L_nG|\,|L_nG/G| < 1 + 2\cdot 1.03\cdot 43n < 90n$ on the circle, and likewise $|G/G^*| < 90n$.

(C) The function $f = \log(G/G^*)$ has an analytic branch in $|z-w_0| < 0.76$. By (B) and the maximum principle for the harmonic function $\operatorname{Re} f = \log|G/G^*|$, $|\operatorname{Re} f| < \log(90n)$ on $|z-w_0| \le 0.65$. For real t, $G^*(t) = \overline{G(t)}$, so $\operatorname{Re} f(t) = 0$ and $f'(t) = 2i\frac{d}{dt}\arg G(t)$. Fix a real t with $|t-w_0| < 0.65$ and put $f_t = \pi(f - f(t))/(4\log(90n))$; then $|\operatorname{Re} f_t| < \pi/4$. As $\tan$ maps $|\operatorname{Re}\zeta| < \pi/4$ into the unit disk, with $\tan 0 = 0$ and $\tan' 0 = 1$, the Schwarz–Pick lemma for $\tan f_t$ on $|z-w_0| < 0.65$ gives $|f_t'(t)| \le 0.65/(0.65^2 - (t-w_0)^2)$, that is, $|\frac{d}{dt}\arg G(t)| \le (2\log(90n)/\pi)\cdot 0.65/(0.65^2-(t-w_0)^2)$ for $|t-w_0| < 0.65$.

(D) For $|t-w_0| \le 0.25$ the coefficient $2n+1-t^2$ in the equation of ϕ_n exceeds $2n$, and $\beta_n = \pi/\sqrt{2n} \le \pi/\sqrt{160} < 0.25$. By Sturm's comparison theorem with $\sin(\sqrt{2n}\,t)$, every interval of length β_n in $[w_0-0.25, w_0+0.25]$ contains a zero of ψ_n. Let s_1 be the largest zero of ψ_n in $[w_0-\beta_n, w_0]$ and s_2 the smallest in $[w_0, w_0+\beta_n]$. As $G(w_0) = a\psi_n(w_0) \ne 0$, $s_1 < w_0 < s_2$, and ψ_n has no zero in (s_1,s_2); so the comparison on $[s_1, s_1+\beta_n]$ gives $s_2 - s_1 < \beta_n$.

(E) $G(s_1) = 2(s_1-w_0) < 0 < 2(s_2-w_0) = G(s_2)$. If a is real, G has a zero in (s_1,s_2), which is excluded. Otherwise $\operatorname{Im} G = (\operatorname{Im} a)\psi_n$ has one strict sign on (s_1,s_2), so $\arg G$ changes by exactly π in modulus there. The function $J(u) = \operatorname{artanh}(u/0.65)$ has derivative $0.65/(0.65^2-u^2)$ and is convex with $J(0) = 0$, so $J(u_1)+J(u_2) \le J(u_1+u_2)$. Integrating the bound of (C) over $[s_1,s_2]$ gives $\pi \le \frac{2\log(90n)}{\pi}(J(w_0-s_1)+J(s_2-w_0)) \le \frac{2\log(90n)}{\pi}J(\beta_n)$, that is, $\pi^2 \le 2\log(90n)\operatorname{artanh}(\beta_n/0.65)$. At $n = 80$ the right side is $2\log 7200\cdot\operatorname{artanh}(0.382099\ldots) = 2\cdot 8.8818\ldots\cdot 0.40251\ldots < 7.2$, and it decreases in n: $\operatorname{artanh}(u)/u$ increases, so $\operatorname{artanh}(\beta_n/0.65) \le \sqrt{80/n}\operatorname{artanh}(\beta_{80}/0.65)$, and $\log(90n)/\sqrt{n}$ decreases. As $\pi^2 > 9.8$, this is a contradiction. □

Lemma 18 (the coefficient a at the zero c)**.** *Let $m \ge 450$ and $2 \le n \le m/2$, let $F = 1+2z+a\psi_n+b\psi_m$ have no zero in the strip $|\operatorname{Im} z| < 0.87$, and let c be a zero of $1+2z+a\psi_n$ with $|\operatorname{Re} c| \le 2$ and $|\operatorname{Im} c| < 0.76$. Then $F(c) \ne 0$, $1 < 9000\,m^4|F(c)|$ and $1+|a| < 110000\,m^{9/2}|F(c)|$.*

Proof. As c lies in the strip, $F(c) \ne 0$; and $|c| < \sqrt{4+0.76^2} < 2.14$. Put $\varpi = 100m/11 \ge 4090$. By Step 4, $|F^{(\ell)}(c)/F(c)| \le \varpi^\ell$ for all ℓ. All quotients below are taken at c. Direct differentiation gives $(E_*F)' = \frac12 F''' - zF'' - F'$, $E_*^2F = \frac14 F^{(4)} - zF''' + (z^2-1)F'' + zF'$ and $(E_*^2F)' = \frac14 F^{(5)} - zF^{(4)} + (z^2-2)F''' + 3zF'' + F'$. With $|c| < 2.2$, $|c^2-2| < 6.84$ and $2.2/\varpi + 6.84/\varpi^2 + 6.6/\varpi^3 + 1/\varpi^4 < 0.001$,

$$\Big|\frac{(E_*F)'}{F}\Big| \le \frac{\varpi^3}{2} + 2.2\varpi^2 + \varpi < 0.501\,\varpi^3, \quad \Big|\frac{(E_*^2F)'}{F}\Big| \le \frac{\varpi^5}{4} + 2.2\varpi^4 + 6.84\varpi^3 + 6.6\varpi^2 + \varpi < 0.251\,\varpi^5.$$

The bound for $1/|F(c)|$. Insert $L_nL_m = 1 + (1/n+1/m)E_* + E_*^2/(nm)$ in the first identity of Step 4, divide by $F(c)$, and use $2(1-1/n) \ge 1$ and $n \ge 2$: $(1-1/m)/|F(c)| \le \varpi + (\frac12 + \frac1m)0.501\,\varpi^3 + 0.251\,\varpi^5/(2m)$. In units of m^4 the three terms are at most 10^{-6}, $0.50223\cdot 0.501\cdot(100/11)^3/m < 0.43$ and $0.251(100/11)^5/2 < 7792.6$; none of them increases with m, and $1/(1-1/m) \le 1.0023$. So $1/|F(c)| < 7811\,m^4$, and $1 < 9000\,m^4|F(c)|$.

The bound for $|a|$. As $L_mF = F + E_*F/m$, the bounds above give $|L_mF/F| \le 1 + (\varpi^2/2 + 2.2\varpi)/m = (100/11)^2m/2 + 21 < 41.33m + 21 < 43m$ and $|(L_mF)'/F| \le \varpi + 0.501\,\varpi^3/m < 376.41m^2 + 9.1m < 390m^2$. By the second identity of Step 4, $a(1-n/m)\psi_n = L_mF - L_m(1+2z)$, where $L_m(1+2z) = 1 + 2(1-1/m)z$ has modulus at most $1+2|c| < 5.4$ at c, and its derivative has modulus at most 2. With $1 < 9000\,m^4|F(c)|$ this gives $|a|(1-n/m)|\psi_n(c)| < 43m|F(c)| + 5.4 < (43m + 48600m^4)|F(c)|$ and $|a|(1-n/m)|\psi_n'(c)| < 390m^2|F(c)| + 2 < (390m^2 + 18000m^4)|F(c)|$. As $1-n/m \ge 1/2$ and $\sqrt{2n} \ge 2$, it follows that $|a||\psi_n(c)| < (86m + 97200m^4)|F(c)|$ and $|a||\psi_n'(c)|/\sqrt{2n} < (390m^2 + 18000m^4)|F(c)|$; both bounds are less than $100000\,m^4|F(c)|$. By Step 4 the larger of $|\psi_n(c)|$ and $|\psi_n'(c)|/\sqrt{2n}$ is at least $1/\sqrt{m+2}$, so $|a| < 100000\,m^4\sqrt{m+2}\,|F(c)|$. With $1 < 9000\,m^4|F(c)|$,

$1+|a| < (9000/\sqrt{m}+100000\sqrt{1+2/m})\,m^{9/2}|F(c)| < 100650\,m^{9/2}|F(c)| < 110000\,m^{9/2}|F(c)|$, since $9000/\sqrt{450} < 424.3$ and $100000\sqrt{1+2/450} < 100222$. □

Lemma 19 (ψ_m near the real axis)**.** *If $m \ge 1$, $|\operatorname{Re} c| \le 2$ and $|\operatorname{Im} c| < 0.76$, then $|\psi_m(c)| \le e^{8+0.76\sqrt{2m}}$.*

Proof. The Hermite equation $H_m'' = 2zH_m' - 2mH_m$ gives $\psi_m'' = 2z\psi_m' - 2m\psi_m$. For real t let $I(t) = \psi_m(t)^2 + \psi_m'(t)^2/(2m)$. Then $I'(t) = 2t\,\psi_m'(t)^2/m \le 4tI(t)$ for $t \ge 0$, so $I(t) \le I(0)e^{2t^2}$ for $t \ge 0$, and for $t \le 0$ as I is even. By the facts at $t = 0$ quoted in Step 1, and as ψ_m has the parity of m, $I(0) = \binom{m}{m/2}2^{-m} \le 1$ for even m and $I(0) = \psi_{m-1}(0)^2 \le 1$ for odd m; so $I \le e^8$ on $[-2,2]$. The vector $V = (\psi_m, \psi_m'/\sqrt{2m})$ satisfies $V' = CV$ with $C = \left(\begin{smallmatrix} 0 & \sqrt{2m} \\ -\sqrt{2m} & 2z \end{smallmatrix}\right)$, whose operator norm is at most $\sqrt{2m} + 2|z|$. Write $c = X + iY$. On the segment from X to c, $|z| < 2.14$, and Gronwall's inequality gives $|\psi_m(c)| \le |V(c)| \le |V(X)|e^{(\sqrt{2m}+4.28)|Y|} \le e^4 e^{0.76\sqrt{2m}+3.26} < e^{8+0.76\sqrt{2m}}$, as $|V(X)| = \sqrt{I(X)} \le e^4$ and $4.28 \cdot 0.76 < 3.26$. □

The last inequality of Step 4. Its constant is $2200000 = 20 \cdot 110000$ (after cancelling $|F(c)| \ne 0$). At $m = 450$ the logarithm $\log 2200000 + 5\log m + 8 + 0.76\sqrt{2m} - 0.197m$ of its right side is, to four decimals, $14.6040 + 30.5462 + 8 + 22.8 - 88.65 = -12.6998$, and its derivative $5/m + 0.76/\sqrt{2m} - 0.197$ decreases and is $-289/1800$ at $m = 450$.

Lemma 20 (circles of the second type)**.** *Let $j \ge 0$ be an integer, ξ real, $\rho > 0$, Q_j as in (a) of Section 4, and $\delta_j = \rho\,\phi_j'(\xi)$, the coefficient of v in $\phi_j(\xi + \rho v)$. Then for $|v| \le 1$, $|\phi_j(\xi + \rho v) - \delta_j v| \le \cosh(\sqrt{Q_j})|\phi_j(\xi)| + (\sinh(\sqrt{Q_j})/\sqrt{Q_j} - 1)|\delta_j|$.*

Proof. Let $u(v) = \phi_j(\xi + \rho v)$. Then $u'' = -qu$ with $q(v) = \rho^2(2j + 1 - \xi^2 - 2\xi\rho v - \rho^2 v^2)$, so $|q| \le Q_j$ on $|v| \le 1$, and Taylor's formula with integral remainder along the radius from 0 to v gives

$$u(v) = u(0) + \delta_j v - v^2 \int_0^1 (1 - \lambda)\, q(\lambda v)\, u(\lambda v)\, d\lambda.$$

Let $u_0(v) = u(0) + \delta_j v$, and let u_{k+1} be the integral term with u_k in place of u. As $\int_0^1 (1-\lambda)\lambda^\ell d\lambda/\ell! = 1/(\ell+2)!$, induction gives $|u_k(v)| \le Q_j^k(|u(0)||v|^{2k}/(2k)! + |\delta_j||v|^{2k+1}/(2k+1)!)$ on $|v| \le 1$. So $\sum_k u_k$ converges uniformly and solves the integral equation. The difference of two solutions satisfies it without the term u_0, and the same induction bounds it by its maximum times $Q_j^k/(2k)!$ for every k; so $u = \sum_k u_k$, and $|u(v) - \delta_j v| = |u(0) + \sum_{k\ge1} u_k(v)| \le |u(0)| \sum_{k\ge0} Q_j^k/(2k)! + |\delta_j| \sum_{k\ge1} Q_j^k/(2k+1)!$. The two sums are $\cosh\sqrt{Q_j}$ and $\sinh(\sqrt{Q_j})/\sqrt{Q_j} - 1$. □

A.2 The case $m/2 \le n < m$ of Lemma 8

This part proves the estimates that Steps 1 and 2 of the case $m/2 \le n < m$ of the proof of Lemma 8 cite, and it gives the numbers of its Step 3. We keep the assumptions and the notation of that case. Let j be one of n and m, and j' the other one. Put $\beta_n = a$ and $\beta_m = b$, so that $U_j = \beta_j\psi_j/F$. Then $|j - j'| = m - n$, $j \ge m/2 \ge 225$, $j' \le m$ and $\sqrt{2m} \ge 30$; and for every point z_0, with $\Omega = \sqrt{2m + 16(\operatorname{Re} z_0)^2}$ as in Step 1, we have $\Omega^2 \ge 2m \ge 900$ and $|\operatorname{Re} z_0| \le \Omega/4$.

