hexagon:2611.01101v1

GAUSSIAN CONTINUED FRACTIONS OF THE NON-REAL ZEROS OF THE GOODSPEED FUNCTION

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Abstract. We investigate Gaussian continued fractions of the phase coordinates associated with the non-real zeros of the Goodspeed function

$$G(z) = \frac{1}{\pi^2}\left(\sin^2(\pi z) + \sin^2\left(\pi\frac{\Gamma(z)+1}{z}\right)\right).$$

An exact logarithmic correction identifies the first Gaussian continued-fraction digit at sufficiently high zeros and yields exponential bounds for the next digit.

For a canonical sequence of non-real zeros accumulating at the origin, we establish four distinct asymptotic laws for the continued-fraction digits. One of these integer-valued digit sequences determines the first two nontrivial coefficients of the underlying holomorphic germ.

We further prove that the subsequent Gaussian digit eventually belongs to an explicitly determined alphabet of 31 Gaussian integers, and calculate the limiting frequency of every state. The frequency law follows from the asymptotic geometry of the zero sequence and equidistribution of a fractional-power phase.

These results provide a continued-fraction and symbolic description of a distinguished non-real zero family, independently of conjectures concerning the global classification of Goodspeed zeros.

1. Introduction

The non-real zeros of an analytic function may encode discrete mathematical structure beyond their locations in the complex plane. In this article, we show that Gaussian continued fractions of phase coordinates associated with the Goodspeed function [7] exhibit explicit asymptotic digit-growth laws and an unexpected finite symbolic structure. For a canonical sequence of non-real zeros accumulating at the origin, one Gaussian partial quotient eventually belongs to an alphabet of exactly 31 Gaussian integers, each occurring with an explicitly determined limiting frequency. Moreover, the asymptotic behaviour of another integer-valued digit sequence determines Euler's constant.

These results connect the local geometry of complex zeros with Gaussian-lattice approximation and equidistribution. They also suggest that continued fractions may provide useful discrete invariants of analytically defined zero sequences.

The function is defined by

$$G(z) = \frac{1}{\pi^2}\left(\sin^2(\pi z) + \sin^2\left(\pi\frac{\Gamma(z)+1}{z}\right)\right). \tag{1}$$

Writing

$$W(z) = \frac{\Gamma(z)+1}{z},$$

Date: 26 November, 2026.

2020 *Mathematics Subject Classification.* Primary 11A55; Secondary 30C15, 11K06, 33B15.

Key words and phrases. Goodspeed function, Gaussian continued fractions, non-real zeros, equidistribution, symbolic digit frequencies, inverse reconstruction.

one observes that, for real x, the two summands in (1) are nonnegative. Consequently, a positive real zero must be a positive integer n for which

$$\frac{(n-1)!+1}{n} \in \mathbb{Z}.$$

By Wilson's theorem [8], these integers are precisely 1 and the primes. Thus the positive real zeros of G are precisely

$$\{1\} \cup \mathbb{P}.$$

The non-real zero set, however, is governed by a different phenomenon. Wherever the quotient is defined, introduce

$$S(z) = \frac{\sin(\pi W(z))}{\sin(\pi z)}.$$

At every non-real zero of G, one has

$$S(z) \in \{i, -i\}.$$

These two sheet equations give rise to distinguished analytic zero branches and associated phase coordinates.

The present article investigates the Gaussian continued fractions of these phase coordinates. For

$$\mu \in \{1, -1, i, -i\},$$

we consider

$$\phi_\mu(z) = z - \mu W(z) - \frac{1}{2}. \tag{2}$$

The coordinates ϕ_μ were introduced in the geometric classification framework of [7]. Here we use them for a different purpose: to investigate the discrete information obtained by successively approximating their values by Gaussian integers.

Unlike ordinary real continued fractions, Gaussian continued fractions involve the two-dimensional lattice $\mathbb{Z}[i]$. At each step, a complex number is replaced by the reciprocal of its displacement from a nearest Gaussian integer. The resulting partial quotients therefore reflect both the size and direction of small complex displacements. Gaussian continued fractions originated with Hurwitz [10]; for a modern treatment of continued-fraction expansions over the Gaussian integers, see Dani and Nogueira [2].

The metrical and symbolic properties of Gaussian continued fractions have been investigated from several perspectives. Hiary and Vandehey [9] studied the invariant measure of the Hurwitz complex continued-fraction map, while Abrams [1] investigated finite partitions associated with complex continued-fraction algorithms. Further developments concerning symbolic dynamics and normality appear in [4].

The setting of the present article is different. Rather than following successive digits of a generic complex number under the continued-fraction transformation, we examine fixed digit positions across a canonically indexed sequence of complex zeros. Their limiting frequencies arise from equidistribution of the indexing parameter and the geometry of a particular circular arc, rather than from the invariant measure of the Hurwitz map.

This suggests a natural question in the present setting: can the Gaussian continued-fraction digits of $\phi_\mu(z)$ reveal quantitative or symbolic structure in the non-real zeros of G?

We give affirmative answers in two distinct geometric regimes.

1.1. **Main results.** Our first results concern zeros in the upper half-plane whose imaginary parts are sufficiently large. For such zeros, the trigonometric sheet equation admits an exact logarithmic normalization.

More precisely, if $G(z) = 0$ and

$$\Im z > \frac{\operatorname{arsinh}(1)}{\pi},$$

then, with

$$\mu = \operatorname{sgn} \Im W(z),$$

there exists an integer k such that

$$\phi_\mu(z) = k + \frac{i}{\pi} \operatorname{Log} \Lambda(e^{2\pi i z}), \tag{3}$$

where

$$\Lambda(q) = \frac{2}{1 - q + \sqrt{1 - 6q + q^2}},$$

with the branches normalized at $q = 0$.

The correction in (3) has modulus less than $1/2$. Consequently, the first Gaussian partial quotient is exactly k. The next digit, denoted by a_1, satisfies

$$|a_1| \sim \frac{\pi}{2} e^{2\pi \Im z} \tag{4}$$

as $\Im z \to \infty$ through the upper non-real Goodspeed zeros.

Thus the first two Gaussian digits record an integer analytic location and an exponentially scaled displacement. These conclusions hold without assigning the zero to a global analytic Type.

Our second investigation concerns a canonical sequence of non-real zeros accumulating at the origin. This sequence is constructed locally from the holomorphic germ

$$H(z) = z^2 W(z) = \Gamma(1 + z) + z$$

and the sheet equation

$$W(z) - c = -i\mathcal{A}(z), \qquad \mathcal{A}(z) = \frac{1}{\pi} \operatorname{arsinh}\big(\sin(\pi z)\big),$$

where c is a sufficiently large positive integer.

The resulting upper zeros z_c^+ satisfy

$$\Re z_c^+ \sim c^{-1/2}, \qquad \Im z_c^+ \sim \frac{1}{2c^2}. \tag{5}$$

In particular, they approach the origin along a quartic cusp.

For each orientation μ, write

$$\phi_\mu(z_c^+) = [a_0^{(\mu)}(c); a_1^{(\mu)}(c), a_2^{(\mu)}(c), \ldots]_{\mathbb{Z}[i]}.$$

We prove that the first two digits eventually stabilize in the form

$$a_0^{(\mu)}(c) = -\mu c, \qquad a_1^{(\mu)}(c) = -2.$$

The third digit, however, exhibits four different asymptotic laws:

$$\begin{aligned} a_2^{(1)}(c) &\sim \frac{-1 + i}{8}\sqrt{c}, \\ a_2^{(-1)}(c) &\sim \frac{-1 - i}{8}\sqrt{c}, \\ a_2^{(-i)}(c) &\sim -\frac{1}{8}\sqrt{c}, \\ a_2^{(i)}(c) &\sim -\frac{3}{4\pi^2} c^{3/2}. \end{aligned} \tag{6}$$

These are asymptotic equivalences in the complex plane.

The exceptional $c^{3/2}$ growth arises from a cubic cancellation in the analytic correction

$$K_i(z) = z - \mathcal{A}(z) = \frac{\pi^2}{3} z^3 + O(z^5).$$

For the other three orientations, the corresponding corrections have simple zeros at the origin. The different orders of vanishing become different growth exponents after the reciprocal operations of the continued-fraction algorithm.

The exceptional orientation also provides an inverse-reconstruction phenomenon. We show that $a_2^{(i)}(c)$ is eventually a real integer and that its asymptotic expansion determines the first two non-constant coefficients of H. In particular, Euler's constant can be recovered from a normalized limit involving this integer-valued digit sequence.

