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The Zilber–Pink conjecture for algebraic tori

Draft generated by Tangli Ge

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Abstract

We prove the full Zilber–Pink conjecture for algebraic tori over $\mathbb{C}$. For every irreducible complex subvariety $X \subseteq \mathbb{G}_m^n$, the intersections of X with torsion cosets have only finitely many maximal atypical components relative to the smallest torsion coset containing X. More precisely, X has only finitely many optimal subvarieties. The proof first establishes non-density over $\overline{\mathbb{Q}}$ in a fixed height neighborhood of the algebraic subgroups of codimension greater than $\dim X$, under the hypothesis that X lies in no proper algebraic subgroup. Specialization then gives the exact intersection statement over $\mathbb{C}$.

From Tangli Ge: I started a project which aims to digest and canonicalize this proof, extract the ideas and theories that could work for more general things, and raise interesting questions based on the understanding of the proof. If you are interested in joining the group, please contact me by email `get2@vcu.edu`.

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1 Introduction

Let $T = \mathbb{G}_m^n$ over $\mathbb{C}$. A *torsion coset* is a translate of a connected subtorus by a torsion point. For an irreducible subvariety $X \subseteq T$ of dimension d, let $S = \langle X \rangle$ be the smallest torsion coset containing X. Given a torsion coset $H \subseteq S$, a component A of $X \cap H$ is *atypical* in S if

$$\dim A > \dim X + \dim H - \dim S.$$

Following Habegger and Pila [32, §2], write $\delta(V) = \dim\langle V \rangle - \dim V$ for the defect of an irreducible subvariety V. An irreducible $A \subseteq X$ is *optimal* if $\delta(A) < \delta(B)$ for every irreducible B with $A \subsetneq B \subseteq X$.

Theorem 1.1 (Zilber–Pink for algebraic tori over $\mathbb{C}$)**.** *Let $X \subseteq \mathbb{G}_m^n$ be an irreducible subvariety over $\mathbb{C}$. As H ranges over the torsion cosets in $\langle X \rangle$, only finitely many atypical components of $X \cap H$ are maximal under inclusion. More precisely, X has only finitely many optimal subvarieties.*

If X lies in no proper algebraic subgroup, then its intersection with the union of all algebraic subgroups of codimension greater than $\dim X$ is not Zariski dense in X.

The non-density assertion is Pink's formulation [45, Conjecture 5.1]. Write $T^{[k]}$ for the union of all algebraic subgroups of codimension at least k, including disconnected ones. A point belongs to $T^{[k]}$ precisely when its coordinates satisfy k independent multiplicative relations $x^{a_1} = \cdots = x^{a_k} = 1$ with $a_i \in \mathbb{Z}^n$. Thus, on a curve contained in no proper algebraic subgroup, only finitely many points satisfy two independent multiplicative relations. In what follows we call X *weakly transverse* if it lies in no proper algebraic subgroup and *transverse* if it lies in no translate of a proper subtorus.

Zilber arrived at the conjecture on intersections with tori (CIT) through the model theory of complex exponentiation [53, 54]. In his original formulation, for X defined over $\mathbb{Q}$, finitely many proper algebraic subgroups of T together contain every component A of an intersection $X \cap H$ with an algebraic subgroup H for which

$$\dim A > \dim X + \dim H - n.$$

The finite collection depends only on X, even as H varies. Zilber showed that CIT, together with Schanuel's conjecture for the complex exponential, implies the uniform Schanuel conjecture [54, Proposition 5]. For a fixed $\overline{\mathbb{Q}}$-subvariety $V \subset \mathbb{C}^n \times T$ of dimension less than n, this gives finitely many proper rational linear subspaces $L \subset \mathbb{C}^n$. Whenever $(z, \exp z) \in V$, one has $z \in L + 2\pi i\, k$ for one of these L and some $k \in \mathbb{Z}^n$, with $k = 0$ if L has codimension one [53, Proposition 5]. Pila revisited this implication in [42, §6].

CIT also implies toric Manin–Mumford and Mordell–Lang [53]. The latter says that for a subgroup Γ of finite rank, $X \cap \Gamma$ is a finite union of cosets of subgroups of Γ. Laurent proved this for subvarieties of tori in all dimensions [36], following work of Lang and Liardet on curves [35, 37]. Indeed, the torsion points are exactly $T^{[n]}$, and the conjecture implies that an irreducible subvariety with dense torsion points is itself a torsion coset. For a finitely generated group one can adjoin its generators as extra coordinates and express membership by multiplicative relations in a larger torus [45, §5]. Pink's mixed Shimura conjecture places these statements and the André–Oort conjecture in a common setting [45, Conjecture 1.3].

1.1 Known cases in algebraic tori

Two complementary cases have guided the subject. For a curve, the first unlikely intersections are with subgroups of codimension two. For a variety of codimension two, they are with subgroups of dimension one. The latter case goes back to Schinzel's 1965 conjecture on coprime polynomials $F, G \in \mathbb{Z}[x_1, \dots, x_n]$. If F and G vanish at $(\xi^{a_1}, \dots, \xi^{a_n})$ with $\xi \in \overline{\mathbb{Q}}^{\times}$ of infinite order and $a_i \in \mathbb{Z}$, Schinzel predicted a nonzero integer relation among the a_i with coefficients bounded in terms of F, G alone [48]. Geometrically, the corresponding one-parameter subgroups must lie in finitely many proper subtori. Schinzel proved the case $n \leq 3$ [49], and Bombieri and Zannier proved it for all n [51]. Bombieri, Masser and Zannier allowed torsion translates of one-dimensional subtori in their codimension-two theorem over $\overline{\mathbb{Q}}$ [15, Theorem 1.6]. Amoroso, Sombra and Zannier refined the dependence on the heights of defining equations for torsion translates [4]. For one-parameter subgroups, Amoroso, Andriamandratomanana and Simon gave an explicit bound for the relation in terms of both degrees and coefficient heights [1].

For a transverse curve C over $\overline{\mathbb{Q}}$, Bombieri, Masser and Zannier proved in 1999 that $C \cap T^{[1]}$ has bounded height and $C \cap T^{[2]}$ is finite [12]. The passage from the first assertion to the second uses a Lehmer-type lower bound of Amoroso and David [2]. Bombieri, Masser and Zannier extended the finiteness theorem for transverse curves to $\mathbb{C}$ [13], then obtained finiteness for weakly transverse curves over $\overline{\mathbb{Q}}$ when $n \leq 5$ [14]. Maurin proved it for all n using Rémond's generalized Vojta inequality [38, 46]. Bombieri, Habegger, Masser and Zannier gave a second proof using bounded height on auxiliary surfaces [11]. The specialization theorem of Bombieri, Masser and Zannier transfers the finite-subgroup containment formulation of CIT from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ in prescribed ranges of dimension and codimension. In particular, it gives the full curve and codimension-two cases over $\mathbb{C}$ [17, Corollaries 1 and 2].

In higher dimension the exceptional locus matters. Bombieri, Masser and Zannier call a positive-dimensional subvariety anomalous if it lies in a coset of a subtorus and its dimension exceeds the expected intersection dimension. Their structure theorem gives finitely many possible subtorus

directions for maximal anomalous subvarieties and shows that the complement X^{oa} of their union in X is Zariski open [15]. If only torsion cosets are used in defining anomalous subvarieties, the complement is the larger set X^{ta}. Bombieri, Masser and Zannier conjectured that X^{ta} is open and that $X^{\mathrm{ta}} \cap T^{[d+1]}$ is finite [15].

The structure theorem controls the directions of the cosets, while their translates may vary. Zilber's weak CIT and Kirby's semiabelian version give finite sets of directions for positive-dimensional atypical components in algebraic families, using Ax's theorem [6, 54, 34]. Aslanyan deduced finiteness for those optimal subvarieties whose smallest containing coset is already a torsion coset [5].

For a plane P over $\overline{\mathbb{Q}}$, Bombieri, Masser and Zannier bounded the height on $P^{\mathrm{oa}} \cap T^{[2]}$ and proved finiteness on $P^{\mathrm{oa}} \cap T^{[3]}$. Their argument also showed in every dimension that the points of $X^{\mathrm{ta}} \cap T^{[d+1]}$ of height at most B form a finite set for each B [16, Theorems 1.1–1.2 and Lemma 8.1]. After partial bounds for higher-dimensional varieties [27], Habegger proved the bounded height conjecture of Bombieri, Masser and Zannier. For every d-dimensional X over $\overline{\mathbb{Q}}$, the set $X^{\mathrm{oa}} \cap T^{[d]}$ has bounded height, and $X^{\mathrm{oa}} \cap T^{[d+1]}$ is finite [29, Corollaries 1.3 and 1.4]. In particular $X \cap T^{[d+1]}$ is not dense whenever X^{oa} is nonempty. Habegger subsequently made the height bound explicit [30]. A weakly transverse X of positive dimension can have $X^{\mathrm{oa}} = \varnothing$, for instance when it lies in a proper translate of a subtorus.

Products of curves illustrate what can still be proved when this open set is empty. For curves $C_1, C_2 \subset \mathbb{G}_m^N$ with $N \geq 3$, the question whether $x \in C_1$ has a power $x^m \in C_2$ amounts to intersecting $C_1 \times C_2$ with the graphs of power maps, of codimension N. For $C_1 = C_2$ weakly transverse over $\overline{\mathbb{Q}}$, Bays and Habegger proved finiteness for $m \geq 2$ when some conjugate of the curve meets $(S^1)^N$ in a finite set [9]. For two curves over $\overline{\mathbb{Q}}$, Amoroso, Masser and Zannier bounded $h(x)$ uniformly for $x \in C_1$ with $x^m \in C_2$ and $[m]C_1 \neq C_2$ [3, Corollary 1.3]. Boxall proved finiteness when at least one curve does not descend to $\overline{\mathbb{Q}}$, assuming C_1 lies in no one-dimensional algebraic subgroup and $C_1 \cup C_2$ in no two-dimensional algebraic subgroup, and excluding m for which $[m]C_1 = C_2$ [19].