Lemma 21. *If $|\operatorname{Im} z_0| \le y_*$, then $|F^{(\ell)}(z_0)/F(z_0)| \le (11\Omega)^\ell$ for $1 \le \ell \le 4$.*

Proof. Let $D = \Omega^{-1}\partial_z$, as in Step 1 of the proof of Proposition 3, and let $\|g\|_r$ be the maximum of $|g|$ on the disk $|z - z_0| \le r/\Omega$. Since $7/11 + 7/30 < 0.87$, F has no zero on the disk $|z - z_0| \le 7/\Omega$ (Step 1 of that case). On it $|z| \le \Omega/4 + y_* + 7/\Omega$, so that $|2z/\Omega| \le 2(1/4 + 7/330 + 7/900) = 1381/2475 < 0.56$.

Growth. Put $k_0 = 0$, $k_1 = 1$, $k_2 = n$, $k_3 = m$, $G_0 = F$ and $G_{\nu+1} = 2\Omega^{-2}(E_* + k_\nu)G_\nu = (D^2 - (2z/\Omega)D + 2k_\nu/\Omega^2)G_\nu$. The operators $E_* + k_\nu$ commute, and $E_* + k_\nu$ annihilates the term of F of index k_ν (Step 1 of that case); so $G_4 = 0$. Also $0 \le 2k_\nu/\Omega^2 \le 1$. Let $\mathbf{G}$ be the vector of the eight functions G_ν and $DG_\nu/1.75$ ($0 \le \nu \le 3$), and $|\mathbf{G}|$ the largest modulus of its entries. Since

$$D\,G_\nu = 1.75 \cdot \frac{DG_\nu}{1.75}, \qquad D\,\frac{DG_\nu}{1.75} = \frac{G_{\nu+1}}{1.75} + \frac{2z}{\Omega} \cdot \frac{DG_\nu}{1.75} - \frac{2k_\nu}{1.75\,\Omega^2}\,G_\nu$$

and $2/1.75 + 0.56 < 1.75$, every entry of $D\mathbf{G}$ has modulus at most $1.75|\mathbf{G}|$ on the disk $|z - z_0| \le 7/\Omega$. Along a radius, the derivative of $\mathbf{G}$ in $\rho = \Omega|z - z_0|$ is $D\mathbf{G}$ times a number of modulus one, and Gronwall's inequality gives $|\mathbf{G}(z)| \le e^{1.75\Omega|z-z_0|}|\mathbf{G}(z_0)|$; so $\|F\|_7 \le e^{12.25}|\mathbf{G}(z_0)|$.

Cauchy's estimates. For $1/2 \le r \le 7$, Cauchy's estimates on the disks of radius $1/(2\Omega)$ give $\|Dg\|_{r-1/2} \le 2\|g\|_r$ and $\|D^2 g\|_{r-1/2} \le 8\|g\|_r$; hence $\|G_{\nu+1}\|_{r-1/2} \le (8 + 2\cdot 0.56 + 1)\|G_\nu\|_r \le 11\|G_\nu\|_r$. So $\|G_\nu\|_{2-\nu/2} \le 11^\nu \|F\|_2$ for $\nu \le 3$, and Cauchy's estimate on the disk $|z - z_0| \le (2 - \nu/2)/\Omega$ gives $|DG_\nu(z_0)| \le 11^\nu \|F\|_2/(2 - \nu/2)$. The largest of these eight bounds is the one for $DG_3(z_0)/1.75$, namely $(2 \cdot 11^3/1.75)\|F\|_2 = (10648/7)\|F\|_2$. Thus $|\mathbf{G}(z_0)| < 1600\,\|F\|_2$.

Harnack's inequality. The function $\Upsilon = \log(e^{12.25}|\mathbf{G}(z_0)|/|F|)$ is harmonic and, by the growth bound, nonnegative on the disk $|z - z_0| \le 7/\Omega$. At a point z_1 with $|z_1 - z_0| \le 2/\Omega$ and $|F(z_1)| = \|F\|_2$ it satisfies $\Upsilon(z_1) < 12.25 + \log 1600 < 19.628$. Harnack's inequality gives $\Upsilon(z_0) \le \frac{7+2}{7-2}\Upsilon(z_1) < 35.34$.

The derivatives of $\log F$. Let $\log F$ be a branch on $|z - z_0| < 7/\Omega$, and $g = \log(e^{12.25}|\mathbf{G}(z_0)|) - \log F$, so that $\operatorname{Re} g = \Upsilon \ge 0$. Carathéodory's inequality $|D^\ell g(z_0)| \le 2\,\ell!\,\operatorname{Re} g(z_0)/7^\ell$ gives, for $\mu_\ell = D^\ell \log F(z_0)$,

$$|\mu_\ell| \le \frac{\ell!\,\Delta}{7^{\ell-1}} \quad (\ell \ge 1), \qquad \Delta = \frac{2\Upsilon(z_0)}{7} < \frac{2 \cdot 35.34}{7} < 10.1.$$

By Faà di Bruno's formula, at z_0, $DF/F = \mu_1$, $D^2F/F = \mu_1^2 + \mu_2$, $D^3F/F = \mu_1^3 + 3\mu_1\mu_2 + \mu_3$ and $D^4F/F = \mu_1^4 + 6\mu_1^2\mu_2 + 4\mu_1\mu_3 + 3\mu_2^2 + \mu_4$. So their moduli are at most Δ, $\Delta^2 + \frac{2}{7}\Delta$, $\Delta^3 + \frac{6}{7}\Delta^2 + \frac{6}{49}\Delta$ and $\Delta^4 + \frac{12}{7}\Delta^3 + \frac{36}{49}\Delta^2 + \frac{24}{343}\Delta$. Each coefficient of the ℓth bound is at most the corresponding coefficient of $(\Delta + 3/7)^\ell$ (binomial theorem). Since $\Delta + 3/7 < 11$ and $D^\ell F/F = \Omega^{-\ell} F^{(\ell)}/F$, the lemma follows. □

Lemma 22. *If* $|\operatorname{Im} z_0| = y_*$, *then* $|U_j(z_0)| \le 4200\,\Omega^4/(j\,(m - n))$, *for* $j = n$ *and* $j = m$.

Proof. For $j \in \{n, m\}$ let $P_j = L_j(1 + 2z) = 1 + 2(1 - 1/j)z$ and $\Xi_j = P_{j'}/P_{nm}$; thus $\Xi_m = \Xi$. The identity of Step 1 and its form with n and m exchanged read

$$(j' - j)\,\beta_j\psi_j = (1 - \Xi_j)(E_* + j')F - \frac{\Xi_j}{j}\,E_*(E_* + j')F. \tag{13}$$

The quotient Ξ_j. The zero $x_* = -1/(2(1 - 1/n)(1 - 1/m))$ of P_{nm} satisfies $|x_*| \le 1/(1 - 1/m) \le 450/449$. Since $P_{j'} - P_{nm} = -E_*P_{j'}/j = 2(1 - 1/j')z/j$ and $P_{nm} = 2(1 - 1/j')(1 - 1/j)(z - x_*)$, we have $\Xi_j - 1 = z/((j - 1)(z - x_*))$. On the two lines $|z - x_*| \ge y_*$, so $|z/(z - x_*)| \le 1 + |x_*|/y_* \le 1 + (450/449)(11/7) < 2.575$; with $1/(j - 1) \le (225/224)/j$ and $2.575 \cdot 225/224 < 2.59$ this gives $|\Xi_j - 1| < 5/j$ and $|\Xi_j| < 1 + 5/j \le 46/45$.

The quotients of F. We have $|z_0|/\Omega \le 1/4 + 7/330 < 0.272$. Lemma 21, $E_*^2 = \partial^4/4 - z\partial^3 + (z^2 - 1)\partial^2 + z\partial$ (a direct computation) and $\Omega^2 \ge 900$ give, at z_0, $|E_*F/F| \le (11\Omega)^2/2 + 11|z_0|\Omega < (60.5 + 11 \cdot 0.272)\Omega^2 < 64\,\Omega^2$ and

$$\Big|\frac{E_*^2 F}{F}\Big| \le \Big(\frac{11^4}{4} + 11^3 \cdot 0.272 + 11^2\Big(0.272^2 + \frac{1}{900}\Big) + \frac{11 \cdot 0.272}{900}\Big)\Omega^4 < 4035\,\Omega^4,$$

where the four terms are 3660.25, 362.032, less than 9.087 and less than 0.004.

Conclusion. Divide (13) by F, and use $E_*(E_*+j')F = E_*^2F + j'E_*F$ and $j' \le m \le \Omega^2/2$. At z_0,

$$(m-n)|U_j| \le |1-\Xi_j|\Big(\Big|\frac{E_*F}{F}\Big| + j'\Big) + \frac{|\Xi_j|}{j}\Big(\Big|\frac{E_*^2F}{F}\Big| + j'\Big|\frac{E_*F}{F}\Big|\Big) \le \frac{5}{j}\cdot 64.5\,\Omega^2 + \frac{46}{45j}(4035+32)\,\Omega^4,$$

which is at most $(322.5/900 + 46\cdot 4067/45)\Omega^4/j < (0.36 + 4157.38)\Omega^4/j < 4200\,\Omega^4/j$. □

Lemma 23. *Inequality* (4) *holds for $j = n$ and for $j = m$.*

Proof. Put $\Pi(z) = (1 - 4iz/\sqrt{2m})^4$; its zero $-i\sqrt{2m}/4$ has imaginary part at most -7.5. Let s run through the zeros of ψ_j in $[w_0-3, w_0+3]$; they are real and simple. For $z = X+iY$ with $|Y| \le y_*$ and $\Omega = \sqrt{2m+16X^2}$ we have $4y_*/\sqrt{2m} \le 14/165$, and

$$\Big|1 - \frac{4iz}{\sqrt{2m}}\Big|^2 = \Big(1 + \frac{4Y}{\sqrt{2m}}\Big)^2 + \frac{16X^2}{2m} \ge \Big(1 - \frac{4y_*}{\sqrt{2m}}\Big)^2\frac{\Omega^2}{2m}, \quad \text{so} \quad |\Pi(z)| \ge \Big(\frac{151}{165}\Big)^4\frac{\Omega^4}{4m^2}.$$

With Lemma 22, on the two lines

$$|U_j/\Pi| \le 4\cdot 4200\,(165/151)^4\,m^2/(j\,(m-n)) < 23952\,m^2/(j\,(m-n)). \tag{14}$$

The strip factors. For real s put $\zeta = \pi(z-s)/(4y_*)$. If $|\operatorname{Im} z| \le y_*$, then $|\operatorname{Im}\zeta| \le \pi/4$, so $\operatorname{Re} e^{2\zeta} \ge 0$ and $|\tanh\zeta| = |e^{2\zeta}-1|/|e^{2\zeta}+1| \le 1$, with equality exactly on the two lines; and $|\tanh\zeta| \to 1$ as $|\operatorname{Re} z| \to \infty$, uniformly. The zeros of $\tanh\zeta$ are the points of $s + 4y_*i\mathbb{Z}$ and its poles have $|\operatorname{Im} z| \ge 2y_*$; so in the strip $|\operatorname{Im} z| < 0.87$ its only zero is the simple zero s, and it has no pole.

The maximum principle. Hence $V_j = U_j/(\Pi\prod_s \tanh(\pi(z-s)/(4y_*)))$ is analytic in the strip $|\operatorname{Im} z| < 0.87$, where F has no zero and U_j vanishes at each s. The rational function U_j is bounded for large $|z|$, since $\deg \beta_j\psi_j \le \deg F$ if $\beta_j \ne 0$ (and $U_j = 0$ if $\beta_j = 0$). As $|\Pi| \to \infty$, $V_j \to 0$ as $|\operatorname{Re} z| \to \infty$, uniformly in $|\operatorname{Im} z| \le y_*$; on the two lines $|V_j| = |U_j/\Pi|$. The maximum principle on the rectangles $|\operatorname{Re} z| \le X_1$, $|\operatorname{Im} z| \le y_*$, with $X_1 \to \infty$, and (14) give $|V_j| < 23952\,m^2/(j(m-n))$ on $|\operatorname{Im} z| \le y_*$. At w_0 we have $|\Pi(w_0)| = (1+2/m)^2 \le (226/225)^2$ and $23952\,(226/225)^2 < 24166 < 25000$; this proves (4). □

Lemma 24. *For $j = n$ and $j = m$, let s run through the real zeros of ψ_j in $[w_0-3, w_0+3]$. Then $-\log\prod_s|\tanh(\pi(w_0-s)/(4y_*))| > 0.608\sqrt{2j-11.25} - 0.315$, where the left side is $+\infty$ if $\psi_j(w_0) = 0$.*

Proof. Let $\psi_j(w_0) \ne 0$. Put $\alpha = \pi/(4y_*) = 11\pi/28$, $\eta = \pi/\sqrt{2j-11.25} \le \pi/\sqrt{438.75} < 0.151$ and $\omega(v) = \log\coth(\alpha v)$ for $v > 0$, so that s contributes $\omega(|w_0 - s|) > 0$ to the left side. Since $\omega'(v) = -2\alpha/\sinh(2\alpha v)$ and $\omega''(v) = 4\alpha^2\cosh(2\alpha v)/\sinh^2(2\alpha v)$, the function ω is decreasing and convex; and the series of $\log(1 \pm e^{-2\alpha v})$ give $\omega(v) = 2\sum_{l\ \text{odd}} e^{-2\alpha l v}/l$.

