A PROOF OF LINCKELMANN'S CONJECTURE ON OUTER DERIVATIONS OF BLOCKS

Abstract. Let B be a block of a finite group algebra over a field of positive characteristic. We prove that if B has positive defect, then $\mathrm{HH}^1(B) \neq 0$, as conjectured by Linckelmann. In fact, the degree-one Batalin–Vilkovisky operator $\mathrm{HH}^1(B) \to Z(B)$ is nonzero. A minimal-counterexample argument reduces the proof to a question about fusion in quasisimple groups. We handle the remaining small defect groups using a local criterion involving the projective centre of a central quotient.

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1. Introduction

Let k be a field of characteristic $p > 0$ and let G be a finite group. A block of kG is the algebra $B = kGb$ for a primitive central idempotent b. Its defect groups form a conjugacy class of p-subgroups of G, defined via the Brauer map in Section 2. A block has positive defect if its defect groups are nontrivial. Equivalently, B is nonsemisimple.

The first Hochschild cohomology $\mathrm{HH}^1(B)$ is the space of k-linear derivations of B modulo inner derivations. Fleischmann, Janiszczak and Lempken proved that $\mathrm{HH}^1(kG) \neq 0$ when p divides $|G|$ [FJL]. Linckelmann asked whether $\mathrm{HH}^1(B) \neq 0$ for every block B of positive defect [Lin18, Question 7.4]. Antrobus, Briggs and Rubio y Degrassi used the degree-one Batalin–Vilkovisky operator to prove nonvanishing in several families, including alternating groups, general linear groups and groups of Lie type in defining characteristic [ABR]. Antrobus subsequently proved the conjecture for $p > 5$ and for blocks with abelian defect at every prime [Ant]. Antrobus, Rubio y Degrassi and Wen proved it for every positive-defect block of a sporadic simple group [ARW, Theorem 3.17].

The standard symmetrizing form on kG sends (u, v) to the coefficient of the identity in uv. Distinct block summands are orthogonal, so this form restricts to a symmetrizing form on B. It identifies $\mathrm{HH}^0(B) = Z(B)$ and $\mathrm{HH}^1(B)$ with the respective duals of $\mathrm{HH}_0(B)$ and $\mathrm{HH}_1(B)$. We write Δ_B^1 for the dual of the Connes operator $\mathrm{HH}_0(B) \to \mathrm{HH}_1(B)$ under these identifications, the Batalin–Vilkovisky operator in degree one.

Theorem A. *Let B be a positive-defect block of a finite group algebra over a field of positive characteristic. Then*

$$\Delta_B^1 : \mathrm{HH}^1(B) \longrightarrow Z(B) \quad \textit{is nonzero.}$$

In particular, $\mathrm{HH}^1(B) \neq 0$.

Section 2 recalls Antrobus's fusion criterion and proves the theorem for defect groups of order at most p^3. In Section 3 we show that a minimal counterexample would produce a block of a quasisimple group with the same defect group for which the fusion criterion fails. The appendices show that the defect group of such a quasisimple block has order at most p^3.

We write p' for orders prime to p, $R' = [R, R]$ for the commutator subgroup, C_n for a cyclic group of order n, and $n_p = p^{v_p(n)}$ for the p-part of a nonzero integer n. For a finite abelian group A, we write A_p for its Sylow p-subgroup.

2. Local criteria for nonvanishing

For $g \in G$, let $K_g^G = \sum_{u \in g^G} u$ be its conjugacy class sum. Say that g is *detected in its centralizer* if there is a homomorphism $C_G(g) \to (\mathbb{F}_p, +)$ which is nonzero on g. The centralizer

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decomposition and the formula for the operator [BKL, Theorem 3.4 and Proposition 4.4], together with compatibility with block projection [ABR, Proposition 2.6], give

$$\operatorname{im}\Delta^1_{kGb} = \operatorname{span}_k\{bK_g^G : g \text{ is detected in its centralizer}\}. \tag{1}$$

The degree-one summand supported on g^G consists of homomorphisms $f : C_G(g) \to k^+$. Its image consists of the elements $f(g)K_g^G$ as f varies. Multiplication by b projects these images onto the block. The existence of such an f with $f(g) \neq 0$ can be tested over $\mathbb{F}_p$. Every f factors through the elementary abelian quotient of $C_G(g)$ by its commutators and pth powers, and linear functionals to $\mathbb{F}_p$ separate points of this quotient.

Let $D \neq 1$ be a defect group of b. Over an algebraic closure $\bar{k}$, write $b = \sum_i c_i$ as a sum of block idempotents. Since Br_D retains the coefficients on $C_G(D)$, $0 \neq \mathrm{Br}_D(b) = \sum_i \mathrm{Br}_D(c_i)$ in $\bar{k}C_G(D)$. Choose c_i with nonzero Brauer image. Assuming Theorem A over $\bar{k}$, (1) for this positive-defect block gives a detected g with $0 \neq c_iK_g^G = c_i(bK_g^G)$. Thus $bK_g^G \neq 0$ in kG, and the same $\mathbb{F}_p$-valued homomorphism gives $\Delta^1_B \neq 0$ by (1). Henceforth k is algebraically closed.

2.1. **A sufficient condition on fusion.** For a p-subgroup $U \leq G$, the Brauer map $\mathrm{Br}_U : (kG)^U \to kC_G(U)$ discards the terms outside $C_G(U)$. It is an algebra homomorphism on the subalgebra fixed by conjugation by U. A *b-Brauer pair* (U, c) consists of U and a block idempotent c of $kC_G(U)$ with $c\,\mathrm{Br}_U(b) = c$. Fix a maximal pair (D, e_D). Its first component D is a defect group of b. Equivalently, the defect groups are the p-subgroups D maximal subject to $\mathrm{Br}_D(b) \neq 0$. For each $Q \leq D$ there is a unique block e_Q such that $(Q, e_Q) \leq (D, e_D)$ in the usual order on Brauer pairs. The *block fusion system* $\mathcal{F}$ has the subgroups of D as objects. An isomorphism $Q \to Q'$ belongs to $\mathcal{F}$ precisely when it is induced by conjugation by an element of G sending (Q, e_Q) to $(Q', e_{Q'})$. The other morphisms are composites of such isomorphisms and inclusions.

A subgroup $Q \leq D$ is *fully centralized* if $|C_D(Q)|$ is maximal among its $\mathcal{F}$-conjugates. Write $\Phi(R) = R^p[R, R]$ for the Frattini subgroup of a finite p-group R. Following Antrobus, we call $a \in D$ a *fusion non-Schur element* if $\langle a \rangle$ is fully centralized and $a \notin \Phi(C_D(a))$. Write $\mathrm{NS}(\mathcal{F})$ for the condition

$$\text{there exists a fusion non-Schur element in } D. \tag{2}$$

Antrobus constructs from such an a an element g detected in its centralizer with $bK_g^G \neq 0$ [Ant, proof of Theorem 1.5]. Thus (2) implies $\Delta^1_B \neq 0$ by (1). Every positive-defect block at $p > 5$ satisfies (2) [Ant, Theorem 1.6].

Lemma 2.1. *Let $D \neq 1$, let $D' = [D, D]$, and let $\exp(D)$ be the largest order of an element of D. If $Z(D) \not\subseteq \Phi(D)$ or $\exp(D') < \exp(D)$, then $\mathrm{NS}(\mathcal{F})$ holds. In particular it holds for abelian D, and for $p = 2$ whenever $\Phi(D)$ is abelian.*

Proof. A central element outside $\Phi(D)$ has centralizer D, so it satisfies (2). This covers the nontrivial abelian case. For the exponent criterion [Ant, Proposition 5.1(iii)], write $\exp(D) = p^m$, choose an element of this order, and replace it by a fully centralized fusion conjugate a. Write $R = C_D(a)$. If $a \in \Phi(R)$, the image of a in R/R' is a pth power. Thus $a = y^p v$ with $y \in R$ and $v \in R' \leq D'$. The factors commute because a centralizes y. Both are killed by p^{m-1}, contrary to the order of a. Finally, for $p = 2$ the quotient of D by the subgroup generated by squares has exponent at most two and hence is abelian. The squares therefore generate $\Phi(D)$. If $\Phi(D)$ is abelian, these commuting generators have orders at most $\exp(D)/2$. The same bound holds for the exponent of $D' \leq \Phi(D)$. □

We will also use the exponent criterion for wreath products. Write $U \wr C_p = U^p \rtimes C_p$, where a generator σ of C_p cyclically permutes the factors. The commutator subgroup lies in the base U^p. Every pth power lies there as well, whereas $((u, 1, \ldots, 1)\sigma)^p = (u, \ldots, u)$. Taking u of largest order gives

$$\exp((U \wr C_p)') \leq \exp(U) < p\exp(U) = \exp(U \wr C_p) \tag{3}$$

for every finite p-group U. The strict inequality between derived and full exponent is preserved under products of nontrivial p-groups having that property, since the exponent of the product is the largest factor exponent.

2.2. **Detection using a central quotient.** For a finite-dimensional algebra A identify $Z(A)$ with $\mathrm{End}_{A\text{-}A}(A)$ by evaluation at 1. The *projective centre* is the ideal of maps factoring through a projective A-bimodule. The quotient by this ideal is the *stable centre*.

Lemma 2.2. *Let $x \neq 1$ be a p-element, $U = \langle x \rangle$, and $H = C_G(x)$. Let (U, c) be a b-Brauer pair. Since U is central in H, the image $\bar{c}$ of c in $k(H/U)$ is a block idempotent* [Ant, Proposition 4.7]*. If the projective centre of $k(H/U)\bar{c}$ is nonzero, then $\Delta^1_{kGb} \neq 0$.*

Proof. We first explain how a class sum in kHc can be used in kGb. Every group element v has a unique expression as a product of commuting elements $v_p v_{p'}$ of p-power and prime-to-p orders, respectively. Let $\pi_x : Z(kH) \to Z(kH)$ keep the terms with $v_p = x$. Since x is central in H, this is a projection onto a p-section. It commutes with multiplication by any sum of block idempotents of kH [Ant, Proposition 2.2]. In particular it commutes with multiplication by $\mathrm{Br}_U(b)$. For a p'-element $t \in H$, the p-part of $h(xt)h^{-1}$ is hxh^{-1}. Consequently, the members of $(xt)^G$ whose p-part equals x are precisely the elements of $x\, t^H$, and all belong to H. Multiplicativity of the Brauer map and $c\,\mathrm{Br}_U(b) = c$ now give

$$c\,\pi_x(\mathrm{Br}_U(bK^G_{xt})) = c\,\mathrm{Br}_U(b)\,xK^H_t = xcK^H_t. \tag{4}$$

Thus $cK^H_t \neq 0$ implies $bK^G_{xt} \neq 0$.

Set $\bar{H} = H/U$. The projective centre of $k\bar{H}$ is the image of the relative trace $v \mapsto \sum_{h\in\bar{H}} hvh^{-1}$ [KL, Lemma 3.2(iv)]. Its projection by $\bar{c}$ is the projective centre of $k\bar{H}\bar{c}$, since bimodule factorizations extend to and restrict from the factors of a product algebra. The trace of $\bar{t} \in \bar{H}$ is $|C_{\bar{H}}(\bar{t})|K^{\bar{H}}_{\bar{t}}$. By hypothesis there is consequently some $\bar{t}$ such that $p \nmid |C_{\bar{H}}(\bar{t})|$ and $\bar{c}K^{\bar{H}}_{\bar{t}} \neq 0$. In particular $\bar{t}$ has order prime to p.

