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A counterexample to local Tor-length bounds

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Abstract

We exhibit complete-intersection ideals generated by regular sequences of lengths three and four in a four-dimensional regular local ring. Their common quotient has length eight, and their Tor modules have lengths $(8, 22, 22, 8)$, with all higher Tor modules zero. Thus both the proposed binomial lower bound in degree one and the proposed total-length bound fail. The calculation uses a graded Koszul complex on a sixteen-dimensional algebra. The local-ring statement and the Tor-length table are verified in Lean.

1 The counterexample

Fernandez Boix and Lima Pereira [1] formulate local analogues of the Buchsbaum–Eisenbud–Horrocks rank bounds. Let $(R, \mathfrak{m})$ be a regular local ring of dimension n, and let $k \le n$. Their published Question 1 asks whether

$$\ell_R \operatorname{Tor}_i^R(R/I, R/J) \ge \binom{k}{i} \ell_R(R/(I+J))$$

for every $0 \le i \le k$, when I contains a regular sequence of length k, J contains one of length n, and $I + J$ is $\mathfrak{m}$-primary. Under these hypotheses, Question 2 asks for the total-length bound

$$\sum_{i=0}^{n} \ell_R \operatorname{Tor}_i^R(R/I, R/J) \ge 2^k \ell_R(R/(I+J)).$$

Conjecture 3 predicts equality in the first formula for every $0 \le i \le k$ when I and J are generated by regular sequences of lengths $k \le n-1$ and n.

These assertions appear on pp. 17–19 of the published paper. They are Conjecture 4.2 and Questions 5.1 and 5.4 in arXiv:2607.26118v1 [2].

The following example violates all three assertions with $n = 4$ and $k = 3$. Throughout, ℓ_R denotes composition length over R.

*The author conducted this research independently.

Theorem 1. *Let*

$$R = \mathbb{Q}[x, y, z, w]_{(x,y,z,w)}, \qquad J = (x^2, y^2, z^2, w^2)R,$$

and put

$$I = (x^2 + xz,\ w^2 + yw,\ y^2 + z^2 + xy + zw)R.$$

The displayed generators of I and J are regular sequences, and $I + J$ is $\mathfrak{m}$-primary, where $\mathfrak{m} = (x, y, z, w)R$. Moreover,

$$\ell_R(R/(I + J)) = 8, \qquad \left(\ell_R \operatorname{Tor}_i^R(R/I, R/J)\right)_{i=0}^{3} = (8, 22, 22, 8),$$

and $\operatorname{Tor}_i^R(R/I, R/J) = 0$ for $i \geq 4$. In particular,

$$\ell_R \operatorname{Tor}_1^R(R/I, R/J) = 22 < 24 = 3\ell_R(R/(I + J))$$

and

$$\sum_{i=0}^{4} \ell_R \operatorname{Tor}_i^R(R/I, R/J) = 60 < 64 = 2^3 \ell_R(R/(I + J)).$$

2 A graded Koszul calculation

Proof of Theorem 1. The ring R is regular local of dimension four, and its coordinate squares form a regular sequence. Put $f_1 = x^2 + xz$ and $f_2 = w^2 + yw$. The polynomial f_1 is monic in x, and the image of f_2 modulo f_1 is monic in w, so they form a regular sequence in $\mathbb{Q}[x, y, z, w]$. Their quotient is free over $\mathbb{Q}[y, z]$, with basis $1, x, w, xw$. Set $s = y^2 + z^2$ and $h = yz$. Multiplication by $y^2 + z^2 + xy + zw$ is triangular in this basis, with diagonal

$$s, \quad s - h, \quad s - h, \quad s - 2h.$$

The determinant is nonzero in the domain $\mathbb{Q}[y, z]$. The adjugate identity therefore shows that the multiplication map is injective. The three elements therefore form a regular sequence before localization, and hence in R. Their ideal is proper because all three vanish at the origin.

Write $C = R/J$. The sixteen squarefree monomials form a $\mathbb{Q}$-basis of C. In this algebra, set

$$u = xy + zw, \qquad v = xz, \qquad t = yw.$$

The generators of I have residues (v, t, u). We use the order (u, v, t) in the Koszul complex; permuting the list gives an isomorphic complex.