Sturm's theorem. On $[w_0-3, w_0+3] = [-7/2, 5/2]$ the function $\phi_j = e^{-t^2/2}\psi_j$ satisfies $\phi_j'' + (2j+1-t^2)\phi_j = 0$ with $2j+1-t^2 \ge 2j - 11.25 = (\pi/\eta)^2$. By Sturm's comparison theorem, applied to ϕ_j and $\sin(\pi(t-t_1)/\eta)$: if $[t_1, t_1+\eta] \subset [w_0-3, w_0+3]$, then ψ_j has a zero in $(t_1, t_1+\eta)$ or vanishes at both ends.

Pairing. Let $s_1^+ < s_2^+ < \dots$ be the zeros of ψ_j larger than w_0 and $s_1^- > s_2^- > \dots$ those smaller than w_0; put $d_k^\pm = |s_k^\pm - w_0|$ and $N = \lfloor 3/\eta \rfloor$, so $N\eta > 3 - \eta$. Sturm's theorem on $[w_0, w_0+\eta]$ and $[w_0-\eta, w_0]$ gives $d_1^\pm < \eta$, as w_0 is not a zero; on $[s_1^-, s_1^- + \eta]$ it gives $d_1^+ + d_1^- \le \eta$, as the first zero after s_1^- is s_1^+. If $1 \le k < N$ and $d_k^+ \le d_1^+ + (k-1)\eta$, then $s_k^+ + \eta < w_0 + (k+1)\eta \le w_0 + 3$, and Sturm's theorem on $[s_k^+, s_k^+ + \eta]$ gives $d_{k+1}^+ \le d_k^+ + \eta$; the same holds on the left. By induction, for

$k \le N$ the zeros $s_k^{\pm}$ exist and lie in $[w_0-3, w_0+3]$, and $d_k^+ + d_k^- \le (2k-1)\eta$. By convexity and monotonicity $\omega(d_k^+) + \omega(d_k^-) \ge 2\omega((k-\frac12)\eta)$; the other zeros in $[w_0-3, w_0+3]$ contribute positive terms. With the series of ω and geometric sums, the left side of the lemma is at least

$$2\sum_{k=1}^{N} \omega((k-\tfrac12)\eta) = 2\sum_{l \text{ odd}} \frac{1-e^{-2\alpha l N\eta}}{l\,\sinh(\alpha l\eta)}.$$

Five terms. All terms are positive; we keep $l = 1,3,5,7,9$. For them $1-e^{-2\alpha l N\eta} \ge 1-e^{-2\alpha\cdot 2.849} > 0.998$, since $N\eta > 3-\eta > 2.849$. For $t > 0$, $1/\sinh t \ge 1/t - t/6$: for $t^2 < 6$ compare the series, $\sinh t/t \le \sum_k (t^2/6)^k$ since $(2k+1)! \ge 6^k$; for $t^2 \ge 6$ the right side is not positive. With $\Sigma = 1 + \frac19 + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} = \frac{117469}{99225} > 1.18386$, the left side of the lemma is therefore at least

$$2\cdot 0.998\Big(\frac{\Sigma}{\alpha\eta} - \frac{5\alpha\eta}{6}\Big) \ge \frac{2\cdot 0.998\,\Sigma}{\alpha\pi}\sqrt{2j-11.25} - \frac{5\alpha\eta}{3}.$$

Here $\alpha < 1.2342$ and $\alpha\pi = 11\pi^2/28 < 3.8774$, so $2\cdot 0.998\,\Sigma/(\alpha\pi) > 2.36299/3.8774 > 0.6094 > 0.608$ and $5\alpha\eta/3 < 5\cdot 1.2342\cdot 0.151/3 < 0.3107 < 0.315$. □

The numbers of Step 3. Put $\mathcal{D}_m(n) = 0.608\sqrt{2n-11.25} - 0.315 - \log(50000\,m^2/(n(m-n)))$ for real $n \in [m/2, m-1]$. Its second derivative $-0.608\,(2n-11.25)^{-3/2} - n^{-2} - (m-n)^{-2}$ is negative, so its minimum is at an end. At $n = m/2$ the function is $0.608\sqrt{m-11.25} - 0.315 - \log 200000$, which increases with m and at $m = 450$ exceeds $12.4203 - 12.2061 > 0.214$. At $n = m-1$, $50000\,m/(m-1) \le 50000\cdot 450/449 < 50200$ gives $\mathcal{D}_m(m-1) \ge 0.608\sqrt{2m-13.25} - 0.315 - \log(50200\,m)$. At $m = 450$ this exceeds $17.7902 - 16.9331 > 0.857$, and its derivative $0.608/\sqrt{2m-13.25} - 1/m$ is positive, because $0.369664\,m^2 - 2m + 13.25$ has a negative discriminant. So $\mathcal{D}_m(n) > 0$ for all $m \ge 450$ and $m/2 \le n \le m-1$, with the smallest margin $\mathcal{D}_{450}(225) > 0.214$.

A.3 The index pairs $(2,4)$, $(3,4)$, $(4,5)$ and $(5,6)$

We prove the bounds for these pairs stated in the paragraph "The indices $m \le 6$" of Section 4. They hold for all $a, b \in \mathbb{C}$, so we may write $F = 1 + 2z + aH_n + bH_m$.

Nodes. As in Section 3 put $w = z + 1/2$, which does not change imaginary parts; $F(w_1)$ means the value of F at $z = w_1 - 1/2$. With $q_j(w) = H_j(w-1/2)/w$,

$$F = w\,(2 + a\,q_n(w) + b\,q_m(w)), \qquad q_2 = 4w - 4 - w^{-1}, \qquad q_3 = 8w^2 - 12w - 6 + 5w^{-1}, \tag{15}$$
$$q_4 = 16w^3 - 32w^2 - 24w + 40 + w^{-1}, \qquad q_5 = 32w^4 - 80w^3 - 80w^2 + 200w + 10 - 41w^{-1}.$$

For nonzero $w_1 \ne w_2$ with sum S and product p, the quotient $(w_1^k - w_2^k)/(w_1 - w_2)$ is 1, S, $S^2 - p$, $S^3 - 2pS$ for $k = 1,2,3,4$ and $-1/p$ for $k = -1$, so the divided differences $[q_j] = (q_j(w_1) - q_j(w_2))/(w_1 - w_2)$ are

$$[q_2] = 4 + \tfrac{1}{p}, \qquad [q_3] = 8S - 12 - \tfrac{5}{p}, \qquad [q_4] = 16(S^2-p) - 32S - 24 - \tfrac{1}{p},$$
$$[q_5] = 32(S^3 - 2pS) - 80(S^2-p) - 80S + 200 + \tfrac{41}{p}. \tag{16}$$

Distinct nonzero real w_1, w_2 are a *pair of nodes* for (n,m) if $[q_n] = [q_m] = 0$. Then F/w takes the same value at w_1 and w_2 for all a, b, by (15). So, by the paragraph "Equal-value points" of Section 4, if F has no real zero, its argument changes between $w_1 - 1/2$ and $w_2 - 1/2$ by an odd multiple of π

if $w_1 w_2 < 0$ and by an even one if $w_1 w_2 > 0$, and $F(w_1)/F(w_2) = w_1/w_2$, that is, for the zeros ω_j of F in the variable w, with multiplicity,

$$\prod_{j=1}^{N} \frac{w_1 - \omega_j}{w_2 - \omega_j} = \frac{w_1}{w_2}. \tag{17}$$

The pair $(2,4)$**.** If $b \neq 0$, then $F/(16b) = z^4 + c_2 z^2 + c_1 z + c_0$ with $c_2 = a/(4b) - 3$, $c_1 = 1/(8b)$ and $c_0 = (c_1 - c_2)/2 - 3/4$, so that

$$4c_0 + 2c_2 - 2c_1 + 3 = 0. \tag{18}$$

Conversely, a quartic $z^4 + c_2 z^2 + c_1 z + c_0$ with $c_1 \neq 0$ that satisfies (18) is $F/(16b)$ for $b = 1/(8c_1)$, $a = 4b(c_2 + 3)$; its zeros have sum zero.

Put $f(z) = 12z^4 + 16z^3 + 12z^2 - 3$. Since $f'(t) = 24t(2t^2 + 2t + 1)$ and $f(0) = -3$, f has two real zeros $-r_1 < 0 < r_2$ and two conjugate nonreal zeros; let z_* be the one with $\operatorname{Im} z_* > 0$, and $h_{24} = \operatorname{Im} z_*$. The values $f(-\frac{27}{40}) = \frac{24243}{640000}$, $f(-\frac{2}{3}) = -\frac{1}{27}$, $f(\frac{3}{8}) = -\frac{237}{1024}$, $f(\frac{2}{5}) = \frac{157}{625}$ give $2/3 < r_1 < 27/40$ and $3/8 < r_2 < 2/5$. The zeros of f have product $-1/4$ and sum $-4/3$, so $|z_*|^2 = 1/(4r_1 r_2) \in (25/27, 1)$ and $\operatorname{Re} z_* = (r_1 - r_2 - 4/3)/2 \in (-8/15, -31/60)$. So $h_{24}^2 = |z_*|^2 - (\operatorname{Re} z_*)^2$ lies between $25/27 - 64/225 = 433/675 > 16/25$ and $1 - 1/4$: $16/25 < h_{24}^2 < 3/4$ and $4/5 < h_{24} < \sqrt{3}/2$. For $u \in \mathbb{C}$ the quartic $(z-u)^3(z+3u)$ has $c_2 = -6u^2$, $c_1 = 8u^3$, $c_0 = -3u^4$, and the left side of (18) is $-f(u)$. So $(z - z_*)^3(z + 3z_*)$ is $F/(16b)$ for

$$b = 1/(64 z_*^3), \qquad a = (12 - 24 z_*^2)\, b, \tag{19}$$

and this F, with zeros of heights h_{24} and $3h_{24}$, has height h_{24}; so has the polynomial with the conjugate coefficients.

Lemma 25. *Let* $\mathcal{D}(z) = 4z^2 + 4z + 2$, $\mathcal{N}(z) = 4z^4 - 4z^2 + 3$ *and, for real* d, $f_d(z) = (2z + d)^2 \mathcal{D}(z) - \mathcal{N}(z)$. *If* $d \neq 0$, *then* f_d *has no zero* z *with* $\operatorname{Im} z \geq h_{24}$.

Proof. We have $f_0 = f$ and $f' = 12z\mathcal{D}$. The zeros $-1/2 \pm i/2$ of $\mathcal{D}$ have height $1/2 < h_{24}$, so $\mathcal{R} = \mathcal{N}/\mathcal{D}$ is analytic for $\operatorname{Im} z \geq h_{24}$, and there $f_d(z) = 0$ exactly when $\mathcal{R}(z) = (2z + d)^2$.

(a) A polynomial on the line. For real ρ_1, ρ_2, $X = (\rho_1 + i\rho_2)^2$ and $y_0 > 0$, expansion gives

$$(\operatorname{Im} X)^2 - 4y_0^2(\operatorname{Re} X + y_0^2) = 4(\rho_2^2 - y_0^2)(\rho_1^2 + y_0^2). \tag{20}$$

Put $v_* = h_{24}^2$ and $z_s = s - 1/2 + i h_{24}$ for real s. By (20) with $X = \mathcal{R}(z_s)$ and $y_0 = 2h_{24}$,

$$\Delta(s) := \frac{|\mathcal{D}(z_s)|^4}{4v_*}\Big(16v_*(\operatorname{Re}\mathcal{R}(z_s) + 4v_*) - \big(\operatorname{Im}\mathcal{R}(z_s)\big)^2\Big) = \frac{|\mathcal{D}(z_s)|^4}{v_*}(4v_* - \rho_2^2)(4v_* + \rho_1^2) \tag{21}$$

for either square root $\rho_1 + i\rho_2$ of $\mathcal{R}(z_s)$; so $\Delta(s) = 0$ exactly when these roots have imaginary part $\pm 2h_{24}$. As $|\mathcal{D}|^2 \mathcal{R} = \mathcal{N}\overline{\mathcal{D}}$, Δ is a polynomial in s; up to terms of degree at most one (the display below is used only for Δ'', in (b)), exact expansion [1] gives

$$\begin{aligned}\Delta(s) = {} & 768s^{10} - 1536s^9 + 768(8v_* + 1)s^8 - 256(24v_* + 1)s^7 + 32(528v_*^2 + 120v_* + 25)s^6 \\ & - 2304v_*(4v_* - 1)s^5 + 16(1344v_*^3 + 144v_*^2 + 76v_* + 27)s^4 \\ & - 16(384v_*^3 - 336v_*^2 - 16v_* + 11)s^3 + (13056v_*^4 - 3840v_*^3 - 1504v_*^2 - 240v_* + 99)s^2.\end{aligned}$$

(b) $\Delta'' > 0$. The coefficient of s^k in Δ'' is $(k+2)(k+1)$ times that of s^{k+2} in Δ, and $16/25 < v_* < 3/4$. So the coefficients of s^8, s^7 in Δ'' are 69120, -110592; those of s^6, s^4, s^2

increase with v_* and exceed their values at $16/25$, which are above 260000, 300000, 90000; those of s^5, s^3 have absolute values at most $10752 \cdot 19 < 205000$ and $46080 \cdot \frac{3}{4} \cdot 2 < 70000$; that of s, $-96(v_*^2(384v_* - 336) + 11 - 16v_*)$, has absolute value at most $96(\frac{9}{16} \cdot 90.24 + 1) < 5500$; and the constant one increases (half its derivative is at least $52224(\frac{16}{25})^3 - 11520(\frac{3}{4})^2 - 3008 \cdot \frac{3}{4} - 240 > 0$), so it is at least its value $1026.3\ldots$ at $16/25$. With $\sigma = |s|$,

$$\begin{aligned}\Delta''(s) \;\geq\; & \sigma^6(69120\sigma^2 - 110592\sigma + 50000) + \sigma^4(210000\sigma^2 - 205000\sigma + 55000) \\ & + \sigma^2(245000\sigma^2 - 70000\sigma + 6000) + (84000\sigma^2 - 5500\sigma + 900),\end{aligned}$$

and these quadratics are positive, as $110592^2 < 4 \cdot 69120 \cdot 50000$, $205000^2 < 4 \cdot 210000 \cdot 55000$, $70000^2 < 4 \cdot 245000 \cdot 6000$ and $5500^2 < 4 \cdot 84000 \cdot 900$.