The coset $\bar{t}$ has a unique lift t of prime-to-p order. Indeed, the p'-part of any lift is such a lift, and multiplying it by a nonidentity element of the central p-group U produces an element with nontrivial p-part. This uniqueness is compatible with conjugation. Hence $t^H \to \bar{t}^{\bar{H}}$ is a bijection and $C_H(t)$ is the inverse image of $C_{\bar{H}}(\bar{t})$. The class sum cK^H_t maps to $\bar{c}K^{\bar{H}}_{\bar{t}} \neq 0$. Moreover U is a central Sylow p-subgroup of $L = C_H(t) = C_G(xt)$. The last equality follows because x, t are powers of xt. The group transfer $L \to U$ sends a central element of U to its $[L : U]$th power, by the coset formula for transfer. Composing with $U \to U/U^p \cong (\mathbb{F}_p, +)$ therefore detects the generator x, and also detects xt since t has prime-to-p order. Equations (4) and (1) finish the proof. □

2.3. **Defect groups with commutator subgroup of order at most p.** The fusion system of a block is saturated. We recall the extension property that we use [AKO, Part I, Section 2]. If $\varphi : Q \xrightarrow{\sim} Q'$ is a fusion isomorphism with fully centralized target, it extends to a fusion morphism $N_\varphi \to D$, where

$$N_\varphi = \{g \in N_D(Q) : \varphi c_g \varphi^{-1} \in \mathrm{Aut}_D(Q')\}.$$

Here c_g is conjugation by g on Q, and $\mathrm{Aut}_D(Q')$ consists of conjugations by $N_D(Q')$. In particular $C_D(Q) \leq N_\varphi$.

Theorem 2.3. *Suppose p is odd and B has extraspecial defect group P of order p^3 and exponent p. Then $\Delta^1_B \neq 0$.*

Proof. Let $Z = Z(P) = P' = \langle x \rangle$ have order p. The commutator induces a nondegenerate alternating pairing on the plane $V = P/Z$ with values in Z. Indeed its radical is $Z(P)/Z = 0$. The induced isomorphism $\bigwedge^2 V \to Z$ identifies the action of any automorphism on Z with the determinant of its action on V. For $a \notin Z$, the space perpendicular to aZ in V is the line through aZ, so $C_P(a) = \langle a, Z \rangle$ is elementary abelian of order p^2. If some such a has no central fusion conjugate, all its fusion conjugates have centralizers of order p^2. Then a satisfies (2). We may henceforth assume that every $a \notin Z$ has a fusion conjugate in Z.

Fix a maximal subgroup A of P and $a \in A \setminus Z$. Then $A = C_P(a)$. A fusion isomorphism sending a into Z has fully centralized target, so it extends over A to an isomorphism $\beta : A \to A'$ for another maximal subgroup A'. In the plane A, conjugation by P gives the full group of p transvections (including the identity) acting trivially on Z and A/Z. Indeed its order is $|P : C_P(A)| = p$, and any such matrix is determined by one off-diagonal entry. In particular

its common fixed line is Z. Conjugating $\mathrm{Aut}_P(A')$ back to A by β gives another subgroup of $\mathrm{Aut}_{\mathcal{F}}(A)$ of the same form, with fixed line $\beta^{-1}(Z) = \langle a \rangle$. In the basis given by these two lines the subgroups are respectively all upper and all lower unitriangular matrices. These generate $\mathrm{SL}_2(\mathbb{F}_p)$ by elementary row operations. Thus $-I_A \in \mathrm{Aut}_{\mathcal{F}}(A)$.

Every maximal subgroup of P is self-centralizing, so A is fully centralized. Since $A \lhd P$ and $-I_A$ commutes with $\mathrm{Aut}_P(A)$, the extension formula above gives $N_{-I_A} = P$. Consequently $-I_A$ extends to $\sigma_A \in \mathrm{Aut}_{\mathcal{F}}(P)$. Its induced map on V has determinant -1, since σ_A inverts Z. It is -1 on the line A/Z, so the other eigenvalue is 1. These eigenvalues are distinct for odd p. For two distinct maximal subgroups, the resulting maps on V have different (-1)-eigenlines. Their quotient therefore gives a nonidentity map on V, induced by a fusion automorphism of P which fixes Z pointwise.

Let $(Z, c) \leq (P, e_P)$ and set $H = C_G(x)$. The group Z is fully centralized, so kHc has defect $C_P(Z) = P$ and fusion system $C_{\mathcal{F}}(Z)$ [Ant, Propositions 2.7 and 2.9]. Its morphisms are the fusion maps extendible over Z so as to fix Z pointwise. Since Z is central in H, the image block $k(H/Z)\bar{c}$ has defect group V and fusion system $C_{\mathcal{F}}(Z)/Z$ [Ant, Proposition 4.7]. The inertial quotient of a block with abelian defect is the fusion automorphism group of its full defect group. Here it is the image E on V of those members of $\mathrm{Aut}_{\mathcal{F}}(P)$ that fix Z pointwise. We have proved $E \neq 1$. Moreover $E \leq \mathrm{SL}(V)$, and E has order prime to p since the Sylow axiom of saturation gives $\mathrm{Aut}_V(V) = 1$ as a Sylow p-subgroup. Each $e \in E$ acts semisimply over an algebraic closure. If it fixes a nonzero vector then its two eigenvalues are 1, since its determinant is 1. Thus $e = 1$, proving that E acts freely on $V \setminus \{0\}$.

By Kessar–Linckelmann [KL, Theorem 1.2], this free action makes the stable centre of $k(H/Z)\bar{c}$ symmetric. By the theorem of Okuyama–Tsushima [OT, Theorems 3 and 5], a block with abelian defect has symmetric ordinary centre exactly when its inertial quotient is trivial. Here $E \neq 1$, so the ordinary centre is not symmetric. Its quotient by the projective centre is symmetric, forcing the projective centre to be nonzero. Apply Lemma 2.2. □

Corollary 2.4. *If $D \neq 1$ and $|D'| \leq p$, then $\Delta_B^1 \neq 0$. In particular this holds for every nontrivial defect group of order at most p^3.*

Proof. The abelian case follows from Lemma 2.1. Otherwise $|D'| = p$. It is central because a p-group acts trivially on a normal subgroup of order p. The same lemma settles the case $\exp(D) > p$. If $\exp(D) = p$, noncommutativity forces p odd and $\Phi(D) = D'$. If any central element lies outside $\Phi(D)$ the lemma again applies. This leaves $Z(D) = D' = \Phi(D)$ of order p, which is precisely the definition of an extraspecial group.

The commutator now makes $D/Z(D)$ a nondegenerate alternating space of dimension $2n$ over $\mathbb{F}_p$. For $n = 1$ the preceding theorem applies. For $n \geq 2$ take $a \notin Z(D)$ and put $R = C_D(a)$. The image of R is the hyperplane perpendicular to $aZ(D)$. Its radical is the line spanned by $aZ(D)$, so the restricted form has rank $2n - 2 > 0$. Hence $R' \neq 1$ and therefore $R' = D'$. As R has exponent p, this gives $\Phi(R) = D' = \Phi(D)$. Map a to a fully centralized fusion conjugate a'. The extension axiom gives an injection $\psi : R \to D$ carrying a to a'. Every homomorphism of finite p-groups carries their powers and commutators to powers and commutators. Thus $\psi(\Phi(R)) \leq \Phi(D)$, with equality by injectivity and the common order p. Now $a \notin \Phi(R)$, whence $a' \notin \Phi(D)$ by injectivity. Since $\Phi(C_D(a')) \leq \Phi(D)$, we obtain (2). Finally a nonabelian p-group needs at least two generators, so $|D : \Phi(D)| \geq p^2$. If $|D| \leq p^3$, this implies $|D'| \leq |\Phi(D)| \leq p$. □

3. Reduction to a single component

Throughout this section k is algebraically closed. We show that a counterexample to degree-one detection forces a very specific obstruction in a quasisimple group. Recall that a finite group L is quasisimple if $L = [L, L]$ and $L/Z(L)$ is nonabelian simple. Earlier reductions to quasisimple groups use additional hypotheses on the fusion system or the prime [ST, Theorem 1.5] [Ant, Proposition 4.11]. Here the minimality argument uses the rank of the degree-one operator.

Theorem 3.1. *Fix a prime p. If there exists a positive-defect block B with $\Delta^1_B = 0$, there exists a positive-defect block d of a quasisimple group L with $p \nmid |Z(L)|$ such that $\mathrm{NS}(\mathcal{F}_d)$ fails. The defect group of d can be taken to be the defect group of a positive-defect block with $\Delta^1 = 0$.*

For a finite-dimensional k-algebra A let $\beta(A)$ denote the rank of $B_0 : \mathrm{HH}_0(A) \to \mathrm{HH}_1(A)$, the degree-zero Connes operator. For our blocks this equals $\operatorname{rank} \Delta^1$. Morita equivalence preserves β by invariance of the Hochschild mixed complex [Lod, Chapters 1–2]. We need one further observation.

Lemma 3.2. *Let $A \to C$ be a unital algebra homomorphism whose image contains the Jacobson radical of C. Then $\beta(A) \geq \beta(C)$.*

Proof. The images of $\operatorname{rad} C$ and of primitive idempotents span $\mathrm{HH}_0(C) = C/[C, C]$. Indeed, modulo the radical, each simple factor is a full matrix algebra over k and its cocentre is spanned by the class of one diagonal matrix unit, which lifts to an idempotent. The Connes operator kills idempotents, by naturality for $k \times k \to C$ sending $(1, 0)$ to the given idempotent. Thus every class in its image comes from the radical and lifts by naturality. □

A surjection of algebras in particular satisfies the preceding lemma. We will apply this to quotients in which the action of a normal subgroup has been made semisimple.

Write $Z_0(H) = O_{p'}(Z(H))$, the prime-to-p part of the centre of a finite group H, and put $\mu(H) = [H : Z_0(H)]$. For $U \leq H$ put $J_U = \operatorname{rad}(kU)$.

Lemma 3.3. *Let $U \trianglelefteq H$. Each block summand of $kH/J_U kH$ is Morita equivalent to a block of a finite group H_1 with $\mu(H_1) \leq [H : U]$.*

Proof. Conjugation by H preserves J_U since it acts by algebra automorphisms on kU. Thus $J_U kH = kHJ_U$ is a two-sided ideal. Set $A = kH/J_U kH$. Decomposing kH into cosets of U gives a grading of A by H/U, with the image of any coset representative a unit in that degree. Its degree-one component is the product of the simple matrix factors of kU/J_U. The sums of their central idempotents over H-orbits split A into ideals. In one such ideal any one factor idempotent e is full, since its conjugates sum to the identity there. Thus the ideal is Morita equivalent to eAe.