The Pila–Zannier strategy. Pila and Zannier introduced a different proof of Manin–Mumford for abelian varieties [44]. In real coordinates relative to a period basis, torsion points lift to rational points in a bounded fundamental domain. A large Galois orbit supplies many such points. The Pila–Wilkie theorem gives only $O_\eta(H^\eta)$ rational points of height at most H outside the union of the positive-dimensional connected semialgebraic subsets of the lifted set, for every $\eta > 0$ [43]. When the orbit exceeds this bound, the proof recovers a translate of a positive-dimensional subgroup. For a torus, the logarithms of a point satisfying two multiplicative relations lie on a rational affine subspace in suitable real coordinates. Capuano, Masser, Pila and Zannier counted the corresponding rational points on Grassmannians to give a new proof of the transverse-curve theorem, with stronger uniformity [21]. Earlier uniform estimates for particular lines used fewnomials [22]. Orr obtained explicit height estimates for certain lines using G-functions [41].

Habegger and Pila developed this strategy for unlikely intersections on powers of the modular curve and on abelian varieties [31, 32]. They isolated the large Galois orbit estimates and functional transcendence statements needed for their general results, and proved unconditionally the finiteness theorem for curves in abelian varieties over $\overline{\mathbb{Q}}$ [32, Theorem 1.1]. Barroero, Kühne and Schmidt proved the curve theorem for all semiabelian varieties over $\overline{\mathbb{Q}}$ [8]. Barroero and Dill established general equivalences between non-density and finiteness formulations [7, Theorem 12.4]. For tori, Bombieri, Masser and Zannier had already recorded Ullmo's observation that their formulation is equivalent to Pink's non-density conjecture when both are asserted in all dimensions and codimensions [17, p. 320]. Capuano surveys these formulations and the known cases in [20].

1.2 A height neighborhood of the algebraic subgroups

The arithmetic step in the proof of Theorem 1.1 concerns subvarieties defined over $\overline{\mathbb{Q}}$. On $T(\overline{\mathbb{Q}})$ take the sum of the absolute logarithmic Weil heights of the coordinates and its sublevel set

$$h(x) = \sum_{j=1}^{n} h(x_j), \qquad \mathcal{B}_\epsilon = \{x \in T(\overline{\mathbb{Q}}) : h(x) \le \epsilon\}.$$

Theorem 1.2 (Arithmetic non-density)**.** *Let $X \subset T_{\overline{\mathbb{Q}}}$ be irreducible and weakly transverse, and put $d = \dim X$. There exist $\epsilon_X > 0$ and a proper Zariski closed subset $X' \subsetneq X$ such that*

$$X(\overline{\mathbb{Q}}) \cap (T^{[d+1]}\mathcal{B}_{\epsilon_X}) \subseteq X'(\overline{\mathbb{Q}}).$$

For a hypersurface, $T^{[d+1]}$ consists of the torsion points. Translation by torsion preserves h, so in this case the theorem is the hypersurface case of the toric Bogomolov theorem of Zhang, also proved by Bombieri and Zannier and by Bilu [52, 18, 10]. For transverse curves Habegger proved finiteness in a height cone about $T^{[2]}$. In a cone, the error in $x = ab$ may satisfy $h(b) \le \epsilon(1 + h(a))$ with $a \in T^{[2]}$ [28, Theorem 1.1]. Weak transversality allows a different behavior. On the curve $x = (2, 3, t)$ take $a = (1, 1, t)$ and $b = (2, 3, 1)$. The error has constant height $\log 6$, so every positive-radius cone contains the points with $h(t)$ sufficiently large [28, p. 99]. In higher dimension, Maurin proved non-density for a positive-radius height cone about $\Gamma T^{[d+1]}$ for any subgroup Γ of finite rank when X is geometrically nondegenerate [39]. This hypothesis requires each torus quotient $\pi : T \to T'$ to satisfy $\dim \pi(X) = \min(d, \dim T')$.

For an absolute tube the condition is simply $h(b) \le \epsilon$. Habegger conjectured that $X^{\mathrm{ta}} \cap (T^{[d+1]}\mathcal{B}_\epsilon)$ is finite for some $\epsilon > 0$ [28, Conjecture 1.8]. For weakly transverse curves $X^{\mathrm{ta}} = X$. In this case fixed-radius finiteness also follows from Maurin's finite-rank cone theorem and Rémond's reduction to the smallest containing coset. Translating the curve into the subtorus underlying that coset gives a transverse curve, and Rémond's inclusion sends the absolute tube into a cone about finite-rank translates in the smaller torus [39, 47].

In arbitrary dimension Rémond proved non-density for weakly transverse X with finite stabilizer under a prescribed bound $h(x) \le B$, with the radius allowed to depend on B [47, Theorem 4.9]. Theorem 1.2 asks for a single radius on the whole of a weakly transverse X, including varieties that are geometrically degenerate in their smallest containing coset. For exact intersections the specialization argument of [17] then applies, and we deduce Theorem 1.1 in Section 6.

Proof strategy. We adapt the height-scale argument from the abelian Zilber–Pink manuscript [40, §§2–8] to reduce Theorem 1.2 to a statement about the normalized logarithms of a generic sequence. If all height scales diverge, a uniform lower bound for geometric degrees allows us to apply Dill's height inequality [23, Theorem 1.1]. If one scale is bounded, Theorem 2.1 gives a monomial map with large arithmetic self-intersection and small height at the marked point. A small section and Evertse's product theorem then cut one auxiliary factor over a field of bounded degree, contradicting its minimality.

2 An arithmetic intersection estimate for monomial maps

Use split tori over $\overline{\mathbb{Q}}$. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. On each factor $T_\ell = \mathbb{G}_m^{n_\ell}$ use the embedding by 1, its coordinate characters and their inverses, with hyperplane bundle A_ℓ. Write $h(x) = \sum_k h(x_k)$ for the sum of the absolute logarithmic heights. On the torus this height and the projective height

bound one another up to fixed multiplicative constants for the max metrics. For x complex, write $\operatorname{Log}|x| = (\log|x_k|)_k$ and denote its logarithmic tangent space by $T_x^{\log}Z = d\log(T_xZ)$. A real linear map on cocharacter spaces acts on complex tangents by complexification.

Let $Z_i = X \times \prod_{\ell=1}^{s} Z_{i\ell} \subset \prod_{\ell=0}^{s} T_\ell$ be geometrically irreducible products. The low factor $X \subset T_0$ is fixed, with $d_0 = \dim X > 0$. Set $d_\ell = \dim Z_{i\ell}$, assumed fixed, $d = \sum_{\ell=0}^{s} d_\ell$. Some high factors can be points. Use $H_{i0} = 1$ and $H_{i\ell} \geq 1$. Choose integers $M_i = b_{i0}$, $b_{i\ell} \geq 1$ with

$$\min_\ell b_{i\ell} \longrightarrow \infty, \qquad b_{i\ell}H_{i\ell}/M_i \longrightarrow 1, \qquad B_i = \prod_{\ell=0}^{s} b_{i\ell}^{d_\ell}.$$

For example, choose $M_i/\max_\ell H_{i\ell} \to \infty$ and round $M_i/H_{i\ell}$. Separation of consecutive scales will only be used in the product-theorem application. Put $D_i = \operatorname{diag}(b_{i\ell}I_{n_\ell})$ and $N_i = \sum_\ell b_{i\ell}A_\ell$ on the source. All intersection numbers of adelic bundles are absolute (normalized by field degree). For an integral matrix F, let $\bar{E}_F$ be the sum of the pullbacks of the canonical $\overline{\mathcal{O}}([0]+[\infty])$ from its output coordinates to its projective graph. On the torus the distinguished section in one coordinate f has Green function $|\log|f||$. This metrized bundle is twice the standard canonically metrized $\mathcal{O}(1)$, up to the change of frame by a character.

Theorem 2.1 (Mixed arithmetic estimate for an adapted map)**.** *Assume the following.*

1. *The degrees of the high factors are bounded. Each is a geometric component of a system of boundedly many equations of bounded degrees over a number field of bounded absolute degree. The coefficients in block ℓ have projective height $O(H_{i\ell})$, with uniform constants.*

2. *A fixed real matrix $\Phi : \bigoplus_\ell \mathbb{R}^{n_\ell} \to W$ and an integer $r \geq 2$ satisfy the following difference condition on a tail. At some r smooth points of Z_i the map*

$$(t_1, \ldots, t_r) \longmapsto (\Phi(t_2 - t_1), \ldots, \Phi(t_r - t_1))$$

is injective on the direct sum of their logarithmic tangent spaces.

After passing to a subsequence, there are integral monomial matrices G_i with a fixed number of rows and a fixed real matrix Λ such that $G_iD_i^{-1} \to \Lambda\Phi$. On the projective graphs one has

$$(\bar{E}_{G_i}^{d+1} \mid Z_i) \geq cM_iB_i, \qquad 0 < (E_{G_i}^{d} \mid Z_i) \leq CB_i. \tag{1}$$

One may begin with any subsequence. The field degree bound in the assumptions is on the equations defining the high factors.

2.1 Logarithmic tangent spaces and residues

Let U be a smooth dense open of an irreducible e-fold $Y \subset T$. Take a smooth projective compactification $\widetilde{Y}$ of U with simple normal crossings boundary. For a boundary component D put $v_D = (\operatorname{ord}_D x_k)_k$. At an e-fold intersection p of ordered components put $w_p = v_{D_1} \wedge \cdots \wedge v_{D_e}$. In dimension zero set $w_p = 1$.