(c) The only zero of Δ. Put $s_0 = \operatorname{Re} z_* + 1/2$, so that $z_{s_0} = z_*$. From $f(z_*) = 0$, $\mathcal{R} = 4z^2 - f/\mathcal{D}$ and $f' = 12z\mathcal{D}$ we get $\mathcal{R}(z_*) = 4z_*^2$ and $\mathcal{R}'(z_*) = 8z_* - 12z_* = -4z_*$. The analytic square root ρ of $\mathcal{R}$ near z_* with $\rho(z_*) = 2z_*$ has $\rho'(z_*) = \mathcal{R}'(z_*)/(2\rho(z_*)) = -1$, so $\rho_2(s) = \operatorname{Im}\rho(z_s)$ has $\rho_2(s_0) = 2h_{24}$ and $\rho_2'(s_0) = 0$. By (21), near s_0 the polynomial Δ is $4v_* - \rho_2^2$, which vanishes with its derivative at s_0, times a smooth function. So $\Delta(s_0) = \Delta'(s_0) = 0$, and by (b), $\Delta(s) > 0$ for $s \neq s_0$.

(d) No zero on the line. If d is real and $f_d(z_s) = 0$, then $2z_s + d$ is a square root of $\mathcal{R}(z_s)$ with imaginary part $2h_{24}$. By (a) and (c), $z_s = z_*$, and $(2z_* + d)^2 = \mathcal{R}(z_*) = 4z_*^2$ gives $d = 0$ or $d = -4z_* \notin \mathbb{R}$.

(e) End of the proof. The leading coefficient of f_d is 12 for all d, so the zeros of f_d depend continuously on d, and by (d) their number in $\operatorname{Im} z > h_{24}$ is constant for $d > 0$ and for $d < 0$. For large $|d|$ it is zero, by Hurwitz's theorem: $f_d/d^2 \to \mathcal{D}$ and $f_d(dt)/d^4 \to 4t^2(3t+1)(t+1)$ locally uniformly, so two zeros of f_d are near $-1/2 \pm i/2$, of height near $1/2$, and the other two are dt with t near -1 and near $-1/3$, real because each is the only zero in a disk symmetric under conjugation and f_d is real. □

Lemma 26. *Every polynomial $F = 1 + 2z + aH_2 + bH_4$ satisfies $\mathcal{H}(F) \leq h_{24} = 0.8286\ldots < \sqrt{3}/2$, with equality exactly for the polynomial* (19)*, whose zeros are $z_*, z_*, z_*, -3z_*$, and for the one with the conjugate coefficients.*

Proof. (i) If $b = 0$, then $\mathcal{H}(F) \leq 1/2$, as for $(2, 3)$ in Section 3 ("A relation between the zeros"). Put $\eta = \sup_{a,b\in\mathbb{C}} \mathcal{H}(F)$; by (19), $\eta \geq h_{24} > 4/5$.

(ii) If $(a', b') \neq (0, 0)$, then $G = a'H_2 + b'H_4$ has a zero of height less than $1/2$. This is clear if $b' = 0$. If $b' \neq 0$, the normal form without the term $1 + 2z$ gives $G/(16b') = (z^2 - X_1)(z^2 - X_2)$ with $(X_1 - \frac{1}{2})(X_2 - \frac{1}{2}) = -\frac{1}{2}$. If $X_1 - 1/2$ is real, so is $X_2 - 1/2$, with the opposite sign; so one X_k exceeds $1/2$, and G has a real zero. Otherwise, conjugating G if necessary, $X_1 - \frac{1}{2} = \varrho_1 e^{i\varphi}$ and $X_2 - \frac{1}{2} = \varrho_2 e^{i(\pi-\varphi)}$ with $0 < \varphi < \pi$ and $\varrho_1\varrho_2 = \frac{1}{2}$. Let y_0 be the height of G, and suppose $y_0 > 0$. The zeros $\pm\sqrt{X_1}$ have $|\operatorname{Im}| \geq y_0$, so by (20), $\varrho_1^2 \sin^2\varphi - 4y_0^2\varrho_1\cos\varphi - 4y_0^2(y_0^2 + \frac{1}{2}) \geq 0$. This quadratic in ϱ_1 is negative at 0, so ϱ_1 is at least its positive root $\lambda(\varphi) = 2y_0\bigl(y_0\cos\varphi + \sqrt{y_0^2 + \frac{1}{2}\sin^2\varphi}\,\bigr)/\sin^2\varphi$. Likewise $\varrho_2 \geq \lambda(\pi - \varphi)$, and $\frac{1}{2} = \varrho_1\varrho_2 \geq \lambda(\varphi)\lambda(\pi - \varphi) = 4y_0^2(y_0^2 + \frac{1}{2})/\sin^2\varphi \geq 4y_0^2(y_0^2 + \frac{1}{2})$. So $8y_0^4 + 4y_0^2 - 1 \leq 0$, that is, $y_0^2 \leq (\sqrt{3} - 1)/4 < 1/4$.

(iii) η is attained. Let $F_j = 1 + 2z + a_jH_2 + b_jH_4$ with $\mathcal{H}(F_j) \to \eta$. If $M_j = \max(|a_j|, |b_j|)$ were unbounded, a subsequence of F_j/M_j would tend locally uniformly to some $a'H_2 + b'H_4 \neq 0$, and by (ii) and Hurwitz's theorem F_j would have a zero of height less than $1/2 < \eta$ for large j. So a subsequence of (a_j, b_j) tends to some (a, b); by Hurwitz's theorem each zero of the limit F (not constant) is a limit of zeros of the F_j, so $\mathcal{H}(F) = \eta$. From now on F is any polynomial of the pair with $\mathcal{H}(F) = \eta = y$; by (i), $b \neq 0$ and F has degree four. Its *contacts* are its zeros of height y, with multiplicity.

(iv) At least three contacts. By (15), a nonreal zero ζ of F satisfies, with $\omega = \zeta + 1/2$, $a\,q_2(\omega) + b\,q_4(\omega) = -2$, and if it is double, also $a\,q_2'(\omega) + b\,q_4'(\omega) = 0$, where $q_2' = 4 + w^{-2}$ and $q_4' = 48w^2 - 64w - 24 - w^{-2}$. Suppose that there are at most two contacts. Move each contact ζ to $\zeta + i\epsilon\,\operatorname{Im}\zeta/|\operatorname{Im}\zeta|$ ($\epsilon > 0$ small) and solve these equations at the moved contacts for (a_ϵ, b_ϵ); there is a solution that tends to (a, b). With one simple contact, the row $(q_2(\omega), q_4(\omega))$ is not zero, as its product with (a, b) is -2; solve for a coefficient whose entry is not zero. With two simple contacts, the two rows are independent: if not, they are equal (both have product -2 with (a, b)), so $[q_2] = [q_4] = 0$, and (16) gives $p = -1/4$ and $16(S^2 - 2S - 1) = 0$; then ω_1, ω_2, the zeros of $w^2 - Sw - 1/4$ with S real, are real, but contacts are not. With a double contact, the rows $(q_2(\omega), q_4(\omega))$ and $(q_2'(\omega), q_4'(\omega))$ are independent: if not, the second is zero (its product with (a, b) is 0), but $q_2'(\omega) = 0$ gives $\omega^2 = -1/4$, and then $q_4'(\omega) = -32 - 64\omega = 0$ gives $\omega = -1/2$. For small ϵ, F_ϵ has degree four, the moved contacts as zeros of height $y + \epsilon$ with the same multiplicities, and its other zeros near the zeros of F of height greater than y; so $\mathcal{H}(F_\epsilon) > \eta$, which is impossible.

(v) The position of the zeros. The imaginary parts of the four zeros have sum zero and absolute values at least y, and at least three are $\pm y$. Up to conjugation, one or two are negative. If one is, it is minus the sum of the others, at most $-3y$, so the others are y and it is $-3y$. If two are, the positive ones and the absolute values of the negative ones have equal sums, and three of these four numbers are y, so all are. So, after conjugating F if necessary: (A) two zeros lie on each line $\operatorname{Im} z = \pm y$, or (B) three lie on $\operatorname{Im} z = y$ and one on $\operatorname{Im} z = -3y$.

(vi) Case (A) is impossible. Let $\alpha = x + iy$ be the mean of the two upper zeros; as the zeros have sum zero, they are $\alpha \pm t_1$, $-\alpha \pm t_2$ with real t_1, t_2. With $U = t_1^2 + t_2^2$ and $V = t_1^2 - t_2^2$ (so $U \ge |V|$), expansion gives $c_2 = -2\alpha^2 - U$, $c_1 = -2\alpha V$, $c_0 = \alpha^4 - \alpha^2 U + (U^2 - V^2)/4$, and (18) becomes $4\alpha^4 - 4\alpha^2(1+U) + 4\alpha V + U^2 - V^2 - 2U + 3 = 0$. Put $\beta = 1 + U - 2x^2 + 2y^2$, $X = 4x^2$, $Y = 4y^2$. The imaginary part of the left side is $4y(V - 2x\beta)$, so $V = 2x\beta$; with this and $U = \beta - 1 + X/2 - Y/2$ the real part is $-X(\beta^2 - 2\beta + 2 + Y) + \beta^2 - 4\beta + 6 + 2Y$. As $\beta^2 - 2\beta + 2 + Y > 0$,

$$X = 2 + \frac{2 - \beta^2}{\beta^2 - 2\beta + 2 + Y}, \qquad 1 - X = \frac{2\beta - 4 - Y}{\beta^2 - 2\beta + 2 + Y}, \tag{22}$$

and $U \ge |V| = \sqrt{X}|\beta|$ reads $2\beta + X - Y - 2 \ge 2\sqrt{X}|\beta|$. Now $Y > 2$, as $y > 4/5$, and $X \le 2 + 2/(1+Y) < Y + 2$. So $2\beta \ge Y + 2 - X > 0$, hence $\beta > 0$, and $2\beta(1 - \sqrt{X}) \ge Y + 2 - X > 0$, hence $\sqrt{X} < 1$. By (22),

$$Y + 2 - X \le 2\beta(1 - \sqrt{X}) = \frac{2\beta(2\beta - 4 - Y)}{(1 + \sqrt{X})(\beta^2 - 2\beta + 2 + Y)} < \frac{4}{1 + \sqrt{X}},$$

because $4(\beta^2 - 2\beta + 2 + Y) - 2\beta(2\beta - 4 - Y) = 8 + 4Y + 2Y\beta > 0$. But with $\sigma = \sqrt{X} \in [0, 1)$ we have $Y + 2 - X > 4 - \sigma^2$ and $(4 - \sigma^2)(1 + \sigma) - 4 = \sigma(4 - \sigma - \sigma^2) \ge 0$, so $Y + 2 - X > 4/(1 + \sigma)$, a contradiction.

(vii) In case (B) an upper zero is repeated. Let $\alpha = x + iy$ be the mean of the three upper zeros $\alpha + t_k$, with real t_k of sum zero; the fourth zero is -3α. Write $\prod_k (t - t_k) = t^3 - \ell_1 t - \ell_0$, with ℓ_1, ℓ_0 real. Then $F/(16b) = ((z - \alpha)^3 - \ell_1(z - \alpha) - \ell_0)(z + 3\alpha)$, and (18) becomes $(12\alpha - 2)\ell_0 = -f(\alpha) + (12\alpha^2 + 4\alpha - 2)\ell_1$. As $\operatorname{Im}\alpha > 0$ and $12\alpha^2 + 4\alpha - 2 = (12\alpha - 2)(\alpha + \frac{1}{2}) - 1$, this is $\ell_0 = -f(\alpha)/(12\alpha - 2) + (\alpha + \frac{1}{2} - 1/(12\alpha - 2))\ell_1$, where the coefficient of ℓ_1 has imaginary part $y + 12y/|12\alpha - 2|^2 > 0$. So its imaginary part determines ℓ_1, and then its real part determines ℓ_0: for each α with $\operatorname{Im}\alpha > 0$ there is exactly one real pair $(\ell_1(\alpha), \ell_0(\alpha))$, continuous in α. If the t_k were distinct, then $4\ell_1^3 - 27\ell_0^2 > 0$ would persist at $\alpha + i\epsilon$ for small $\epsilon > 0$, and the quartic with the zeros $-3(\alpha + i\epsilon)$ and $\alpha + i\epsilon + t$, for the three real zeros t of $t^3 - \ell_1(\alpha + i\epsilon)t - \ell_0(\alpha + i\epsilon)$, would satisfy

(18) and have a coefficient of z near $c_1 \neq 0$; so it would be $F_\epsilon/(16b_\epsilon)$ for a polynomial F_ϵ of the pair, of height $y + \epsilon > \eta$. Hence the upper zeros are $\alpha + t, \alpha + t, \alpha - 2t$ for a real t.

