A countable group that is hyperlinear and not sofic

Peter Burton (peterburton1728@gmail.com) - Claude Fable 5.2 written draft

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This note was written by Claude Fable 5.2, an AI system made by Anthropic, from the Lean development that it accompanies and under the direction of Peter Burton. It has not been fully checked by a human. The Lean development is the proof of record: where this note and the Lean files differ, the Lean files are right. The author is actively working on preparation of a human-verified, human-facing exposition of the proof.

Abstract. We describe a countable group that is hyperlinear and not sofic, which answers in the negative Pestov's question whether every hyperlinear group is sofic. The group is a quotient of a semidirect product of matrix groups, defined over a nilpotent extension of the ring $\mathbb{Z}[1/2]$, by a central infinite cyclic subgroup. Its hyperlinearity is an instance of a theorem of Thom. Its non-soficity is new: a subgroup with Kazhdan's property (T) is normalized by eight elements whose action on three central elements cannot be followed by any sofic approximation. The proof is formalized in Lean 4; this note follows the formal proof and names its files.

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1. The theorem

Let G be a group. For permutations p, q of a nonempty finite set Y let $d(p,q) = |\{y : p(y) \neq q(y)\}|/|Y|$ be the normalized Hamming distance. For complex matrices A, B indexed by a nonempty finite set Y let $\|A-B\|_2^2 = \frac{1}{|Y|}\sum_{i,j}|A_{ij}-B_{ij}|^2$ be the squared normalized Hilbert-Schmidt distance.

Definition. G is *sofic* if for every finite $F \subseteq G$ and every $\varepsilon > 0$ there are a nonempty finite set Y and a map σ from G to the permutations of Y with $d(\sigma(gh), \sigma(g)\sigma(h)) \leq \varepsilon$ for all $g, h \in F$ and $d(\sigma(g), \sigma(h)) \geq 1-\varepsilon$ for all distinct $g, h \in F$. G is *hyperlinear* if for every finite $F \subseteq G$ and every $\varepsilon > 0$ there are a nonempty finite set Y and a map σ from G to the unitary matrices indexed by Y with $\|\sigma(gh) - \sigma(g)\sigma(h)\|_2^2 \leq \varepsilon$ for all $g, h \in F$ and $\|\sigma(g) - \sigma(h)\|_2^2 \geq 2-\varepsilon$ for all distinct $g, h \in F$.

Theorem 1. There is a countable group that is hyperlinear and not sofic.

In Lean this is `HyperlinearNotSofic.exists_countable_hyperlinear_not_sofic` in `Challenge.lean`, with the two definitions above.

Pestov's guide [P] defines sofic and hyperlinear groups as subgroups of metric ultraproducts of symmetric groups and of unitary groups, and characterizes them by local criteria (Theorems 3.5 and 3.6 of [P]) that differ from the definition above in three ways: multiplicativity is asked only when $gh \in F$, the image of the identity is asked to be close to the identity, and the separation is the constant $1/4$. Since soficity occurs in Theorem 1 under a negation, the comparison matters. The file `ModelTests.lean` proves that the soficity defined above is equivalent to the criterion of Theorem 3.5 of [P], and that the hyperlinearity defined above implies the criterion of Theorem 3.6 of [P]. The first uses the amplification argument of Elek and Szabó [ES], in the formalization of [GA]. The equivalence of the criteria with the ultraproduct definitions is not formalized.

Every sofic group is hyperlinear [ES]. Pestov's guide continues "By contrast, the converse implication is unknown" and asks, as Open question 3.4: "Is every hyperlinear group sofic?" Theorem 1 says that the answer is no.

Until 2026 no group was known to be non-sofic. The first was announced by OpenAI in August 2026 [O], and its mechanism was analysed by Kun and Thom [KT2]. A non-hyperlinear group

was announced in September 2026 [ALT]. Neither answers Pestov's question, which needs a single group on the hyperlinear side and on the non-sofic side. Natural candidates were known: Thom [T] proved that the group $K_0(\mathbb{Z}[1/p])/\mathbb{Z}$ of de Cornulier [C], a finitely presented group with property (T) that is not residually finite, is hyperlinear, and more generally that a quotient of a hyperlinear group by a central subgroup is hyperlinear; in August 2026 these examples were still "not known to be sofic" [AT]. The group of this note is a central quotient of the same kind, but it is not that group, and we prove nothing about that group.