Lemma 2.2. *In $\bigwedge^e \operatorname{Lie}(T)$ the complex span of the w_p equals the span of the wedges of bases of $T_y^{\log}Y$ at smooth points.*

Proof. Test the two spans against a constant complex alternating e-form α, viewed as an invariant logarithmic form. Its pullback ω is a global log top form on $\widetilde{Y}$. In SNC parameters t_j, write $x_k = u_k \prod_j t_j^{v_{jk}}$. The deepest iterated residue is $\langle \alpha, w_p \rangle$. If $\omega \neq 0$, logarithmic Hodge–de Rham

degeneration gives a nonzero class in $F^e H^e(U,\mathbb{C})$. This follows directly from $E_1^{e,0} = H^0(\Omega^e(\log D))$ and $F^{e+1}H^e = 0$. Its class is in the image of $H^e(T)$, which is pure of weight $2e$. Strictness makes that image meet $W_{2e-1}H^e(U)$ trivially. Hence the class survives in $\mathrm{Gr}^W_{2e} H^e(U)$. In the weight spectral sequence its top-weight piece embeds by residues in $H^0(D^{[e]})(-e)$, where $D^{[e]}$ is the disjoint union of deepest strata. Indeed $E_1^{-e,2e}$ has no incoming differential and the sequence degenerates at E_2. Thus some deepest residue is nonzero. The converse follows from the displayed local expression. Annihilators agree, proving the lemma. These mixed Hodge theory facts are given in Deligne and described in [33, §3]. □

Write $S = \mathrm{span}_{x\in X_{\mathrm{sm}}} T_x^{\log} X$. A high flag in a copy of Z_i gives a real subspace E spanned by the boundary valuations in their individual blocks. Let U_E denote $\Phi|_{\mathbb{R}^{n_0}}$ composed with $W \to W/\Phi E$.

Lemma 2.3. *One can choose a full flag in every high factor of one copy with the following properties:*

$$\dim \Phi E = \sum_{\ell>0} d_\ell, \qquad \mathrm{rank}(U_E|_{T_x^{\log} X}) = d_0 \text{ generically}, \qquad \dim_{\mathbb{C}} U_E(S) > d_0. \tag{2}$$

Proof. Choose a nonzero maximal minor witnessing the difference hypothesis. Keep the r low tangents P_a fixed. The minor is multilinear in the top wedge of each high factor of each copy individually. Substitute Lemma 2.2 in all these slots. A term in the expansion is nonzero. Its flags E_a keep the difference map injective on $\bigoplus_a (P_a \oplus E_a)_{\mathbb{C}}$. Thus Φ is injective on each summand and $\bigcap_a \Phi(P_a \oplus E_a) = 0$ because the kernel of the difference operator is diagonal. If $U_{E_a}(S) = U_{E_a}(P_a)$ for every a, then $\Phi(S)$ belongs to this common intersection. Yet $\Phi(S) \neq 0$ because $d_0 > 0$ and each $\Phi|_{P_a}$ is injective. At least one copy satisfies all of (2). The nonzero low minor there gives generic rank. □

2.2 Families of boundary flags and integral monomial maps

We use the coefficient height bound in Theorem 2.1 to control the leading units at the terminal points of a boundary flag.

For each high factor separately use the coefficient tuples as parameters of a fixed finite-type family. After a finite stratification and finite covers over each stratum, the geometric components admit smooth projective models of dense opens with relative SNC boundaries. To see this, over a generic point split the geometric generic components by a finite extension, resolve projective closures and principalize the complement of smooth opens in characteristic zero. Split the boundary components and intersections over a further finite extension. Spread out, shrinking to make the models smooth proper, the boundaries relative SNC, the lists of components exhaustive, and the rational inverse of each birational map defined over a fiberwise dense open. The fact that the geometric component list spreads follows by geometric integrality after shrinking and constructibility of the complement of the list in the family. Repeat on the closed complement by Noetherian induction. For a general stratification result see [24].

The deepest strata are finite sections after passing to these covers and shrinking. Near a section through p choose local equations $t_1, \dots, t_e$ for the labeled boundary components. They give

$$x_k = u_k \prod_{a=1}^{e} t_a^{v_{ak}}, \qquad h(u_k(p)) \ll 1 + h(\text{parameter}), \tag{3}$$

with the v_{ak} fixed on each piece. Indeed choose the equations at the generic point of the section, extend near it, and remove vertical bad loci. Each unit evaluated on the section is a fixed rational function of a lift of the coefficient tuple. It solves a polynomial with coefficients rational functions

of the base parameters and nonzero leading coefficient on a further open. The height bound for a root gives (3). The finite cover degree bounds the field degree of the lift, including geometrically zero-dimensional factors. Finitely many charts suffice, or one shrinks and repeats. Thus (3) holds at *every* available deepest flag, and the constants for block ℓ depend on its own $H_{i\ell}$.

After extraction, use Lemma 2.3 to choose flags with fixed incidence types and fixed valuation vectors in every high block. Work at the chosen complex embedding. All selected points and units are defined over a compositum of bounded degree. Write $u_{i\ell}$ for the tuple of terminal units in block ℓ (for a point block, its coordinates). The real vectors $H_{i\ell}^{-1}\operatorname{Log}|u_{i\ell}|$ are bounded by (3), since absolute field degree is bounded. Take limits ξ_ℓ along a subsequence. This extraction precedes the choice of the low rows. With $V = W/\Phi E$ and quotient π put

$$U = \pi\Phi_0, \qquad c = \pi \sum_{\ell>0} \Phi_\ell \xi_\ell \in V. \tag{4}$$

This gives a *single* displacement vector for all surviving rows, including those still to be chosen.

Concatenate the flags, giving high normals $v_1, \dots, v_e$ in their original blocks, $e = \sum_{\ell>0} d_\ell$. For step a choose a real row λ_a on W annihilating Φv_j for $j < a$ with $\lambda_a \Phi v_a = 1$; independence in (2) gives such rows. Later choose $d_0 + 1$ real rows on V and lift them to W through π. For each of these rows round $b_{i\ell}\lambda\Phi_\ell$ separately in each block to an integer row $g_{i\ell}(\lambda)$, *inside* the subspace annihilating every preceding normal in that block (all its normals for a low row). This subspace is rational in the input character coordinates. Its integral lattice has finite covering radius because the normals are now fixed. Every row error is $O(1)$. For the low block round $M_i\lambda\Phi_0$ without constraints. We have exact zero along every earlier normal, and along the detected normal in block ℓ we have order $b_{i\ell} + O(1)$. No rationality of ΦE in the output is required. A surviving low row λ on V then satisfies

$$\frac{1}{M_i} \sum_{\ell>0} g_{i\ell}(\lambda) \operatorname{Log}|u_{i\ell}| \longrightarrow \lambda c, \qquad M_i^{-1} g_{i0}(\lambda) \longrightarrow \lambda U. \tag{5}$$

The total error from rounding in the first limit is $O(\sum_{\ell>0} H_{i\ell}/M_i) = o(1)$.

2.3 Arithmetic restriction along a boundary flag

Set $d_0 = q$. On the selected high models order the prime divisors according to their normals. Inside their product with X let S_a denote the component of the intersection of the first a selected divisors containing the prescribed terminal stratum. Locally, at that stratum, the inclusions $S_a \subset S_{a-1}$ are given by successive SNC parameters. Use any projective model of X smooth on the low chart below.

For a row g write $f_g(x) = x^g$ and equip $L_g = f_g^*\mathcal{O}([0] + [\infty])$ with its canonical metric. Form the graph of all $d + 1$ rows over the chosen product of compactifications. Its intersection numbers agree with those on the original projective graph by the projection formula on a common proper birational model.

At step a use the graph over S_{a-1} of the remaining restricted characters. The local ring of S_{a-1} at the generic point of S_a is a DVR. Each character extends there as a map to $\mathbb{P}^1$, so the graph is isomorphic to the source over this generic point. In the SNC parameters the detecting character has order $g_a v_a$. Each later character has order zero there, since its exponent row annihilates every earlier normal. Its restriction to S_a is therefore a nonzero rational function.

Apply the arithmetic divisor formula to the distinguished section of $\bar{L}_{g_a}$ (see [26, Theorem 1.4.3] for the local induction formula). The prime over S_a occurs with multiplicity $|g_a v_a|$. Every other component of the effective divisor has nonnegative mixed height by nefness. The local integrals are nonnegative because the Green function is $|\log|f_{g_a}||$ and the remaining metrics are semipositive.

The detecting character is constant, equal to 0 or ∞, on the retained prime. Forgetting this output gives a proper birational map to the graph over S_a of the remaining characters, since both graphs have function field $k(S_a)$ over the chosen flag field k. The remaining bundles pull back under this map. The projection formula and induction on a give

$$(\bar{L}_{g_1} \cdots \bar{L}_{g_{d+1}} \mid Z_i) \;\geq\; \Big(\prod_{a=1}^{e} |g_a v_a|\Big)(\bar{L}'_0 \cdots \bar{L}'_q \mid X). \tag{6}$$

The primed bundles come from the functions $f'_{ij}(x) = a_{ij}x^{r_{ij}}$ on X, where $r_{ij} = g_{i0}(\lambda_j)$ and $a_{ij} = \prod_{\ell>0} u_{i\ell}^{g_{i\ell}(\lambda_j)}$. Their normalized exponents and normalized archimedean log constants have the limits in (5). The first factor in (6) is $\gg \prod_{\ell>0} b_{i\ell}^{d_\ell}$. This use of successive generic divisors permits indeterminacy of the monomial map at the deepest intersection. It also permits different graph resolutions at every index.

Lemma 2.4 (Translated logarithmic image)**.** *Let $U : \mathbb{R}^{n_0} \to V$ satisfy (2) for X. For every $c \in V$ there is a smooth point z of rank q such that*

$$U \operatorname{Log}|z| + c \notin U(T_z^{\log} X). \tag{7}$$

Proof. Suppose instead that the membership holds on the rank-q locus. On a local holomorphic chart write $w = U \log z$ and take any local holomorphic conormal row $a(z)$ of the image tangent $P_z = \operatorname{im} dw$. Then $a(z)(w + \overline{w} + 2c) = 0$. Antiholomorphic differentiation gives $a(z)\overline{dw} = 0$, so $\overline{P}_z \subset P_z$ and equality holds by dimension. The holomorphic Gauss map therefore takes values in the real Grassmannian. In a real Grassmann chart its holomorphic matrix coordinates are real, hence locally constant. Irreducibility gives constancy throughout the connected smooth rank-q locus. All tangent images there would be in one fixed q-plane, contrary to (2). □

Choose z as in (7) and put $v = U \operatorname{Log}|z| + c$. The real rows annihilating v span, after complexification and restriction, the dual of P_z because $P_z \cap \mathbb{C}v = 0$. Choose q real rows $\lambda_1, \ldots, \lambda_q$ from this annihilator giving a basis on P_z. Choose a real λ_0 with $\lambda_0 v \neq 0$. These are exactly the low rows to which (5) applies. In local branches of the logarithms of f'_{ij} on X, set $w_{ij} = M_i^{-1} \log f'_{ij}$ and subtract imaginary constants so that their imaginary parts vanish at z. For $1 \leq j \leq q$ their tuple converges locally in C^1 to holomorphic coordinates centered at z. Their real parts at z converge to zero by (5). On a smaller chart the image therefore contains a single fixed complex polydisk about 0 for every large i, with an inverse branch in the chart. For example, take coordinates in which the limit is the identity and apply the inverse function theorem with a uniform derivative bound on a smaller polydisk. Shrink the source chart also so that $M_i^{-1} |\log |f'_{i0}|| \geq \eta > 0$.