(viii) End of the proof. Put $z_0 = \alpha + t$ and $d = -3t$. The zeros of F are $z_0, z_0, z_0 + d, -3z_0 - d$, and expanding $(z - z_0)^2(z - z_0 - d)(z + 3z_0 + d)$ shows that the left side of (18) is $-f_d(z_0)$. As $\operatorname{Im} z_0 = y \geq h_{24}$, Lemma 25 gives $d = 0$. Then $f(z_0) = 0$ and $\operatorname{Im} z_0 > 0$, so $z_0 = z_*$, $F/(16b) = (z - z_*)^3(z + 3z_*)$, and a, b are given by (19). Hence $\eta = h_{24}$; and every polynomial of the pair with height h_{24} has height η, so by (iv)–(viii) it is (19) or, if F was conjugated in (v), the polynomial with the conjugate coefficients. □

The pair $(3, 4)$**.** Let $t_1 < t_2 < t_3$ be the zeros of $5t^3 + 9t^2 - 45t - 25$. For $S = (t_k + 3)/2$ and $p = 5/(4t_k)$, (16) gives $[q_3] = 4t_k - 4t_k = 0$ and $[q_4] = \frac{4}{5t_k}(5t_k^3 + 9t_k^2 - 45t_k - 25) = 0$. So the zeros of $w^2 - Sw + p$, if real, distinct and nonzero, are a pair of nodes for $(3, 4)$. In the variable z they are the zeros $l_k < r_k$ of $4t_k(z + \frac{1}{2})^2 - 2t_k(t_k + 3)(z + \frac{1}{2}) + 5$; put $I_k = [l_k, r_k]$. Sign changes give the intervals for t_k in the table below, and exact rational arithmetic [1] shows that, for every t in the interval of t_k, this quadratic with t in place of t_k changes sign on the intervals given for l_k and r_k, which therefore contain them:

k	$10^{12}\, t_k$	l_k	r_k
1	$(-3815393485570, -3815393485569)$	$(-1.311446, -1.311445)$	$(-0.096252, -0.096251)$
2	$(-517404417703, -517404417702)$	$(-1.553003, -1.553002)$	$(1.794300, 1.794301)$
3	$(2532797903271, 2532797903272)$	$(-0.308319, -0.308318)$	$(2.074717, 2.074718)$

So the l_k, r_k are real, distinct and not $-1/2$; and $-1/2$ lies in I_1 and I_2 and to the left of I_3; $|I_1| < 1.215195 < 10/7$ and $|I_3| < 2.383037$, so $|I_3|^2 < 300/49 = 12(5/7)^2$. Let θ_k be the signed angle θ of Section 4 for I_k. The right side of (6) increases with $(x - \mu)^2$, and at $x = \mu$ it is $(y^2 - s^2)/(2|y|s)$, which increases with $|y|$; so at a point of height at least y_0 the angle θ_I of an interval I of half-length ς satisfies $\cot|\theta_I| \geq (y_0^2 - \varsigma^2)/(2y_0\varsigma) = \cot\bigl(2\arctan(\varsigma/y_0)\bigr)$. As $|I_1|/(2y_0) < 1$ and $|I_3|/(2y_0) < \sqrt{3}$ for $y_0 = 5/7$, we get $|\theta_1| < \pi/2$ and $|\theta_3| < 2\pi/3$ at every point of height at least $5/7$.

Lemma 27. *Every polynomial $F = 1 + 2z + aH_3 + bH_4$ satisfies $\mathcal{H}(F) < 5/7$.*

Proof. If $b = 0$, then F is a trinomial or has a real zero, and $\mathcal{H}(F) \leq K < 5/7$ by Schmeisser's theorem [9]. Let $b \neq 0$, and suppose that every zero of F has height at least $5/7$. Then F has degree four and no real zero. As $F/(16b) = z^4 + \frac{a}{2b}z^3 - 3z^2 + \ldots$, the two zeros of F'' have product $-1/2$. If the four zeros of F lay in one half-plane $\pm \operatorname{Im} z \geq 5/7$, then by the Gauss–Lucas theorem, applied twice, so would the zeros of F'', and their product would have modulus at least $25/49 > 1/2$.

The sum $\sum_j \theta_k(z_j)$ over the zeros z_j of F is the change of the argument of F along I_k (Section 4). As $-1/2$ lies in I_1 and I_2 but not in I_3, it is an odd multiple of π for $k = 1, 2$ and an even one for $k = 3$, by the paragraph "Nodes". With two zeros on each side of the real axis, $|\theta_1| < \pi/2$ would put $\sum_j \theta_1(z_j)$ in $(-\pi, \pi)$. So, conjugating F if necessary, z_1, z_2, z_3 are upper and z_4 is lower. Each θ_k lies in $(0, \pi)$ at upper points and in $(-\pi, 0)$ at lower ones, so $\sum_j \theta_1(z_j)$ and $\sum_j \theta_2(z_j)$ lie in $(-\pi, 3\pi)$ and both equal π; as $|\theta_3| < 2\pi/3$, $\sum_j \theta_3(z_j)$ lies in $(-2\pi/3, 2\pi)$ and equals 0. Hence $g_{34} = 5\theta_1 - 4\theta_2 + 2\theta_3$ satisfies $\sum_j g_{34}(z_j) = \pi$.

Claim: $-\pi + \arctan\frac{1}{14} < g_{34}(z) < \arctan\frac{1}{44}$ *for* $\operatorname{Im} z \geq 5/7$. Let $I_k^+ \supset I_k$ and $I_k^- \subset I_k$ be the intervals formed by the outer and by the inner ends of the intervals in the table. For $\operatorname{Im} z > 0$ the angle $\theta_I(z)$ is the integral of a positive function over I, so it increases with I, and $g^- \leq g_{34} \leq g^+$ for $g^\pm = 5\theta_{I_1^\pm} - 4\theta_{I_2^\mp} + 2\theta_{I_3^\pm}$. For $I = [l, r]$ and real x put $Z_I(x) = (x - r + \frac{5}{7}i)(x - l - \frac{5}{7}i)$; at $z = x + \frac{5}{7}i$

this is $(z-r)\overline{(z-l)}$, whose argument is $\theta_I(z)$. So $Z^{\pm} = Z_{I_1^{\pm}}^5\, \overline{Z_{I_2^{\mp}}}^4\, Z_{I_3^{\pm}}^2$ is a monic polynomial of degree 22 in x with argument $g^{\pm}(x+\frac{5}{7}i)$ modulo 2π. Exact rational arithmetic [1] shows that the real polynomials $\operatorname{Re} Z^+ - 44 \operatorname{Im} Z^+$ and $\operatorname{Re} Z^- - 14 \operatorname{Im} Z^-$ have no real zero (Sturm counts) and are positive at $x=0$, so they are positive. As $\operatorname{Re} Z^+ - 44 \operatorname{Im} Z^+ = -\sqrt{1937}\,|Z^+| \sin(g^+ - \arctan\frac{1}{44})$, and g^+ is continuous on the line $\operatorname{Im} z = 5/7$ and tends to 0 at both ends, $g^+ - \arctan\frac{1}{44}$ stays in $(-\pi, 0)$ there, and so does its supremum on the line; in the same way g^- and its infimum on the line exceed $\arctan\frac{1}{14} - \pi$. Each θ_I is harmonic for $\operatorname{Im} z > 0$, continuous on $\operatorname{Im} z \ge 5/7$ and tends to zero at infinity, so the maximum principle on large half-disks, as in the proof of Lemma 7, extends both bounds to $\operatorname{Im} z \ge 5/7$.

As $g_{34}(z_4) = -g_{34}(\bar{z}_4)$, the claim gives $\sum_j g_{34}(z_j) < 3\arctan\frac{1}{44} + \pi - \arctan\frac{1}{14} < \pi$, against $\sum_j g_{34}(z_j) = \pi$. The last inequality holds as these angles lie in $(0, \pi/2)$ and $\tan(3\arctan\frac{1}{44}) = (3\cdot 44^2 - 1)/(44^3 - 3\cdot 44) = 5807/85052 < 1/14$, since $14 \cdot 5807 = 81298$. □

The pair $(4,5)$**.** By (16), $[q_5] + 41[q_4] = 16(2S^3 + 36S^2 - 87S - 49 - 4(S+9)p)$. So for $S \ne -9$, $[q_4] = [q_5] = 0$ exactly when $p = (2S^3 + 36S^2 - 87S - 49)/(4S + 36)$ and, for this p, $p\,[q_4] = 4(S^6 + 14S^5 - 111S^4 + 228S^3 - 143S^2 - 6S + 41)/(S+9)^2$ vanishes. The sextic changes sign on $(-20.1032178007, -20.1032178006)$ and on $(2.910004, 2.910005)$; let S_1, S_2 be zeros there and p_1, p_2 the corresponding p, which are not zero because $S+9$ and $2S^3+36S^2-87S-49$ do not vanish on these intervals. Let $\alpha_i < \beta_i$ be the zeros of $w^2 - S_i w + p_i$. Exact rational arithmetic [1] shows that, for every S in the interval of S_i, the quadratic $(4S+36)(w^2 - Sw) + 2S^3 + 36S^2 - 87S - 49$ changes sign on each of the following intervals of w; so (α_1, β_1), (α_2, β_2) are pairs of nodes for $(4,5)$, and

$$-20.11 < \alpha_1 < -20.09, \quad -\tfrac{1}{125000} < \beta_1 < 0, \quad 0.43 < \alpha_2 < 0.45, \quad 2.46 < \beta_2 < 2.48. \tag{23}$$

Lemma 28. *Every polynomial $F = 1 + 2z + aH_4 + bH_5$ satisfies $\mathcal{H}(F) < 17/20$.*

Proof. Put $y_0 = 17/20$ and $\mathcal{K}(w) = (w-\alpha_1)^2(w-\alpha_2)/((w-\beta_1)^2(w-\beta_2))$. Suppose that every zero of F has height at least y_0. Then F has no real zero; let $\omega_1, \dots, \omega_N$, $N \le 5$, be its zeros in the variable w. By (17) for the two pairs of nodes, the first one twice, and by (23), $\prod_j |\mathcal{K}(\omega_j)| = |\alpha_1/\beta_1|^2 |\alpha_2/\beta_2| > (20\cdot 125000)^2 \cdot 0.43/2.48 > (20\cdot 125000)^2/6 > 240^5$. As $240 > 1$ and $N \le 5$, it is enough to show that $|\mathcal{K}(w)| < 240$ for $|\operatorname{Im} w| \ge y_0$.

Let $\mathcal{K}_0(w) = (w+20.10)^2(w-0.44)/(w^2(w-2.47))$. On the line $w = x + iy_0$, the inequality $|\mathcal{K}_0| < 225$ is the positivity of the sextic $225^2(x^2+y_0^2)^2((x-2.47)^2+y_0^2) - ((x+20.10)^2+y_0^2)^2((x-0.44)^2+y_0^2)$, with leading coefficient $225^2 - 1$. Exact rational arithmetic [1] shows that it has no real zero (a Sturm count); it is positive at $x = 0$. As $\mathcal{K}_0$ has only real poles and tends to one at infinity, its modulus on the line is bounded by a number less than 225, and the maximum modulus principle on large half-disks gives $|\mathcal{K}_0| < 225$ for $\operatorname{Im} w \ge y_0$, and by symmetry for $\operatorname{Im} w \le -y_0$. For real μ, μ' and $|\operatorname{Im} w| \ge y_0$ we have $|w-\mu|/|w-\mu'| \le 1 + |\mu - \mu'|/|w-\mu'| \le 1 + |\mu-\mu'|/y_0$. By (23), $|\alpha_1 + 20.10|$, $|\alpha_2 - 0.44|$, $|\beta_2 - 2.47| < 0.01 = y_0/85$ and $|\beta_1| < 1/125000$. Applying the bound to the factors $|w-\alpha_1|/|w+20.10|$ (twice), $|w-\alpha_2|/|w-0.44|$, $|w-2.47|/|w-\beta_2|$ and $|w|/|w-\beta_1|$ (twice) of $|\mathcal{K}/\mathcal{K}_0|$ gives $|\mathcal{K}(w)| < 225(86/85)^4(1+1/106250)^2 = 235.78\ldots < 240$ for $|\operatorname{Im} w| \ge y_0$. □

The pair $(5,6)$**.** Let $w^2 - Sw + p$ have distinct nonzero real zeros w_1, w_2, and let $r_l + s_l w$ be the remainder of $H_l(w - 1/2)$ on division by it. Then $q_l(w_k) = r_l/w_k + s_l$, so $[q_l] = -r_l/p$, and w_1, w_2 are a pair of nodes for $(5,6)$ exactly when $r_5 = r_6 = 0$. Division gives

$$r_5 = (64S - 80)p^2 - (32S^3 - 80S^2 - 80S + 200)p - 41,$$
$$r_6 = -64p^3 + (192S^2 - 384S - 240)p^2 - (64S^4 - 192S^3 - 240S^2 + 800S + 60)p + 31.$$