Put $P = \mathrm{Stab}_H(e)/U$ and $S = e(kU/J_U) \cong M_d(k)$. Only the degrees in P occur in the corner. For each $v \in P$ choose a representative h in the stabilizer. Conjugation by the unit eh acts on S by an inner automorphism. Multiplying eh by a unit of S gives a homogeneous unit w_v commuting with S, with $w_1 = e$. The component has the form Sw_v, and $w_v w_t w_{vt}^{-1} = \alpha(v, t)e$ for a scalar $\alpha(v, t) \in k^\times$. Associativity says α is a normalized cocycle. Hence $eAe \cong M_d(k) \otimes k_\alpha P$.

To realize this twisted algebra in a group algebra, put $m = |P|$ and $f(v) = \prod_{t \in P} \alpha(v, t)$. The cocycle identity gives $\alpha(v, t)^m = f(v)f(t)/f(vt)$. Choose $a(v)^m = f(v)$, with $a(1) = 1$. Replacing w_v by $w_v/a(v)$ makes all values of α roots of unity of order dividing m. They generate a finite cyclic p'-group $Z \leq k^\times$. On $H_1 = Z \times P$ put $(z, v)(z', t) = (zz'\alpha(v, t), vt)$. The summand of kH_1 for the inclusion character of Z identifies with $k_\alpha P$ as follows. The primitive central idempotent of kZ belonging to this character is $e_Z = |Z|^{-1} \sum_{z \in Z} z^{-1}(z, 1)$, where the coefficient z^{-1} is a scalar in k. The elements $e_Z(1, v)$ are a basis of $e_Z kH_1$ and have multiplication constants $\alpha(v, t)$. Finally $\mu(H_1) \leq |P| \leq [H : U]$. □

Suppose now that $B = kGb$ is a positive-defect block with $\beta(B) = 0$ for which $\mu(G)$ is minimal. Throughout this section write $Z_0 = Z_0(G)$. Two consequences of minimality will be used.

Lemma 3.4. *If $Z_0 < U \trianglelefteq G$, then $(kG/J_U kG)b$ is semisimple and $D \leq U$ for every defect group D of B. Every block of a normal subgroup covered by B is G-stable. Here B covers a block with idempotent c if $bc \neq 0$.*

Proof. Each block of the displayed quotient is a quotient of B. A nonsemisimple summand would, by Lemmas 3.2 and 3.3, give a positive-defect counterexample with parameter at most $[G : U] < \mu(G)$. Recall that a module is relatively U-projective if it is a direct summand of a module induced from U. A vertex of an indecomposable module is a minimal p-subgroup for which this holds. Green's vertex theorem gives a simple B-module V with vertex D [Gre,

Theorem 12]. Vertices are unique up to conjugacy, and if V is relatively U-projective then U contains a conjugate of D. The ideal $J_U kG$ is nilpotent since $(J_U kG)^m = J_U^m kG$ for every m. Thus V factors through the displayed semisimple summand, so is projective over $kG/J_U kG$. On restriction of scalars this algebra is the induced module $kG \otimes_{kU} (kU/J_U)$. Hence V is relatively U-projective, forcing its vertex into U by normality.

For the last assertion let c be a covered block, with stabilizer T. Let $f = \sum_{c' \in G \cdot c} c'$ be the sum of the distinct conjugates of c. They are orthogonal and permuted by G, so f is a central idempotent of kG. Since $bc \neq 0$ and $fc = c$, we have $bf \neq 0$ and hence $bf = b$ by the primitivity of b. The ideal of kGf generated by c contains all its conjugates, whose sum is f, so c is full in kGf. Likewise, bc is full in B: conjugation by the units bg, for $g \in G$, shows that the ideal it generates contains all bc', whose sum is b. Furthermore $cgc = 0$ for $g \notin T$, by orthogonality of c and gcg^{-1}. Taking the corner therefore gives $c(kG)c = c(kT)c$. Since c is central in kT, this corner is the summand kTc of kT. The full-corner equivalence for kGf sends b to the primitive central idempotent bc of kTc. Consequently B is Morita equivalent to a block of T. As $Z_0 \leq Z_0(T)$, a proper stabilizer would again contradict minimality. □

Choose a minimal nontrivial normal subgroup N/Z_0 of G/Z_0. Lemma 3.4 gives $D \leq N$. Since Z_0 has p'-order, $D \cap Z_0 = 1$. Thus D embeds in N/Z_0. If N/Z_0 were abelian, D would be abelian. Choose $a \in D \setminus D^p$, which is possible because $D \neq 1$. Every fusion conjugate of a in D has centralizer D, so $\langle a \rangle$ is fully centralized. Also $a \notin D^p = \Phi(D) = \Phi(C_D(a))$. Hence a satisfies (2). This contradicts $\beta(B) = 0$.

Thus N/Z_0 is a direct product of isomorphic nonabelian simple groups, permuted transitively by G. If U_i is the preimage in N of the ith factor, put $L_i = U_i'$. As the factor is perfect, $U_i = Z_0 L_i$, whence $L_i' = L_i$ and $L_i/Z(L_i)$ is that simple factor. Different L_i commute: $[U_i, U_j] \leq Z_0$, so for fixed $v \in U_j$ the commutator with v is a homomorphism on U_i and kills U_i'. We have obtained

$$N = Z_0 L_1 \cdots L_s.$$

Let c be a block of N covered by B. It is G-stable by Lemma 3.4. Thus c is a central idempotent in kG and $bc = b$. In particular c is the unique covered block of N. Conjugation by N on L_i is inner because Z_0 is central and the other L_j commute with L_i. Hence the block idempotents of kL_i are central in kN. They are orthogonal and sum to 1, so there is a unique block idempotent d_i of kL_i with $d_i c = c$. Since G fixes c, conjugation permutes these idempotents along with their components. The d_i are permuted transitively. The defect of c is D, by [Ant, Proposition 4.9(i)] and $D \leq N$.

Suppose some d_i has defect zero. By transitivity they all do, so every $kL_i d_i$ is a full matrix algebra over k. Since the factors commute and c is a central idempotent, multiplication gives a k-algebra homomorphism

$$kZ_0 \otimes_k \bigotimes_{i=1}^{s} kL_i d_i \longrightarrow kNc, \qquad z \otimes a_1 \otimes \cdots \otimes a_s \longmapsto z a_1 \cdots a_s c.$$

It is surjective. Indeed, write $n = z\ell_1 \cdots \ell_s$ for $n \in N$, with $z \in Z_0$ and $\ell_i \in L_i$. Then $z \otimes \ell_1 d_1 \otimes \cdots \otimes \ell_s d_s$ maps to nc, since $d_i c = c$ for every i. The elements nc span kNc. As Z_0 is abelian of p'-order and k is algebraically closed, kZ_0 is a product of copies of k. The source is therefore a product of full matrix algebras, and its quotient kNc is semisimple. This contradicts the fact that c has defect group $D \neq 1$. In particular all d_i have positive defect.

Proposition 3.5. *In a minimal counterexample, $s = 1$.*

Proof. Suppose $s > 1$. Fix $L = L_1$, put $T = N_G(L)$ and let $M = Z_0 L_2 \cdots L_s$. Then $M \trianglelefteq T$, $LM = N$, $[G : T] = s$, and $L \cap M$ is central of p'-order. We construct a representation of B by matrices over an algebra attached to T/M. The point of passing to T is that T fixes the first component, so its action can retain the radical from L while killing the radicals from the other components.

The coefficient algebra.

Choose a simple B-module V and a simple kT-submodule $W \subseteq V|_T$. The nilpotent ideal $J_M kT$ kills W. Let C be the block summand of $kT/J_M kT$ acting on W, and let I be the image of $J_L kT$ in C. This is a nilpotent ideal. Since $d_1 c = c$ and $cb = b$, we have $d_1 b = b$ and d_1

acts as 1 on V, hence on W. The idempotent d_1 is T-invariant by its uniqueness for L and c. Its image is therefore central in the quotient, so it acts as the identity on the whole block C containing W.

As a left kL-module, C is projective. In fact it is a summand of kT/J_MkT, a sum over T/N of copies of the kL-module

$$kN/J_MkN \cong kL \otimes_{k(L\cap M)} (kM/J_M).$$

The isomorphism is multiplication (choose representatives of $L/(L \cap M)$), and $k(L \cap M)$ is semisimple. No simple kLd_1-module is projective, since d_1 has positive defect: a projective simple of a symmetric algebra is also injective and forms a semisimple block by itself. It follows that $\mathrm{soc}_{kL}(C) \subseteq J_LC = I$. Indeed a simple submodule having nonzero image in the semisimple top would split off, by composing projection to the top with a retraction onto its image. In particular

$$Ia = 0 \quad (a \in C) \qquad \Longrightarrow \qquad a \in \mathrm{soc}_{kL}(C) \subseteq I \subseteq J(C), \tag{5}$$

where $J(C) = \operatorname{rad} C$. This also shows $I \neq 0$, since $C \neq 0$ has nonzero socle.

Matrix entries from the first component. Use the free right C-module $kG \otimes_{kT} C$ to define $\rho : kG \to M_s(C)$ by left multiplication. We first show that $\rho(kG)$ contains $M_s(I)$. Choose left coset representatives $g_1 = 1, g_2, \ldots, g_s$ of T. Since N is normal and contained in T, every element of L acts diagonally. For $u \in J_L$ its jth entry is the image of $g_j^{-1}ug_j$. If $j > 1$, this belongs to J_{L_i} for a component L_i of M and hence to J_M: $L_i \trianglelefteq M$, so its radical generates a nilpotent ideal of kM. Only the first diagonal entry can survive. Multiplying on the right by $\rho(kT)$ gives the entire ideal I in position $(1, 1)$ alone. For an elementary matrix aE_{11} so obtained, $\rho(g_i)(aE_{11})\rho(g_j^{-1}) = aE_{ij}$ in the chosen coset basis. This proves the inclusion.

The block action. We claim that $e = \rho(b)$ is the identity matrix. It commutes with $M_s(I)$, so I annihilates every off-diagonal entry of e on the left, by commuting with the diagonal matrices aE_{ii} for $a \in I$. These entries therefore lie in $J(C)$ by (5). The entry e_{11} commutes with all of C, since $\rho(kT)$ preserves the first coordinate and its complement, and induces all of C there. Every e_{ii} equals e_{11} by commutation with $\rho(g_i)$, since $\rho(g_i)$ carries the first basis vector to the ith one and permutes the other coordinates with invertible entries. The centre of the block C is local, with residue field k; its radical consists of central nilpotents and hence is contained in $J(C)$. Thus e reduces to a scalar idempotent modulo $M_s(J(C))$. Either e or $1 - e$ belongs to this nilpotent ideal, so e is 0 or 1. Finally W is a quotient of the regular module C, and $kG \otimes_{kT} W$ surjects onto V by Frobenius reciprocity. Since $bV = V$, we get $e = 1$.