On that polydisk the mixed current $\bigwedge_{j=1}^{q} dd^c |\operatorname{Re} w_{ij}|$ is a fixed positive multiple of Lebesgue measure on the product of its purely imaginary coordinate intervals. This follows from the one-variable distributional second derivative of $|t|$, followed by exterior product in independent coordinates. Consequently

$$\int_{\text{chart}} |\log |f'_{i0}|| \bigwedge_{j=1}^{q} dd^c |\log |f'_{ij}|| \;\gg\; M_i^{q+1}. \tag{8}$$

The arithmetic divisor formula on the residual graph using the distinguished section for row 0 bounds its intersection below by the local integral times the place weight. Over a field K_i defining the graph and the selected terminal flag, the normalized weight of this archimedean place is at least $1/[K_i : \mathbb{Q}]$. Section 2.2 bounds these field degrees uniformly. Combining (6) and (8) proves

$$(\bar{E}_{G_i}^{d+1} \mid Z_i) \gg M_i B_i. \tag{9}$$

One may append all input coordinate characters to G_i, assigning them zero limiting rows in $\Lambda\Phi$. This ensures generic finiteness on the source. It preserves (9) by nefness and preserves $G_i D_i^{-1} \to \Lambda\Phi$. This construction works inside every subsequence of the original data, after extraction.

3 Geometric degree estimates

We give the degree bound also for unbounded-degree subproducts because this is the quantifier in Dill's height inequality [23]. Suppose $Y = \prod_\ell Y_\ell$ is a product in the fixed tori, $d = \sum e_\ell$ and $\mathcal{D} = \prod_\ell \deg Y_\ell$. Choose from the full flags of each factor a basis $\mathcal{F}_\ell$ of the Plücker span in Lemma 2.2.

Size of a flag wedge. If d integral characters have an invertible valuation matrix A at a deepest flag of a d-fold, their generic mapping degree to $\mathbb{G}_m^d$ is at least $|\det A|$. Indeed their local form is $u(t)t^A$ for a vector of holomorphic units. Using logarithms of these units and A^{-1}, choose a tuple of holomorphic units ν with $\nu^A = u$. The parameters $z_j = t_j\nu_j(t)$ are biholomorphic coordinates at the origin, and the map is z^A. Inside a sufficiently small product of punctured disks, kernel translates of any interior point still lie there and there are $|\det A|$ of them. The map is locally open there. Choose a generic image value avoiding the image of the complement of the birational open and the nongeneric fiber locus. All these preimages count towards the algebraic generic degree. Apply this to each nonzero coordinate minor of a flag wedge on Y_ℓ. Projection to that coordinate subtorus has degree $\le C \deg Y_\ell$, by intersecting with the coordinate fibers in the chosen projective embedding. Hence every integral coordinate in $\mathcal{F}_\ell$ is $O(\deg Y_\ell)$, on any SNC model.

For a real matrix g in fixed coordinates on source and target define

$$P_Y(g) = \sum_{|J|=d} \sum_{w_\ell \in \mathcal{F}_\ell} \left\langle e_J^*, (\wedge^d g)(w_0 \wedge \cdots \wedge w_s) \right\rangle^2. \tag{10}$$

Use the block inclusions in this wedge. This bounded-degree integer polynomial has coefficient size $O(\mathcal{D}^2)$ and a bounded number of summands. By multilinearity and Lemma 2.2, it is positive exactly when g is injective on some smooth product tangent. Let $\mathcal{B}$ be a fixed closed rational box on which $P_Y > 0$. For integral F and positive block weights b_ℓ, we claim

$$\min_{\mathcal{B}} P_Y \gg \mathcal{D}^{-C}, \qquad (E_F^d \mid Y) \gg \mathcal{D}^{-C} \prod_\ell b_\ell^{e_\ell} \quad (F \operatorname{diag}(b_\ell I)^{-1} \in \mathcal{B}). \tag{11}$$

For the first inequality apply real quantifier elimination to the graph of the minimum of a universal bounded-degree polynomial on $\mathcal{B}$. It produces finitely many polynomials in the value and the coefficients. At a singleton fiber, a nonzero specialized polynomial must vanish there, since otherwise signs are constant on a neighborhood. After clearing fixed denominators its integral coefficient size is $\mathcal{D}^{O(1)}$. Dividing out a power of the variable gives the elementary lower bound for its positive root. For $g = F \operatorname{diag}(b_\ell I)^{-1}$, one squared minor in (10) is $\gg \mathcal{D}^{-C}$ since the number of summands is bounded. On inserting the block weights its determinant acquires the factor $\prod_\ell b_\ell^{e_\ell}$. The corresponding output multidegree bounds the absolute determinant from above by the flag calculation and is at most E_F^d. This proves (11), after adjusting C. This is the toric analogue of the argument in [40, Lemma 4.1].

Upper degrees. A raw monomial output is a ratio of two homogeneous monomials of the same block degrees $O(b_{i\ell})$. By padding with the first homogeneous coordinate one gets, on its graph, a map from the output hyperplane bundle into a fixed multiple of N_i which is invertible on the torus. The map is defined by cross multiplication of the two globally generating output coordinates. Multiply these maps for all output rows and double for the symmetric bundle. Thus $tN_i - E_F$ is

effective on the graph for a fixed t. Both E_F and tN_i are nef. Telescoping top powers and restricting mixed nef classes to an effective divisor gives $(E_F + N_i)^d = O(N_i^d) = O(B_i)$ on the bounded-degree products. In particular it gives the upper bound in (1) for G_i. The appended input rows give positive geometric degree. This proves Theorem 2.1.

4 Separated height scales

Write $L_v(x) = (\log|x_j|_v)_j$ and $n_v = [K_v : \mathbb{Q}_{v|\mathbb{Q}}]/[K : \mathbb{Q}]$. Then $h(x) = \frac{1}{2}\sum_v n_v \|L_v(x)\|_1$. A fixed real matrix acts on these vectors with field-independent norm. For a finite nonempty ordered list of blocks, take in each a split torus T_ℓ, a nonzero subtorus W_ℓ, a fixed irreducible parent Y_ℓ over $\overline{\mathbb{Q}}$, marked points $y_{i\ell} \in Y_\ell(\overline{\mathbb{Q}})$, and a nonnegative integer bound m_ℓ for the local dimension at $y_{i\ell}$ of $Y_\ell \cap y_{i\ell}W_\ell$. Assume $H_{i\ell} \geq 1$ and $h(y_{i\ell}) = O(H_{i\ell})$, with successive ratios $H_{i\ell}/H_{i,\ell+1} \to 0$. Each high scale tends to infinity and $h(f(y_{i\ell}))/H_{i\ell}$ has positive liminf for every fixed integral homomorphism nonzero on W_ℓ. If present, the low scale equals one, $W_0 = T_0$, and its marks eventually avoid each fixed proper torsion coset. Write $V_* \subset \bigoplus \mathrm{Lie}_{\mathbb{R}} W_\ell$ for a real plane, Φ for the orthogonal projector on its complement in the *full* ambient space. Here $\mathrm{Lie}_{\mathbb{R}}$ denotes the real cocharacter space. Ranks on logarithmic tangents are taken after complexification.

Proposition 4.1 (Exclusion of separated height scales)**.** *Under the preceding hypotheses, the following two conditions cannot both hold:*

$$\operatorname{codim}_{\oplus \mathrm{Lie}\, W_\ell} V_* > \sum_\ell m_\ell, \qquad \sum_v n_v \left\|\Phi\big((L_v(y_{i\ell})/H_{i\ell})_\ell\big)\right\|_1 \longrightarrow 0. \tag{12}$$

We first exclude (12) for all-high data in every dimension. We then exclude it for data with a low block, using the all-high conclusion whenever the low block drops out. We may choose a counterexample with least total ambient dimension within the stage under consideration. We explain first why (12) follows from failure of Theorem 1.2.

4.1 Reduction to a sequence of separated scales

For $n = 0$ the unlikely locus is empty. For $n > 0$, if every positive neighborhood in Theorem 1.2 meets X densely, enumerate the proper closed subsets of X over $\overline{\mathbb{Q}}$. Choose $x_i = z_i e_i$ outside the first i subsets with $h(e_i) \leq 1/i$ and z_i on a torsion coset of codimension $> d$. This gives a generic sequence in the sense that each proper closed subset is eventually avoided.

If $H_i = h(x_i) \to \infty$, define

$$q_i(a) = \frac{1}{2H_i}\sum_v n_v |\langle a, L_v(x_i)\rangle|, \qquad a \in \mathbb{R}^n.$$

These real seminorms are uniformly 1-Lipschitz for the maximum norm on their domain. A subsequence converges locally uniformly to a seminorm q, with $\sum_j q(e_j) = 1$. In particular

$$M_{\mathrm{slow}} = \mathbb{Z}^n \cap \ker q$$

is a saturated sublattice of rank strictly smaller than n. Complete a lattice basis of it to a unimodular character basis, giving torus coordinates $x_i = (x_i^-, x_i^+)$. Then $h(x_i^-) = o(H_i)$. For any fixed integral character (a^-, a^+) with $a^+ \neq 0$ one has $h(x_i^a)/H_i \to q(a) > 0$, interpreting a in the chosen coordinates. This includes characters on the high block combined with fixed characters on the slow block. The same positive-liminf property follows for a fixed homomorphism nonzero

on the high block by choosing one output character which sees that block. Passing to a further subsequence preserves all of these assertions. Iterate on x_i^- if its height is unbounded, after extracting a subsequence on which it diverges. If the slow lattice has rank zero, stop with no low block. If its rank is positive and the slow height is bounded, this furnishes the unique low block with scale one. The descent in dimension is strict, every high scale tends to infinity, and successive scales in increasing order have ratio tending to zero. Only the integral kernel is used: the real kernel of a limiting seminorm may be irrational.