The principal symbolic result concerns the next partial quotient $a_3^{(i)}(c)$. After subtracting the diverging integer digit, the corresponding complete quotient approaches the circular arc

$$g(t) = \frac{1}{t + ib}, \qquad -\frac{1}{2} \le t \le \frac{1}{2}, \qquad b = \frac{9}{8\pi^2}. \tag{7}$$

We prove that the relevant centred real phases are uniformly distributed modulo one. The intersection of the arc (7) with the Gaussian nearest-integer rounding grid then yields an explicit finite symbolic law.

Specifically, the digits $a_3^{(i)}(c)$ eventually belong to an alphabet of exactly 31 Gaussian integers. Every member of this alphabet occurs with a strictly positive limiting frequency, and each frequency is calculated exactly as the length of an explicitly determined parameter interval.

The phenomenon is illustrated in Figure 1. It is important to distinguish this distribution across the canonical indices c from a Gauss–Kuzmin law describing successive digits within one continued fraction.

The results combine the local analytic structure of the Wilson–Gamma function with nearest-lattice approximation and classical equidistribution methods. The high-zero correction has a universal character arising from the trigonometric sheet equation, whereas the canonical tail supplies a specific analytic sequence on which the resulting digit laws can be evaluated.

No conjecture concerning the global classification of Goodspeed zeros is required.

1.2. **Organization of the paper.** Section 2 fixes the Gaussian continued-fraction convention and constructs the canonical non-real zero tail.

Section 3 establishes the exact logarithmic correction and the exponential digit bounds at high zeros.

Section 4 derives the four asymptotic digit laws and the recovery of coefficients of the Wilson–Gamma germ.

Section 5 proves the 31-state theorem, calculates the limiting frequencies, and gives explicit examples.

Section 6 discusses the scope of these results and possible extensions.

2. Preliminaries

We recall the analytic construction underlying the canonical non-real zero sequence and fix the continued-fraction conventions used throughout. No global classification of the zeros of the Goodspeed function is assumed.

2.1. **Phase coordinates and Gaussian continued fractions.** Let

$$W(z) := \frac{\Gamma(z) + 1}{z}, \qquad G(z) := \frac{\sin^2(\pi z) + \sin^2(\pi W(z))}{\pi^2},$$

on their natural holomorphic domain

$$\Omega_G = \mathbb{C} \setminus \{0, -1, -2, \ldots\}.$$

The associated sheet function is

$$S(z) := \frac{\sin(\pi W(z))}{\sin(\pi z)},$$

with removable singularities filled in where appropriate.

At every non-real zero of G, the denominator of S is nonzero and

$$S(z) \in \{i, -i\}.$$

Indeed,

$$\pi^2 G(z) = \big(\sin(\pi z) + i\sin(\pi W(z))\big)\big(\sin(\pi z) - i\sin(\pi W(z))\big).$$

For

$$\mu \in \{1, -1, i, -i\},$$

define the Goodspeed phase coordinate

$$\phi_\mu(z) := z - \mu W(z) - \frac{1}{2}.$$

Our purpose is to study Gaussian continued fractions of the values $\phi_\mu(z)$ at selected non-real zeros of G.

For a real number x, write

$$[x] := \left\lceil x - \frac{1}{2} \right\rceil.$$

This is nearest-integer rounding with half-integer ties directed toward $-\infty$. Define the corresponding nearest-Gaussian-integer map by

$$\mathcal{N}_-(w) := [\Re w] + i[\Im w].$$

Thus

$$w - \mathcal{N}_-(w) \in \left(-\frac{1}{2}, \frac{1}{2}\right] + i\left(-\frac{1}{2}, \frac{1}{2}\right].$$

Definition 2.1 (Gaussian continued fraction)**.** For $\xi \in \mathbb{C}$, put $\xi_0 = \xi$ and define recursively

$$a_j = \mathcal{N}_-(\xi_j), \qquad \xi_{j+1} = \frac{1}{\xi_j - a_j},$$

whenever $\xi_j \notin \mathbb{Z}[i]$. The algorithm terminates when some $\xi_j \in \mathbb{Z}[i]$.

We denote the resulting finite or infinite expansion by

$$\xi = [a_0; a_1, a_2, \ldots]_{\mathbb{Z}[i]}.$$

The numbers $a_j \in \mathbb{Z}[i]$ are its Gaussian partial quotients, and the ξ_j are its complete quotients.

This is a nearest-Gaussian-integer continued-fraction algorithm of Hurwitz type [10, 2], with the tie-breaking convention fixed in Definition 2.1.

The convergents p_j/q_j are defined by

$$p_{-2} = 0, \qquad p_{-1} = 1,$$
$$q_{-2} = 1, \qquad q_{-1} = 0,$$

and

$$p_j = a_j p_{j-1} + p_{j-2}, \qquad q_j = a_j q_{j-1} + q_{j-2}.$$

Our convention is deterministic, including at rounding boundaries. In particular, it need not commute with complex conjugation at those boundaries.

The first digit records the nearest Gaussian lattice point, while subsequent digits measure increasingly fine reciprocal displacements. If

$$\xi = a_0 + r, \qquad r \neq 0,$$

then

$$\xi_1 = \frac{1}{r}.$$

Thus small coordinate remainders may produce large subsequent partial quotients. A relation between coordinate displacement and geometric distance in the z-plane additionally requires control of the derivative of ϕ_μ.

2.2. **The canonical non-real zero tail.** The relevant local analytic structure is most conveniently expressed through

$$H(z) := z^2 W(z) = \Gamma(1+z) + z.$$

Unlike W, the function H is holomorphic at the origin, with

$$H(0) = 1.$$

Write

$$H(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + O(z^4),$$

where the coefficients are real. For the Gamma specialization,

$$h_1 = 1 - \gamma, \qquad h_2 = \frac{\gamma^2 + \zeta(2)}{2},$$

and γ denotes Euler's constant.

Define the odd holomorphic germ

$$\mathcal{A}(z) := \frac{1}{\pi} \operatorname{arsinh}\big(\sin(\pi z)\big),$$

using the branches normalized at the origin. Its expansion is

$$\mathcal{A}(z) = z - \frac{\pi^2}{3} z^3 + \frac{\pi^4}{6} z^5 + O(z^7).$$

For a positive integer c, consider the local branch equation

$$W(z) - c = -i\mathcal{A}(z). \tag{8}$$

Since

$$\sinh(\pi \mathcal{A}(z)) = \sin(\pi z),$$

every nonzero solution of (8) sufficiently near the origin is a zero of G. More precisely,

$$S(z) = (-1)^{c+1} i.$$

The following construction provides the canonical zero sequence needed throughout this article.

Lemma 2.2 (Canonical non-real zero germ)**.** *There exist $\varepsilon > 0$ and a unique holomorphic function Z on $|t| < \varepsilon$ satisfying*

$$Z(0) = 1$$

and

$$Z(t)^2 = H(tZ(t)) + it^2 Z(t)^2 \mathcal{A}(tZ(t)). \tag{9}$$

For every sufficiently large positive integer c, put

$$t_c = c^{-1/2}, \qquad z_c^+ := t_c Z(t_c).$$

Then z_c^+ is a simple non-real zero of G in the upper half-plane, and

$$W(z_c^+) - c = -i\mathcal{A}(z_c^+), \qquad S(z_c^+) = (-1)^{c+1} i.$$

Moreover,

$$\begin{aligned} z_c^+ = t_c + \frac{h_1}{2} t_c^2 + \left(\frac{h_1^2}{8} + \frac{h_2}{2}\right) t_c^3 \\ + \left(\frac{h_1 h_2 + h_3}{2} + \frac{i}{2}\right) t_c^4 + O(t_c^5). \end{aligned} \tag{10}$$

In particular,

$$\Re z_c^+ \sim c^{-1/2}, \qquad \Im z_c^+ \sim \frac{1}{2c^2},$$

and

$$\frac{\Im z_c^+}{(\Re z_c^+)^4} \longrightarrow \frac{1}{2}.$$

The conjugate points $z_c^- := \overline{z_c^+}$ *are simple zeros in the lower half-plane.*

Proof. Put $z = tZ$ and $c = t^{-2}$ in (8). Multiplication by t^2Z^2 gives

$$Z^2 - H(tZ) - it^2Z^2\mathcal{A}(tZ) = 0.$$

The left side is holomorphic near $(t, Z) = (0, 1)$, vanishes there, and has derivative 2 with respect to Z at that point. The holomorphic implicit-function theorem therefore gives a unique germ $Z(t)$ with $Z(0) = 1$.

Write

$$Z(t) = 1 + \alpha t + \beta t^2 + (\delta + i\eta)t^3 + O(t^4),$$

with real coefficients initially to be determined. Substituting the expansions of H and $\mathcal{A}$ into (9) and comparing successive powers of t gives

$$\alpha = \frac{h_1}{2}, \qquad \beta = \frac{h_1^2}{8} + \frac{h_2}{2},$$

and

$$\delta = \frac{h_1h_2 + h_3}{2}, \qquad \eta = \frac{1}{2}.$$

This proves (10).