Identifying the radical. It remains to identify I. Because L and M commute and generate N, we have $J_N = J_LkN + J_MkN$: the two ideals on the right are nilpotent, and the quotient is an image of the semisimple algebra $(kL/J_L) \otimes_k (kM/J_M)$. Thus C/I is a direct summand of kT/J_NkT. For each simple C/I-module S, the module $kG \otimes_{kT} S$ is annihilated by J_N and belongs to B because $\rho(b) = 1$. By Lemma 3.4 it is projective over kG/J_NkG. This last algebra is a free left kT/J_NkT-module by the coset decomposition and normality of N. As a bimodule over kT/J_NkT, it splits as the summand kT/J_NkT and the span of the images of elements outside T. Tensoring this decomposition with S shows that S is a summand of the restricted induced module. Thus S is projective over kT/J_NkT and over its summand C/I. Every simple C/I-module is projective, so C/I is semisimple and $I = J(C)$.

Now $\rho(B)$ contains all of $M_s(J(C))$. Lemma 3.2 and Morita invariance give $0 = \beta(B) \geq \beta(C)$. But $I \neq 0$, so C is nonsemisimple. By Lemma 3.3 it is Morita equivalent to a positive-defect block of a group with parameter at most $[T : M] < [G : Z_0]$. This contradicts minimality. □

Proof of Theorem 3.1. By Proposition 3.5, $N = Z_0L$ with L quasisimple of p'-centre. As N/L has p'-order and $D \leq N$, we have $D \leq L$. The covered block d of L has defect D. For a normal subgroup containing D, Antrobus proves that (2) for a covered block implies (2) for B [Ant, Proposition 4.9(v)]. If $\mathrm{NS}(\mathcal{F}_d)$ held, this implication and [Ant, Theorem 1.5 and its proof] would give $\Delta^1_B \neq 0$. Thus d has the asserted properties. □

Proof of Theorem A. Suppose some positive-defect block has $\Delta^1 = 0$. Theorem 3.1 then supplies a positive-defect block B_0 with $\Delta^1_{B_0} = 0$ and a quasisimple block with the same defect group D

for which (2) fails. Theorem D.1, proved in the appendices, gives $|D| \leq p^3$. But Corollary 2.4 applies directly to B_0 and gives $\Delta^1_{B_0} \neq 0$. This is a contradiction. □

Appendix A. Detecting elements by torus characters

The goal of the appendices is Theorem D.1, which bounds the defect groups of quasisimple blocks that fail (2). A block has *maximal defect* if its defect groups are Sylow p-subgroups. In that case one element detected in its group centralizer works for every such block. For smaller defect we work with the defect group of the specified block.

For a finite group of Lie type, $\mathbf{G}$ denotes a connected reductive algebraic group, F its Frobenius (or Steinberg) endomorphism, and $\mathbf{G}^F$ its finite group of fixed points. The defining characteristic divides the size q of the ground field. A cocharacter of a torus is a homomorphism from the multiplicative group into it; a character goes in the opposite direction. Their pairing is the integer m for which their composite is $t \mapsto t^m$. This converts the search for an element and a detecting homomorphism into arithmetic with roots.

Suppose x is a p-element of G and some homomorphism $C_G(x) \to C_p$ is nonzero on x. Any block whose defect group contains a conjugate of x satisfies NS: take a fully centralized block conjugate and restrict the transported homomorphism to its centralizer in the defect group. In particular one such x works for every maximal-defect block. A witness in a central p'-quotient works upstairs too, by lifting x to the unique p-element in its coset. Conversely, the centralizer of this lift maps onto the centralizer downstairs by uniqueness of the lift. A detecting map upstairs kills the central kernel and therefore descends.

We next construct detecting homomorphisms in a simply connected semisimple group $G = \mathbf{G}^F$. Write X and Y for the character and cocharacter lattices of an F-stable maximal torus $\mathbf{T}$. For a root $\alpha \in X$ its coroot is $\alpha^\vee \in Y$, and $\langle \alpha, \alpha^\vee \rangle = 2$. Write s_α for the reflection $v \mapsto v - \langle \alpha, v \rangle \alpha^\vee$ on Y. In the ordinary split and graph-twisted types, the Frobenius on the cocharacters of a rational maximal torus of Weyl type w is conjugate to $qw\tau$, where τ is the defining diagram automorphism. Every Weyl type occurs. For $x = v(\zeta)$ with ζ of order m prime to q, a root γ centralizes x exactly when $m \mid \langle \gamma, v \rangle$. The semisimple centralizer $\mathbf{C} = C_{\mathbf{G}}(x)$ is connected. A torus character λ perpendicular to its coroots extends to $\mathbf{C}$, because the coroot subtori generate the maximal torus of $[\mathbf{C}, \mathbf{C}]$. Indeed the quotient of $\mathbf{C}$ by its derived group is a torus and is also the quotient of $\mathbf{T}$ by $\mathbf{T} \cap [\mathbf{C}, \mathbf{C}]$. If it takes $\mathbf{C}^F$ into the cyclic group μ_h of hth roots of unity and $\lambda(x)$ has order h_p, then $\lambda(x)$ has nonzero image in $\mu_h/(\mu_h)^p \cong C_p$ when $p \mid h$. Here we use the fact that the p-part of a cyclic group of order h has exponent h_p. The root-system description of the semisimple centralizer and the rational parametrization of maximal tori are given in [MT, Chapters 14 and 25].

Lemma A.1. *Let $\mathbf{G}$ be semisimple simply connected, and use an F-stable maximal torus. The following constructions give a p-element x and a character of $C_{\mathbf{G}}(x)^F$ detecting x modulo pth powers. A condition on the torus action refers to F on cocharacters.*

1. *For odd $p \mid q - \epsilon$, with $\epsilon = \pm 1$, take a long root α on whose coroot line F acts as ϵq.*
2. *For $p = 2$ and q odd, take long roots α, β meeting at angle $120°$. Suppose F acts on their coroot span as Qs_α, with $Q = q$ or $-q$.*
3. *For $p = 5$ and $q \equiv \pm 2 \pmod 5$, take orthogonal long roots α, β. Suppose F sends $\alpha^\vee$ to $q\beta^\vee$ and $\beta^\vee$ to $-q\alpha^\vee$.*

Let T_0 be the torus generated by $\alpha^\vee$ in (1), and by $\alpha^\vee, \beta^\vee$ in (2) and (3). Then $x \in T_0$. The roots of $C_{\mathbf{G}}(x)$ are precisely those perpendicular to the chosen root in (1), or to both chosen roots in (2) and (3).

Proof. Write $x = v(\theta)$ with $|\theta| = m$. To see that x is rational it suffices to show $(F-1)v \in mY$, where Y is the cocharacter lattice. A root γ centralizes x precisely when $m \mid \langle \gamma, v \rangle$. For a long root, pairing with its coroot gives $0, \pm 1$ on all roots other than that root and its negative.

In (1), take $v = \alpha^\vee$ and $m = (q - \epsilon)_p$. The pairings with v are $0, \pm 1, \pm 2$, and $m \geq 3$. Thus the centralizing roots are perpendicular to α. The character α extends to the centralizer, takes its fixed points into $\mu_{q-\epsilon}$, and has value θ^2 at x, of order m.

In (2), put $m = (q^2 - 1)_2 \geq 8$ and $v = Q\alpha^\vee + (Q+1)\beta^\vee$. Then $(F-1)v = (q^2-1)\beta^\vee$. The reflection formulas $s_\alpha(\alpha^\vee) = -\alpha^\vee$ and $s_\alpha(\beta^\vee) = \beta^\vee + \alpha^\vee$ give the displayed equality. The three

positive roots of the A_2 generated by α, β have pairings $Q-1, Q+2, 2Q+1$. The first is not divisible by m because $(Q-1)_2 < m$, and the other two are odd. For any other root outside this A_2, put $a = \langle \gamma, \alpha^\vee \rangle$, $b = \langle \gamma, \beta^\vee \rangle$. Both belong to $\{0, \pm 1\}$. If $Qa + (Q+1)b$ is divisible by m, reducing first modulo two gives $a = 0$, and then $(Q+1)_2 < m$ gives $b = 0$. These are exactly the two orthogonality conditions. In particular the centralizing coroots are perpendicular to β, so β extends to the centralizer. Its values on the fixed points lie in μ_{q^2-1} because F^2 acts on its line by q^2. Its value θ^{Q+2} at x has order m.

In (3), put $m = (q^2+1)_5$ and $v = \alpha^\vee + q\beta^\vee$. Here $(F-1)v = -(q^2+1)\alpha^\vee$. For the four roots $\pm\alpha, \pm\beta$ the pairings are $\pm 2, \pm 2q$. For all other roots they have the form $a + qb$, with $a, b \in \{0, \pm 1\}$. Since $q \equiv \pm 2 \pmod 5$, divisibility by five forces $a = b = 0$. The character α extends to the centralizer and takes its fixed points into μ_{q^2+1}, since F^2 acts on its line by $-q^2$. Its value is θ^2 at x, of order m. □

We record a common check for passage to a central p-quotient. Here $Z(\mathbf{C})^\circ$ denotes the identity component of the centre of $\mathbf{C}$. Use T_0 from Lemma A.1 and put $\mathbf{C} = C_{\mathbf{G}}(x)$. Every root of $\mathbf{C}$ is trivial on T_0, so T_0 commutes with the torus and root subgroups generating $\mathbf{C}$. Thus $T_0 \subseteq Z(\mathbf{C})^\circ$. The dimension of $Z(\mathbf{C})^\circ$ is the torus rank minus the rank of the roots of $\mathbf{C}$. If the perpendicular roots have rank $\operatorname{rank}(\mathbf{G}) - \dim T_0$, equality of dimensions gives $T_0 = Z(\mathbf{C})^\circ$. In (1) an element of $T_0 \cap Z(\mathbf{G})$ is killed by two, since the root evaluates on its coroot as the square map. In (2) the intersection is killed by three, by the A_2 Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. In either case T_0 contains no nontrivial central p-element. A conjugator taking x to xz for a central p-element z must normalize $C_{\mathbf{G}}(x)$ and hence T_0. Since $x \in T_0$, this forces $z = 1$. Thus centralizers lift in the quotient of $\mathbf{G}^F$ by a central p-subgroup, and the detecting root character descends because all roots kill the centre.

Appendix B. Blocks of classical groups

Proposition B.1. *Let S be simple of ordinary type $A, {}^2A, B, C, D, {}^2D$, and J a quasisimple cover with p'-centre, $p = 2, 3, 5$. Every positive-defect block of J satisfies* NS.

A reduction to smaller rank. We record the reduction used for ordinary groups of Lie type in this section and in Section E. Let $p \nmid q$.

Write $G = \mathbf{G}^F$ with $\mathbf{G}$ simply connected and $Z = Z(G)_p$. For ordinary covers it suffices to treat G/Z: averaging over any further central p'-kernel identifies each quotient block and its local Brauer pairs with their lifts in the trivial central-character summand. The block labels used here come from the partition of characters into Lusztig series. The semisimple parameter s belongs to the dual group, whose roots and coroots are exchanged. A unipotent block contains a unipotent character, lying in the series with $s = 1$. For a block of G/Z lift to G and let s be its semisimple p'-label in the dual group. Form

$$\mathbf{L}^* = C_{\mathbf{G}^*}(Z(C_{\mathbf{G}^*}(s)^\circ)^\circ), \qquad L = \mathbf{L}^F, \qquad N = N_G(\mathbf{L}, e_s^L),$$

where $\mathbf{L}$ is dual to $\mathbf{L}^*$ and e_s^L is the series idempotent. Here N denotes the stabilizer of this idempotent in the rational normalizer of $\mathbf{L}$. The displayed $\mathbf{L}^*$ is the smallest Levi containing the connected centralizer. The element s is called *isolated* if its connected centralizer lies in no proper Levi, and *quasi-isolated* if its full centralizer lies in no proper Levi. Thus a quasi-isolated parameter which is not isolated still gives a proper $\mathbf{L}^*$. The full centralizer normalizes it and N/L is an abelian p'-quotient of its rational component group [BDR17, §7.A].