For a Zariski-generic sequence on a weakly transverse irreducible X, enlarge the ℓth block to the tuple of all coordinate blocks through block ℓ and let Y_ℓ be the closure of its projection. The distinguished W_ℓ is the last torus factor. Set $m_\ell = \dim Y_\ell - \dim Y_{\ell-1}$, so $\sum m_\ell = \dim X$. Genericity gives the required local W_ℓ-fiber bound at the marked tuples. The low projected points eventually avoid every fixed proper torsion coset of the low torus, by pulling it back to the original ambient torus. If $x_i = z_i e_i$ and z_i lies in a torsion coset of direction C_i and codimension $> \dim X$, then $L_v(z_i) \in \mathrm{Lie}_{\mathbb{R}} C_i$ at all places, regardless of the field of definition. Take a subsequence of constant codimension and a limit V of $\mathrm{diag}(1/H_{i\ell}) \mathrm{Lie}_{\mathbb{R}} C_i$. If $h(e_i) \to 0$, projecting the original normalized x_i-logs to the orthogonal complement of the scaled C_i has averaged norm at most $Ch(e_i)$, since $\min H_{i\ell} \geq 1$ (the fixed unimodular coordinate change costs a constant). Their averaged total norms are bounded. Operator-norm convergence of projectors gives vanishing averaged projection to $V^\perp$. In the enlarged data, put V into the distinguished blocks: the earlier copied coordinates tend to zero in averaged norm relative to the later scale. This proves (12), including the copied coordinates. If all initial heights are bounded one starts with a single low block. In that case the same Grassmannian argument applies with scale one.

4.2 Quotients and logarithmic tangent ranks

In dealing first with all-high sequences, and then with those having a low block, use least full ambient dimension for a counterexample to (12). For any fixed nonzero product direction $C = \prod_\ell C_\ell$ the inequalities

$$a_\ell = \dim_{y_{i\ell}} (Y_\ell \cap y_{i\ell} C_\ell), \qquad \sum_\ell a_\ell \geq \mathrm{rank}(\Phi|_{\mathrm{Lie}\, C}) \tag{13}$$

cannot hold infinitely often. Indeed, after fixing a_ℓ, quotient by C_ℓ and put $t_\ell = \dim(C_\ell W_\ell / W_\ell)$. The new local bounds are $m_\ell + t_\ell - a_\ell$. For the geometric justification restrict Y_ℓ to the locally closed locus where the W_ℓ-fiber dimension is at most m_ℓ and the C_ℓ-fiber dimension is at least a_ℓ. Over $T_\ell/(C_\ell W_\ell)$ its fibers have dimension at most $m_\ell + t_\ell$ and each nonempty fiber to T_ℓ/C_ℓ has dimension at least a_ℓ. Stratifying the constructible image and taking one stratum containing a subsequence gives the new parents, with the asserted local bounds. More explicitly, write Q for the constructible image and t for a point of $T_\ell/(C_\ell W_\ell)$. The fiber dimension theorem bounds $\dim Q_t$ by $m_\ell + t_\ell - a_\ell$ whenever this fiber is nonempty. Each irreducible locally closed stratum is open in its own closure, so that closure has the same fiber germ at a selected mark. The codimension of the image of V_* in the distinguished quotient decreases by $\mathrm{rank}(\Phi|_{\mathrm{Lie}\, C}) - \sum t_\ell$. This preserves the strict inequality in (12) under (13). Small projection is preserved because the new projector composed with the quotient kills V_*. If a distinguished quotient factor becomes zero, the new exceptional plane has zero coordinate there. Removing it preserves the codimension and can only reduce the sum of the nonnegative bounds. At least one factor survives by strictness. On surviving high factors every test homomorphism nonzero on the distinguished quotient pulls back to one nonzero on the original distinguished factor. On a surviving low factor a proper torsion coset pulls back to a proper torsion coset. These observations preserve all the sequence conditions under projection, and a nontrivial quotient reduces the ambient dimension. This proves the exclusion by minimality or the previous

all-high stage. The fiber-dimension argument uses constructible images, as in [40, Lemma 3.1]: the component of an original C_ℓ-fiber of dimension at least a_ℓ through any point of the locally closed locus stays in the closed high-fiber locus, and its intersection with the open low-fiber locus is dense in that component.

If a complex g fails to be injective on every log tangent of a product Z, the proof of [40, Lemma 3.2] gives a product subtorus with a strict fiber surplus over the g-rank. Its functional-transcendence step on a torus can be seen directly. Scale the factor blocks generically so that on every rational subspace arising from a subtorus the scaled map has the largest rank achievable by such scalings (there are countably many subtori). Call the map h. In a logarithmic chart choose a very general smooth point where the restriction of h to the logarithmic image of Z has maximal differential rank. Let J be its fiber germ there, which is smooth and positive-dimensional. Put $U_1 = \overline{\exp J}^{\mathrm{Zar}}$ and let the smallest coset containing U_1 have direction C^*. In coordinates on C^* the logarithms have no nonzero rational relation modulo constants. Ax's differential theorem [6] gives

$$\dim J + \dim C^* \le \dim \overline{(J, \exp J)}^{\mathrm{Zar}} \le \dim C^* - \operatorname{rank}(h|_{\operatorname{Lie} C^*}) + \dim U_1.$$

In the middle term we take the algebraic closure of the graph restricted to J. The tangent intersection with $\operatorname{Lie} C^*$ at the selected point therefore exceeds its h-rank. Degenerate $\operatorname{Lie} C^*$ by successively weighted block scalings to the direct sum of the graded pieces of its coordinate filtration. Explicitly, for $S = \operatorname{Lie} C^*$ and $F_j = \bigoplus_{k \ge j} \operatorname{Lie} T_k$, the limiting space is $S_0 = \bigoplus_j \operatorname{pr}_j(S \cap F_j)$, as follows from a rational basis adapted to the filtration. This is a rational product direction. The product tangent is invariant under those scalings, so the tangent-intersection dimension can only increase in the limit. The generic-maximal choice of h bounds the rank of the original g on the limiting product direction by the preceding h-rank. At a very general point the rank of every rational quotient on Z is generic. Thus the excess gives an excess in fiber dimension, also at the marked point by upper semicontinuity.

For clarity about the finite sets, let D_ℓ be a maximal-dimensional fiber component through the mark, with smallest ordinary coset of direction $B_\ell \subseteq C_\ell$. Take an enlargement $E_\ell \subseteq Y_\ell$ of maximum dimension among enlargements of no greater ordinary-coset defect, and write H_ℓ for its ordinary direction. This makes E_ℓ weakly optimal. Since $B_\ell \subseteq H_\ell$, we have

$$\dim E_\ell - \dim D_\ell \ge \dim H_\ell - \dim B_\ell, \qquad \operatorname{rank}(g|_{\oplus \operatorname{Lie} H_\ell}) - \operatorname{rank}(g|_{\oplus \operatorname{Lie} B_\ell}) \le \sum_\ell (\dim H_\ell - \dim B_\ell).$$

Thus any strict (or weak) fiber surplus over the rank for C persists in the parents with direction $\prod H_\ell$. If the original sum of fiber dimensions is positive then one H_ℓ is nonzero. Barroero–Dill's semiabelian finite-directions theorem supplies a *fixed finite set* of such directions per parent [7, Theorem 9.2]. For each nonzero product direction H on these finite lists, (13) implies that on a tail the sum of the local H_ℓ-fiber dimensions in the parents is strictly smaller than $\operatorname{rank}(\Phi|_{\operatorname{Lie} H})$. A nonzero minor gives a neighborhood of Φ on which $\operatorname{rank}(g|_{\operatorname{Lie} H}) \ge \operatorname{rank}(\Phi|_{\operatorname{Lie} H})$ (for rank zero this holds everywhere). Intersecting the finitely many tails and neighborhoods yields one tail and one neighborhood for every marked subproduct, regardless of degree. A failure of tangent injectivity would give a strict surplus for one of these directions, contradicting the inequality on that tail.

For the difference assertion use $r = 1 + \sum n_\ell$. A failing difference test gives product directions in the r copies by the same fiber-surplus argument. Let C_a be the product direction in copy a, $J_a = \Phi(\operatorname{Lie} C_a)$ and A_a its sum of local fiber dimensions. Then

$$\sum_a A_a > \sum_a \dim J_a - \dim \bigcap_a J_a.$$

For some nonzero C_a one has $A_a \geq \dim J_a$. If some C_a vanishes, the intersection term is zero and there is a strict surplus in another copy. If every direction is nonzero, strict deficits in all r copies would sum to at least r. For the selected nonzero direction, if $A_a = 0$, the direction lies in $\ker \Phi$ and is itself a fixed obstruction to (13) at every index (choose it from one such index). If $A_a > 0$, replacement by a weakly optimal direction retains a nonzero direction and the weak inequality. The finite sets and (13) give the contradiction. This checks the difference hypothesis of Theorem 2.1 with the full low factor still present.