For $t = t_c > 0$ sufficiently small, the expansion gives

$$\Im(tZ(t)) = \frac{1}{2}t^4 + O(t^5) > 0.$$

Equation (9) is equivalent, for $t \neq 0$, to (8). Consequently

$$\sin(\pi W(z_c^+)) = (-1)^{c+1}i\sin(\pi z_c^+),$$

and hence $G(z_c^+) = 0$.

The zero is simple. Indeed,

$$W'(z) + i\mathcal{A}'(z) = -2z^{-3} + O(z^{-2})$$

near the origin. Thus the selected branch equation has a simple zero at z_c^+ for large c. The complementary factor in the factorization of G is nonzero there, since $\sin(\pi z_c^+) \neq 0$. The simplicity of z_c^+ follows.

Finally, G is real-symmetric, so complex conjugation gives the corresponding simple lower zero. The remaining asymptotic assertions follow directly from (10). □

The construction uses only the holomorphic germ H and the local sheet equation. In particular, it does not require any global classification of Goodspeed zeros.

2.3. **Initial continued-fraction digits.** At the canonical upper zeros, the exact branch identity yields

$$\phi_\mu(z_c^+) = -\mu c - \frac{1}{2} + K_\mu(z_c^+), \qquad K_\mu(z) := z + i\mu\mathcal{A}(z). \tag{11}$$

The four correction germs have the expansions

$$K_\mu(z) = \begin{cases} (1+i)z + O(z^3), & \mu = 1, \\ (1-i)z + O(z^3), & \mu = -1, \\ 2z + O(z^3), & \mu = -i, \\ \dfrac{\pi^2}{3}z^3 + O(z^5), & \mu = i. \end{cases} \tag{12}$$

Equation (11) is the specialization of the original Goodspeed phase coordinate $\phi_\mu = z - \mu W - \frac{1}{2}$ to the canonical upper zero branch $W(z_c^+) = c - i\mathcal{A}(z_c^+)$. The four choices of μ are evaluated at the same zeros and do not represent four different Family assignments.

Lemma 2.3 (Stabilization of the first Gaussian digits)**.** *For every $\mu \in \{1, -1, i, -i\}$ and every sufficiently large integer c, the Gaussian continued fraction of $\phi_\mu(z_c^+)$ begins with*

$$a_0(c) = -\mu c, \qquad a_1(c) = -2.$$

Its third complete quotient is given exactly by

$$\xi_2(c) = \frac{1}{2} - \frac{1}{4K_\mu(z_c^+)}. \tag{13}$$

Proof. By (10), the real part of z_c^+ is positive and tends to zero, while its imaginary part is of smaller order.

Equation (12) therefore shows that

$$\Re K_\mu(z_c^+) > 0, \qquad K_\mu(z_c^+) \longrightarrow 0$$

for each of the four orientations.

It follows from (11) and the prescribed tie convention that the nearest Gaussian integer is

$$a_0(c) = -\mu c.$$

The first remainder is

$$r_c = -\frac{1}{2} + K_\mu(z_c^+),$$

so

$$\xi_1(c) = \frac{1}{r_c} \longrightarrow -2.$$

Since -2 lies in the interior of its Gaussian rounding square, we have $a_1(c) = -2$ eventually.

Finally,

$$\xi_2(c) = \frac{1}{\xi_1(c) + 2} = \frac{r_c}{1 + 2r_c} = \frac{1}{2} - \frac{1}{4K_\mu(z_c^+)},$$

as claimed. □

The identity (13) is the starting point for the four asymptotic digit-growth laws derived in Section 4. The exceptional cancellation

$$K_i(z) = z - \mathcal{A}(z) = \frac{\pi^2}{3}z^3 + O(z^5)$$

will be particularly important: it produces a different growth scale from the other three orientations and ultimately underlies the finite symbolic law of Section 5.

3. The logarithmic phase correction

We now examine the Gaussian continued fractions of the phase coordinates at Goodspeed zeros of sufficiently large imaginary part. Unlike the canonical-tail results of Section 2, the statements here apply to every non-real zero above an explicit height, without assuming any analytic Type classification.

Put

$$y_* := \frac{\operatorname{arsinh}(1)}{\pi}, \qquad r_* := e^{-2\pi y_*} = 3 - 2\sqrt{2}.$$

For $\mu \in \{1, -1\}$, define

$$\mathcal{H}_\mu := \{z \in \mathbb{C} : \Im z > 0,\ \mu \Im W(z) > 0\}.$$

On this domain set

$$D_\mu(z) := \frac{\operatorname{Log}(1 - e^{2\pi i \mu W(z)}) - \operatorname{Log}(1 - e^{2\pi i z})}{i\pi},$$

where Log denotes the principal logarithm, and put

$$B_\mu(z) := \phi_\mu(z) + D_\mu(z).$$

Both logarithms are analytic on $\mathcal{H}_\mu$, since their exponential arguments have modulus less than one.

The exponential representation of the sine function gives the exact identity

$$S(z) = \mu i \exp(i\pi B_\mu(z)). \tag{14}$$

Consequently, at a Goodspeed zero in $\mathcal{H}_\mu$,

$$B_\mu(z) \in \mathbb{Z}.$$

Indeed, $S(z) = \pm i$, so $\exp(i\pi B_\mu(z)) = \pm 1$.

The following result expresses the correction D_μ entirely in terms of z.

Theorem 3.1 (Exact logarithmic correction and first Gaussian digit)**.** *Let $z = x + iy$ be a zero of G with $y > y_*$, and choose*

$$\mu = \operatorname{sgn} \Im W(z).$$

Then $\mu \in \{1, -1\}$, and

$$k := B_\mu(z) \in \mathbb{Z}.$$

Put

$$q := e^{2\pi i z}, \qquad |q| = e^{-2\pi y} < r_*,$$

and define

$$\Lambda(q) := \frac{2}{1 - q + \sqrt{1 - 6q + q^2}}, \tag{15}$$

where the square root is the analytic branch equal to 1 at $q = 0$.

Then

$$D_\mu(z) = \frac{\operatorname{Log} \Lambda(q)}{i\pi}, \qquad \phi_\mu(z) = k + \frac{i}{\pi} \operatorname{Log} \Lambda(q). \tag{16}$$

Moreover,

$$0 < |D_\mu(z)| < y_* < \frac{1}{2}. \tag{17}$$

In particular, with the rounding convention of Definition 2.1,

$$\mathcal{N}_-(\phi_\mu(z)) = k. \tag{18}$$

Thus the first Gaussian partial quotient equals the exact logarithmic location coordinate at every such zero.

Proof. At a zero of G,

$$|\sin(\pi W(z))| = |\sin(\pi z)|.$$

Since

$$|\sin(\pi z)| \geq \sinh(\pi y) > 1$$

for $y > y_*$, the value $W(z)$ cannot be real. Hence the stated choice of μ is well defined and $z \in \mathcal{H}_\mu$.

Write

$$p = e^{2\pi i \mu W(z)}, \qquad \lambda = e^{i\pi D_\mu(z)}.$$

Then $|p| < 1$, and the definition of D_μ gives

$$\lambda = \frac{1-p}{1-q}.$$

Since $B_\mu(z) = k$, we also have

$$\mu W(z) = z - \frac{1}{2} + D_\mu(z) - k,$$

and therefore

$$p = -q\lambda^2.$$

Combining these identities yields

$$q\lambda^2 - (1-q)\lambda + 1 = 0. \tag{19}$$

Put $r = |q| < r_*$. Since

$$|p| = r|\lambda|^2 < 1,$$

we have $|\lambda| < r^{-1/2}$.

On the circle $|\lambda| = r^{-1/2}$,

$$|q\lambda^2 + 1| \leq 2,$$

whereas

$$|(1-q)\lambda| \geq \frac{1-r}{\sqrt{r}} > 2,$$

the last inequality following from $r < 3 - 2\sqrt{2}$. Rouché's theorem shows that (19) has exactly one root inside this circle.

That root is the analytic solution tending to 1 as $q \to 0$, namely $\Lambda(q)$ in (15). Thus $\lambda = \Lambda(q)$.

The discriminant $1 - 6q + q^2$ has positive real part throughout $|q| < r_*$, and consequently $\Lambda(q)$ has positive real part. Furthermore, both $1-p$ and $1-q$ lie in the open right half-plane. Their principal logarithms have arguments in $(-\pi/2, \pi/2)$. It follows that

$$\operatorname{Log}(1-p) - \operatorname{Log}(1-q) = \operatorname{Log}\Lambda(q),$$

establishing (16).