Lemma B.2. *Suppose $\mathbf{L}^*$ is proper, N/L is cyclic, and the block of G/Z has positive defect. Write the fixed points of the algebraic derived group of $\mathbf{L}$ as $\prod_i H_i$, with one factor for each Frobenius orbit of simple components. If every positive-defect block of each $H_i/Z(H_i)_p$ satisfies (2), then the block of G/Z does as well.*

Proof. Let b be the block of G above the given block of G/Z. The correspondence of Bonnafé–Dat–Rouquier associates to b a block c of N. We may choose a common defect group $D \leq N$ for b and c so that the correspondence identifies their Brauer pairs on the same subgroups of D [BDR17, Theorem 7.7]. Cyclicity of N/L supplies the extension hypothesis in this correspondence [Ruh20, Theorem 2 and Lemma 3]. Since N/L has order prime to p and Z is

a central p-subgroup, $Z \leq D \leq L$. The blocks of G/Z and N/Z below b and c thus have the same fusion system on $P = D/Z$ [Ant, Proposition 4.7]. The block of N/Z covers a block of L/Z with defect group P. A witness for this block of L/Z gives one for the block of N/Z by [Ant, Proposition 4.9(i),(v)]. Put $M = [\mathbf{L}, \mathbf{L}]^F Z/Z$. If P has nontrivial image A in the abelian quotient $(L/Z)/M$, choose $u \in P$ whose image is nonzero under a homomorphism $A \to C_p$. Conjugation in L/Z preserves the image in its abelian quotient, so this homomorphism detects every fusion conjugate of u in P. In particular it restricts to a detecting homomorphism on the centralizer of a fully centralized conjugate. Otherwise $P \leq M$ and the same normal-subgroup result reduces to its covered block. The derived Levi is simply connected; each Frobenius orbit of its simple factors contributes the fixed points H_i of one factor under the return map. Hence $M = (\prod H_i)/K$ for a central p-subgroup K, and each H_i has smaller semisimple rank than $\mathbf{G}$. The central p-subgroup $T = (\prod Z(H_i)_p)/K$ lies in P. If $T = P$ the defect group is abelian. Otherwise the block below it in $M/T = \prod_i H_i/Z(H_i)_p$ has defect group $P/T \neq 1$. A block of a direct product is a tensor product of blocks, with componentwise fusion. Choose a fully centralized witness for a positive-defect factor and take the identity in the other factors. Its centralizer is the product of the factor centralizers, so projection to the chosen factor gives the required detection. The central-quotient and witness-lifting results [Ant, Propositions 4.7 and 4.3] lift it through T. □

Proof of Proposition B.1. Maximal defect in defining characteristic. Choose an F-stable Borel subgroup $\mathbf{B}$ with unipotent radical $\mathbf{U}$. Its fixed point group $U = \mathbf{U}^F$ is a Sylow subgroup of G. Let $\mathbf{V}$ be the subgroup generated by all nonsimple positive root groups. It is connected and normal in $\mathbf{U}$, and $\mathbf{U}/\mathbf{V}$ is the direct product of the additive groups indexed by the simple roots. There is an F-fixed tuple with every coordinate nonzero. Indeed, on any orbit of simple roots, the return map on the first coordinate has the form $z \mapsto cz^e$ with $c \neq 0$ and $e > 1$ a power of the defining characteristic. The equation $z^{e-1} = c^{-1}$ has a solution over the algebraic closure, and the remaining coordinates are then determined around the orbit. Lang's theorem for $\mathbf{V}$ lifts this tuple to $u \in U$. The simple-root coordinates characterize regular unipotent elements in $\mathbf{U}$, and $\mathbf{B}$ is the unique Borel containing u [MT, Theorem 14.23]. Consequently, if $gug^{-1} \in U$ then $g\mathbf{B}g^{-1} = \mathbf{B}$. All such conjugates have nonzero image in the elementary abelian quotient $(\mathbf{U}/\mathbf{V})^F$. This quotient kills $\Phi(U)$, which contains $\Phi(C_U(v))$ for every $v \in U$. Thus any fully centralized block conjugate of u satisfies (2). The centre has p'-order, so its quotients have the same Sylow groups and group fusion. This argument also works for exceptional Steinberg maps: they permute simple-root orbits and preserve nonsimple supports.

Maximal defect in cross characteristic. At odd $p \nmid q$, Weir's Sylow theorem [Wei55] gives direct products of iterated wreath products of cyclic p-groups for matrix symplectic, orthogonal, linear and unitary groups. Thus (3) and the cyclic base case give the strict exponent inequality of Lemma 2.1. The symplectic and orthogonal isometry groups have the same Sylow groups, up to isomorphism, as the corresponding simple groups at odd primes. In type A the determinant and central quotients have p'-order if $p \nmid q - \delta$, where $\delta = 1$ for linear groups and -1 for unitary groups. Otherwise take $x = \mathrm{diag}(\zeta, \zeta^{-1}, 1^{n-2})$ with $|\zeta| = (q - \delta)_p$. In dimension $n \geq 4$, the repeated eigenvalue forces any scalar preserving the multiset to be 1. Thus the ordinary centralizer preserves each of the first two lines. The ratio of its scalar actions on these lines is a character into a cyclic group of order $q - \delta$. In the unitary case the two lines are chosen nondegenerate for the standard Hermitian form, so their scalars have norm one. The ratio kills scalar matrices and its value at x is ζ^2, a generator of the Sylow p-subgroup of the target. In dimension three a nontrivial scalar would make the three eigenvalues one coset of a group of order three. This requires $p = |\zeta| = 3$. In that case $(q - \delta)_3 = 3$, $(q^2 - 1)_3 = 3$, and $(q^3 - \delta)_3 = 9$ by the lifting-the-exponent formula. Dividing the special group order $q^3(q^2 - 1)(q^3 - \delta)$ by the scalar centre of order three gives a Sylow subgroup of order nine in the projective group. It is abelian. In dimension two the odd Sylow group is cyclic.

At $p = 2$ and odd q, the rank-one projective Sylow group is dihedral (possibly Klein four). In the nonabelian case its commutator subgroup is generated by the square of a rotation, so Lemma 2.1 applies. In linear or unitary dimension $n \geq 3$ use the two-root construction above on the first three coordinates, with $Q = \delta q$. A torus with the required twist exists: in the unitary

case the longest Weyl element composed with the diagram automorphism acts as -1. The eigenvalues of x are θ^Q, θ, $\theta^{-(Q+1)}$, 1^{n-3}. A scalar preserving them has 2-power order, being a ratio of eigenvalues. A nontrivial group of such scalars would contain -1. The two eigenvalues of order m must then be swapped, so $v_2(Q-1) = v_2(m) - 1$. This forces $v_2(Q+1) = 1$, leaving the unique eigenvalue of order $m/2 \geq 4$ without a partner. Thus the projective centralizer lifts to the ordinary one; the root character β kills scalars and still detects x.

In symplectic dimension at least four take r of order $t = (q-\epsilon)_2 \geq 4$ on a plane with rotation torus of order $q-\epsilon$, and extend it by the identity. Its projective centralizer lifts since 1 occurs and -1 does not. Projection to the plane torus modulo $\{\pm 1\}$ detects r. In orthogonal dimension at least seven choose two planes of signs $\epsilon, -\epsilon$, with nondegenerate complement, and take $x = (r, -I_2, I)$. Choose ϵ so that $4 \mid q - \epsilon$. The spinor norm on the rotation group of a plane of sign η is the quotient of its cyclic group of order $q-\eta$ by the squares. This follows from the degree-two spin covering of that torus. The rotation r generates the Sylow 2-subgroup on the first plane, so it is a nonsquare. On the second plane $(q+\epsilon)_2 = 2$, so $-I_2$ is also a nonsquare. Thus $x \in \Omega$. The $1, -1$ multiplicities differ, so its projective centralizer again lifts. Its restriction to the first plane is a rotation, since reflections invert r. The same projection detects x. The lower-rank identifications are $B_2 = C_2$, $D_3 = A_3$ and $B_1 = C_1 = A_1$.

Blocks below maximal defect. We induct on the semisimple rank, with the small fixed-point groups treated in Section D. For an ordinary classical group the factors H_i in Lemma B.2 again have ordinary classical type. Indeed a classical Levi preserves the weight spaces of its defining torus in the natural module. Frobenius permutes the spaces and can replace a linear factor by its dual; on an orthogonal factor it preserves the natural representation. In particular the return map on a D_4 factor cannot be triality, which interchanges the natural and half-spin representations.

Humphreys' theorem [Hum71] gives full or zero defect in defining characteristic. Assume $p \nmid q$. In type A the connected centralizer of $s \neq 1$ is the proper eigenspace Levi of a projective matrix. Its component group embeds in the cyclic group of scalars taking a lift to a conjugate. Use the rank reduction. For $s = 1$, at two all unipotent characters of a classical group belong to its principal block [CE04, Theorem 21.14]. At odd p, unipotent characters of the special group are restrictions of unipotent characters of the corresponding general group. If $p \nmid q - \delta$, the determinant index is prime to p and a covering block has the same defect, a product of the wreath groups above [FS82, (7A)]. If $p \mid q - \delta$ every partition has empty 1-core, so the same classification places all general-group unipotent characters in the principal block; its unique covered block is principal.

In types $B, C, D, {}^2D$ first take $p = 2$, so q is odd. A nonidentity label s of odd order has an odd-order geometric lift to a symplectic or special orthogonal matrix. Some eigenvalue λ differs from ± 1. The pair λ, λ^{-1} contributes a general linear factor to the connected centralizer, whose scalar torus remains positive dimensional under the central isogeny. Thus the connected centralizer lies in a proper Levi. Its component group embeds in the centre of the simply connected dual cover, a 2-group, and has exponent dividing $|s|$ by Bonnafé's component-group theorem [BDR17, §7.A]. Hence $N = L$ and the rank reduction applies. For $s = 1$ use the principal block as above.

Odd cross characteristic in the other classical types. For odd $p \nmid q$ in types $B, C, D, {}^2D$ the following argument applies to every block of every ordinary cover, including faithful spin blocks. Choose $x \in C_J(D)$ whose coefficient in the block idempotent b is nonzero. Such an element exists since $\mathrm{Br}_D(b) \neq 0$. If a Sylow p-subgroup P of $C_J(x)$ contains D, the coefficient of x survives in $\mathrm{Br}_P(b)$ as well. Maximality of D gives $P = D$.