4.3 Divergent height scales

Take a compact rational box $\mathcal{B}$ about Φ in the neighborhood obtained in Section 4.2. Each marked subproduct, of any degree, therefore satisfies (11) for every normalized map in the box. In the notation of Dill's theorem this gives fixed θ, ω such that

$$(E^{\dim Z} \mid Z) \geq \theta^{-1} \prod_{\ell} \Big((\deg Z_\ell)^{-\omega} a_\ell^{\dim Z_\ell} \Big).$$

Let m be the number of high blocks, and let $W_{\ell j}$ denote the $N_\ell + 1$ homogeneous coordinates of the fixed embedding of parent ℓ. Here E uses one copy of each ordinary output hyperplane bundle. Each parent has positive dimension, since its marked heights diverge. We check the remaining hypotheses of [23, Theorem 1.1]. For the original fixed parents let $\mathcal{X}$ be the graph closure of F, U^0 their product torus open, $a_\ell \sim M/H_{i\ell}$, and $N_a = \sum a_\ell A_\ell$ on the source and by pullback on $\mathcal{X}$. Each row f_j can be expressed in $[D_j : N_j]$ with both expressions monomials of block degrees exactly ta_ℓ for an integer t independent of row and index. Use the signed input coordinates and the zeroth coordinate to homogenize and pad. The graph relations define $\mathrm{pr}_j^* \mathcal{O}(1) \hookrightarrow tN_a$ by D_j/Y_{j0} or N_j/Y_{j1} on the corresponding target affine charts; these agree wherever both are defined. The inclusion is invertible over U^0. With k the fixed number of rows put $P = N_a + E$. This is very ample on the graph by its product Segre–Veronese embedding. Under the product of the preceding inclusions the generating coordinates Ξ of this embedding become monomials of multidegrees $(1 + kt)a_\ell$ on the source. Dill permits the cardinality of Ξ to grow. The bundle $P - E = N_a$ is generated by the *fixed* number $\prod_\ell (N_\ell + 1)$ of coordinates $\prod_\ell W_{\ell,j_\ell}^{a_\ell}$. They have coefficients in $\{0, 1\}$ and need no further injection denominator. Thus $t_1 = 1 + kt$, $t_2 = 1$, the generator count and $\delta_\ell = 1$ are fixed. At each place the norm of the full source Veronese coordinate tuple equals that of the preceding pure powers. The Segre coordinate sup norm is multiplicative. Consequently the difference $h(\Xi(x)) - h(\mathcal{Z}(x))$ in Dill is exactly the sum of the Weil heights of the monomial output coordinates. Every constant in Dill, including each threshold on $h(W_\ell(x_\ell))$, depends only on the fixed parent varieties, fixed coordinate system, $\mathcal{B}$ and ambient dimensions. These constants are independent of the field degrees of the marks.

Choose $M/\max H_{i\ell} \to \infty$, round each a_ℓ, and round $a_\ell \Phi_\ell$ entrywise to F_ℓ. Then $F \operatorname{diag}(a_\ell I)^{-1}$ eventually lies in $\mathcal{B}$, the input weighted height is $\gg M$, and the output height is $o(M)$ from the small-projection hypothesis and the $O(\sum H_{i\ell}) = o(M)$ rounding loss. For $m \geq 2$ all input thresholds and successive degree ratios of Dill's Theorem 1.1 now hold. This contradicts his inequality.

For $m = 1$ a dummy factor gives a direct application with $m' = 2$. Adjoin the fixed parent $\mathbb{P}^1$ with open $\mathbb{G}_m$, weight 1 and the identity output, and mark it at $2^{\lfloor \sqrt{M} \rfloor}$. Its height tends to infinity but is $o(M)$. Every marked closed subvariety in this factor is either the marked point or $\mathbb{P}^1$. In the former case the numerical inequality is unchanged. In the latter it is multiplied by the positive binomial factor for $(E \boxtimes \mathcal{O}(1))^{d+1}$. The graph, coordinate, and height conditions checked above are preserved; $a_1 \to \infty$ supplies the required degree separation. The theorem contradicts the same two height asymptotics without any duplication of genuine sequences.

5 A bounded height scale

We now take a least-ambient-dimension counterexample with a low block. We give a bounded-degree choice that uses only point heights and ordinary algebraic geometry. Embed each factor torus in the affine chart $[1 : x_1 : \cdots : x_n : x_1^{-1} : \cdots : x_n^{-1}] \subset \mathbb{P}^{2n}$, so that the torus is closed in the affine chart. Include the coordinate–inverse equations whenever an ideal is considered there.

5.1 Equations for the auxiliary factors

Lemma 5.1 (Reduced equations for components)**.** *Fix the number of affine variables, a bound for the number of input equations, and a bound for their degrees. For systems with coefficients in a number field k and projective coefficient heights $\leq H$ with $H \geq 1$, the following are uniform. Every geometric irreducible component W of the reduced zero set has, over an extension of k of bounded relative degree, defining equations of bounded number and degree and coefficient heights $O(H)$ which* generate its reduced ideal in the affine chart*. Its projective closure has bounded geometric degree. The bounds depend only on the fixed numbers and degrees.*

Proof. Use the product of projective coefficient spaces as the parameter variety, treating the finitely many zero/degree patterns separately. Over the generic point of an irreducible stratum split the geometric components after a finite extension of the function field. Take the projective closures of the generic affine components, choose generators of their homogeneous ideals, and spread these generators, on the normalization in that finite extension after shrinking. Each resulting projective family has geometrically integral generic fiber. By generic flatness and openness (or constructibility followed by shrinking) of geometric integrality it has flat, geometrically integral fibers over a further nonempty open. In particular the dehomogenized specialized generators generate the reduced prime ideal of each affine chart fiber where it is nonempty. The affine zero set equals the union of these chart pieces over the generic point. The two set differences are constructible, and their images miss the generic point; shrinking ensures equality on all fibers. Likewise shrink to retain nonempty chart fibers for the components. If some become nested or coincide, every irreducible component of their union is still one of the listed geometrically integral varieties. Replacing the base by an open, the cover is finite and surjective. To remove an exceptional locus on the cover shrink away its image, which is proper closed. Noetherian induction on the complement gives a finite collection of strata, finite covers, fixed projective coordinates for the generators and finitely many templates. All these data descend to one fixed number field. Each coefficient on the cover is a fixed rational function and satisfies a polynomial over the base function field. On suitable strata/affine charts its leading coefficient is nonzero. The specialized polynomial coefficient heights are $O(1 + h(\tau))$ by heights of evaluation of fixed rational functions at the input tuple τ. The root height inequality gives the same bound for every lifted coefficient. The degrees of the covers and the fixed number field bound the relative field degree of the chosen lifts. Geometric degrees are bounded by the fixed templates or by ordinary Bézout.

□

5.2 A minimal product through the marked points

Along subsequences call a product $Z_i = \prod Z_{i\ell}$ admissible if each factor is geometrically irreducible, contains its marked point inside its parent, has bounded degree, and has reduced affine ideal generators in bounded number and degrees, over fields of bounded absolute degree, of coefficient height $O(H_{i\ell})$. Take the compositum of the defining fields from the finitely many factors and of a fixed field of definition of the parents. The fixed parents are themselves admissible. Minimize

the total dimension over all subsequences and such choices and then fix individual dimensions on a subsequence.

For any fixed D, B, on a tail an equation of degree $\le D$ through a marked factor, over a relative degree $\le B$ extension of the common field, cannot cut its $Z_{i\ell}$ properly. Suppose otherwise. Fix the factor on a subsequence and let k_i' be the field of the offending equation. Consider the evaluation rows of all monomials of degree at most D on all conjugates of the marked point over k_i' in the fixed affine chart. Each resulting projective row has height $O_D(H_{i\ell})$. Select a maximal independent set; its size is bounded by the fixed number of monomials. Gaussian elimination gives kernel vectors with prescribed unit free coordinates, of coefficient heights $O_D(H_{i\ell})$ by minors. Their uniqueness and Galois invariance of the kernel put them in k_i'. The given cutting equation lies in this kernel, so at least one kernel basis element still cuts $Z_{i\ell}$ properly. Adjoin that small equation to the generators of $Z_{i\ell}$. Lemma 5.1 gives a geometric component through the mark with all the admissible bounds and strictly smaller dimension. This contradicts the minimum.

Projective closures admit the finite projections required later. Smoothness, conormal equations and all local estimates can be checked in the fixed affine chart. Over characteristic zero the generator differentials span the conormal at all smooth points. If infinitely many marks were singular choose a Jacobian minor of size the codimension, nonzero at a generic smooth point and vanishing at the singular mark. It is a bounded-degree proper cut over the common field, contradicting the previous paragraph. The case of codimension zero is smooth. Finally, at the low scale the number, degrees and coefficient heights of the reduced ideal generators are bounded, as are their degrees over $\mathbb{Q}$. Northcott gives a finite list of low varieties, so the low factor becomes fixed after extraction. No dense set of height-controlled points on the high factors has been used.

A fixed low point. For removal of a fixed zero-dimensional low factor let $p \in \mathbb{G}_m^a(\overline{\mathbb{Q}})$ be its mark. It avoids every proper torsion coset, precisely multiplicative independence of its coordinates modulo torsion. Put all of them into one S-unit group. Dirichlet gives

$$\frac{1}{2}\sum_v n_v|\ell \cdot L_v(p)| \ \ge c_p\|\ell\|, \qquad \ell \in \mathbb{R}^a$$

by real-linear independence of the relevant logarithmic vectors and compactness of the unit sphere. Thus if the exceptional plane $V \subset L \oplus H$ does not project onto the full low space $L = \mathbb{R}^a$, a nonzero low row kills V yet gives a positive constant at the mark, contradicting the small projected norm. If there are no high blocks, surjectivity contradicts the positive codimension in (12). Write $b = \dim H$ for the dimension of the distinguished high sum, $\gamma = \operatorname{codim}_{L\oplus H} V$, $t = \dim(V \cap L)$. Upon taking k copies with a common L coordinate the fiber product has dimension $a + k(b - \gamma)$ by surjectivity. Its image in H^k has kernel dimension t and hence codimension $k\gamma - a + t$. Choose fixed k so that this exceeds $k\sum_{\ell>0} m_\ell$ (possible because $\gamma > \sum_\ell m_\ell$). Choose k subsequences of the high factors separated by ratios tending to infinity from the fastest scale of the preceding copy to the slowest scale of the next one. At every place of a common extension use $L_v(p)$ as the shared low coordinate; the full linear map sending $(l, h_1, \dots, h_k)$ to $(\Phi(l, h_j))_j$ has a bounded inverse on a complement of its kernel. Project away l and sum over places to see that the high tuple approaches the new exceptional plane in averaged norm. The projected plane lies in the distinguished H^k even though the marks may include extra ambient high coordinates. Equivalently take the map with each h_j in the full ambient high space; its kernel still has all h_j in H since $\ker \Phi = V$. The new all-high data contradict the all-high sequence theorem.