It remains to prove the uniform bound. Write

$$u(q) := \Lambda(q) - 1.$$

Equation (19) is equivalent to

$$u = q(1+u)(2+u).$$

Applying the Lagrange inversion formula [3] gives, for $n \geq 1$,

$$[q^n] \operatorname{Log}\Lambda(q) = \frac{1}{n}[t^{n-1}](1+t)^{n-1}(2+t)^n > 0.$$

Thus the Taylor series of $\operatorname{Log}\Lambda$ has strictly positive coefficients. For $|q| = r < r_*$, it follows that

$$|\operatorname{Log}\Lambda(q)| \leq \log\Lambda(r).$$

Since $\Lambda(r)$ increases on $(0, r_*)$ and

$$\Lambda(r_*) = 1 + \sqrt{2},$$

we obtain

$$|\operatorname{Log}\Lambda(q)| < \log(1+\sqrt{2}) = \operatorname{arsinh}(1).$$

Dividing by π proves the upper bound in (17).

The correction cannot vanish. Indeed, $D_\mu(z) = 0$ would give $\phi_\mu(z) = k$, hence

$$\mu W(z) = z - k - \frac{1}{2}.$$

For $\mu = \pm 1$, this implies

$$\sin^2(\pi W(z)) = \cos^2(\pi z)$$

and therefore

$$\pi^2 G(z) = 1,$$

contrary to $G(z) = 0$.

Finally,

$$\phi_\mu(z) = k - D_\mu(z),$$

and $|D_\mu(z)| < 1/2$ places $\phi_\mu(z)$ strictly inside the Gaussian rounding square centred at k. Hence $\mathcal{N}_-(\phi_\mu(z)) = k$. □

The correction in Theorem 3.1 is universal in the following sense: once the sheet equation $S(z) = \pm i$ and the chosen orientation are fixed, it depends on the zero only through $q = e^{2\pi i z}$. The special Gamma-theoretic structure of W determines which values of z are actually Goodspeed zeros.

A combinatorial observation. The algebraic function appearing in the logarithmic correction has the alternative representation

$$\Lambda(q) = \frac{1 - q - \sqrt{1 - 6q + q^2}}{2q}.$$

Consequently,

$$\Lambda(q) = 1 + 2q + 6q^2 + 22q^3 + 90q^4 + \cdots$$

is the ordinary generating function of the large Schröder numbers [3]. Thus the universal logarithmic correction arising from the Goodspeed sheet equation is the logarithm of a classical combinatorial generating function. The present use of this function concerns its analytic role in the zero equation rather than its enumerative interpretation.

We next quantify the resulting continued-fraction digit.

Theorem 3.2 (Exponential growth of the next Gaussian digit)**.** *Under the hypotheses of Theorem 3.1, suppose additionally that*

$$r := e^{-2\pi y} \le \frac{1}{2000}, \qquad \delta_r := 2r + 9r^2.$$

Then

$$\frac{2r}{\pi}(1 - \delta_r) \le |D_\mu(z)| \le \frac{2r}{\pi}(1 + \delta_r). \tag{20}$$

The first complete Gaussian quotient after the integer digit k is

$$\xi_1 = -\frac{1}{D_\mu(z)}.$$

It satisfies

$$\left|\xi_1 + \frac{i\pi}{2q} - i\pi\right| < 21r. \tag{21}$$

Consequently, the next Gaussian partial quotient $a_1 = \mathcal{N}_-(\xi_1)$ satisfies

$$\left| a_1 - \left(-\frac{i\pi}{2q} + i\pi \right) \right| \leq \frac{1}{\sqrt{2}} + 21r, \tag{22}$$

and

$$\left| |a_1| - \frac{\pi}{2} e^{2\pi y} \right| < 4. \tag{23}$$

In particular, uniformly along any sequence of upper Goodspeed zeros whose imaginary parts tend to infinity,

$$|a_1| \sim \frac{\pi}{2} e^{2\pi y}.$$

Proof. Put

$$\ell(q) := \operatorname{Log} \Lambda(q).$$

Its Taylor expansion, obtained from the Lagrange-inversion formula in the preceding proof, begins

$$\ell(q) = 2q + 4q^2 + \frac{38}{3} q^3 + 48q^4 + O(q^5). \tag{24}$$

All its coefficients are positive.

We first establish the uniform remainder bound

$$|\ell(q) - 2q - 4q^2| \leq 18|q|^3, \qquad |q| \leq \frac{1}{16}. \tag{25}$$

Indeed, direct substitution into (19) and an elementary logarithmic estimate give

$$\Lambda(1/16) < \frac{289}{250}, \qquad \log(289/250) < \frac{29}{200}.$$

Using positivity of the Taylor coefficients, with $R = 1/16$, gives

$$\begin{aligned} |\ell(q) - 2q - 4q^2| &\leq \left(\frac{|q|}{R} \right)^3 \left(\ell(R) - 2R - 4R^2 \right) \\ &\leq 18|q|^3. \end{aligned}$$

For $r \leq 1/2000$, write

$$\ell(q) = 2q(1 + 2q + e(q)), \qquad |e(q)| \leq 9r^2.$$

The triangle inequality then gives

$$2r(1 - \delta_r) \leq |\ell(q)| \leq 2r(1 + \delta_r).$$

Since

$$|D_\mu(z)| = \frac{|\ell(q)|}{\pi},$$

this proves (20).

Next,

$$\xi_1 = -\frac{1}{D_\mu(z)} = -\frac{i\pi}{\ell(q)} = -\frac{i\pi}{2q} \frac{1}{1 + 2q + e(q)}.$$

Elementary algebra gives

$$\left| \frac{1}{1 + 2q + e(q)} - 1 + 2q \right| \leq \frac{13r^2 + 18r^3}{1 - 2r - 9r^2}.$$

Consequently,

$$\left| \xi_1 + \frac{i\pi}{2q} - i\pi \right| \leq \frac{\pi}{2} \frac{13 + 18r}{1 - 2r - 9r^2}\, r < 21r,$$

where the final inequality holds for $r \leq 1/2000$. This proves (21).

Nearest-Gaussian rounding introduces an error of modulus at most $1/\sqrt{2}$, so (22) follows.

Finally, since $|q| = r$,

$$\left|-\frac{i\pi}{2q}\right| = \frac{\pi}{2r} = \frac{\pi}{2}e^{2\pi y}.$$

Thus

$$\left||a_1| - \frac{\pi}{2r}\right| \le \frac{1}{\sqrt{2}} + 21r + \pi < 4.$$

This establishes (23) and its asymptotic consequence. □

Remark 3.3 (Geometric meaning and limitations). The magnitude of the digit a_1 records the height of a sufficiently high Goodspeed zero, while its leading complex direction depends on

$$q = e^{2\pi ix}e^{-2\pi y}$$

and hence on the real part modulo one. In this sense the first two Gaussian digits encode an integer location and a finer exponential-scale correction.

These statements do not identify the analytic Type of the zero. Nor does a small phase-coordinate correction, by itself, provide a corresponding bound on geometric displacement in the z-plane; such a conclusion requires independent derivative or inverse-function estimates.

The integer $k = B_\mu(z)$ is the exact analytic location coordinate. Its identification with an originally assigned Type 2 continuation index, where applicable, remains a separate classification question.

4. Canonical-tail asymptotics

We now return to the canonical upper zero tail $\{z_c^+\}$ constructed in Lemma 2.2. The imaginary parts of these zeros tend to zero, so the high-altitude hypotheses of Section 3 do not apply. Instead, the exact identity (13) yields four different asymptotic regimes.

For each $\mu \in \{1, -1, i, -i\}$, let

$$\phi_\mu(z_c^+) = [a_0^{(\mu)}(c); a_1^{(\mu)}(c), a_2^{(\mu)}(c), \ldots]_{\mathbb{Z}[i]},$$

and denote the corresponding complete quotients by $\xi_j^{(\mu)}(c)$.

By Lemma 2.3, for all sufficiently large c,

$$a_0^{(\mu)}(c) = -\mu c, \qquad a_1^{(\mu)}(c) = -2,$$

and

$$\xi_2^{(\mu)}(c) = \frac{1}{2} - \frac{1}{4K_\mu(z_c^+)}. \tag{26}$$

Since Gaussian rounding changes a complex number by at most $1/\sqrt{2}$, the growth of $a_2^{(\mu)}(c)$ is governed by the corresponding complete quotient.