For the ordinary quasisimple covers in these types we can compute the isomorphism type of D in a full symplectic or orthogonal isometry group I. Indeed, the simply connected group maps onto J with central 2-group kernel and maps onto $\mathrm{Sp}(V)$ or $\Omega(V)$ with central 2-group kernel. Here $\Omega(V)$ is the image of the spin group in the orthogonal group $O(V)$. If $\pi : H \to H/K$ is either quotient and $y \in H$, then the homomorphism

$$\pi^{-1}(C_{H/K}(\pi y)) \longrightarrow K, \qquad h \longmapsto hyh^{-1}y^{-1},$$

has kernel $C_H(y)$. Its image and K are 2-groups, so the centralizers of y and πy have isomorphic Sylow p-subgroups. Also $[O(V) : \Omega(V)]$ is a power of two. In even defining characteristic we use $B_n(q) \cong C_n(q)$. Hence, for an appropriate matrix $y \in I$, the group D is isomorphic to a Sylow p-subgroup of $C_I(y)$.

Write $y = su$ for its commuting semisimple and unipotent parts. The semisimple centralizer decomposes as a direct product of isometry groups on the primary spaces for the eigenvalues $1, -1$, general linear groups on pairs of reciprocal primary spaces, and unitary groups on self-reciprocal primary spaces over extension fields. The centralizer of y is the product of the centralizers of the components of u in these factors. In each factor the unipotent centralizer modulo a normal subgroup of order a power of the defining characteristic is a direct product of general linear, unitary, symplectic or orthogonal groups, with a possible extra elementary abelian 2-factor in characteristic two. These assertions follow from [DLO, Chapters 2 and 4–7]. More precisely, the structure formulas in the preprint version are Theorems 2.2.8, 4.1.2–4.1.4, 5.2.1, 6.1.2 and 7.1.1. In even characteristic Theorem 5.2.1 for $O(V)$ assumes that its unipotent argument lies in Ω. This holds on the 1-primary space: $y \in \Omega(V)$ has $\operatorname{rank}(y-1)$ even by the Dickson invariant, and $y - 1$ is invertible on the complementary even-dimensional space (its eigenvalues occur in inverse pairs). The restriction of y to the 1-primary space is unipotent.

Passing through the indicated normal subgroups preserves Sylow p-subgroups up to isomorphism, as does discarding each 2-factor. Weir's theorem [Wei55] now expresses D as a product of iterated wreath products of cyclic p-groups. By (3) and the cyclic base case, $\exp(D') < \exp(D)$ for $D \neq 1$, so Lemma 2.1 gives the required witness.

Base cases and exceptional covers. The small rational factors $\mathrm{SL}_2(2)$, $\mathrm{SL}_2(3)$, $\mathrm{SU}_3(2)$, $\mathrm{Sp}_4(2)$ occurring in the induction have abelian cross-characteristic Sylow subgroups after the indicated central p-quotient (orders 1, 3, 4, 5 or 9). For an exceptional Schur cover J with p'-centre, the map to its simple quotient induces an isomorphism between Sylow p-subgroups and preserves conjugacy of their elements, by the unique lifting of p-elements explained in Section A. In the maximal-defect case use the preceding maximal-defect arguments for the principal block of the simple quotient. Its fusion system consists of all group conjugations between subgroups of a Sylow subgroup. A fully centralized witness there remains fully centralized for the fusion of any block of J with that Sylow subgroup as defect group: the latter conjugacy class is a subset and contains the witness. For these covers [KM13, proof of Proposition 8.2] gives maximal defect, all characters of height zero, or $3.\Omega_7(3)$ at two with defect order at most eight. The second case has abelian defect [MNSFT24, Theorem A]; the nonabelian groups of order eight have derived exponent two. This finishes the induction and proves the proposition. □

Appendix C. Maximal defect in simple groups

Write

$$\mathcal{X}_3 = \{\mathrm{Ru}, J_4, {}^2F_4(2)'\} \cup \{{}^2F_4(2^f) : f > 1 \text{ odd},\ 3 \nmid f\}, \qquad \mathcal{X}_5 = \{\mathrm{Th}\},$$

and set $\mathcal{X}_p = \varnothing$ otherwise.

Theorem C.1. *If S is nonabelian simple, $p \mid |S|$ and $S \notin \mathcal{X}_p$, then some p-element of S is detected in its centralizer.*

Proof. Fix a Sylow p-subgroup P of S. In any S-conjugacy class of p-elements we may choose $x \in P$ such that $C_P(x)$ is a Sylow p-subgroup of $C_S(x)$. Indeed, a Sylow subgroup of $C_S(x)$ contains its central p-element x and is conjugate into P. For such a choice, any homomorphism $f\colon C_P(x) \to \mathbb{F}_p$ with $f(x) \neq 0$ gives a detecting map on $C_S(x)$. Compose transfer to the abelianization of $C_P(x)$ with f. Since x is central in $C_S(x)$, its image is $[C_S(x) : C_P(x)]f(x)$, which is nonzero. Conversely, a detecting map on $C_S(x)$ restricts to one on $C_P(x)$. In particular an abelian $P \neq 1$ suffices: take $x \in P \setminus P^p$.

For $p > 5$ use [ABR, Theorem 5.1]. In A_n an odd p-cycle is detected by restriction to its support, since permutations commuting with it act on that support by powers of the cycle. For $p = 2$ take $(1234)(56)$ and restrict to its unique four-cycle support when $n \geq 6$. The Sylow 2-subgroups of A_5 are abelian. For the classical groups and for groups in defining characteristic apply the maximal-defect argument of Section B to the principal block. Full centralization in its

fusion system is exactly the Sylow-centralizer condition of the first paragraph of this proof, so transfer gives the required homomorphism on the group centralizer. The sporadic groups outside the stated exceptions are covered by [ABR, Proposition 5.5]. In the Tits group an element of class $16A$ generates its own centralizer, giving detection at two [Atlas]. Its Sylow groups at five and thirteen are abelian. The Suzuki and small Ree groups have abelian cross-characteristic Sylow subgroups [ABR, proof of Theorem 5.1].

In $S = {}^2F_4(q)$, $q = 2^f > 2$ with f odd, the Sylow 5-subgroups are abelian by the torus calculation in [RZ24, proof of Lemma 15]. For $p = 3$ outside $\mathcal{X}_3$ we have $a = v_3(q+1) = 1 + v_3(f) \geq 2$. There is an element of order three with centralizer $\mathrm{SU}_3(q)$ [RZ24, Lemma 13]. Use a unitary basis over $\mathbb{F}_{q^2}$ for the standard hermitian form. The diagonal entries of norm one form a cyclic group of order $q+1$. Thus the determinant-one diagonal 3-group $T \cong C_{3^a}^2$ and a coordinate cycle s form a Sylow subgroup P of S since $|S|_3 = 3^{2a+1}$. For $t = \mathrm{diag}(a,b,c) \in T$ and $i = 1, 2$, the cube of ts^i is abc times the identity, hence is 1. Every element of order 3^a therefore lies in T, and is nonscalar since a determinant-one scalar has order at most three. A nontrivial coordinate cycle fixes only scalar diagonal matrices. Consequently the centralizer in P of every element of order 3^a is T. Such an element lies outside T^3 by its order. This applies also to the representative chosen in the first paragraph and gives detection.

For the other exceptional types, first suppose p is odd and $p \mid q - \epsilon$ for $\epsilon = \pm 1$. The highest long root α is fixed by the defining diagram automorphism. Twisting the standard torus by 1 or s_α gives action ϵq on its line, so Lemma A.1(1) applies. Only E_6 at three can require a central p-quotient. The roots perpendicular to the highest root contain the A_5 on nodes $1, 3, 4, 5, 6$ in Bourbaki numbering. The quotient check after the lemma applies.

This settles all cases at three. At five with $q \equiv \pm 2 \pmod 5$, the Sylow groups in $G_2, {}^3D_4, F_4$ are abelian by [ABR, Lemma 5.3]. For $E_6, {}^2E_6$ and E_7 the Sylow 5-subgroups are abelian whenever q has order four modulo five [FMZ, proof of Proposition 7.6]. It remains to treat E_8. Choose a standard subsystem D_4 in E_8 and use its usual coordinates, in which the roots are $\pm e_i \pm e_j$. In this subsystem put $\alpha = e_1 - e_2$, $\beta = e_1 + e_2$. The signed permutation $e_1 \mapsto e_2$, $e_2 \mapsto -e_1$, $e_3 \mapsto -e_3$, $e_4 \mapsto e_4$ belongs to $W(D_4)$ and sends $\alpha \mapsto \beta \mapsto -\alpha$. Its torus realizes Lemma A.1(3). All relevant centres have order prime to five.

At two, each remaining root system has a pair of long roots α, β meeting at $120°$ and fixed by the diagram automorphism. In G_2 take simple roots of the long-root A_2. In type ${}^3D_4$ take α_2 and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = s_1 s_3 s_4(\alpha_2)$. In $F_4, E_6, {}^2E_6, E_7, E_8$ take the Bourbaki pairs $(1,2), (1,3), (2,4), (1,3), (8,7)$, respectively. The order-three diagram automorphism fixes the two displayed triality roots, and the order-two diagram automorphism of E_6 fixes nodes 2 and 4. Hence twisting by s_α gives Lemma A.1(2), with $Q = q$. For the central involution in E_7 use the quotient check after the lemma. In fact the roots perpendicular to α_1, α_3 include the four simple roots on nodes $2, 5, 6, 7$ and the highest root of the E_6 on nodes $1, \ldots, 6$. Its coefficients in that system are $(1, 2, 2, 3, 2, 1)$, so these five perpendicular roots are independent. □

Appendix D. Quasisimple blocks and the remaining finite families

Theorem D.1. *Let L be quasisimple with centre of order prime to p and let D be a nontrivial defect group of a block of kL. If the block fails* NS*, then $|D| \leq p^3$. For an ordinary group of Lie type, or one of its central quotients, the stronger assertion holds that a failure requires maximal defect and $L/Z(L) \in \mathcal{X}_p$.*

Here an ordinary group of Lie type is the fixed-point group of the simply connected simple algebraic group under a Steinberg endomorphism. The stronger assertion for these groups is needed when a proper Levi subgroup occurs. We give the finite-family cases here and complete the induction in Section E.

Write $S = L/Z(L)$. In maximal defect, any p-element of S detected in its centralizer can be conjugated into the image of D. Its coset has a unique p-element lift, since the kernel is central of p'-order. Pull back the detecting map to the centralizer of that lift and restrict it after fusion to a fully centralized representative in D. Theorem C.1 therefore handles $S \notin \mathcal{X}_p$. Every group on the list has Sylow p-subgroups of order p^3. For the sporadic groups and the Tits group see [Atlas]. For $S = {}^2F_4(q)$ use $|S|_3 = 3(q+1)_3^2$ and $(q+1)_3 = 3$.