2. The group

Let $R = \mathbb{Z}[1/2]$. Let A be the set of quadruples (s, u, v, c) with $s, v \in R$ and $u, c \in R^3$, with componentwise addition and the product

$$(s, u, v, c)\,(s', u', v', c') = (ss',\ su' + s'u,\ sv' + s'v,\ sc' + s'c + v'u).$$

This is a ring with unit $(1, 0, 0, 0)$. Write $U(u) = (0, u, 0, 0)$, $V = (0, 0, 1, 0)$ and $C(c) = (0, 0, 0, c)$. Then $U(u)V = C(u)$, all other products of such elements vanish, and $1 + C(c)$ is central in A and invertible. The ring A is finitely generated. A matrix $B \in \mathrm{GL}_3(R)$ acts on A by the ring automorphism $(s, u, v, c) \mapsto (s, Bu, v, Bc)$.

Let St be the group of all $B \in \mathrm{GL}_3(R)$ with $Be_0 = e_0$, where e_0, e_1, e_2 is the standard basis of R^3. It acts on $\mathrm{GL}_3(A)$ entry by entry. Put $\tilde{P} = \mathrm{GL}_3(A) \rtimes \mathrm{St}$. For $c \in R^3$ let $\zeta(c) \in \mathrm{GL}_3(A)$ be the scalar matrix $(1 + C(c)) \cdot I$. Then $\zeta(c)\zeta(c') = \zeta(c + c')$, each $\zeta(c)$ is central in $\mathrm{GL}_3(A)$, and $B\,\zeta(c)\,B^{-1} = \zeta(Bc)$ for $B \in \mathrm{St}$. So the *line* $\{\zeta(te_0) : t \in R\}$ is central in $\tilde{P}$. Put $z = \zeta(e_0)$ and

$$G = \tilde{P}/\langle z \rangle.$$

G is countable. Let Γ be the image in G of the elementary group $\mathrm{EL}_3(A)$, which is generated by the matrices $I + xE_{ik}$ with $x \in A$ and $i \neq k$. We use the following elements of G, written by representatives in $\tilde{P}$:

$$a = \zeta(e_1), \qquad b = \zeta(e_2), \qquad j = \zeta(e_0/2),$$

$$X = I + \tfrac{1}{2}E_{01}, \qquad S_1 = I - E_{21}, \qquad S_2 = I - E_{12}, \qquad D = \mathrm{diag}(1, \tfrac{1}{2}, 2).$$

Lemma 2. (i) a, b, j lie in Γ and are central in Γ; $j^2 = 1$ and $j \neq 1$. (ii) Conjugation by each of X, S_1, S_2, D maps Γ onto itself. (iii) For $m \geq 1$: $S_1^m a S_1^{-m} = ab^{-m}$ and $S_1^m b S_1^{-m} = b$; $S_2^m b S_2^{-m} = ba^{-m}$ and $S_2^m a S_2^{-m} = a$; $Da^2D^{-1} = a$ and $DbD^{-1} = b^2$; $XaX^{-1} = ja$. (iv) Γ is infinite, finitely generated, and has Kazhdan's property (T).

Proof. (iii) follows from $B\zeta(c)B^{-1} = \zeta(Bc)$: for instance $Xe_1 = e_1 + e_0/2$ and $D(2e_1) = e_1$. For (i), $j^2 = \zeta(e_0) = z$, and $j \neq 1$ because $e_0/2 \notin \mathbb{Z}e_0$; the scalar matrices $\zeta(c)$ are products of commutators of elementary matrices, because $[I + U(u)E_{ik}, I + VE_{ki}] = I + C(u)E_{ii}$. (ii) holds because St permutes the elementary matrices. In (iv), property (T) is the theorem of Ershov and Jaikin-Zapirain [EJZ] that $\mathrm{EL}_n(A)$ has property (T) for a finitely generated ring A and $n \geq 3$, which passes to quotients; we use its formalization in [GA]. (Lean: `HNS/HeisAlg`, `Witness`, `Marks`, `Quotient`, `WitnessFacts`, `PropT`.)

3. The group is hyperlinear

Write $e(x) = \exp(2\pi i x)$.

Reduction. For $\ell \geq 0$ put $n = 2^{\ell+1}+1$. Since $2\cdot(-2^\ell) \equiv 1$ modulo n, there is a ring homomorphism $\rho : R \to \mathbb{Z}/n$. All the constructions of Section 2 make sense over $\mathbb{Z}/n$ and give a finite group $H = \tilde{P}(\mathbb{Z}/n)$ and a homomorphism $\tilde{P} \to H$, again written ρ. The line $L = \{\zeta(te_0) : t \in \mathbb{Z}/n\}$ is a central cyclic subgroup of H of order n, and $\rho(\zeta(te_0)) = \zeta(\rho(t)e_0)$.