Theorem 4.1 (Four asymptotic digit laws)**.** *As $c \to \infty$ through the positive integers, the Gaussian partial quotients $a_2^{(\mu)}(c)$ satisfy*

$$\lim_{c\to\infty} \frac{a_2^{(1)}(c)}{\sqrt{c}} = \frac{-1+i}{8}, \tag{27}$$

$$\lim_{c\to\infty} \frac{a_2^{(-1)}(c)}{\sqrt{c}} = \frac{-1-i}{8}, \tag{28}$$

$$\lim_{c\to\infty} \frac{a_2^{(-i)}(c)}{\sqrt{c}} = -\frac{1}{8}, \tag{29}$$

$$\lim_{c\to\infty} \frac{a_2^{(i)}(c)}{c^{3/2}} = -\frac{3}{4\pi^2}. \tag{30}$$

For $\mu = \pm i$, the digits $a_2^{(\mu)}(c)$ are real integers for all sufficiently large c. More precisely,

$$\begin{aligned} \Im\xi_2^{(-i)}(c) &= \frac{1}{16c} + O(c^{-3/2}), \\ \Im\xi_2^{(i)}(c) &= \frac{9}{8\pi^2} + O(c^{-1/2}). \end{aligned} \tag{31}$$

In the exceptional orientation $\mu = i$, there is a refined expansion

$$\begin{aligned} \xi_2^{(i)}(c) = &-\frac{3}{4\pi^2}c^{3/2} + \frac{9h_1}{8\pi^2}c \\ &+ \left(-\frac{27h_1^2}{32\pi^2} + \frac{9h_2}{8\pi^2} - \frac{3}{8}\right)\sqrt{c} \\ &+ D_0 + \frac{9i}{8\pi^2} + O(c^{-1/2}), \end{aligned} \tag{32}$$

where

$$D_0 = \frac{1}{2} + \frac{3h_1}{16} + \frac{3}{8\pi^2}\big(h_1^3 - 3h_1h_2 + 3h_3\big)$$

is real.

Consequently,

$$\begin{aligned} a_2^{(i)}(c) = &-\frac{3}{4\pi^2}c^{3/2} + \frac{9h_1}{8\pi^2}c \\ &+ \left(-\frac{27h_1^2}{32\pi^2} + \frac{9h_2}{8\pi^2} - \frac{3}{8}\right)\sqrt{c} + O(1). \end{aligned} \tag{33}$$

Proof. Put $t = c^{-1/2}$ and write $z_c = z_c^+$ for brevity. By (10),

$$z_c = t + \alpha t^2 + \beta t^3 + (\delta + i/2)t^4 + O(t^5), \tag{34}$$

where

$$\alpha = \frac{h_1}{2}, \qquad \beta = \frac{h_1^2}{8} + \frac{h_2}{2}, \qquad \delta = \frac{h_1h_2 + h_3}{2}.$$

In particular,

$$z_c = t(1 + O(t)), \qquad \Im z_c = \frac{1}{2}t^4 + O(t^5).$$

For $\mu = 1$, equation (12) gives

$$K_1(z) = (1 + i)z + O(z^3).$$

Hence (26) implies

$$\xi_2^{(1)}(c) = -\frac{1}{4(1 + i)z_c} + O(1) = \frac{-1 + i}{8}t^{-1} + O(1).$$

Similarly,

$$\xi_2^{(-1)}(c) = \frac{-1 - i}{8}t^{-1} + O(1).$$

Since nearest-Gaussian rounding introduces a bounded error, these identities prove (27) and (28).

For $\mu = -i$, we have

$$K_{-i}(z) = z + \mathcal{A}(z) = 2z - \frac{\pi^2}{3}z^3 + O(z^5),$$

and therefore

$$\xi_2^{(-i)}(c) = \frac{1}{2} - \frac{1}{8z_c} + O(z_c). \tag{35}$$

It follows that

$$\xi_2^{(-i)}(c) = -\frac{1}{8}t^{-1} + O(1).$$

Moreover,

$$\Im\left(-\frac{1}{8z_c}\right) = \frac{\Im z_c}{8|z_c|^2} = \frac{t^2}{16} + O(t^3).$$

The remaining analytic terms in (35) have real Taylor coefficients and contribute $O(t^4)$ to the imaginary part. Thus

$$\Im\xi_2^{(-i)}(c) = \frac{1}{16c} + O(c^{-3/2}).$$

This proves the third limit and the first identity in (31).

The exceptional orientation $\mu = i$ requires a more detailed expansion. Using the series for $\mathcal{A}$,

$$K_i(z) = z - \mathcal{A}(z) = \frac{\pi^2}{3}z^3 - \frac{\pi^4}{6}z^5 + O(z^7).$$

Consequently,

$$\begin{aligned}\xi_2^{(i)}(c) &= \frac{1}{2} - \frac{1}{4K_i(z_c)} \\ &= \frac{1}{2} - \frac{3}{4\pi^2}z_c^{-3} - \frac{3}{8}z_c^{-1} + O(z_c).\end{aligned} \tag{36}$$

From (34),

$$z_c^{-1} = t^{-1} - \alpha + O(t)$$

and

$$\begin{aligned}z_c^{-3} = t^{-3} - 3\alpha t^{-2} + (6\alpha^2 - 3\beta)t^{-1} \\ - 10\alpha^3 + 12\alpha\beta - 3\delta - \frac{3i}{2} + O(t).\end{aligned} \tag{37}$$

Substituting these expressions into (36), and then using the stated values of α, β, δ, gives (32).

In particular,

$$\Im\xi_2^{(i)}(c) = \frac{9}{8\pi^2} + O(c^{-1/2}).$$

The limiting imaginary parts for $\mu = i$ and $\mu = -i$ both lie strictly between $-1/2$ and $1/2$. Therefore their nearest Gaussian integers have zero imaginary part for all sufficiently large c. This establishes the eventual reality assertion.

Finally, rounding the real part of (32) changes it by at most $1/2$. This proves (33), and its leading term proves (30). □

The different growth scales arise from the orders of vanishing of the correction germs K_μ at the origin. For $\mu = 1, -1, -i$, these germs have simple zeros, whereas K_i has a zero of order three. The reciprocal operation in (26) magnifies this difference into the respective $c^{1/2}$ and $c^{3/2}$ laws.

4.1. **Recovery of the Wilson–Gamma germ.** The exceptional orientation $\mu = i$ has a further consequence: the asymptotic behaviour of a single integer-valued digit sequence retains analytic information about H.

Corollary 4.2 (Recovery of two germ coefficients from Gaussian digits)**.** *Let*

$$A_0 := \frac{3}{4\pi^2}, \qquad b_c := a_2^{(i)}(c).$$

The following limits exist:

$$L_1 := \lim_{c\to\infty} \frac{b_c + A_0 c^{3/2}}{c}, \tag{38}$$

and

$$L_2 := \lim_{c\to\infty} \frac{b_c + A_0 c^{3/2} - L_1 c}{\sqrt{c}}. \tag{39}$$

They determine the first two nonconstant coefficients of $H(z) = 1 + h_1 z + h_2 z^2 + \cdots$ *by*

$$h_1 = \frac{8\pi^2}{9} L_1 \tag{40}$$

and

$$h_2 = \frac{8\pi^2}{9}\left(L_2 + \frac{3}{8}\right) + \frac{3}{4} h_1^2. \tag{41}$$

In the Gamma specialization, this recovers Euler's constant through

$$\gamma = 1 - \frac{8\pi^2}{9} L_1.$$

Proof. Equation (33) gives

$$b_c + A_0 c^{3/2} = \frac{9h_1}{8\pi^2} c + O(\sqrt{c}).$$

Dividing by c and passing to the limit yields

$$L_1 = \frac{9h_1}{8\pi^2},$$

which proves (40).

Subtracting $L_1 c$ from the refined expansion and dividing by $\sqrt{c}$ gives

$$L_2 = -\frac{27h_1^2}{32\pi^2} + \frac{9h_2}{8\pi^2} - \frac{3}{8}.$$

Rearranging proves (41). Finally, $h_1 = 1 - \gamma$ by the definition of H. □

Remark 4.3 (Information retained under integer quantization). The sequence $\{b_c\}$ consists of ordinary integers for all sufficiently large c. Nevertheless, its asymptotic growth determines h_1 and h_2 exactly.

The general problem of recovering real parameters from integer-part sequences has precedents in the theory of Beatty sequences; see, for example, Graham and O'Bryant [6]. The distinctive feature here is that the rounded data arise as Gaussian partial quotients of analytically defined phase coordinates, and that their asymptotic coefficients determine the local Wilson–Gamma germ.

This is a coarse reconstruction from the canonically indexed digit sequence, not a new determination of the full Wilson–Gamma germ. In particular, the bounded rounding error in (33) prevents the same direct limiting argument from isolating the constant term D_0.

For the Gamma specialization, $h_2 = (\gamma^2 + \zeta(2))/2$, so the recovery is consistent with the classical identity $\zeta(2) = \pi^2/6$. No new independent evaluation of $\zeta(2)$ is claimed.