Let $\mathcal{T}(S) = 82S^{10} - 830S^9 + 2765S^8 - 1580S^7 - 9965S^6 + 21026S^5 - 14945S^4 + 11860S^3 - 18445S^2 + 7740S + 2592$ and $p(S) = (-205S^3 + 635S^2 - 255S - 225)/(32S^6 - 240S^5 + 600S^4 - 540S^3 + 180S^2 - 532S + 540)$. Exact arithmetic [1] shows that the denominator of $p(S)$ is prime to $\mathcal{T}$, and that $r_5(S, p(S))$ and $r_6(S, p(S))$, times the second and the third power of this denominator, are polynomials divisible by $\mathcal{T}$. So at a real zero S of $\mathcal{T}$ the zeros of $w^2 - Sw + p(S)$, if real, distinct and nonzero, are a pair of nodes for $(5, 6)$. For each row of the table below, exact rational arithmetic [1] shows that $\mathcal{T}$ has a zero S_i within $2 \cdot 10^{-6}$ of S_i^0, and that, for every S in a narrow interval about S_i on which the denominator of p keeps one sign, the quadratic $w^2 - Sw + p(S)$ times this denominator changes sign on the intervals of radius $2 \cdot 10^{-6}$ about α_i^0 and β_i^0. This gives five pairs of nodes $\alpha_i < \beta_i$ with $|\alpha_i - \alpha_i^0|, |\beta_i - \beta_i^0| < 2 \cdot 10^{-6}$.

i	1	2	3	4	5
$10^6 S_i^0$	−2124505	1102130	1596804	1828872	2688410
$10^6 \alpha_i^0$	−2048845	−1805569	−626158	−555650	360845
$10^6 \beta_i^0$	−75660	2907699	2222962	2384521	2327565
k_i	2	−2	−3	4	1

Lemma 29. *Every polynomial $F = 1 + 2z + aH_5 + bH_6$ satisfies $\mathcal{H}(F) < 43/50$.*

Proof. Put $y_0 = 43/50$. With k_i from the table, let γ_i, δ_i be α_i, β_i if $k_i > 0$ and β_i, α_i if $k_i < 0$, define γ_i^0, δ_i^0 in the same way, and put $\mathcal{K}(w) = \prod_{i=1}^5 \big((w-\alpha_i)/(w-\beta_i)\big)^{k_i} = \prod_i \big((w-\gamma_i)/(w-\delta_i)\big)^{|k_i|}$. Suppose that every zero of F has height at least y_0. Then F has no real zero; let $\omega_1, \dots, \omega_N$, $N \le 6$, be its zeros in the variable w. Raising (17) for the ith pair to the power k_i, multiplying, and using $|\gamma_i/\delta_i| > (10^6|\gamma_i^0|-2)/(10^6|\delta_i^0|+2)$, we get $\prod_j |\mathcal{K}(\omega_j)| = \prod_i |\gamma_i/\delta_i|^{|k_i|} > 38.8 > (181/100)^6 = 35.16\dots$, the product of the rational lower bounds being $38.89\dots$. As $181/100 > 1$ and $N \le 6$, it is enough to show that $|\mathcal{K}(w)| < 181/100$ for $|\operatorname{Im} w| \ge y_0$.

Let $\mathcal{K}_0 = \prod_i ((w - \gamma_i^0)/(w - \delta_i^0))^{|k_i|}$. On the line $w = x + iy_0$ the inequality $|\mathcal{K}_0| < 9/5$ is the positivity of $81 \prod_i ((x - \delta_i^0)^2 + y_0^2)^{|k_i|} - 25 \prod_i ((x - \gamma_i^0)^2 + y_0^2)^{|k_i|}$, a polynomial of degree 24 (as $\sum_i |k_i| = 12$) with leading coefficient 56. Exact rational arithmetic [1] shows that it has no real zero (a Sturm count); it is positive at $x = 0$. As in the proof of Lemma 28, $|\mathcal{K}_0| < 9/5$ for $|\operatorname{Im} w| \ge y_0$, and each of the 24 factors $(w - \gamma_i)/(w - \gamma_i^0)$ and $(w - \delta_i^0)/(w - \delta_i)$ of $\mathcal{K}/\mathcal{K}_0$, counted with the exponents $|k_i|$, has modulus at most $1 + 2 \cdot 10^{-6}/y_0 = 1 + 1/430000$ there. Hence $|\mathcal{K}(w)| < \frac{9}{5}(1 + 1/430000)^{24} = 1.80010\dots < 181/100$ for $|\operatorname{Im} w| \ge y_0$. □

A.4 The pair $(2, 5)$

This part proves the statements of the paragraph on $(2, 5)$ in Section 4, with its S, X, s_* and θ_*. Let $I_* = [X - s_*, X + s_*]$, let θ_{I_*} be the signed angle θ of the paragraph "Equal-value points" for I_*, and put $Z_\pm = \frac{1}{2}\big(S \pm \sqrt{S^2 + 1}\big)$, so that $X \pm s_* = Z_\pm - \frac{1}{2}$.

Lemma 30. *Let $F = 1 + 2z + aH_2 + bH_5$ with $a, b \in \mathbb{C}$. (a) $0 < S < 0.21$, $-1/2 < X < 0$ (so $|X| < 1/2$), $1/2 < s_* < 0.511$ and $Z_- < 0 < Z_+$. (b) The endpoints $X \pm s_*$ of I_* are equal-value points for $(2, 5)$. (c) If $b \ne 0$, then F has a zero in the open strip $|\operatorname{Im} z| < 0.87$.*

Proof. (a) The cubic $2S^3 - 5S^2 - 4S + 1$ takes the values 1, -0.041978 and $-9/32$ at $S = 0$, 0.21 and $1/4$. Its derivative $6S^2 - 10S - 4$ is convex and equals -4 at 0 and $-49/8$ at $1/4$, so it is negative on $[0, 1/4]$. Hence the cubic has exactly one zero in $(0, 1/4)$, and it lies in $(0, 0.21)$. So $-1/2 < X < 0$ and $1/4 < s_*^2 = (1 + S^2)/4 < 1.0441/4 < 0.511^2$; finally $Z_+ Z_- = -1/4$ and $Z_+ > Z_-$.

(b) In the notation of Appendix A.3 ($q_j(w) = H_j(w-1/2)/w$), $Z_\pm$ have sum S and product $p = -1/4$, and (16) gives $[q_2] = 4 - 4 = 0$ and $[q_5] = 32(S^3 + \frac{S}{2}) - 80(S^2 + \frac{1}{4}) - 80S + 200 - 164 = 16(2S^3 - 5S^2 - 4S + 1) = 0$. So $Z_\pm$ are a pair of nodes for $(2,5)$, and $X \pm s_*$ are equal-value points.

(c) *Step 1: the angles.* Suppose that $b \neq 0$ and that F has no zero in the strip. Let $z_1, \dots, z_5$ be the zeros of F, counted with multiplicity, and put $\theta_k = |\theta_{I_*}(z_k)|$; then $|\operatorname{Im} z_k| \geq 0.87$. As $H_5 = 32z^5 - 160z^3 + 120z$ and $1 + 2z + aH_2$ has degree at most two, $F/(32b) = z^5 - 5z^3 + \cdots$, and therefore

$$\sum_{k=1}^{5} z_k = 0, \qquad \sum_{k=1}^{5} z_k^2 = 0^2 - 2 \cdot (-5) = 10. \tag{24}$$

As F has no real zero, (a), (b) and the paragraph "Equal-value points" show that $\sum_k \theta_{I_*}(z_k)$ is an odd multiple of π. Replacing a, b by $\bar{a}, \bar{b}$ conjugates the zeros and negates the angles, so we may assume that

$$\sum_{k=1}^{5} \theta_{I_*}(z_k) \geq \pi. \tag{25}$$

The quotient $(y^2 - s^2)/(2ys)$ increases with $y > 0$ and decreases with $s > 0$. So (6), with $(x-\mu)^2 \geq 0$, and (a) give $\cot\theta_k \geq (0.87^2 - s_*^2)/(1.74\, s_*) > (0.7569 - 0.511^2)/(1.74 \cdot 0.511) > 5/9$, that is,

$$\theta_k < \theta_* < \pi/2 \qquad (k = 1, \dots, 5). \tag{26}$$

As $\theta_{I_*}(z_k)$ has the sign of $\operatorname{Im} z_k$ and $2\theta_* < \pi$, at least three zeros are upper; by (24) not all five are.

Step 2: three upper angles with sum greater than π. If exactly three zeros are upper, their angles have such a sum by (25). Otherwise let $z_1, \dots, z_4$ be upper and z_5 lower, and put $\Sigma_* = \sum_{k\leq 4} (\operatorname{Im} z_k)^2$. By (24), $|\operatorname{Im} z_5| = \sum_{k\leq 4} \operatorname{Im} z_k$, so $(\operatorname{Im} z_5)^2 \geq \Sigma_*$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^5$ be the vectors of the real and of the imaginary parts of $z_1, \dots, z_5$. By (24), $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to $\mathbf{1} = (1, \dots, 1)$ and to each other, and $|\mathbf{x}|^2 = 10 + |\mathbf{y}|^2 = 10 + \Sigma_* + (\operatorname{Im} z_5)^2$. Hence $\operatorname{Re} z_5 = \mathbf{x} \cdot \mathbf{p}$, where $\mathbf{p}$ is the orthogonal projection of $(0,0,0,0,1)$ onto the orthogonal complement of $\mathbf{1}$ and $\mathbf{y}$, and $|\mathbf{p}|^2 = \frac{4}{5} - (\operatorname{Im} z_5)^2/|\mathbf{y}|^2$. By the Cauchy–Schwarz inequality and $(\operatorname{Im} z_5)^2 \geq |\mathbf{y}|^2/2$,

$$(\operatorname{Re} z_5)^2 \leq |\mathbf{x}|^2 |\mathbf{p}|^2 = 10\Big(\frac{4}{5} - \frac{(\operatorname{Im} z_5)^2}{|\mathbf{y}|^2}\Big) + \frac{4\Sigma_* - (\operatorname{Im} z_5)^2}{5} \leq 3 + \frac{3}{5}\Sigma_*. \tag{27}$$

By (6), $2s_* |\operatorname{Im} z_k| \cot\theta_k = (\operatorname{Re} z_k - X)^2 + (\operatorname{Im} z_k)^2 - s_*^2$ for every k. We add these identities for $k \leq 4$, subtract the one for $k = 5$ times the weights $\operatorname{Im} z_k / |\operatorname{Im} z_5|$ ($k \leq 4$), which have sum one, and insert $\sum_{k\leq 4} \operatorname{Re} z_k = -\operatorname{Re} z_5$ and $\sum_{k\leq 4} (\operatorname{Re} z_k)^2 = |\mathbf{x}|^2 - (\operatorname{Re} z_5)^2$. This gives

$$\sum_{k=1}^{4} 2s_* \operatorname{Im} z_k \,(\cot\theta_k - \cot\theta_5) = 10 + 2\Sigma_* - 2(\operatorname{Re} z_5)^2 + 4X \operatorname{Re} z_5 + 3X^2 - 3s_*^2.$$

By (27), $|X| < 1/2$ and $2\sqrt{v} \leq 2 + v/2$ ($v \geq 0$), the right side is at least $4 + \frac{4}{5}\Sigma_* - 2\sqrt{3 + \frac{3}{5}\Sigma_*} - 3s_*^2 \geq \frac{1}{2} + \frac{1}{2}\Sigma_* - 3s_*^2$. This is positive, since $\Sigma_* \geq 4 \cdot 0.87^2$, $s_* < 0.511$ and $\frac{1}{2} + 1.5138 - 0.7834 > 0$. So $\theta_k < \theta_5$ for some $k \leq 4$, and by (25) the other three upper angles have sum greater than π.

Step 3: a rectangle. Let z_1, z_2, z_3 be upper zeros as in Step 2, and z_4, z_5 the other two zeros. As $\theta_1 + \theta_2 + \theta_3 > \pi$, (26) gives $\theta_j > \pi - 2\theta_* > 0$ for $j \leq 3$. Since $\cot(\pi - 2\theta_*) = (1 - \cot^2\theta_*)/(2\cot\theta_*) = (1 - 25/81)/(10/9) = 28/45$, (6) gives $(\operatorname{Re} z_j - X)^2 < s_*^2 + \frac{56}{45} s_* \operatorname{Im} z_j - (\operatorname{Im} z_j)^2$ for $j \leq 3$. The right side increases with s_*, and, as $\frac{56}{45} \cdot 0.511 < 0.64 < 1.74$, it decreases with $\operatorname{Im} z_j \geq 0.87$. At $s_* = 0.511$ it is less than $0.0575 < 0.28^2$ at $\operatorname{Im} z_j = 0.87$ and negative at $\operatorname{Im} z_j = 1$. With $|X| < 1/2$ this gives

$$|\operatorname{Re} z_j| < \tfrac{4}{5}, \qquad 0.87 \leq \operatorname{Im} z_j < 1 \qquad (j = 1, 2, 3). \tag{28}$$

Step 4: the other two zeros are lower. By (24), $\Delta_* = (z_4-z_5)^2 = 2(z_4^2+z_5^2)-(z_4+z_5)^2 = 20-2(z_1^2+z_2^2+z_3^2)-(z_1+z_2+z_3)^2$. By (28), $\operatorname{Re}(z_j^2) \le (\operatorname{Re} z_j)^2 < 16/25$ and $|\operatorname{Im}(z_j^2)| = 2|\operatorname{Re} z_j|\operatorname{Im} z_j < 8/5$ for $j \le 3$, while $\operatorname{Re}((z_1+z_2+z_3)^2) < (12/5)^2$ and $|\operatorname{Im}((z_1+z_2+z_3)^2)| < 2\cdot\frac{12}{5}\cdot 3$. Hence $\operatorname{Re}\Delta_* > 20 - 6\cdot\frac{16}{25} - \frac{144}{25} = 10.4$ and $|\operatorname{Im}\Delta_*| < 6\cdot\frac{8}{5} + \frac{72}{5} = 24$. So $(\operatorname{Re}(z_4-z_5))^2 \ge \operatorname{Re}\Delta_* > 10$ and $|\operatorname{Im}(z_4-z_5)| = |\operatorname{Im}\Delta_*|/(2|\operatorname{Re}(z_4-z_5)|) < 24/(2\sqrt{10}) < 4$. If one of z_4, z_5 were upper, the other would be the only lower zero, and by (24) its height would be the sum of the imaginary parts of the four upper zeros, at least $4\cdot 0.87$; then $|\operatorname{Im}(z_4-z_5)| \ge 5\cdot 0.87 > 4$. So z_4 and z_5 are lower.