Representations. Fix $r \geq 1$ and let $\chi_k(\zeta(te_0)) = e(kt/n)$ for $0 \leq k < r$, a character of L. Choose a representative $\mathrm{rep}(y) \in H$ in every coset $y \in H/L$, and for $h \in H$ write $h\,\mathrm{rep}(y) = \mathrm{rep}(hy)\,\lambda(h,y)$ with $\lambda(h,y) \in L$. Let $\pi(h)$ be the matrix indexed by $(H/L) \times \{0,\dots,r-1\}$ that sends the basis vector at (y,k) to $\chi_k(\lambda(h,y))$ times the basis vector at (hy,k). This is the direct sum over k of the representations of H induced from the characters χ_k. Three facts (`HNS/Monomial`, `HNS/CosetRep`):

1. π is a homomorphism from H to the unitary matrices.
2. If $p^{-1}q \notin L$ then $\|\pi(p) - \pi(q)\|_2^2 = 2$, because the two monomial matrices have disjoint supports.
3. If $c \in L$ then $\|\pi(p) - \pi(pc)\|_2^2 = \frac{1}{r}\sum_{k<r} |1 - \chi_k(c)|^2$.

The maps. Choose a lift $s(g) \in \tilde{P}$ of every $g \in G$ and put $\sigma(g) = \pi(\rho(s(g)))$, a unitary matrix. It depends on ℓ and r.

Multiplicativity. For $g, h \in G$ there is $m \in \mathbb{Z}$ with $s(g)s(h) = s(gh)z^m$. By facts 1 and 3,

$$\|\sigma(gh) - \sigma(g)\sigma(h)\|_2^2 = \frac{1}{r}\sum_{k<r} |1 - e(km/n)|^2 \leq \Big(\frac{2\pi r|m|}{n}\Big)^2,$$

which tends to 0 as $\ell \to \infty$ for fixed r and m.

Separation. Let $g \neq h$ and $x = s(g)^{-1}s(h)$, so $x \notin \langle z\rangle$. If x does not lie on the line of $\tilde{P}$, then some nonzero element $\delta \in R$ depending on x has the property that $\rho(x) \notin L$ whenever $\rho(\delta) \neq 0$. A nonzero element of R is nonzero modulo $2^{\ell+1}+1$ for all large ℓ, so by fact 2 the squared distance of $\sigma(g)$ and $\sigma(h)$ is exactly 2 for all large ℓ. If $x = \zeta(ce_0)$ lies on the line, then $c \notin \mathbb{Z}$, and $c = \alpha/2^B$ with an integer α not divisible by 2^B. Take $r = 2^B$ and $\ell + 1 \geq B$. Since $2^{\ell+1} \equiv -1$ modulo n, we have $\rho(c) = -\alpha 2^{\ell+1-B}$ and

$$\chi_k(\rho(x)) = e\Big(-\frac{k\alpha 2^{\ell+1-B}}{n}\Big) = e\Big(-\frac{k\alpha}{2^B}\Big)\,e\Big(\frac{k\alpha}{2^B n}\Big).$$

The first factors sum to 0 over $0 \leq k < 2^B$, because $e(-\alpha/2^B)$ is a root of unity of order dividing 2^B other than 1. The second factors are within $2\pi|\alpha|/n$ of 1. Hence $|\sum_{k<2^B} \chi_k(\rho(x))| \leq 2^B \cdot 2\pi|\alpha|/n$, and by fact 3

$$\|\sigma(g) - \sigma(h)\|_2^2 = 2 - \frac{2}{2^B}\,\mathrm{Re}\sum_{k<2^B} \chi_k(\rho(x)) \;\geq\; 2 - \frac{4\pi|\alpha|}{n}.$$

Conclusion. Given a finite $F \subseteq G$ and $\varepsilon > 0$, choose B such that every line coordinate c arising from a pair of elements of F is of the form $\alpha/2^B$, put $r = 2^B$, and choose ℓ so large that the finitely many estimates above hold with error at most ε. This proves that G is hyperlinear. (Lean: `HNS/PreMap`, `Reduction`, `CharSum`, `HyperWit`; the statement is `HNS.wit_hyperlinear_models`.)

This is the construction of the second proof of Proposition 3.2 of [T], carried out for the group G.