The next section uses the finer consequence of (32):

$$\Re \xi_2^{(i)}(c) = -A_0 c^{3/2} + \frac{9h_1}{8\pi^2} c + O(\sqrt{c}),$$

while

$$\Im \xi_2^{(i)}(c) \longrightarrow \frac{9}{8\pi^2}.$$

The fractional parts of the real complete quotients will be shown to be equidistributed modulo one. Combined with Gaussian rounding, this produces the finite symbolic law of Section 5.

5. A finite symbolic law

The exceptional orientation $\mu = i$ exhibits a further phenomenon. Although the third Gaussian partial quotient $a_2^{(i)}(c)$ diverges at order $c^{3/2}$, the subsequent digit $a_3^{(i)}(c)$ eventually belongs to a finite set of Gaussian integers. Moreover, every member of this set occurs with an explicit limiting frequency.

Throughout this section, abbreviate

$$\xi_j(c) := \xi_j^{(i)}(c), \qquad a_j(c) := a_j^{(i)}(c),$$

and put

$$A_0 := \frac{3}{4\pi^2}, \qquad b := \frac{9}{8\pi^2}. \tag{42}$$

In particular,

$$0 < b < \frac{1}{2}.$$

5.1. **Equidistribution of the canonical phase.** By Theorem 4.1, the imaginary part of $\xi_2(c)$ eventually lies strictly between $-1/2$ and $1/2$. Hence $a_2(c)$ is a real integer for all sufficiently large c.

Define the centred fractional part

$$t_c := \Re\xi_2(c) - a_2(c) \in \left(-\frac{1}{2}, \frac{1}{2}\right].$$

Lemma 5.1 (Equidistribution of the centred phase)**.** *The sequence $\{t_c\}$ is uniformly distributed in $(-1/2, 1/2]$.*

Equivalently, for every interval $I \subseteq (-1/2, 1/2]$,

$$\lim_{C\to\infty} \frac{1}{C} \#\{c \le C : t_c \in I\} = |I|,$$

where the finitely many indices for which t_c has not yet been defined are omitted.

Proof. Equation (32) gives

$$\Re\xi_2(c) = -A_0 c^{3/2} + Bc + D\sqrt{c} + E + O(c^{-1/2}), \tag{43}$$

where B, D, E are real constants. Specifically,

$$B = \frac{9h_1}{8\pi^2},$$

$$D = -\frac{27h_1^2}{32\pi^2} + \frac{9h_2}{8\pi^2} - \frac{3}{8}, \qquad E = D_0.$$

Fix a nonzero integer h and define

$$f_h(x) := h\big(-A_0 x^{3/2} + Bx + D\sqrt{x} + E\big).$$

Then

$$f_h''(x) = -\frac{3hA_0}{4\sqrt{x}} - \frac{hD}{4x^{3/2}}.$$

On every sufficiently large dyadic interval $N < x \le 2N$, this derivative has constant sign and satisfies

$$|f_h''(x)| \asymp_h N^{-1/2}.$$

The classical van der Corput second-derivative estimate [5] therefore gives

$$\sum_{N<c\le 2N} e^{2\pi i f_h(c)} = O_h(N^{3/4} + N^{1/4}).$$

Replacing $f_h(c)$ by $h\Re\xi_2(c)$ changes each exponential by $O_h(c^{-1/2})$, according to (43). The total additional error on the dyadic interval is $O_h(\sqrt{N})$. Consequently,

$$\sum_{N<c\le 2N} e^{2\pi i h\Re\xi_2(c)} = O_h(N^{3/4}).$$

Summing over dyadic intervals gives

$$\sum_{c\le C} e^{2\pi i h\Re\xi_2(c)} = o(C).$$

By Weyl's criterion [11], the sequence $\Re\xi_2(c)$ is uniformly distributed modulo one.

Since $a_2(c)$ is the nearest real integer to $\Re\xi_2(c)$ eventually, the centred fractional parts t_c are uniformly distributed on $(-1/2, 1/2]$. □

5.2. **The limiting circular arc.** The next complete quotient is

$$\xi_3(c) = \frac{1}{\xi_2(c) - a_2(c)}.$$

By (31),

$$\xi_2(c) - a_2(c) = t_c + ib + O(c^{-1/2}).$$

Since $b > 0$, the denominators are bounded away from zero for all sufficiently large c. It follows uniformly that

$$\xi_3(c) = g(t_c) + O(c^{-1/2}), \qquad g(t) := \frac{1}{t+ib}. \tag{44}$$

The curve

$$\mathcal{C}_b := \left\{ \frac{1}{t+ib} : -\frac{1}{2} \le t \le \frac{1}{2} \right\}$$

is a circular arc in the lower half-plane. Indeed, writing $g(t) = u(t) + iv(t)$ gives

$$u(t) = \frac{t}{t^2+b^2}, \qquad v(t) = -\frac{b}{t^2+b^2},$$

and hence

$$u^2 + \left(v + \frac{1}{2b}\right)^2 = \frac{1}{4b^2}. \tag{45}$$

The arc is symmetric about the imaginary axis:

$$g(-t) = -\overline{g(t)}.$$

Its Gaussian nearest-integer digits are determined by the rounding cells intersected by $\mathcal{C}_b$.

5.3. **The 31-state theorem.** For $d > 0$ satisfying $2db < 1$, define

$$I_d := \frac{1 - \sqrt{1-4d^2b^2}}{2d}, \qquad O_d := \frac{1 + \sqrt{1-4d^2b^2}}{2d}. \tag{46}$$

These are the positive solutions of

$$\Re g(t) = d.$$

For $j = 0, 1, \dots, 8$, put

$$V_j := \sqrt{\frac{b}{j+1/2} - b^2}. \tag{47}$$

These are the positive solutions of

$$\Im g(t) = -j - \frac{1}{2}.$$

For a Gaussian integer a, define its limiting frequency, when it exists, by

$$\nu(a) := \lim_{C\to\infty} \frac{1}{C}\#\{c_0 \le c \le C : a_3(c) = a\},$$

where c_0 is any sufficiently large fixed starting index.

Theorem 5.2 (The 31-state symbolic law)**.** *For the canonical upper Goodspeed zeros z_c^+ and the orientation $\mu = i$, the Gaussian digit $a_3(c)$ eventually belongs to the 31-element alphabet*

$$\begin{aligned}\mathscr{A}_3 = \{&\pm 4 - ji : 2 \le j \le 7\}\\ &\cup \{\pm 3 - ji : j \in \{1, 2, 7, 8\}\}\\ &\cup \{\pm 2 - ji : j \in \{0, 1, 8, 9\}\}\\ &\cup \{\pm 1 - 9i, -9i\}.\end{aligned} \tag{48}$$

For every $a \in \mathbb{Z}[i]$, the limiting frequency exists and is

$$\nu(a) = \left|\{t \in [-1/2, 1/2] : \mathcal{N}_-(g(t)) = a\}\right|, \tag{49}$$

where the right-hand side denotes Lebesgue length.

Every $a \in \mathscr{A}_3$ has strictly positive frequency, and

$$\nu(a) = 0 \qquad (a \notin \mathscr{A}_3).$$

The exact frequencies are given in Table 1. In particular,

$$\sum_{a\in\mathscr{A}_3} \nu(a) = 1.$$

Proof. We first determine the Gaussian rounding cells intersected by $\mathcal{C}_b$.

On the positive half of the arc, $0 \le t \le 1/2$, the real part $u(t)$ is nonnegative. Its maximum is $1/(2b) < 9/2$. The imaginary part increases from $-1/b$ to $-b/(1/4 + b^2)$, crossing exactly the horizontal half-integer lines indexed by $j = 0, \dots, 8$.

The vertical rounding boundaries that meet this half-arc are

$$u = \frac{1}{2}, \quad u = \frac{3}{2}, \quad u = \frac{5}{2}, \quad u = \frac{7}{2}.$$

Their intersections are described by (46). The horizontal intersections are the points in (47).