Step 5: the lower angles. By (24) and (28), $|\operatorname{Im} z_4| + |\operatorname{Im} z_5| = \operatorname{Im}(z_1+z_2+z_3) < 3$, so $0.87 \le |\operatorname{Im} z_k| < 2.13$ for $k = 4, 5$, and $\sum_k (\operatorname{Im} z_k)^2 < 3 + 2.13\,(|\operatorname{Im} z_4| + |\operatorname{Im} z_5|) < 3 + 2.13\cdot 3 = 9.39$. The real part of (24) gives $\sum_k (\operatorname{Re} z_k)^2 = 10 + \sum_k (\operatorname{Im} z_k)^2 < 20$, so $|\operatorname{Re} z_k - X| < \sqrt{20} + \frac{1}{2} < 5$. By (6), $s_* > 1/2$ and $|\operatorname{Im} z_k| \ge 0.87$, $\cot\theta_k < (25 + 2.13^2)/(2\cdot\frac{1}{2}\cdot 0.87) < 34 < 35$, that is, $\theta_k > \arctan\frac{1}{35}$.

Step 6: the contradiction. By (26) and Step 5, the sum in (25) is $\theta_1+\theta_2+\theta_3-\theta_4-\theta_5 < 3\theta_* - 2\arctan\frac{1}{35}$. As $\tan\theta_* = 9/5 > \sqrt{3}$, the angle $3\theta_* - \pi$ lies in $(0, \pi/2)$, as $2\arctan\frac{1}{35}$ does, and by the triple-angle formula its tangent is $(3\cdot\frac{9}{5} - (\frac{9}{5})^3)/(1 - 3(\frac{9}{5})^2) = \frac{27}{545} < \frac{1}{20} < \frac{35}{612} = \frac{2/35}{1-1/35^2} = \tan(2\arctan\frac{1}{35})$. So the sum is less than π, contrary to (25). □

A.5 The pair $(3, 5)$: steps of Section 5

This part proves the steps of Section 5 that the text does not prove. We use the notation of these two places. In Lemmas 31–33 the interval $I = [\nu_1, \nu_2]$, its midpoint m_I, its half-length s_I, the angle θ_I and $M = \operatorname{arccot}(13/50)$ are as in Section 5. Since $m_I = (\nu_1+\nu_2)/2$ and $s_I = (\nu_2-\nu_1)/2$, the intervals of Lemma 10 give

$$-0.927 < m_I < -0.926, \qquad 0.6685 < s_I < 0.6695, \qquad s_I^2 < 0.4489. \tag{29}$$

At a nonreal point $x+iy$ the angle $|\theta_I(x+iy)|$ lies in $(0,\pi)$, and the cotangent formula of the proof of Lemma 12 reads

$$\cot|\theta_I(x+iy)| = \frac{(x-m_I)^2 + y^2 - s_I^2}{2|y|\,s_I}. \tag{30}$$

For fixed $s > 0$ the function $y \mapsto (y^2-s^2)/(2ys) = (y/s - s/y)/2$ increases on $y > 0$. For fixed $y > 0$ it decreases in $s > 0$. Hence (30) gives, for every point z with $|\operatorname{Im} z| \ge y > 0$,

$$\cot|\theta_I(z)| \;\ge\; \frac{y^2 - s_I^2}{2y\,s_I}. \tag{31}$$

Formulas (30) and (31) hold for every bounded real interval, with its midpoint and half-length in place of m_I and s_I. The function cot decreases on $(0,\pi)$; we use this without further mention.

Lemma 31. *Let P be a quintic of height at least 0.87 with three upper zeros z_1, z_2, z_3 and two lower zeros z_4, z_5. Then*

$$|z_1|^2 + |z_2|^2 + |z_3|^2 < 14, \qquad \sum_{k=1}^{5} |z_k|^2 < \frac{350}{9}, \qquad |z_k|^2 < \frac{280}{9} < 5.6^2 \quad (k = 1, \dots, 5).$$

Proof. Each upper zero. Put $\theta_j = \theta_I(z_j)$ for $j = 1, 2, 3$. The proof of Lemma 12 shows that $|\theta_I(z_k)| < M$ for all five zeros and that $\theta_1+\theta_2+\theta_3 \ge \pi$. Hence

$$\pi - 2M < \theta_j < M < \frac{\pi}{2} \qquad (j = 1, 2, 3). \tag{32}$$

So $\cot\theta_j > 0$ and $\csc\theta_j > 0$. By the proof of Lemma 12, z_j lies on the circle with center $m_I + is_I\cot\theta_j$ and radius $s_I\csc\theta_j$. So $|z_j| \le \sqrt{m_I^2 + s_I^2\cot^2\theta_j} + s_I\csc\theta_j$, and (29) gives

$$|z_j| < \varrho(\theta_j), \qquad \varrho(\theta) = \sqrt{0.93^2 + 0.67^2\cot^2\theta} + 0.67\csc\theta. \tag{33}$$

Convexity. On $(0,\pi/2)$ the functions cot and csc are positive, decreasing and convex, since $\cot'' = 2\cot\cdot\csc^2$ and $\csc'' = \csc\cdot(\cot^2+\csc^2)$. The function $v \mapsto \sqrt{0.93^2 + 0.67^2v^2}$ is convex and increasing for $v \ge 0$. Hence ϱ is positive, decreasing and convex on $(0,\pi/2)$. So is ϱ^2, because $v \mapsto v^2$ is convex and increasing for $v \ge 0$. We have $3M > \pi$, since $\cot M = 13/50 < 1/\sqrt{3} = \cot(\pi/3)$. The interval $[\pi - 2M, M]$ has length $3M - \pi$. On it the convex function ϱ^2 lies below its chord:

$$\varrho(\theta)^2 \le \varrho(M)^2 + \frac{M-\theta}{3M-\pi}\bigl(\varrho(\pi-2M)^2 - \varrho(M)^2\bigr) \qquad (\pi - 2M \le \theta \le M).$$

The bracket is nonnegative, because ϱ decreases. Apply this to $\theta_1, \theta_2, \theta_3$ and add. The numbers $M - \theta_j$ have sum $3M - (\theta_1+\theta_2+\theta_3) \le 3M - \pi$. With (33) this gives

$$|z_1|^2 + |z_2|^2 + |z_3|^2 < \varrho(\pi-2M)^2 + 2\varrho(M)^2. \tag{34}$$

The two values. Since $\cot M = 13/50$, the double-angle formulas give

$$\cot(\pi-2M) = \frac{1-\cot^2 M}{2\cot M} = \frac{1-(13/50)^2}{26/50} = \frac{2331}{1300}, \qquad \csc(\pi-2M) = \frac{1+\cot^2 M}{2\cot M} = \frac{2669}{1300}.$$

Also $0.93^2 + 0.67^2(13/50)^2 = 0.89524564 < 0.947^2$ and $\csc^2 M = 1 + (13/50)^2 = 2669/2500 < 1.04^2$. Hence

$$\varrho(M) < 0.947 + 0.67\cdot 1.04 = 1.6438 < 1.65.$$

Next, $0.93^2 + 0.67^2(2331/1300)^2 < 2.3082 < 1.52^2$ and $0.67\cdot 2669/1300 < 1.3756$. Hence

$$\varrho(\pi - 2M) < 1.52 + 1.3756 = 2.8956 < 2.92.$$

Put $\Sigma_+ = |z_1|^2 + |z_2|^2 + |z_3|^2$. By (34),

$$\Sigma_+ < 2.92^2 + 2\cdot 1.65^2 = 13.9714 < 14. \tag{35}$$

The lower zeros. Let $\mu = (z_1+z_2+z_3)/3$ be the mean of the upper zeros; thus $3\mu = U$, with U as in the proof of Lemma 12. Each upper zero has imaginary part at least 0.87, so $|\mu| \ge \operatorname{Im}\mu \ge 0.87$. Put

$$\Sigma_0 = \sum_{j=1}^{3}|z_j-\mu|^2, \qquad \Sigma_2 = \sum_{j=1}^{3}(z_j-\mu)^2, \qquad \Pi = \prod_{j=1}^{3}(z_j-\mu).$$

The three numbers $z_j - \mu$ have sum zero. Expanding $|z_j|^2 = |\mu + (z_j-\mu)|^2$ therefore gives $\Sigma_+ = 3|\mu|^2 + \Sigma_0$. In particular

$$\Sigma_0 = \Sigma_+ - 3|\mu|^2 \ge 0. \tag{36}$$

Clearly $|\Sigma_2| \le \Sigma_0$. The inequality between the geometric and the arithmetic mean of the three numbers $|z_j-\mu|^2$ gives $|\Pi| \le (\Sigma_0/3)^{3/2}$. By (35), $\Sigma_0 \le \Sigma_+ < 14 < 27\cdot 0.87^2 \le 27|\mu|^2$, so $(\Sigma_0/3)^{1/2} < 3|\mu|$. Hence

$$|\Sigma_2| \le \Sigma_0, \qquad |\Pi| \le |\mu|\,\Sigma_0. \tag{37}$$

By (8), $z_4 + z_5 = -3\mu$ and $z_1^3 + \cdots + z_5^3 = 0$. The sum of the cubes of three numbers with sum zero is three times their product. So

$$z_1^3 + z_2^3 + z_3^3 = \sum_{j=1}^{3} \bigl(\mu + (z_j - \mu)\bigr)^3 = 3\mu^3 + 3\mu\Sigma_2 + 3\Pi.$$

Also

$$z_4^3 + z_5^3 = (z_4 + z_5)^3 - 3z_4z_5(z_4 + z_5) = -27\mu^3 + 9\mu\, z_4z_5.$$

The two sums add up to zero, and $\mu \neq 0$. This gives

$$z_4z_5 = \frac{8}{3}\mu^2 - \frac{1}{3}\Sigma_2 - \frac{\Pi}{3\mu}, \qquad (z_4 - z_5)^2 = (z_4 + z_5)^2 - 4z_4z_5 = -\frac{5}{3}\mu^2 + \frac{4}{3}\Sigma_2 + \frac{4\Pi}{3\mu}.$$

By (37), $|z_4 - z_5|^2 \le \frac{5}{3}|\mu|^2 + \frac{8}{3}\Sigma_0$. The parallelogram identity gives

$$|z_4|^2 + |z_5|^2 = \tfrac{1}{2}\bigl(|z_4 + z_5|^2 + |z_4 - z_5|^2\bigr) \le \tfrac{16}{3}|\mu|^2 + \tfrac{4}{3}\Sigma_0.$$

By (36), $3|\mu|^2 \le \Sigma_+$. Hence, with (35),

$$\sum_{k=1}^{5} |z_k|^2 \le \Sigma_+ + \tfrac{16}{3}|\mu|^2 + \tfrac{4}{3}(\Sigma_+ - 3|\mu|^2) = \tfrac{7}{3}\Sigma_+ + \tfrac{4}{3}|\mu|^2 \le \tfrac{25}{9}\Sigma_+ < \tfrac{350}{9}.$$

Each zero. By (8), $z_k = -\sum_{j\neq k} z_j$. The Cauchy–Schwarz inequality gives $|z_k|^2 \le 4\sum_{j\neq k}|z_j|^2$, that is, $5|z_k|^2 \le 4\sum_{j=1}^{5}|z_j|^2 < 1400/9$. So $|z_k|^2 < 280/9 < 31.36 = 5.6^2$. □

Lemma 32. *Let P be a quintic of height $y \ge 0.87$ with three upper zeros. Let $x + \sqrt{q} + iy$ and $x - \sqrt{q} + iy$, with x real and $q \ge 0$, be two upper contacts of P, counted with multiplicity; for $q = 0$ this means one contact of multiplicity at least two. Then*

$$(x - m_I)^2 < 3s_I^2 - y^2, \qquad q < 3s_I^2 - y^2 < 0.591,$$

which is (10). *Moreover $y^2 < 3s_I^2 < 1.35$ and $-1.70 < x < -0.15$.*

Proof. An angle bound. Put $M_y = 2\arctan(s_I/y)$ and $\Delta = y^2 - s_I^2$. By (29), $\Delta > 0$. The formula for the cotangent of a double angle gives $\cot M_y = (1 - s_I^2/y^2)/(2s_I/y) = \Delta/(2ys_I) > 0$. Every zero of P has height at least y. So (31) gives

$$|\theta_I(z_k)| \le M_y < \frac{\pi}{2} \qquad (k = 1, \ldots, 5). \tag{38}$$

Let θ_1 and θ_2 be the angles θ_I at the two contacts, and θ_3 the angle θ_I at the third upper zero, all counted with multiplicity. As in the proof of Lemma 12, $\sum_k \theta_I(z_k) \ge \pi$. The two lower zeros contribute negative terms, so $\theta_1 + \theta_2 + \theta_3 > \pi$. With (38) this gives

$$\pi - M_y < \theta_1 + \theta_2 \le 2M_y < \pi.$$

Hence

$$\cot(\theta_1 + \theta_2) < \cot(\pi - M_y) = -\cot M_y = -\frac{\Delta}{2s_Iy}. \tag{39}$$

The inequality in polynomial form. Put $t_1 = (x - m_I)^2$ and $t_2 = q$. By (30),

$$\cot\theta_1 = \frac{t_1 + t_2 + \Delta + 2(x - m_I)\sqrt{q}}{2s_Iy}, \qquad \cot\theta_2 = \frac{t_1 + t_2 + \Delta - 2(x - m_I)\sqrt{q}}{2s_Iy}.$$

Thus $\cot\theta_1 + \cot\theta_2 = (t_1+t_2+\Delta)/(s_I y) > 0$ and $4s_I^2y^2\cot\theta_1\cot\theta_2 = (t_1+t_2+\Delta)^2 - 4t_1t_2$. We multiply (39) by the positive number $4s_I^2y^2(\cot\theta_1+\cot\theta_2)$, and we use $\cot(\theta_1+\theta_2) = (\cot\theta_1\cot\theta_2 - 1)/(\cot\theta_1+\cot\theta_2)$. This gives

$$(t_1+t_2+\Delta)^2 - 4t_1t_2 - 4s_I^2y^2 < -2\Delta\,(t_1+t_2+\Delta),$$

that is,

$$\Upsilon(t_1,t_2) < 0, \qquad \text{where} \quad \Upsilon(t_1,t_2) = (t_1-t_2)^2 + 4\Delta(t_1+t_2) + 3\Delta^2 - 4s_I^2y^2. \tag{40}$$

We need two facts:

$$\Delta^2 + 4s_I^2y^2 = (y^2+s_I^2)^2, \qquad y^2 + s_I^2 < 4\Delta. \tag{41}$$

The first is an identity. The second says $5s_I^2 < 3y^2$. It holds, since $5s_I^2 < 5\cdot 0.4489 = 2.2445$ by (29) and $3y^2 \geq 3\cdot 0.7569 = 2.2707$.