4. A criterion for non-soficity

Theorem 3. Let P be a countable group and $\Gamma \leq P$ an infinite, finitely generated subgroup with Kazhdan's property (T). Let a, b, j be central elements of Γ with $j^2 = 1 \neq j$. Suppose that there are elements $t_{1,m}, t_{2,m}$ $(m = 1, 2, 3)$, t_D, t_X of P that normalize Γ and whose conjugations $\beta(g) = tgt^{-1}$ satisfy: $\beta_{1,m}(a) = ab^{-m}$ and $\beta_{1,m}(b) = b$; $\beta_{2,m}(b) = ba^{-m}$ and $\beta_{2,m}(a) = a$; $\beta_D(a^2) = a$ and $\beta_D(b) = b^2$; $\beta_X(a) = ja$. Then P is not sofic.

In Lean this is `HNS.not_sofic_of_actors`. By Lemma 2 it applies to $P = G$ with $t_{i,m} = S_i^m$, $t_D = D$ and $t_X = X$, and gives the second half of Theorem 1 (`HNS.wit_not_sofic`). We now describe its proof. Steps 1 to 3 follow the mechanism of [O] as analysed in [KT2], which rests on Kun's theorem [K] and on the cluster groups of [KT1]; the statements we use from [O] are proved in its Lean development, of which this package compiles a slice. In [O] and [KT2] the obstruction comes from conjugating a Kazhdan subgroup properly into itself. Here the eight elements normalize Γ, and the obstruction is the one of Lemma 4 below.

Suppose that P is sofic. Then P has a sofic approximation: maps $\sigma_n : P \to \mathrm{Sym}(V_n)$ that are asymptotically multiplicative and under which every non-trivial element has asymptotically no fixed points. Fix a finite symmetric generating set S of Γ.

Step 1: expander cores. By Kun's theorem [K], in the partition form proved in [O], there are $\gamma > 0$ and partitions of the sets V_n such that every part expands at rate γ under $\sigma_n(S)$ and the total boundary of the parts is $o(|V_n|)$. Each part is pruned to a *core*: the cores are disjoint, each expands at rate $\gamma/4$ through its internal edges, and they cover all but $o(|V_n|)$ points (`HNS/Prune`, `Complete`, `Cores`, `KunCores`). On a core K the generators are completed to permutations ρ_s of K.

Step 2: cluster groups. Fix n and a set of *bad* points of V_n of small density, off which the model is multiplicative on short words, short non-trivial elements have no fixed point, and points stay in their cores. A core is *typical* if it meets the bad set in a small proportion. On a typical core, short words in the ρ_s act as in Γ off a small exceptional set (`HNS/LocalModel`). The *defect* of a permutation p of K is the number of pairs (s, x) with $p(\rho_s x) \neq \rho_s(px)$; it measures how far p is from commuting with the generators. By expansion, two permutations of small defect are either close or almost disjoint, and by the improvement theorem of [O] (compare [KT1]) every permutation of small defect is close to one of much smaller defect (`HNS/Improve`). It follows that the classes of permutations of small defect, under the relation of being close, form a finite group H_K, the *cluster group* of the core (`HNS/ClusterSystem`, `CoreCluster`). A central element g of Γ acts on K by a permutation of small defect; its class $g_K \in H_K$ is its *mark*. Marks multiply as the elements do, and $j_K \neq 1$, because $\sigma_n(j)$ moves almost every point (`HNS/ClusterMarks`).

Step 3: matching and transport. Let t be one of the eight elements and $\tau = \sigma_n(t)$. Off the bad set, $\tau\sigma_n(g) = \sigma_n(\beta(g))\tau$ for short $g \in \Gamma$. Colour the points of a typical core K by the core that contains their image under τ. Few edges of K join points of different colours, so by expansion one colour carries almost all of K (`HNS/Dominant`): τ sends almost all of K into one

core K'. With the weights $\mu(K) = |K|/|V_n|$ this defines a partial injection $K \mapsto K'$ of the set of typical cores that is almost measure preserving (`HNS/Matching`). Transporting permutations of K to K' along τ changes defects, distances and products by little, and so induces an isomorphism $\psi : H_K \to H_{K'}$ with $\psi(g_K) = \beta(g)_{K'}$ for the central elements in question (`HNS/TransportEst`, `TransportIso`, `Stage`, `StageIso`, `StageTransport`, `StageActors`).

Step 4: the finite-order lemma. After steps 1 to 3 the group Γ has disappeared, and what is left is a statement about a weighted family of finite groups.

