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A proposed proof of Cazenave's Conjecture 6(iii)

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Abstract

We give a proposed proof of Cazenave's Conjecture 6(iii). For every finite simple connected graph on at least two vertices, the square of the largest Laplacian eigenvalue is at least four times the second-largest adjacency eigenvalue plus eight. The argument treats graphs of maximum degree three using exact Rayleigh quotients for small subgraphs and bounds on the adjacency eigenvalues.

Cazenave's Conjecture 6(iii) concerns the second-largest adjacency eigenvalue and the largest Laplacian eigenvalue of a connected graph [1, Section 7]. The argument below bounds these eigenvalues using small subgraphs and the remaining structure of graphs of maximum degree three. Kramer proved Conjecture 6(ii) from the same section, which gives $\mu_2^2 \geq 4\lambda_2 + 1$ for the second-largest Laplacian eigenvalue μ_2 [2]. The present note addresses Conjecture 6(iii).

Contribution and review status. Rafael Coba Bruges requested the investigation and submits this note. OpenAI's Codex generated the proof and drafted this note. Independent review and information about prior proofs are welcome.

Theorem 1. *Let G be a finite, simple, connected graph on $n \geq 2$ vertices. Let λ_2 denote the second-largest eigenvalue of its adjacency matrix A and let μ_1 denote the largest eigenvalue of its Laplacian $L = D - A$. Then*

$$\mu_1^2 \geq 4\lambda_2 + 8.$$

Proof. Write Δ for the maximum degree. We use three elementary facts. First, for every subgraph H and every nonzero real vector x indexed by its vertices, Rayleigh's principle, applied after extending x by zero, gives

$$\mu_1(G) \geq \frac{\sum_{uv \in E(H)} (x_u - x_v)^2}{\sum_{u \in V(H)} x_u^2}.$$

Every omitted edge contributes a nonnegative term. In particular, the star at a maximum-degree vertex gives $\mu_1 \geq \Delta + 1$. Second, for any positive vertex weights w, the adjacency spectral radius is at most $\max_v \sum_{u \sim v} w_u / w_v$, by the row-sum bound on the diagonally similar matrix $\operatorname{diag}(w)^{-1} A \operatorname{diag}(w)$. Taking unit weights gives $\lambda_2 \leq \Delta$. Third, by interlacing, deleting any vertex leaves a graph whose largest adjacency eigenvalue bounds $\lambda_2(G)$ from above.

Maximum degree at least four. We have $\mu_1^2 \geq (\Delta + 1)^2 \geq 4\Delta + 8 \geq 4\lambda_2 + 8$.

Row	Edges	Coordinates of x in vertex order	Quotient	Lower bound
1	$01, 02, 03, 14, 15$	$(3, -3, -1, -1, 1, 1)$	$100/22$	$9/2$
2	$01, 02, 12, 03, 13, 24$	$(-4, -4, 7, 3, -2)$	$421/94$	$179/40$
3	$01, 02, 12, 03, 14, 35$	$(-7, 6, 0, 3, -2, -1)$	$434/99$	$35/8$
4	$01, 02, 12, 03, 14, 34$	$(-1, 1, 0, 1, -1)$	$18/4$	$35/8$
5	$01, 02, 03, 14, 25$	$(-4, 2, 2, 1, -1, -1)$	$115/27$	$17/4$
6	$01, 02, 03, 14, 24$	$(-2, 1, 1, 1, -1)$	$35/8$	$17/4$
7	$01, 02, 03, 14$	$(-13, 7, 4, 4, -2)$	$1059/254$	$25/6$

Table 1: Rayleigh witnesses for the largest Laplacian eigenvalue.

Maximum degree three. If $n \leq 4$, deleting any vertex gives a graph of maximum degree at most two, so $\lambda_2 \leq 2$ and $\mu_1 \geq 4$ suffices. Assume $n \geq 5$. Table 1 gives exact Rayleigh quotients for the subgraphs used below. The notation ij means the edge $\{i, j\}$.

If an edge joins two degree-three vertices and belongs to no triangle, its endpoints have four distinct other neighbors. This gives row 1. If two triangles share an edge, they give a diamond. Since G is connected and $n \geq 5$, an edge leaves the diamond. The two endpoints of its shared edge already have degree three, so the outgoing edge is incident to one of the other corners. This gives row 2. Both bounds have square greater than 20, and $4\lambda_2 + 8 \leq 20$. Henceforth suppose neither configuration occurs.