The ideal $(u, v, t)C$ has graded dimensions $3, 4, 1$ in degrees $2, 3, 4$. The three quadrics are independent. Multiplying u by x, y, z, w gives the four squarefree cubics, and $xyzw$ also lies in the ideal. Thus both $(u, v, t)C$ and $C/(u, v, t)$ have dimension eight. Since the latter is supported at the origin, $I + J$ is $\mathfrak{m}$-primary.

The middle Koszul differential is

$$d_2(p, q, r) = (-vp - tq,\ up - tr,\ uq + vr).$$

It raises polynomial degree by two. Its rank on constant inputs is three. For linear inputs, write p_x, p_y, p_z, p_w for the coefficients of p, and use analogous notation for q, r. The first coordinate of $d_2(p, q, r) = 0$ gives

$$p_y = p_w = q_x = q_z = 0,$$

the second gives $p_x = p_z = r_x = r_z = 0$, and the third gives $q_y = q_w = r_y = r_w = 0$. The degree-one block is therefore injective and has rank twelve. In degree two, the inputs $(zw, 0, 0)$, $(0, xz, 0)$ and $(0, 0, yw)$ map to $(0, xyzw, 0)$, $(-xyzw, 0, 0)$ and $(0, 0, xyzw)$. These are three independent vectors in the degree-four component. That block has rank three. All higher blocks vanish, so

$$\operatorname{rank}_{\mathbb{Q}} d_2 = 3 + 12 + 3 = 18.$$

The first differential has image $(u, v, t)C$ and rank eight. The last differential is

$$d_3(c) = (tc, -vc, uc).$$

Its ranks on input degrees zero, one and two are $1, 4, 3$. For degree one, multiplication by u is injective. In degree two, the choices $c = xz, yw, xy$ give $(xyzw, 0, 0)$, $(0, -xyzw, 0)$ and $(0, 0, xyzw)$, respectively. Higher degrees give zero. Hence d_3 also has rank eight.

Tensor the Koszul resolution on the displayed generators of I with C, and use the reordered bases above. The terms have dimensions $(16, 48, 48, 16)$ and the differentials have ranks $(8, 18, 8)$. Its homology dimensions are

$$16 - 8 = 8, \qquad 48 - 8 - 18 = 22, \qquad 48 - 18 - 8 = 22, \qquad 16 - 8 = 8.$$

The resolution has length three, so higher Tor modules vanish. The residue field of R is $\mathbb{Q}$; for these finite-length modules, composition length equals rational vector-space dimension. This proves the asserted table and inequalities. □

3 Relation to the eight-point algebra

The common quotient is the algebra of Cartwright, Erman, Velasco and Viray [3, Proposition 5.1]. Their ideal

$$(x_1^2, x_1x_2, x_2^2, x_3^2, x_3x_4, x_4^2, x_1x_4 + x_2x_3)$$

extends to $I + J$ in R under $(x_1, x_2, x_3, x_4) = (x, z, w, y)$. That paper studies its place on the Hilbert scheme of eight points and proves its nonsmoothability. Erman and Velasco [4, Proposition 4.5 of the author version] subsequently relate the associated exterior-flattening ranks to Koszul Betti numbers. Here the algebra occurs as the common quotient of two complete-intersection ideals, yielding the Tor-length inequalities in Theorem 1.

4 Formal verification

The Lean companion proves Theorem 1 over the displayed localized polynomial ring, using a projective resolution of R/I. It includes the regular-sequence hypotheses, the comparison of composition length with rational dimension, the entire Tor table and the vanishing in higher degrees.

The entry point is the module `Gerstenhaber.LocalTorCounterexample.Counterexample16`. Its main declaration is `actual_local_tor_counterexample`; the table and vanishing statements are `actualTor_length_table` and `actualTor_length_high`. The companion's statement map gives the full declaration names.

The development uses Lean 4.36.0-rc2 and mathlib revision

`07977d2921637684e9ecdf70fc1927570233637d`.

The command `python3 tools/verify.py` in the companion checks the source hashes, compiles the proof and audits the terminal axioms. The only terminal axioms are `propext`, `Classical.choice` and `Quot.sound`. Integer rank certificates are checked inside Lean.

AI assistance

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