Let Ω be a finite set with weights $\mu(\omega) \geq 0$, and write $\mu(E) = \sum_{\omega \in E} \mu(\omega)$. A *near measure-preserving partial injection* with error ε is an injective map φ from a subset $\operatorname{dom}\varphi \subseteq \Omega$ to Ω such that $\mu(\Omega \setminus \operatorname{dom}\varphi) \leq \varepsilon$ and $\sum_{\omega \in \operatorname{dom}\varphi} |\mu(\varphi\omega) - \mu(\omega)| \leq \varepsilon$. Two consequences (`HNS/Mass`):

- (M1) If $E, E' \subseteq \Omega$ and $\varphi(E \cap \operatorname{dom}\varphi) \subseteq E'$, then $\mu(E) \leq \mu(E') + 2\varepsilon$.
- (M2) If $f : \Omega \to [0, M]$ satisfies $f(\varphi\omega) \geq f(\omega)$ on $\operatorname{dom}\varphi$ and $f(\varphi\omega) \geq f(\omega) + 1$ on $E \cap \operatorname{dom}\varphi$, then $\mu(E) \leq (2M+1)\varepsilon$.

For (M2), injectivity gives $\sum_{\operatorname{dom}} \mu(\varphi\omega) f(\varphi\omega) \leq \sum_{\Omega} \mu f$, while the left side is at least $\sum_{\operatorname{dom}} \mu(\omega) f(\varphi\omega) - M\varepsilon \geq \sum_{\Omega} \mu f - 2M\varepsilon + \mu(E \cap \operatorname{dom}\varphi)$. Hence $\mu(E \cap \operatorname{dom}\varphi) \leq 2M\varepsilon$, and $\mu(E) \leq \mu(E \cap \operatorname{dom}\varphi) + \varepsilon$.

Lemma 4 (finite-order lemma). For every $\omega \in \Omega$ let H_ω be a finite group with elements $a_\omega, b_\omega, j_\omega$ such that a_ω commutes with b_ω and with j_ω, and $j_\omega^2 = 1$. Let $T_{1,m}, T_{2,m}$ $(m = 1, 2, 3)$, T_D, T_X be near measure-preserving partial injections of Ω with error ε, each with isomorphisms $\psi_\omega : H_\omega \to H_{\varphi\omega}$ for ω in its domain, such that: along $T_{1,m}$, $a \mapsto ab^{-m}$ and $b \mapsto b$; along $T_{2,m}$, $b \mapsto ba^{-m}$ and $a \mapsto a$; along T_D, $a^2 \mapsto a$ and $b \mapsto b^2$; along T_X, $a \mapsto ja$. Then $\mu\{\omega : j_\omega \neq 1\} \leq 162\,\varepsilon$.

Proof. For an element g of a finite group let $k(g)$ be the exponent of 2 in the order of g. Three facts (`HNS/OrderLemma`):

- (K1) $k(g^2) = \max(k(g) - 1, 0)$.
- (K2) If x and y commute, then for integers $m \neq m'$ with $|m - m'| \leq 3$ at most one of $k(xy^{-m}) + 2 \leq k(y)$ and $k(xy^{-m'}) + 2 \leq k(y)$ holds. Indeed $xy^{-m'} = xy^{-m} \cdot y^{m-m'}$, and $k(y^{m-m'}) \geq k(y) - 1 > k(xy^{-m})$ in the first case, so the product has $k = k(y^{m-m'}) \geq k(y) - 1$.
- (K3) If j commutes with x, $j^2 = 1$, and x and jx have odd order, then $j = 1$: for odd N with $x^N = (jx)^N = 1$ we get $1 = j^N x^N = j$.

Write $k_a(\omega) = k(a_\omega)$ and $k_b(\omega) = k(b_\omega)$, and put

$$E = \{k_a + 2 \leq k_b\}, \qquad F = \{k_b + 2 \leq k_a\}, \qquad Q = \{|k_a - k_b| \leq 1 \text{ and } (k_a > 0 \text{ or } k_b > 0)\}.$$

Shears. For $m = 0, 1, 2, 3$ let $E_m = \{k(ab^{-m}) + 2 \leq k_b\}$, so $E_0 = E$. By (K2) the four sets are pairwise disjoint, and they lie in $\{k_b > 0\}$. Along $T_{1,m}$ the order of a_ω equals the order of $(ab^{-m})_{\varphi\omega}$ and k_b is preserved, so $T_{1,m}$ maps E into E_m, and $\mu(E) \leq \mu(E_m) + 2\varepsilon$ by (M1). Summing, $4\mu(E) \leq \mu\{k_b > 0\} + 6\varepsilon$. In the same way $4\mu(F) \leq \mu\{k_a > 0\} + 6\varepsilon$.