Suppose there is a triangle with at least two degree-three vertices. Each degree-three vertex of G has at most two degree-three neighbors. Indeed, if it had three, each of the three incident edges would belong to a triangle, so the three neighbors would span at least two edges. Two of those triangles would share an edge. Assign weight 1 to degree-three vertices and $3/4$ to all others. The weighted row sums are at most $11/4$ at the degree-three vertices and at most $8/3$ elsewhere, so $\lambda_2 \leq 11/4$.

Fix such a triangle. If every neighbor outside the triangle of a triangle vertex is a leaf, connectivity implies that G is the triangle with its attached leaves. Deleting a triangle vertex leaves maximum degree at most two and gives $\lambda_2 \leq 2$. Otherwise choose a triangle vertex with an external neighbor that has another neighbor. Also choose the external neighbor of a second degree-three vertex of the triangle. The two external neighbors are distinct, since a common one would yield two triangles sharing an edge. The extra step from the first external neighbor cannot return to another triangle vertex, for the same reason. It leads either to the second external neighbor (row 4) or to a new vertex (row 3). Thus $\mu_1 \geq 35/8$, whose square is greater than $4(11/4) + 8$.

In the remaining case, the degree-three vertices are pairwise nonadjacent. The sum of a row of A^2 is the sum of the degrees of that vertex's neighbors, which is at most 6. Hence $\lambda_2 \leq \sqrt{6} < 5/2$. If a degree-three vertex outside all triangles has at least two neighbors that are not leaves, extend two edges of its star by one step. Neither step can end at another neighbor of the center, since that would give a triangle. If the two steps end at distinct vertices, we obtain row 5. If they end at the same vertex, we obtain row 6. In both cases $\mu_1 \geq 17/4$, whose square is greater than $4(5/2) + 8$.

Otherwise each degree-three vertex has at least two leaf neighbors or lies in a triangle whose other two vertices have degree two. Delete all leaves and all such pairs of degree-two triangle vertices. This preserves connectivity of the remaining graph because each deleted leaf or pair attaches to only one retained vertex. The remaining graph has maximum degree at most two, and each original degree-three vertex remains and now has degree at most one. Because it contains

at least one such vertex, it is a path or a single vertex. It follows that the original graph has at most two vertices of degree three.

Delete one degree-three vertex from the original graph. Give weight $4/3$ to the sole remaining degree-three vertex if there is one, and weight 1 to every other vertex. The weighted row sums are at most $7/3$, so interlacing gives $\lambda_2(G) \le 7/3$. The four-vertex star at any degree-three vertex of G has an outgoing edge by connectivity and $n \ge 5$. This gives row 7. Thus $\mu_1^2 \ge (25/6)^2 = 625/36 > 4(7/3) + 8$.

Maximum degree at most two. The graph is a path or a cycle. For the two-vertex path, $\mu_1 = 2$ and $\lambda_2 = -1$, giving equality. For an even cycle, $\mu_1 = 4$ and $\lambda_2 \le 2$. For an odd cycle, put $c = \cos(\pi/n)$. The cycle spectra give $\mu_1 = 2 + 2c$ and $\lambda_2 = 2(2c^2 - 1)$, so the difference $\mu_1^2 - 4\lambda_2 - 8$ is $4(1 - c)(3c + 1) \ge 0$.

For a path of $n \ge 3$ vertices, put $a = \pi/n$ and $b = 2\pi/(n + 1)$. The path spectra give $\mu_1 = 2 + 2\cos a$ and $\lambda_2 = 2\cos b$. Since $b \ge 3a/2$ and $a \le \pi/3$, monotonicity of cosine and concavity of sine give

$$\cos a - \cos b \ge 2\sin(5a/4)\sin(a/4) \ge \tfrac{1}{2}\sin^2 a.$$

For the last step, $\sin(5a/4) \ge \sin a$ because $5a/4 \le \pi/2$, and $\sin(a/4) \ge \sin(a)/4$ by concavity. Therefore $\mu_1^2 - 4\lambda_2 - 8 = 8(\cos a - \cos b) - 4\sin^2 a \ge 0$. □

References

[1] T. Cazenave, *Lemmas and Conjectures in Spectral Graph Theory*, arXiv:2611.03879v1, 2 November 2026. Section 7, Conjecture 6(iii).

[2] A. M. Kramer, *A proof of a conjecture of Cazenave on graph eigenvalues*, 8 November 2026. Hexagon 2611.00232v1.