A direct comparison gives the strict ordering

$$\begin{aligned}0 < I_{1/2} < I_{3/2} < V_8 < I_{5/2} < V_7 < I_{7/2}\\ < V_6 < V_5 < V_4 < V_3 < V_2 < O_{7/2}\\ < V_1 < O_{5/2} < V_0 < \frac{1}{2}.\end{aligned} \tag{50}$$

For completeness, this ordering can be verified by finite rational arithmetic. The classical bounds

$$3.1415926 < \pi < 3.1415927$$

imply

$$\frac{1139863}{10^7} < b < \frac{1139864}{10^7}.$$

Substitution into (46) and (47), followed by squaring positive quantities, gives the following rational enclosures:

Endpoint	Strict enclosure
$I_{1/2}$	$(0.00651, 0.00652)$
$I_{3/2}$	$(0.02009, 0.02010)$
V_8	$(0.02042, 0.02043)$
$I_{5/2}$	$(0.03566, 0.03567)$
V_7	$(0.04696, 0.04697)$
$I_{7/2}$	$(0.05674, 0.05675)$
V_6	$(0.06740, 0.06741)$
V_5	$(0.08793, 0.08794)$
V_4	$(0.11107, 0.11108)$
V_3	$(0.13990, 0.13991)$
V_2	$(0.18055, 0.18056)$
$O_{7/2}$	$(0.22896, 0.22897)$
V_1	$(0.25099, 0.25100)$
$O_{5/2}$	$(0.36433, 0.36434)$
V_0	$(0.46365, 0.46366)$

All the intervals are disjoint and appear in the order asserted in (50). The comparisons are exact rational inequalities; the terminating decimals merely abbreviate rationals with denominator 10^5.

The strict ordering also shows that the arc crosses rounding boundaries without passing through a Gaussian rounding-cell corner. Each crossing is transverse.

Traversing the positive half-arc from $t = 0$ to $t = 1/2$, the successive Gaussian digits are

$$\begin{gathered} -9i, \quad 1-9i, \quad 2-9i, \quad 2-8i, \quad 3-8i, \quad 3-7i, \\ 4-7i, \quad 4-6i, \quad 4-5i, \quad 4-4i, \quad 4-3i, \quad 4-2i, \\ 3-2i, \quad 3-i, \quad 2-i, \quad 2. \end{gathered}$$

These are the 16 positive-half states, including the central state $-9i$.

By the symmetry $g(-t) = -\overline{g(t)}$, the negative half contributes the 15 corresponding states with negative real parts. Thus the full arc meets precisely 31 Gaussian rounding cells.

No other cell has the arc in its closure: the arc is compact, the crossings are transverse, and no crossing occurs at a cell corner. Consequently, the uniform approximation (44) shows that

$$a_3(c) \in \mathscr{A}_3$$

for every sufficiently large c. This establishes eventual alphabet membership, not merely a statement of density one.

It remains to establish the frequencies. The finitely many boundary parameters in (50), together with their negatives, partition $[-1/2, 1/2]$ into intervals on which $\mathcal{N}_-(g(t))$ is constant.

Fix one such state a. Away from arbitrarily small neighbourhoods of the partition endpoints, the arc has positive distance from the Gaussian rounding boundaries. The error $O(c^{-1/2})$ in (44) therefore cannot change the rounded digit for sufficiently large c.

By Lemma 5.1, the proportion of indices for which t_c lies in any fixed interval converges to its length. The proportion lying in the excluded boundary neighbourhoods is bounded by their total length in the limit. Letting those neighbourhoods shrink to zero proves (49).

The strict inequalities in (50) show that every listed interval has positive length. The exact expressions in Table 1 follow by subtracting consecutive endpoints. Their total is one because the intervals partition $[-1/2, 1/2]$ up to finitely many boundary points. □

TABLE 1. The 31 Gaussian states and their exact limiting frequencies. In each row containing $\pm$, the stated frequency belongs to each of the two Gaussian integers separately, rather than to their combined occurrence. The single central state $-9i$ has frequency $2I_{1/2}$.

State(s) a	Exact frequency $\nu(a)$	Approx.
$-9i$	$2I_{1/2}$	0.013035
$\pm 1 - 9i$	$I_{3/2} - I_{1/2}$	0.013577
$\pm 2 - 9i$	$V_8 - I_{3/2}$	0.000332
$\pm 2 - 8i$	$I_{5/2} - V_8$	0.015234
$\pm 3 - 8i$	$V_7 - I_{5/2}$	0.011299
$\pm 3 - 7i$	$I_{7/2} - V_7$	0.009785
$\pm 4 - 7i$	$V_6 - I_{7/2}$	0.010660
$\pm 4 - 6i$	$V_5 - V_6$	0.020526
$\pm 4 - 5i$	$V_4 - V_5$	0.023143
$\pm 4 - 4i$	$V_3 - V_4$	0.028835
$\pm 4 - 3i$	$V_2 - V_3$	0.040650
$\pm 4 - 2i$	$O_{7/2} - V_2$	0.048410
$\pm 3 - 2i$	$V_1 - O_{7/2}$	0.022025
$\pm 3 - i$	$O_{5/2} - V_1$	0.113344
$\pm 2 - i$	$V_0 - O_{5/2}$	0.099321
± 2	$\frac{1}{2} - V_0$	0.036341

Example 5.3 (The central state $-9i$)**.** As a simple illustration of Theorem 5.2, consider the central point $t = 0$ of the limiting circular arc. Since

$$g(0) = -\frac{i}{b} = -\frac{8\pi^2}{9}i \approx -8.772982i,$$

its nearest Gaussian integer is $-9i$.

The condition

$$\mathcal{N}_-(g(t)) = -9i$$

holds precisely when

$$-I_{1/2} < t < I_{1/2},$$

apart from the irrelevant choice of rounding convention at the endpoints, where

$$I_{1/2} = 1 - \sqrt{1 - b^2}, \qquad b = \frac{9}{8\pi^2}.$$

By the equidistribution established in Lemma 5.1, the limiting frequency of this state is therefore

$$\nu(-9i) = 2I_{1/2} = 2\left(1 - \sqrt{1 - \frac{81}{64\pi^4}}\right) \approx 0.013035364.$$

Thus approximately 1.3035% of the canonical indices eventually produce the Gaussian digit $a_3^{(i)}(c) = -9i$, in the sense of limiting frequency.