Dilation. Along T_D, by (K1), $k_a(\varphi\omega) = \max(k_a(\omega) - 1, 0)$ and $k_b(\omega) = \max(k_b(\varphi\omega) - 1, 0)$. Hence $k_b - k_a$ does not decrease along T_D, and it increases by at least 1 at every ω with $k_a(\omega) > 0$ or

$k_b(\omega) > 0$. Let f be $k_b - k_a$ truncated to $[-4, 4]$, plus 4. Then $0 \le f \le 8$, f does not decrease along T_D, and f increases by at least 1 on Q. By (M2), $\mu(Q) \le 17\varepsilon$.

Odd order. The sets $\{k_a > 0\}$ and $\{k_b > 0\}$ are contained in $E \cup F \cup Q$. With the two inequalities for the shears, $4\mu(E) + 4\mu(F) \le 2\mu(E) + 2\mu(F) + 2\mu(Q) + 12\varepsilon$, so $\mu(E) + \mu(F) \le \mu(Q) + 6\varepsilon \le 23\varepsilon$ and $\mu\{k_a > 0\} \le 40\varepsilon$.

Twist. Along T_X the order of a_ω equals that of $(ja)_{\varphi\omega}$, so T_X maps $\{k_a = 0\}$ into $\{k(ja) = 0\}$, and by (M1) $\mu\{k(ja) > 0\} \le \mu\{k_a > 0\} + 2\varepsilon$. By (K3), $j_\omega \ne 1$ implies $k_a(\omega) > 0$ or $k(j_\omega a_\omega) > 0$. Therefore $\mu\{j \ne 1\} \le 2\mu\{k_a > 0\} + 2\varepsilon \le 82\varepsilon$. The Lean proof (`HNS/FiniteOrder`, `FiniteOrderGroups`) keeps the cruder bound $\mu\{k_a > 0\} \le 80\varepsilon$ and so obtains 162ε. □

Step 5: contradiction. Apply Lemma 4 to the set Ω of typical cores of V_n with the weights $\mu(K) = |K|/|V_n|$, the cluster groups H_K, the marks a_K, b_K, j_K, and the eight partial injections and isomorphisms of step 3; the relations between the marks are those of Theorem 3. For large n the error ε is as small as we like, while the typical cores carry almost all of the weight and $j_K \ne 1$ on every one of them. This contradicts Lemma 4, and proves Theorem 3. (Lean: `HNS/StageFinal`, `Negl`, `Words`, `CritStage`, `CritSeq`, `Criterion`.)

The Lean development of [O] defines soficity by models on the sets $\{0, \dots, n-1\}$ that send the identity to the identity and separate from the identity; `HNS/SoficBridge` shows that the definition of Section 1 implies it.

5. What the Lean development contains

The development is the directory `lean` of the GitHub repository peterburton1728/hyperlinear-not-sofic. It is registered in the Palomar registry as PALOMAR-2026-11-26-000010, version 1 [PAL]. Registration means that Palomar's mechanical verification of the proof passed and that its automated review identified no blocking problem; it is not peer review.

The compared statement is `Challenge.lean`; `Solution.lean` repeats its definitions and proves the theorem from the library `HNS/` (44 files). `Audit.lean` prints the axioms of the main declarations: `propext`, `Classical.choice` and `Quot.sound`. `ModelTests.lean` contains the comparison of the definitions with the criteria of [P] and controls: finite groups and $\mathbb{Z}$ are sofic and hyperlinear, and sofic groups are hyperlinear, in the vocabulary of the statement.

Two existing developments are compiled again as part of the package: a slice of [O] (Kun's theorem in partition form, the gap for almost-centralizers of expanders, cluster groups, the improvement theorem) and files of [GA] (the theorem of Ershov and Jaikin-Zapirain; sofic amplification). Their results are not credited to this project; the changes made to them for Lean 4.35 are recorded in the directory `provenance/`.

Not formalized, and not claimed: the equivalence of the local criteria with the ultraproduct definitions; finite generation or finite presentation of G; any statement about the group $K_0(\mathbb{Z}[1/p])/\mathbb{Z}$ or about any group other than G. Whether the result is new was examined only by a search of arXiv listings and of the web up to 25 November 2026, which found no group proved to be hyperlinear and not sofic.

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