Assume henceforth that the defect is nonmaximal and that $p \in \{2, 3, 5\}$, using [Ant, Theorem 1.6] for larger primes. All simples in the block have the same character of $Z(L)$: the primitive idempotents of $kZ(L)$ are central in kL, and exactly one acts as the identity on the block. Let K be the kernel of that character. Multiplication by the central idempotent $e_K = |K|^{-1} \sum_{z \in K} z$ gives the algebra isomorphism $k[L/K] \to kLe_K$, $gK \mapsto ge_K$. It identifies the local Brauer maps because an element centralizing a p-subgroup modulo K centralizes each element of the subgroup: each coset of K contains at most one p-element. Moreover $Z(L/K) = Z(L)/K$, since $L/Z(L)$ is nonabelian simple. We can therefore assume the central character is faithful and the centre is cyclic. Section B treats the ordinary classical groups. The exceptional Lie types are completed in Section E.

Alternating groups. At odd p, a positive-defect group of A_n or $2.A_n$ is a Sylow subgroup of S_{pw} up to isomorphism, where w is the block weight [Ols, Cab]. This description applies to the spin blocks as well. The alternating subgroup has index two in the corresponding symmetric group or double cover, which leaves the odd defect group unchanged. Writing pw in base p describes a Sylow p-subgroup of S_{pw} as a product of iterated wreath products of C_p: split the support into sets of sizes p^t, with multiplicities given by the digits. On p^t letters a Sylow subgroup is obtained from the one on p^{t-1} letters by wreathing with C_p. These subgroups have the required orders by the formula $v_p(n!) = \sum_{j \geq 1} \lfloor n/p^j \rfloor$. Now (3) and Lemma 2.1 apply.

At 2, the defect group of A_n is Sylow in A_{2w}. For $w \leq 2$ it is abelian. For $w \geq 3$ choose it to contain $x = (1234)(56)$. Every element of $C_{A_n}(x)$ preserves the unique four-cycle support; restriction gives a homomorphism to $\langle (1234) \rangle$ taking x to a generator. The additional covers of A_6, A_7 of central order divisible by 3 can occur at 2 and 5. Their Sylow orders at these primes are 8 and 5 respectively. A nonabelian group of order eight has exponent four and commutator subgroup of order two.

Sporadic groups. The two tables of An and Eaton give the following bounds for noncyclic, nonmaximal defect; cyclic defect already satisfies Lemma 2.1. The bounds are taken from [AE11, Table 1] and [AE13, Table 1]. The bounds include faithful blocks of covering groups.

p	$\lvert D \rvert$ at most	exceptions to this bound
5	25	none
3	9	$\mathrm{Co}_1, \mathrm{M} \quad (\lvert D \rvert = 27)$
2	16	$\mathrm{Ly} \quad (\lvert D \rvert = 128)$

For odd p every order is at most p^3, including the two exceptional orders in the middle row. In the exceptional nonmaximal 3-blocks of the simple groups Co_1 and M, the defect groups meet classes 3D and 3C, respectively. Their centralizers are $C_3 \times A_9$ and $C_3 \times \mathrm{Th}$ [ARW, proof of Theorem 3.16]. The class element generates the centre, so projection to the first factor detects it in either case. At $p = 2$, a nonabelian group D of order at most 16 has $|\Phi(D)| \leq 4$. In particular its Frattini subgroup is abelian and Lemma 2.1 applies.

The proof of [ARW, Theorem 3.16] shows directly that the defect group of the indicated 2-block of Ly contains an element of class 8A detected in its centralizer. This is the stronger local assertion needed here.

Lie type and the Tits group. In defining characteristic, Humphreys' theorem [Hum71] gives only zero or maximal defect in the ordinary simply connected group, and the same holds in central p'-quotients. For exceptional covering groups, the character-table check in [KM13, proof of Proposition 8.2] gives height zero for every character of a nonmaximal block, except possibly at $p = 2$ in $3.\Omega_7(3)$ with $|D| \leq 8$. The height-zero theorem [MNSFT24] gives abelian defect in the first case; the order-eight argument applies in the second. The Tits group has trivial Schur multiplier and order $2^{11}3^35^213$. Its nonmaximal positive defects at odd primes are abelian. Its 2-blocks have defects $11, 0, 0$, by the modular character table of the Tits group. The same multiplicities can be read from [Bre] with `Collected(PrimeBlocks(CharacterTable("2F4(2)'"),2).defect)`.

Small groups in Levi subgroups. The rank induction can involve groups below the simplicity range. At a nondefining $p \leq 5$, the following table lists all their nontrivial Sylow groups after

quotienting by the central p-part.

group	$(p,$ Sylow group)
$\mathrm{SL}_2(2)$	$(3, C_3)$
$\mathrm{SL}_2(3)$	$(2, C_2^2)$
$\mathrm{SU}_3(2)$	$(3, C_3^2)$
$\mathrm{Sp}_4(2) = S_6$	$(3, C_3^2),\ (5, C_5)$
${}^2B_2(2)$	$(5, C_5)$
${}^2G_2(3)$	$(2, C_2^3)$

For $\mathrm{SU}_3(2)$ the diagonal third roots of determinant one and a three-cycle generate a subgroup of order 27, whose quotient by the scalars is C_3^2. The small Ree group has $\mathrm{PSL}_2(8)$ normal of index three, whose Sylow 2-subgroup is the additive group of $\mathbb{F}_8$. The other entries follow from the natural actions (the Suzuki group has order 20).

In $G_2(2)$ the only relevant prime is 3 and the Sylow order is 27. Proper defects are abelian. For maximal defect choose a witness x for the normal subgroup $N = \mathrm{PSU}_3(3)$ of index two by Theorem C.1. Transfer from $C_{G_2(2)}(x)$ to $C_N(x)$ followed by the detecting map sends x to $[C_{G_2(2)}(x) : C_N(x)]$ times its nonzero value, so still detects it. Finally, F_4 is not a proper connected Dynkin subdiagram of any irreducible Dynkin diagram, so ${}^2F_4(2)$ does not occur as a derived component in a proper Levi. This completes the finite cases of Theorem D.1.

Appendix E. Unipotent and isolated blocks in exceptional Lie type

Let $G = \mathbf{G}^F$ be of ordinary simply connected exceptional type, including triality and the Suzuki–Ree families. Let $p \in \{2, 3, 5\}$ be different from the defining characteristic, and put $Z = Z(G)_p$. We prove that every positive, nonmaximal-defect unipotent or isolated block of G/Z satisfies NS. A block inflated across a further central p'-quotient has the same Brauer pairs and fusion, via the central idempotent averaging over the kernel. The exceptional Schur covers and the small derived groups were treated in Section D.

For the ordinary split and graph-twisted groups we use the following part of the block parametrization. For an odd prime p let e be the multiplicative order of q modulo p. At $p = 2$ use the order of q modulo four. A Φ_e-torus is a rational torus on whose cocharacter space the eigenvalues of F/q are primitive eth roots of unity. For $e = 1, 2$ this means that F acts by $q, -q$, respectively. An e-split Levi is the centralizer of such a torus. The classification labels the blocks under consideration by pairs $(\mathbf{L}, \lambda)$ consisting of an e-split Levi and an irreducible character of $\mathbf{L}^F$. We need the derived type of $\mathbf{L}$ and the position of its central torus in a defect group. The character λ serves to distinguish labels on the same Levi.

We use the numbered block labels of [KM13], including its supplement. The corrected allocation of unnumbered Harish–Chandra series in [KM25, Proposition 6.9] leaves the numbered labels used below unchanged. For a labeling pair $(\mathbf{L}, \lambda)$ a defect group contains $Z^\circ(\mathbf{L})_p^F$, by [KM13, Theorems 1.2(b) and 7.12(d)] for bad primes. For good-prime labels see [Hol, proof of Theorem 1.2], and for unipotent blocks see [Eng00, Theorem A]. An element x in a defect group detected by a homomorphism $C_G(x) \to C_p$ gives a fusion witness. To see this, transport the homomorphism to a fully centralized block-fusion conjugate of x, then restrict to its centralizer in the defect group.

The Suzuki and small Ree groups have abelian cross-characteristic Sylow subgroups. At $p = 3$ in cross characteristic, the principal block is the only block of G_2, ${}^3D_4$ or ${}^2F_4$ with nonabelian defect [BrM, proof of Proposition 4.1]. Their Sylow 5-subgroups and those of F_4 are abelian, since 5 does not divide the Weyl-group order. In triality at 2, the only quasi-isolated odd-order parameter is 1 and the only unipotent block is principal [EKKS14, Section 5]. The large Ree group has defining characteristic two. We include G_2 at two in the table below.

Odd primes. In the notation $E_6^\delta(q)$, the signs $\delta = +1$ and -1 denote $E_6(q)$ and ${}^2E_6(q)$ respectively. Independently, when $p \mid q - \epsilon$ for $\epsilon = \pm 1$, put $p^a = (q - \epsilon)_p$. The construction for the rows of the table below is as follows. We will find a rational maximal torus $\mathbf{T} \subseteq \mathbf{L}$ and a long root α perpendicular to the roots of $[\mathbf{L}, \mathbf{L}]$, with F acting as ϵq on its coroot line. For $|\zeta| = p^a$, the element $x = \alpha^\vee(\zeta)$ belongs to $Z^\circ(\mathbf{L})_p^F$. Indeed the coroot torus commutes with $\mathbf{T}$

and with every root subgroup of $\mathbf{L}$, so its connected image lies in the centre of $\mathbf{L}$. The relation $\zeta^{\epsilon q} = \zeta$ gives rationality. Thus x lies in a defect group, and Lemma A.1(1) detects it.

The following list gives the nonabelian cases to which we apply this test. In the middle column the numbers are the nonunipotent rows of [KM13, Tables 2–7 and Supplement, Table 1]. The same rows apply after Ennola duality. A 1 in the last column means a torus.

(G,p)	rows	possible derived types of $\mathbf{L}$
$(F_4,3)$	$3,5,7$	1
$(E_6^\delta,3)$	$13,14$	1
$(E_7,3)$	$3,5,6,8,9,10,12,14,15,17,19,20$	$1, A_1, A_1^2, A_1^3, D_4$
$(E_8,3)$	$1,2,3,4,6,7,8,9,10,12$	$1, A_1^2, A_1^3, D_4$
$(E_8,5)$	$1,3,7,10,13,16,45$	1

The abelian exceptions are identified in [KM13, Propositions 3.5, 4.3, 5.3, 6.7 and 6.10]. In particular all relative Weyl groups in its Table 8 and in Supplement, Table 2 have order prime to five. Theorem 1 of the supplement identifies row 45 and its Ennola dual as its only nonabelian cases. Ennola duality replaces q by $-q$ in the torus and Levi data. Proposition 6.11 extends the conclusions for $E_8(q)$ to $q = 2$. In particular the exceptional cases to check there at $p = 3$ are the Ennola duals of rows 12–17 of Table 6, of which only row 12 has nonabelian defect, with a torus as its Levi. At $p = 5$ the applicable rows of Table 8 have abelian defect. Among nonprincipal unipotent blocks of nonabelian defect at odd bad primes the only cases are the D_4 labels of E_6^ϵ, E_7, E_8 at three [FLLMZ, Lemma 7.3 and Table 5]. The E_7 and E_8 labels are covered by the D_4 argument below. We treat the central quotient in E_6 separately.

