Hall deficiency one at the distinct-multiples threshold

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Abstract. In the divisibility matching of Erdős and Pomerance, consider the last interval too short to admit distinct multiples of $1, \ldots, n$. We show that every set violating Hall's condition in this interval has deficiency exactly one. The observation follows from deleting a single right vertex at the first feasible threshold. Matching computations verify it for $1 \le n \le 10\,000$, and a census of all subsets gives the number of violations for $1 \le n \le 22$.

1. The threshold

Let $f(n)$ be the least integer F such that the open interval $(n, n+F)$ contains distinct integers $a_1, \ldots, a_n$ with $k \mid a_k$ for every k. This is the convention of Erdős Problem 710 [1] and OEIS A390246 [2]. The closed right endpoint used by Erdős and Pomerance [3] and by A219787 [2] is $n + f(n) - 1$. For example, $f(10) = 15$ gives right endpoint 24.

For $F \ge 1$, form a bipartite graph with left side $L = \{1, \ldots, n\}$, right side $R_F = \{n+1, \ldots, n+F-1\}$, and an edge from k to b whenever $k \mid b$. A matching covering L selects the required distinct multiples. Such a matching eventually exists, since $a_k = (n+1)k$ works for sufficiently large F. Also $f(n) \ge n+1$, since R_F contains $F-1$ candidates. For $S \subseteq L$, write $N_F(S)$ for its neighbors. Its deficiency is $|S| - |N_F(S)|$.

Proposition. Let $n \ge 1$ and $F = f(n)$. At threshold $F-1$ there exists a set $S \subseteq L$ of positive deficiency, and every such set satisfies

$$|N_{F-1}(S)| = |S| - 1.$$

Proof. At F a matching covers L, so the vertices of each S have $|S|$ distinct partners in $N_F(S)$. Passing to $F-1$ removes just the candidate $n+F-1$. Hence $|N_{F-1}(S)| \ge |N_F(S)| - 1 \ge |S| - 1$. Minimality of F rules out a covering matching at $F-1$, so Hall's theorem [4] supplies a set with fewer neighbors than vertices. Every set with this property attains equality in the bound. □

Equivalently, the maximum matching size at $F-1$ is $n-1$. Deleting the final candidate destroys at most one edge of a covering matching, and minimality excludes a matching of size n. The argument works for any nested sequence of candidate sets that grows by one right vertex at each step with all earlier edges retained.

Example. For $n = 4$, the assignment $(a_1, a_2, a_3, a_4) = (5, 6, 9, 8)$ fits at $F = 6$. At $F = 5$, the set $\{2, 3, 4\}$ has only the two neighbors 6 and 8. This witness and the assignment together certify $f(4) = 6$. For $n = 1$, the single candidate at $F = 2$ disappears at $F = 1$, giving the same deficiency.

More generally, removing d candidates decreases the size of any neighborhood by at most d. In the size-four example, removing both 9 and 8 leaves the set $\{2, 3, 4\}$ with the single neighbor 6. Its deficiency is then two.

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2. Computational evidence

For each $1 \le n \le 10\,000$, we computed a covering matching at $F = f(n)$ and checked its partners for divisibility, distinctness and membership in the interval. At $F - 1$ we computed a maximum matching from scratch. Starting at its unmatched left vertex, alternating paths identify a violating set S. We then counted the multiples of its members in the interval directly. In every case the matching had $n - 1$ edges and $|N_{F-1}(S)| = |S| - 1$. All thresholds agree with A390246, and the first 500 right endpoints agree with A219787 [2]. The largest witness returned by this procedure had 812 elements, at $n = 9048$.

A second calculation determined F by backtracking and enumerated every subset of L through $n = 22$. Table 1 gives the first eight cases. The last column is the maximum of $|S| - |N_{F-2}(S)|$ over all subsets for that n. Across $2 \le n \le 22$ its largest observed value was two. It was one at $n = 6$ and $n = 8$. For $n = 1$, $F - 2 = 0$ lies outside the stated domain.

Table 1. Hall violations one and two steps below the threshold.

n	F	Violating sets at $F-1$	Max. deficiency at $F-1$	At $F-2$
1	2	1	1	—
2	3	2	1	2
3	4	3	1	2
4	6	1	1	2
5	6	5	1	2
6	9	2	1	1
7	9	2	1	2
8	11	5	1	1

At $n = 50\,000$ and $100\,000$ the same matching checks gave $f(n) = 110\,161$ and $226\,701$. The witnesses had respectively 1 348 and 2 861 elements and one fewer neighbor. An earlier computation [5] reports both thresholds and records the matching-size drop at five selected values of n. Tolbert [6] reports a matching computation with a closed right endpoint through 10^6.

Relation to Problem 710. The proposition is an elementary consequence of Hall's theorem. The asymptotic question in [1] concerns the growth of $f(n)$. Howe, Huffman, Kohlmeyer and Kruer [7] report a Lean formalization of the asymptotic formula announced by Turturean. The finite-threshold argument above uses only the existence of a covering matching at F.

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References

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