Example 5.4. Consider the upper canonical branch equation

$$W(z) - 1000 = -i\mathcal{A}(z).$$

Numerical solution gives

$$z_{1000}^{+} \approx 0.03185028725517443 + 5.09399654074228 \cdot 10^{-7}i.$$

For $\mu = i$, the corresponding phase coordinate is

$$\phi_i(z_{1000}^{+}) \approx -0.499894231753922 - 999.999999994942i.$$

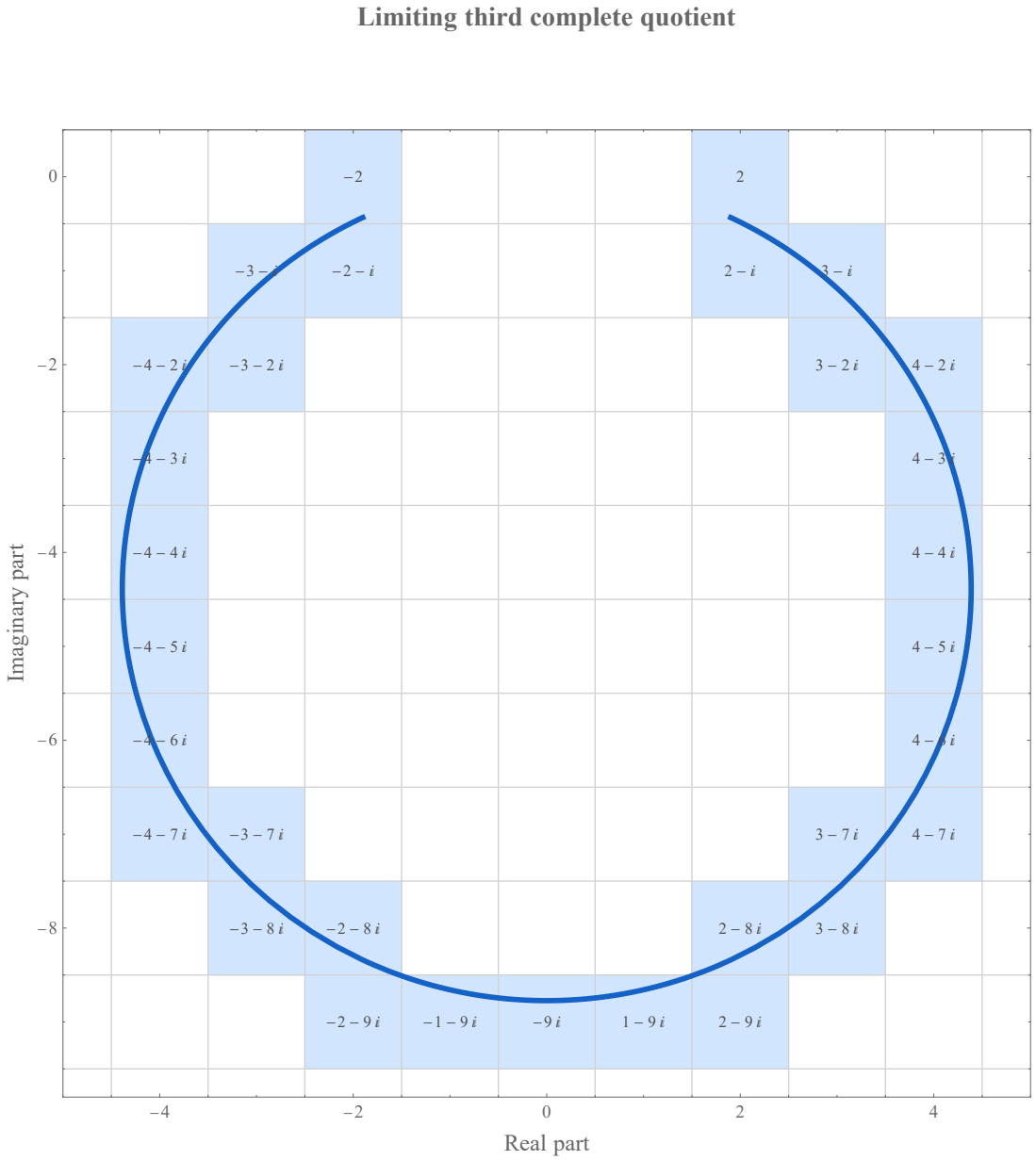

FIGURE 1. The limiting third complete quotient $g(t) = 1/(t + ib)$, where $b = 9/(8\pi^2)$ and $-1/2 \leq t \leq 1/2$, superimposed on the Gaussian nearest-integer rounding grid. The shaded cells are the 31 Gaussian rounding cells intersected by the arc. Each cell corresponds to a possible eventual value of the partial quotient $a_3^{(i)}(c)$. The limiting frequency of a digit is determined by the length of the corresponding parameter interval, rather than by arclength.

Its Gaussian continued fraction begins

$$\phi_i(z_{1000}^+) = [-1000i; -2, -2363, -4 - 3i, \ldots]_{\mathbb{Z}[i]}.$$

Indeed, the third complete quotient is

$$\xi_2^{(i)}(1000) \approx -2363.158363433 + 0.113035250i,$$

so that $a_2^{(i)}(1000) = -2363$ and

$$t_{1000} \approx -0.158363433.$$

The next complete quotient is

$$\xi_3^{(i)}(1000) \approx -4.183317438 - 2.985931305i.$$

Hence

$$a_3^{(i)}(1000) = -4 - 3i,$$

one of the 31 states in Theorem 5.2.

For comparison, the limiting circular arc gives

$$g(t_{1000}) \approx -4.159593 - 2.993978i.$$

Both points lie inside the Gaussian rounding cell centred at $-4 - 3i$, illustrating the asymptotic approximation (44). These calculations are numerical illustrations, not interval certificates for this individual zero.

Remark 5.5 (Arithmetic and geometric dependence of the frequency law). The limiting distribution depends on the parameter

$$b = \frac{9}{8\pi^2},$$

which arises from the quartic imaginary displacement of the canonical zeros and the cubic cancellation in $K_i(z)$.

The coefficients h_1, h_2 of the Wilson–Gamma germ affect the lower-order real phase in (43), but they do not appear in the limiting frequencies. Their presence does not obstruct equidistribution because the leading fractional-power term has nonzero coefficient.

The theorem describes frequencies across the sequence of canonical indices c. It is not a Gauss–Kuzmin law for the digits inside a single Gaussian continued fraction. Nor does it assert that successive digits or digit blocks are independent.

Remark 5.6 (No global Type discrimination). The 31-state law concerns a specific canonically constructed sequence of non-real Goodspeed zeros. Its proof uses neither a global classification nor a proposed equivalence between analytic Types and continued-fraction digit patterns.

In particular, the existence of a finite limiting symbolic alphabet does not imply that different Goodspeed Families possess disjoint symbolic languages. Such a conclusion would require additional information about independently classified zero sequences.

6. Discussion

The preceding results exhibit two rather different continued-fraction regimes for the non-real zeros of the Goodspeed function. At sufficiently large imaginary heights, the phase coordinates admit an exact logarithmic correction, and their first Gaussian partial quotient is an integer location coordinate. The following digit has magnitude asymptotic to

$$\frac{\pi}{2} e^{2\pi \Im z},$$

so that the continued fraction records both an integer phase location and an exponentially finer displacement. These conclusions do not require a classification of the zero into a particular analytic Type.

Near the origin, the canonical non-real zero sequence exhibits a different phenomenon. Although the zeros satisfy

$$\Re z_c^+ \sim c^{-1/2}, \qquad \Im z_c^+ \sim \frac{1}{2c^2},$$

the Gaussian partial quotients of their phase coordinates obey four distinct growth laws. The distinction arises from the order of vanishing of the correction germs K_μ: three have simple zeros at the origin, whereas

$$K_i(z) = z - \mathcal{A}(z) = \frac{\pi^2}{3} z^3 + O(z^5)$$

has a zero of order three. This cancellation is responsible for the exceptional $c^{3/2}$ growth in Theorem 4.1.

The exceptional orientation also illustrates the retention of analytic information under integer quantization. Although $a_2^{(i)}(c)$ is eventually an ordinary integer, its asymptotic expansion determines the first two nonconstant coefficients of

$$H(z) = \Gamma(1+z) + z.$$

In particular, Corollary 4.2 recovers Euler's constant from the asymptotic behaviour of a single sequence of Gaussian partial quotients. This is a reconstruction from canonically indexed data; it is distinct from the full inverse-rigidity results obtained from the unlabelled non-real zero tail in [7].

The principal symbolic consequence is Theorem 5.2. After removing the diverging real Gaussian digit, the next complete quotient approaches

$$g(t) = \frac{1}{t + ib}, \qquad b = \frac{9}{8\pi^2}.$$

Equidistribution of the centred real phase then produces a finite alphabet of exactly 31 Gaussian integers, each with a positive and explicitly calculable limiting frequency. The resulting distribution is determined by the intersections of a circular arc with Gaussian rounding cells. It is a distribution across the canonical indices c, rather than a Gauss–Kuzmin law for successive digits of one fixed continued fraction.

An interesting feature is the different dependence of these conclusions on the Taylor coefficients of H. The coefficient-recovery formulas involve h_1 and h_2, whereas the limiting 31-state frequencies depend only on the constant b, which arises from the quartic imaginary displacement of the canonical zeros and the cubic cancellation in K_i. Thus the coarse symbolic distribution retains less analytic information than the diverging digit sequence from which it is derived.

6.1. **Further questions.** Several directions merit investigation. First, the local construction of the canonical zero sequence uses the holomorphic germ

$$H(z) = z^2 W(z)$$

and its real Taylor coefficients, rather than global properties of the Gamma function. It is therefore natural to consider other real-symmetric holomorphic germs satisfying $H(0) = 1$. One may ask under which hypotheses the four digit-growth laws and the limiting 31-state alphabet persist. A general result of this kind would clarify which features arise from the local trigonometric sheet equation and which depend on the specific Wilson–Gamma function.

Second, the recovery formulas of Corollary 4.2 suggest investigating how much additional information is retained by the full integer sequence $\{a_2^{(i)}(c)\}$. The bounded error introduced by nearest-integer rounding prevents the direct normalized-limit argument from recovering all higher Taylor coefficients. Whether more refined methods can overcome this limitation remains open.

Finally, one may study continued-fraction patterns associated with other independently constructed sequences of Goodspeed zeros. The present results concern the canonical upper tail and do not imply that different Goodspeed Families have disjoint digit languages. In particular, the phase coordinates studied here should not be regarded as an intrinsic criterion for global Type membership.

The results nevertheless show that the non-real zeros of the Goodspeed function support a precise arithmetic and symbolic structure: their analytic geometry determines continued-fraction growth, integer-valued reconstruction formulas, and explicit limiting digit frequencies.

AI Declaration

Extended conversations with OpenAI's ChatGPT (GPT-5.6 Sol), together with its Work Mode, played a substantial role in the exploratory development and preparation of this article. The system assisted with symbolic and numerical experimentation, proof strategies, the formulation and testing of conjectures, and the drafting and refinement of mathematical arguments and exposition.

In particular, these interactions contributed to the investigation of Gaussian continued fractions of the Goodspeed phase coordinates, the derivation of the logarithmic phase correction, the asymptotic analysis of the canonical non-real zero tail, the coefficient-recovery formulas, and the development of the finite symbolic alphabet and its limiting frequency distribution.

AI-assisted calculations, drafts, and proposed arguments were examined and revised during the development of the manuscript. The author has also undertaken further mathematical verification, including independent scrutiny of the principal results. The direction of the research, the interpretation of numerical experiments, the selection and presentation of results, and responsibility for the final mathematical claims rest with the author.

The author acknowledges the extensive use of Wolfram Mathematica for symbolic and numerical experimentation, high-precision computation, and visualization, including the preparation of the figure in this article.

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