At the good prime five in E_6^δ, E_7, [FMZ, proofs of Propositions 7.6–7.7 and Tables 2–3] give $q \equiv \epsilon \pmod 5$ in every nonabelian case. The only unipotent one is principal. The nonunipotent labels have a full e-torus as Levi. The dual centralizer types at $e = 1$ are A_5A_1 in E_6, and A_5A_2, D_6A_1, $A_7.2$, $\Phi_1.E_6.2$ in E_7, with their Ennola duals at $e = 2$. For the torus labels see also [Hol, Remark 3.3 and Theorem 1.2]. At $e = 1$ the relevant Levi entries are those of [KM13, Table 3, row 13, and Table 4, rows 1, 3, 5 and 8]. The same pairs apply at the good prime by the cited remark. Here $e = 1$ for $\epsilon = 1$ and $e = 2$ otherwise. The E_6 case on this list is split for $e = 1$ and twisted for $e = 2$. To check the other signs directly, put $Q = \delta q \equiv -1 \pmod 5$. In the order formula for $E_6^\delta(q)$, only the even degrees among $2,5,6,8,9,12$ contribute a factor of five. Each contributes $(Q+1)_5$, since none of those even degrees is divisible by five. For the rational torus with Frobenius $\epsilon q\tau$ described below, the diagram involution τ has eigenspace dimensions four and two for $+1$ and -1. The order of its group of fixed points, $|\det(F-1)|$, is $(q-\epsilon)^4(q+\epsilon)^2$. In the present case $\delta = -\epsilon$, so its 5-part is $(Q+1)_5^4$, the full 5-part of the group order. This torus therefore contains a Sylow 5-subgroup. All defect groups are then abelian.

For the torus choice, write the given e-split Levi as $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$ for a Φ_e-torus $\mathbf{S}$. The Sylow-torus theorem [BM, Theorem 3.4] puts $\mathbf{S}$ in a Sylow Φ_e-torus, and all such Sylow tori are conjugate by G. In F_4, E_7, E_8 the Weyl group contains -1, so a torus with scalar Frobenius ϵq exists and has full absolute rank. The extending Sylow torus is therefore a maximal torus $\mathbf{T}$ with this same action. It centralizes $\mathbf{S}$, which puts it inside the given $\mathbf{L}$. In E_7 or E_8, any three mutually orthogonal roots have a perpendicular root. Indeed the subsystem perpendicular to one root is D_6 in E_7, or E_7 in E_8 (the usual coordinate realizations). In D_6, perpendicular to $e_1 - e_2$ is $A_1 + D_4$, so any further root in this subsystem is perpendicular to a root of its other factor. In E_8, the roots perpendicular to the first root form E_7, and those perpendicular to the first two form D_6. The subsystem perpendicular to the third root in this D_6 contains D_4. For a set of just one or two roots the same computations give a perpendicular root at an earlier step. For a standard D_4 Levi the simple roots are $2,3,4,5$ in Bourbaki numbering, and root 7 is perpendicular to all four. Weyl conjugacy gives the assertion for any such Levi. For E_6^δ write τ for the diagram involution. The longest Weyl element acts as $-\tau$. When $\delta = \epsilon$ the defining Frobenius is split for $e = 1$ or is the graph twist for $e = 2$. In the latter case composing the graph automorphism with the longest Weyl element gives -1. Thus in both cases there is a torus with scalar Frobenius ϵq. In the unaligned row 14 at three, Table 3 (and its Ennola dual) gives $\mathbf{L}$ itself as a maximal torus with a four-dimensional Φ_e-part. On the standard torus with Frobenius $\epsilon q\tau$, the τ-fixed space also has dimension four, the full Φ_e-rank by the order formula.

Its centralizer is this torus because a perpendicular root would satisfy $\tau(\gamma) = -\gamma$, impossible since τ preserves positive roots. Sylow-torus conjugacy therefore identifies the specified $\mathbf{L}$ with this standard torus over the finite field. The diagram involution fixes the simple root numbered 2, providing the required coroot line.

For the central quotient in E_6 at three, all roots are conjugate to the highest root. Its perpendicular roots have rank five, as noted in the proof of Theorem C.1. The central-quotient argument following Lemma A.1 therefore applies.

For the unipotent 3-block with D_4 Levi in E_6^ϵ, the calculation in [FLLMZ, Lemma 7.3 and Table 5] gives a defect group P with an abelian normal subgroup $A \cong C_{3^a}^2$ of index three and $|Z(P)| = 3$. The ambient central subgroup $Z = Z(G)_3$ also has order three and lies in P, so $Z = Z(P)$. Choose g lifting a generator of P/A, and write w for its action on A, in additive notation. Since A is abelian, $w^3 = 1$ and $P' = (w-1)A$. The group $V = 3^{a-1}A$ is a two-dimensional $\mathbb{F}_3$-space. On V the endomorphism $w-1$ is nilpotent because $(w-1)^3 = w^3 - 1 = 0$. In dimension two its square is therefore zero. Consequently

$$3^{a-1}P' = (w-1)V \subseteq V^w \subseteq Z(P) = Z.$$

Also $V^w \neq 0$, so $Z \subset A$. Since $|V| = 9 > |Z|$, the quotient A/Z still has exponent 3^a. Thus $\exp((P/Z)') \leq 3^{a-1} < \exp(P/Z)$, and Lemma 2.1 applies.

The prime two. Choose $\epsilon = \pm 1$ with $q \equiv \epsilon \pmod 4$ and put $m = (q-\epsilon)_2$. Thus $m \geq 4$. The nonprincipal unipotent blocks of nonabelian defect occur only in E_7 and E_8. Their defect groups have, respectively, normal C_m with quotient C_2, and normal C_m^2 with quotient C_2^2 [KM25, Table 10]. In the nonunipotent isolated series the only blocks below the first numbered row of an individual series are E_8 rows 4 and 6. Their defect groups contain, respectively, C_m^2 and C_{2m} of index two [KM13, proof of Proposition 6.4]. For completeness, the exhaustiveness of these two nonunipotent exceptions, together with the unipotent ones of noncentral defect, is stated in [Ruh, Lemma 7.1, arXiv version]. In every case there is an abelian normal subgroup A with elementary abelian quotient. This property passes to any quotient by taking the image of A. For each resulting nontrivial defect group D the Frattini subgroup lies in the image of A, since an elementary abelian quotient kills all squares and commutators. Hence $\Phi(D)$ is abelian, and Lemma 2.1 gives the assertion.

It remains to treat the first rows. In these rows the defect group is G-conjugate to a Sylow 2-subgroup of $H = \mathbf{H}^F$, where $\mathbf{H} = \pi^{-1}(C_{\mathbf{G}^*}(s))$ and $\pi : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}} \cong \mathbf{G}^*$. This is Ruhstorfer's Proposition E(a), with the construction in Section 2.2 of [Ruh]. Its hypothesis is maximal defect within the s-series, namely $|D| = |C_{\mathbf{G}^*}(s)^F|_2$ for the block upstairs in G. The required equality follows from the maximal-block list of [Ruh, Proposition E and Lemma 7.1, arXiv version]. The defect group orders are also given by [KM13, Theorem 1.2]. The component group of $\mathbf{H}$ has odd order. Consequently taking fixed points of $\mathbf{H}^\circ$ gives the same Sylow 2-subgroups as taking fixed points of $\mathbf{H}$, since the quotient embeds in the component group. The last column below gives the type of $\mathbf{H}^\circ$ over the algebraic closure. The cited tables also specify the finite forms and the permutations of the components.

G	KM13 table and rows	$\mathbf{H}^\circ$ type
G_2	$9:1,2$	A_2
F_4	$2:1,2$	A_2^2
E_6^δ	$3:1,2,6,7,8,12$	A_2^3
E_7	$4:1,2$	A_5A_2
E_8	$5:1,2;\ 3,5;\ 7,8,9$	$A_8;\ E_6A_2;\ A_4^2$

In particular the two G_2 rows include both the split and unitary forms of A_2 [KM13, Lemma 6.13].

First consider products of components of type A_{2i}. Their simply connected cover has odd central kernel, hence odd kernel and cokernel on fixed points. A Frobenius orbit of components contributes one group $\mathrm{SL}_{2i+1}^\delta(Q)$, over a suitable power Q of q, with $\delta = +$ or $-$ denoting the linear or unitary form. Its Sylow 2-subgroup is that of $\mathrm{GL}_{2i}^\delta(Q)$ because adjoining the inverse determinant on one line gives a subgroup of odd index $Q^{2i}(Q^{2i+1} - \delta)/(Q - \delta)$.

Carter and Fong [CF, Theorems 1 and 4] describe the Sylow 2-subgroups of $\mathrm{GL}_{2i}^\delta(Q)$ as products of iterated wreath products by C_2. The initial groups are cyclic when $Q \equiv \delta \pmod 4$

and semidihedral when $Q \equiv -\delta \pmod 4$. For a semidihedral group the generators can be taken to be a rotation r of order 2^t, $t \geq 3$, and an involution acting on r by the power $2^{t-1} - 1$. Its derived group is $\langle r^2 \rangle$ and therefore has smaller exponent than the whole group. For a nontrivial cyclic group the same inequality is immediate. Equation (3) and Lemma 2.1 therefore give the desired property in each product. This applies in all rows involving only components A_{2i}. In these rows $Z(G)_2 = 1$.

In the two remaining rows $\mathbf{H}^\circ$ has type A_5A_2 or E_6A_2. Both factors are F-stable since they have different ranks. In the A_2 factor take a torus of cycle type $(2, 1)$. For its simple roots α, β the action on their coroot span is Qs_α, with $Q = q$ for the split form and $Q = -q$ for the unitary form. Indeed, on the plane of triples of coordinate sum zero, the transposition of the first two coordinates is s_α. Unitary Frobenius adds the common sign -1. Enlarge this torus by an F-stable maximal torus of the other factor. This gives a maximal torus of $\mathbf{G}$ since $\mathbf{H}^\circ$ has full rank.

The 2-element constructed in Lemma A.1(2) belongs to the A_2 factor of $\mathbf{H}$ and is rational, so lies in H and hence in a conjugate of the defect group. All ambient roots have the same length, so the lemma detects this element in its ambient centralizer. In E_7 the roots of the commuting A_5 factor are perpendicular to α, β and already have the largest possible rank, five. Thus the detecting character descends through the central involution by the argument following the lemma. The centre of E_8 is trivial.

Completion of the proof of Theorem D.1. For $p > 5$ use [Ant, Theorem 1.6]. Maximal defect at the other primes is covered by Theorem C.1. For a nonisolated parameter, apply Lemma B.2. Its minimal Levi is proper. The quotient N/L is a quotient of a subgroup of the component group of the dual semisimple centralizer. That component group embeds in the geometric centre of the simply connected cover (lift the semisimple element and take commutators). This centre is cyclic in exceptional type except for triality. In triality an odd-order parameter has component group of odd order inside a 2-group, and at odd primes the small-rank calculation above already covers all blocks. Thus the cyclicity required in the reduction holds. The factors of the proper Levi have smaller semisimple rank. None has a quotient in $\mathcal{X}_p$ because the only Lie types in that list are Ree or Tits, and F_4 is not a proper connected subdiagram of an exceptional Dynkin diagram. The positive-defect factor in the reduction therefore has a witness by induction, with the small fixed-point groups as treated in Section D. This completes the quasisimple classification. □

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