5.3 Small sections and the product theorem

We now close the argument for a positive-dimensional low factor. Use Theorem 2.1 with the auxiliary product of minimum dimension. Choose $M_i/\max_\ell H_{i\ell} \to \infty$ and $b_{i\ell}$ as before. By the mean-norm bound in (12), the placewise triangle inequality and $b_{i\ell}H_{i\ell}/M_i \to 1$, the adapted sequence satisfies

$$h_{\bar{E}_{G_i}}(y_i) = o(M_i), \qquad (\bar{E}_{G_i}^{d+1} \mid Z_i) \gg M_i B_i, \qquad 0 < (E_{G_i}^{d} \mid Z_i) \ll B_i. \tag{14}$$

Proposition 5.2 (A bounded-degree cut over a bounded extension)**.** *For the admissible products and marks above satisfying (14), there are constants B, D independent of i such that, for all sufficiently large i in the extracted sequence, a hypersurface of degree at most D through one marked factor cuts that factor properly. Its equation is defined over an extension of k_i of degree at most B, where k_i is the common field of definition of the factors.*

Proof. Existence over the base field. Abbreviate $F = G_i$, $\bar{E} = \bar{E}_{G_i}$, $M = M_i$, $b_\ell = b_{i\ell}$ and $N_i = \sum b_\ell A_\ell$. All bounds in (14) are absolute. The arithmetic degree can be computed over the field of a selected boundary flag. By base change and the projection formula it has the same value on the original graph over k_i. Take a projective resolution of the graph over k_i isomorphic above its smooth locus. Choose $a, A > 0$ uniform in (14) so that $(\bar{E}^{d+1}) \geq aMB_i$ and $(E^d) \leq AB_i$. These bounds and the arithmetic Siu inequality for nef bundles [50, Theorem 5.2.2(2)] give positive arithmetic volume for $\bar{E} - cM\overline{\mathbf{1}}$, with fixed $0 < c < a/((d+1)A)$, using a rational shift slightly smaller if needed. Consequently for some positive k there is a nonzero section s of kE over k_i of sup norm at most $\exp(-ckM)$ at each infinite place, at most one at every finite place, decreasing c once uniformly. Evaluate only after this choice over $L = k_i(y)$. Final norms and index loci use k_i itself.

The torus charts. For each factor, choose the same independent conormal equations and invertible dependent-coordinate Jacobian minor at y_ℓ for all places of L. Translate affine coordinates to the mark and invert the minor. The equations become

$$V + AT + Q(T, V) = 0$$

where each monomial of Q has total degree at least two. All coefficients are ratios of polynomials of fixed degree in the point coordinates and the finitely many equation coefficients, after normalizing a nonzero coefficient of each equation to one. Their heights are $O(H_\ell)$. At a finite place set S_w equal to the maximum of 1 and their moduli. At infinity multiply this maximum by C/δ, where $0 < \delta < 1/4$ and C depends only on the number of variables, degrees and monomials. Take $\rho_w = S_w^{-3}$. Then $\sum_w n_w \log(\rho_w^{-1}) \ll_\delta H_\ell$. On a closed parameter disk of radius $r < \rho_w$ the map $V \mapsto -AT - Q(T, V)$ is a contraction on the ball $\|V\| \leq S_w r$ (increase C at infinity). Indeed, at finite places the linear term has norm at most $S_w r$ and the Lipschitz constant is at most $S_w^2 r < 1$. At infinite places the fixed combinatorial constants are absorbed by C. Use bounded analytic functions on this disk with the supremum norm, or the corresponding Tate algebra at a finite place. The solutions on smaller disks agree, giving analytic functions on $\|T\| < \rho_w$. All coordinate displacements are < 1 at finite places and $< \delta$ at infinite places. The normalized Jacobian remains invertible. The branch lies in Z_ℓ: the chosen equations define its smooth germ at the mark, so every equation of Z_ℓ vanishes on the implicit formal series with coefficients in L, hence on the convergent analytic branch at each place. The branch belongs to the smooth locus wherever it is used since its Jacobian minor remains invertible and dimensions agree.

The torus coordinates and their inverses are both affine coordinates. For $|y_j|_w \geq 1$ apply the displacement bound to x_j; for $|y_j|_w < 1$ apply it to x_j^{-1}. Thus $\log|x_j|_w - \log|y_j|_w$ is zero at finite places and is $O(\delta)$ at infinite places. The global torus frame e given by the product of the distinguished output sections has weight $-\log\|e\|_w = \sum_j |\log|F_j|_w|$. It varies on the disks by at

most $C'\delta M$ at infinity and by zero elsewhere. In the completion at y, write $s/e^k = \sum_\alpha c_\alpha T^\alpha$ with $c_\alpha \in L$. This is the same algebraic expansion at every place. For any $c_\alpha \neq 0$, Cauchy on the archimedean polydisks and the Gauss norm on the nonarchimedean disks, followed by the product formula, give

$$ckM \leq kh_{\bar{E}}(y) + C'\delta kM + C_\delta \sum_\ell |\alpha_\ell| H_\ell.$$

Open disks suffice by letting their radii grow to ρ_w. Fix δ small, then i large. Every such α has $\sum_\ell |\alpha_\ell|/(kb_\ell) \geq \eta > 0$. A zero-dimensional block supplies no local parameters.

Finite norm. In each output row, after monomial homogenization and padding, write the projective coordinates as two forms of common block degrees $t_j b_\ell$. On the graph their cross relation gives a homomorphism from output $\mathcal{O}(1)$ to the indicated input bundle, invertible over the torus. Tensor the row homomorphisms and square, padding again if necessary. We obtain $E \hookrightarrow tN$, nonvanishing on the torus, $N = \sum_\ell b_\ell A_\ell$, t fixed. The resulting section of ktN descends to the normalization of the source since the proper birational resolution has pushforward structure sheaf equal to the normal structure sheaf. The index at the marked point is unchanged. Choose finite linear projections from each projective factor to $\mathbb{P}^{d_\ell}$, étale at y_ℓ. A generic center works simultaneously at its finitely many conjugates; the nonempty open condition has a k_i-point since k_i is infinite. The product projection, followed in the source by normalization, is finite of degree $e_i = \prod_\ell \deg Z_\ell$, uniformly bounded, and its pullback of the target line bundle of degree ktb_ℓ on each block is ktN.

The field norm of the section is a nonzero polynomial G on the target of degrees $D_\ell = e_i ktb_\ell$, defined over k_i. On an affine trivialization every characteristic coefficient is integral and in the function field of the smooth target, hence regular. Evaluation of the characteristic polynomial at the section shows that the pullback of its norm is divisible by the original section in the local ring. At the marked smooth branch the completions over $\overline{\mathbb{Q}}$ are identified factorwise since the projection is étale factorwise. Thus the target index with weights $1/D_\ell$ is $\geq \eta/(e_i t)$. This uses no flatness of the finite morphism at other branches.

One uniform arithmetic product theorem. Omit point factors, write m for the number of surviving factors and $N_* = \sum d_\ell$. Below the fixed positive index choose $N_* + 2$ positive irrational thresholds in arithmetic progression with fixed rational spacing $\epsilon \leq 1$. Index loci are defined by homogeneous derivatives of weighted order less than the threshold. There is a nested chain of irreducible components through the mark by extending components of the smaller loci to components of the larger loci. They are all proper since $G \neq 0$. Some adjacent components coincide by dimension. Evertse's product theorem [25, Theorem 1, (1.1)–(1.3)] applies for $m \geq 2$, once every $D_\ell/D_{\ell+1} \geq (mN_*/\epsilon)^{N_*}$. For this common component $J = \prod J_\ell$, of codimension c_J, it gives directly

$$[k' : k_i] \prod_\ell \deg J_\ell \leq (mc_J/\epsilon)^{c_J},$$

with all factors defined over the same field k'.

If $m = 1$, relabel the surviving factor as 1. Use $G^*(X, Y) = G(X)Y_0$ on $\mathbb{P}^{d_1} \times \mathbb{P}^1$, of bidegree $(D_1, 1)$. On $Y_0 \neq 0$ the weighted index of G^* equals that of G. Hence the closure $J_1 \times \mathbb{P}^1$ of $J_1 \times \mathbb{A}^1$ is a common component of the two index loci of G^*, where J_1 is the common component for G. Apply Evertse's theorem with two blocks and total ambient dimension $d_1 + 1$. The degree-ratio hypothesis holds for large i since $D_1 \to \infty$. Explicitly, the hypothesis and the resulting bound are

$$D_1 \geq \left(\frac{2(d_1+1)}{\epsilon}\right)^{d_1+1}, \qquad [k' : k_i] \deg J_1 \leq \left(\frac{2c_J}{\epsilon}\right)^{c_J},$$

where $c_J = \operatorname{codim}_{\mathbb{P}^{d_1}} J_1$ and the added factor has degree one.

In either case choose a proper factor J_ℓ. Both $\deg J_\ell$ and $[k' : k_i]$ are bounded independently of i. There is a hypersurface over k' of degree at most $\deg J_\ell$ containing J_ℓ. Indeed a generic k'-linear projection of J_ℓ to projective space of dimension $1 + \dim J_\ell$ has hypersurface image with degree at most its source degree, and its pulled-back equation suffices. Pull back along the factor projection of Z_ℓ, which is surjective. This gives the asserted proper cut through y_ℓ. □

Conclusion of the arithmetic proof. The degree of the hypersurface and the degree of its field of definition over k_i are bounded independently of i. Its equation therefore gives the proper cut excluded by the minimality argument in Section 5. This excludes the case of a positive-dimensional bounded factor. The fixed-point argument of Section 5 and the divergent-scale argument of Section 4.3 cover the other cases. Proposition 4.1 follows, and the reduction in Section 4 proves Theorem 1.2. □

6 From arithmetic non-density to optimal subvarieties

6.1 Exact non-density over the complex numbers

We apply the transfer theorem of Bombieri, Masser and Zannier [17, p. 310, Theorem]. Its hypothesis over $\overline{\mathbb{Q}}$ is that, for every irreducible r-fold $Y \subseteq T$ of positive codimension, $Y \cap T^{[r+1]}$ lies in a finite union of proper algebraic subgroups of T.