Solving the inequality. Let $t_1, t_2 \geq 0$ satisfy (40). For fixed t_2, Υ is a quadratic polynomial in t_1 with leading coefficient 1; its derivative in t_1 is $2(t_1-t_2)+4\Delta$. Suppose first that $t_2 > 2\Delta$. Then the least value of $\Upsilon(\cdot,t_2)$ on $t_1 \geq 0$ is taken at $t_1 = t_2 - 2\Delta$, and by (41)

$$\Upsilon(t_2-2\Delta,t_2) = 8\Delta t_2 - \Delta^2 - 4s_I^2y^2 = 8\Delta t_2 - (y^2+s_I^2)^2.$$

So $8\Delta t_2 < (y^2+s_I^2)^2 < 16\Delta^2$, that is, $t_2 < 2\Delta$. This is a contradiction. Hence $t_2 \leq 2\Delta$. Then the derivative is at least $4\Delta - 2t_2 \geq 0$ for $t_1 \geq 0$, and by (41)

$$0 > \Upsilon(t_1,t_2) \geq \Upsilon(0,t_2) = (t_2+2\Delta)^2 - \Delta^2 - 4s_I^2y^2 = (t_2+2\Delta)^2 - (y^2+s_I^2)^2.$$

So $t_2 + 2\Delta < y^2 + s_I^2$, that is, $t_2 < 3s_I^2 - y^2$. The polynomial Υ is symmetric in t_1 and t_2, so also $t_1 < 3s_I^2 - y^2$. Finally, by (29),

$$3s_I^2 - y^2 < 3\cdot 0.4489 - 0.7569 = 0.5898 < 0.591.$$

Consequences. Since $q \geq 0$, we get $y^2 < 3s_I^2 < 1.3467 < 1.35$. Also $|x - m_I| < \sqrt{0.591} < 0.77$, and (29) gives $-1.697 < x < -0.156$. □

Lemma 33. *Let P be a quintic of height $y \geq 0.87$ whose three upper zeros are all contacts. As in the proof of Lemma 13, write the upper zeros $\mu+\varepsilon_j$ $(j=1,2,3)$ and the lower zeros $-\frac{3}{2}\mu \pm \omega$: here μ is the mean of the upper zeros, so that $\operatorname{Im}\mu = y$, the ε_j are real with sum zero, and $|\operatorname{Im}\omega| \leq y/2$. Write $\mu = \mu_1 + iy$ and $\omega = \omega_1 + i\omega_2$ with $\mu_1, \omega_1, \omega_2$ real. Then*

$$\mu_1^2 + \Big(\omega_1 + \frac{\omega_2}{y}\,\mu_1\Big)^2 < 1.03, \qquad \Big(-\tfrac{3}{2}\mu_1 \pm \omega_1\Big)^2 < 5.15.$$

In particular the real parts $-\frac{3}{2}\operatorname{Re}\mu \pm \operatorname{Re}\omega$ of both lower zeros lie in $(-3,3)$.

Proof. Put $\Sigma_2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$. (Here $z_j - \mu = \varepsilon_j$ for the upper zeros, so this is the Σ_2 of the proof of Lemma 31.)

A bound for Σ_2. Number the ε_j so that $\varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_3$. The upper zeros $\mu+\varepsilon_1$ and $\mu+\varepsilon_3$ are contacts of the form $x \pm \sqrt{q} + iy$, with $x = \mu_1 + (\varepsilon_1+\varepsilon_3)/2$ and $2\sqrt{q} = \varepsilon_3 - \varepsilon_1$. By Lemma 32, $(\varepsilon_3-\varepsilon_1)^2 = 4q < 4(3s_I^2 - y^2)$. The two gaps $\varepsilon_2 - \varepsilon_1$ and $\varepsilon_3 - \varepsilon_2$ are nonnegative and have sum $\varepsilon_3 - \varepsilon_1$, so their squares have sum at most $(\varepsilon_3-\varepsilon_1)^2$. Since $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 0$,

$$(\varepsilon_1-\varepsilon_2)^2 + (\varepsilon_1-\varepsilon_3)^2 + (\varepsilon_2-\varepsilon_3)^2 = 3\Sigma_2.$$

Hence $3\Sigma_2 \le 2(\varepsilon_3-\varepsilon_1)^2$, and

$$\Sigma_2 < \tfrac{8}{3}\,(3s_I^2 - y^2). \tag{42}$$

The imaginary part of the cubic relation. The proof of Lemma 13 gives

$$\varepsilon_1\varepsilon_2\varepsilon_3 = \tfrac{5}{4}\mu^3 + 3\mu\omega^2 - \mu\,\Sigma_2,$$

and the left side is real. Now $\operatorname{Im}\mu^3 = 3\mu_1^2 y - y^3$, $\operatorname{Im}(\mu\omega^2) = y(\omega_1^2-\omega_2^2)+2\mu_1\omega_1\omega_2$ and $\operatorname{Im}(\mu\Sigma_2) = y\Sigma_2$. So the imaginary part of the right side vanishes:

$$\tfrac{5}{4}(3\mu_1^2 y - y^3) + 3y(\omega_1^2-\omega_2^2) + 6\mu_1\omega_1\omega_2 - y\Sigma_2 = 0.$$

We divide by $3y$ and complete the square in ω_1:

$$\Big(\omega_1 + \frac{\omega_2}{y}\,\mu_1\Big)^2 = \frac{\Sigma_2}{3} - \frac{5}{12}(3\mu_1^2 - y^2) + \frac{\omega_2^2}{y^2}(\mu_1^2 + y^2). \tag{43}$$

The bound 1.03. Add μ_1^2 to both sides of (43). On the right, the coefficient of μ_1^2 becomes $\omega_2^2/y^2 - 1/4 \le 0$, and the coefficient of y^2 is $5/12 + \omega_2^2/y^2 \le 2/3$, because $\omega_2^2 \le y^2/4$. With (42), (29) and $y^2 \ge 0.7569$ we get

$$\mu_1^2 + \Big(\omega_1 + \frac{\omega_2}{y}\,\mu_1\Big)^2 \le \frac{\Sigma_2}{3} + \frac{2}{3}y^2 < \frac{8}{3}s_I^2 - \frac{2}{9}y^2 < \frac{8}{3}\cdot 0.4489 - \frac{2}{9}\cdot 0.7569 < 1.0289 < 1.03.$$

The lower real parts. We have

$$-\tfrac{3}{2}\mu_1 \pm \omega_1 = \pm\Big(\omega_1 + \frac{\omega_2}{y}\,\mu_1\Big) - \Big(\tfrac{3}{2} \pm \frac{\omega_2}{y}\Big)\mu_1, \qquad \Big|\tfrac{3}{2} \pm \frac{\omega_2}{y}\Big| \le 2.$$

The Cauchy–Schwarz inequality in $\mathbb{R}^2$ gives $(-\frac{3}{2}\mu_1 \pm \omega_1)^2 < (1+2^2)\cdot 1.03 = 5.15 < 9$. □

B An algebraic description of the second height

The double contact $r = x + iy$ and the simple contact $\zeta = s - iy$ of Section 5 satisfy $\zeta = -r + d$ with $d = x + s$ real (this d is not that of Lemma 15). Set $Q^*(r,d) = L(r)\,Q(r,-r+d)$, with L, Q as in Section 5. It has real coefficients and degree five in r. For a quintic with such contacts, r is a root of $Q^*(\cdot,d)$, and so is $\bar r$, since d and the coefficients are real. Compute

$$\Delta(d,T) = \left.\frac{\operatorname{Res}_r(Q^*(r,d), Q^*(r+v,d))}{v^5}\right|_{v^2=-4T}.$$

Here $\bar r = r + v$ with $v = -2i\,\operatorname{Im} r$, so that $v^2 = -4T$ with $T = (\operatorname{Im} r)^2$ (not the operator T of Section 2). An overall nonzero scalar is immaterial. Over the roots ρ_i of $Q^*(\cdot,d)$ the resultant is a constant times $\prod_{i,j}(\rho_i - \rho_j + v)$; the factors with $i = j$ give v^5, and those for (i,j) and (j,i) multiply to $v^2 - (\rho_i - \rho_j)^2$. So the quotient is a polynomial in v^2. At the extremal point the derivative $\gamma_r(r_0, 0)$ of the simple contact with respect to r is real and positive (proof of Lemma 15), so $d = \gamma(r,0) + r$ is a nonsingular local coordinate. Along real d the contacts remain on opposite lines at a common height $\operatorname{Im} r(d)$, and this height is largest at the extremal point, by Section 5. So T is stationary there. Since $\Delta(d, T(d)) = 0$ along this curve, differentiation gives $\Delta = \partial_d\Delta = 0$ at the extremal point. Form $\operatorname{Res}_d(\Delta, \partial_d\Delta)$. Exact factorization and Sturm counts [1] give a single irreducible factor having a zero in $[7569/10000, 81/100]$, of degree 57 with exactly one zero there. This construction and interval identify h_0^2. The field norm of that root, which is $-a_0/a_{57}$ for the factor $\sum_k a_k T^k$, is negative [1]. So the root is not a square in the number field that it generates, and h_0 has degree 114. The full coefficient arrays and independent resultant checks accompany the proof records [1].

References

[1] D. Goel, *Proof records for the Hermite strip problem: certificates, input data and verifier programs*, supplement to this paper, November 2026, https://doi.org/10.5281/zenodo.24547213

[2] D. Goel, *Turán's Hermite strip problem (1967) is resolved: the sharp half-width is* 0.90366…, focused version of this paper, November 2026, https://doi.org/10.5281/zenodo.24547227

[3] J. Dieudonné, *La théorie analytique des polynômes d'une variable (à coefficients quelconques)*, Mémorial des Sciences Mathématiques **93**, Gauthier-Villars, Paris, 1938, http://www.numdam.org/item/MSM_1938__93__1_0/

[4] A. Giroux, *Estimates for the imaginary parts of the zeros of a polynomial*, Proc. Amer. Math. Soc. **44** (1974), 61–67.

[5] W. K. Hayman, *Research Problems in Function Theory*, Athlone Press, London, 1967.

[6] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory: Fiftieth Anniversary Edition*, Problem Books in Mathematics, Springer, Cham, 2019. We quote Problem 4.6, Update 4.6 and the tables from the draft of September 2018, arXiv:1809.07200

[7] E. Makai and P. Turán, *Hermite expansion and distribution of zeros of polynomials*, Publ. Math. Inst. Hungar. Acad. Sci. **8** (1963), 157–163, https://real.mtak.hu/201433/1/cut_MATKUTINT_8_1_-_2_1963_pp157_-_163.pdf

[8] NIST Digital Library of Mathematical Functions, equation 18.18.28, https://dlmf.nist.gov/18.18.E28

[9] G. Schmeisser, *Optimale Schranken zu einem Satz über Nullstellen Hermitescher Trinome*, J. Reine Angew. Math. **246** (1971), 147–160, https://gdz.sub.uni-goettingen.de/download/pdf/PPN243919689_0246/LOG_0011.pdf

[10] G. Schmeisser, *Estimates for the imaginary parts of the zeros of a polynomial*, Math. Balkanica (N.S.) **16** (2002), 203–217, https://www.math.bas.bg/infres/MathBalk/MB-16/MB-16-203-217.pdf

[11] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc., Providence, 1975.

[12] P. Turán, *Az approximációelmélet egyes nyitott problémáiról* [On some open problems of approximation theory], Mat. Lapok **25** (1974), 21–75, https://real-j.mtak.hu/9373/1/MTA_MatematikaiLapok_1974.pdf