To obtain this formulation from Theorem 1.2, fix T and induct on r. If Y lies in a proper torsion coset, the subgroup generated by that coset is a finite union of translates of its direction and is proper. Otherwise Theorem 1.2, with multiplicative error equal to the identity, puts $Y \cap T^{[r+1]}$ in a proper closed subset of Y. For $r = 0$ that subset is empty. For $r > 0$, each irreducible component Z of the subset has dimension $e < r$, and $T^{[r+1]} \subseteq T^{[e+1]}$. Apply induction to Z and take the union of the resulting finite collections of subgroups.

The hypothesis of the transfer theorem now holds in every dimension and codimension, hence on each of its finite rectangles. The theorem gives the same finite-subgroup containment over $\mathbb{C}$. If Y is weakly transverse, every proper algebraic subgroup meets it in a proper closed subset. Thus $Y \cap T^{[\dim Y+1]}$ is non-dense over $\mathbb{C}$. For $Y = T$ this follows directly since $T^{[\dim T+1]}$ is empty.

6.2 Optimal subvarieties

Proof of Theorem 1.1. Work over an algebraically closed field where exact toric non-density holds in every dimension. Let Y be irreducible, write $S = \langle Y \rangle$, and take a proper optimal $A \subsetneq Y$. Write $w = \dim A$, $k = \dim\langle A \rangle$, $s = \dim S$, and $d = \dim Y$. Then

$$(s - k) + w > d \tag{15}$$

because $\delta(A) < \delta(Y)$. Choose w coordinate characters on the torsion coset $\langle A \rangle$, identified by torsion translation with a k-torus, whose restriction to A is dominant onto $\mathbb{G}_m^w$. The coordinate functions generate the function field of A, so one may take a maximal algebraically independent subcollection. Every nonempty open of A maps to a set containing a nonempty open of $\mathbb{G}_m^w$, so its preimages of torsion tuples are dense in A. Each such preimage lies in a torsion coset of codimension w inside $\langle A \rangle$, hence of codimension $s - k + w$ inside S. This also holds for $w = 0$.

Translate S by torsion and apply exact non-density to Y in S. By (15) *every* proper optimal A is contained in the same proper closed subset of Y, namely the closure of the points of Y lying in torsion cosets of codimension $> d$ in S. In each of its finitely many irreducible components containing A, optimality of A is inherited. Induct on $\dim Y$, including the component itself among its optimal subvarieties. This gives finiteness of all optimal A. A maximal atypical component A

relative to S is optimal. Indeed, it has $\delta(A) < \delta(Y)$, and an enlargement of no greater defect lies in a strictly larger atypical component cut out by the enlargement's special closure. This proves Theorem 1.1. □

References

[1] F. Amoroso, N. H. Andriamandratomanana, and D. Simon, *Sur une conjecture de Schinzel*, arXiv:2502.10549 (2025).

[2] F. Amoroso and S. David, *Le problème de Lehmer en dimension supérieure*, J. Reine Angew. Math. 513 (1999), 145–179.

[3] F. Amoroso, D. Masser and U. Zannier, *Bounded height in pencils of finitely generated subgroups*, Duke Math. J. 166 (2017), 2599–2642.

[4] F. Amoroso, M. Sombra, and U. Zannier, *Unlikely intersections and multiple roots of sparse polynomials*, Math. Z. 287 (2017), 1065–1081.

[5] V. Aslanyan, *Some remarks on atypical intersections*, Proc. Amer. Math. Soc. 149 (2021), 4649–4660.

[6] J. Ax, *On Schanuel's conjectures*, Ann. of Math. 93 (1971), 252–268. doi:10.2307/1970774.

[7] F. Barroero and G. A. Dill, *Distinguished categories and the Zilber–Pink conjecture*, Amer. J. Math. 147 (2025), 715–778, Theorems 9.2, 9.3 and 12.4. Article.

[8] F. Barroero, L. Kühne and H. Schmidt, *Unlikely intersections of curves with algebraic subgroups in semiabelian varieties*, Selecta Math. (N.S.) 29 (2023), Paper No. 18.

[9] M. Bays and P. Habegger, *A note on divisible points of curves*, Trans. Amer. Math. Soc. 367 (2015), 1313–1328.

[10] Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. J. 89 (1997), 465–476.

[11] E. Bombieri, P. Habegger, D. Masser and U. Zannier, *A note on Maurin's theorem*, Rend. Lincei Mat. Appl. 21 (2010), 251–260.

[12] E. Bombieri, D. Masser and U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Int. Math. Res. Not. (1999), no. 20, 1119–1140.

[13] E. Bombieri, D. Masser and U. Zannier, *Finiteness results for multiplicatively dependent points on complex curves*, Michigan Math. J. 51 (2003), 451–466.

[14] E. Bombieri, D. Masser and U. Zannier, *Intersecting curves and algebraic subgroups: conjectures and more results*, Trans. Amer. Math. Soc. 358 (2006), 2247–2257.

[15] E. Bombieri, D. Masser and U. Zannier, *Anomalous subvarieties—structure theorems and applications*, Int. Math. Res. Not. IMRN (2007), no. 19, Art. rnm057, 33 pp.

[16] E. Bombieri, D. Masser and U. Zannier, *Intersecting a plane with algebraic subgroups of multiplicative groups*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008), 51–80.

[17] E. Bombieri, D. Masser and U. Zannier, *On unlikely intersections of complex varieties with tori*, Acta Arith. 133 (2008), 309–323. Published article.

[18] E. Bombieri and U. Zannier, *Algebraic points on subvarieties of* $\mathbb{G}_m^n$, Int. Math. Res. Not. (1995), no. 7, 333–347.

[19] G. Boxall, *Points on a curve with a power on a curve*, preprint, arXiv:2401.05024 (2024).

[20] L. Capuano, *Exploring Zilber's conjecture on atypical intersections in tori and beyond*, J. London Math. Soc. (2) 113 (2026), no. 4, e70536.

[21] L. Capuano, D. Masser, J. Pila and U. Zannier, *Rational points on Grassmannians and unlikely intersections in tori*, Bull. Lond. Math. Soc. 48 (2016), 141–154.

[22] P. B. Cohen and U. Zannier, *Fewnomials and intersections of lines with real analytic subgroups in* $\mathbb{G}_m^n$, Bull. Lond. Math. Soc. 34 (2002), 21–32.

[23] G. Dill, *Generalized Vojta–Rémond inequality*, Theorem 1.1. arXiv:1811.07784.

[24] S. Encinas, A. Nobile and O. Villamayor, *On algorithmic equiresolution and stratification of Hilbert schemes*, Proc. Lond. Math. Soc. 86 (2003), 607–648. arXiv:math/0010228.

[25] J.-H. Evertse, *An explicit version of Faltings' Product Theorem and an improvement of Roth's lemma*, Acta Arith. 73 (1995), 215–248, Theorem 1, p. 218. Published article.

[26] W. Gubler and J. Hertel, *Local heights of toric varieties over non-archimedean fields*, Publ. Math. Besançon Algèbre Théorie Nombres (2017), 5–77, Theorem 1.4.3. Published article.

[27] P. Habegger, *Intersecting subvarieties of* $\mathbb{G}_m^n$ *with algebraic subgroups*, Math. Ann. 342 (2008), 449–466.

[28] P. Habegger, *A Bogomolov property for curves modulo algebraic subgroups*, Bull. Soc. Math. France 137 (2009), 93–125.

[29] P. Habegger, *On the bounded height conjecture*, Int. Math. Res. Not. IMRN (2009), 860–886.

[30] P. Habegger, *Effective height upper bounds on algebraic tori*, in *Around the Zilber–Pink conjecture*, Panor. Synthèses 52, Soc. Math. France, 2017, 167–242.

[31] P. Habegger and J. Pila, *Some unlikely intersections beyond André–Oort*, Compos. Math. 148 (2012), 1–27.

[32] P. Habegger and J. Pila, *O-minimality and certain atypical intersections*, Ann. Sci. Éc. Norm. Supér. (4) 49 (2016), 813–858.

[33] P. Hacking, *The homology of tropical varieties*, Collect. Math. 59 (2008), 263–273, Section 3. arXiv:0711.1847. For the logarithmic Hodge and weight filtrations see P. Deligne, *Théorie de Hodge II*, Publ. Math. IHÉS 40 (1971), 5–57. Article.

[34] J. Kirby, *The theory of the exponential differential equations of semiabelian varieties*, Selecta Math. (N.S.) 15 (2009), 445–486.

[35] S. Lang, *Division points on curves*, Ann. Mat. Pura Appl. (4) 70 (1965), 229–234.

[36] M. Laurent, *Équations diophantiennes exponentielles*, Invent. Math. 78 (1984), 299–327.

[37] P. Liardet, *Sur une conjecture de Serge Lang*, Astérisque 24–25 (1975), 187–210.

[38] G. Maurin, *Courbes algébriques et équations multiplicatives*, Math. Ann. 341 (2008), 789–824.

[39] G. Maurin, *Équations multiplicatives sur les sous-variétés des tores*, Int. Math. Res. Not. IMRN (2011), 5259–5366.

[40] OpenAI, *The abelian Zilber–Pink conjecture*, manuscript of 24 September 2026, especially Propositions 3.4, 5.3, 6.1 and Sections 7–8. Manuscript and source.

[41] M. Orr, *Explicit height bounds for G-functions and unlikely intersections with lines in tori*, preprint, arXiv:2607.09565 (2026).

[42] J. Pila, *Combining the conjectures of Schanuel and Zilber–Pink*, Rend. Lincei Mat. Appl. 36 (2025), 653–670.

[43] J. Pila and A. J. Wilkie, *The rational points of a definable set*, Duke Math. J. 133 (2006), 591–616.

[44] J. Pila and U. Zannier, *Rational points in periodic analytic sets and the Manin–Mumford conjecture*, Rend. Lincei Mat. Appl. 19 (2008), 149–162.

[45] R. Pink, *A common generalization of the conjectures of André–Oort, Manin–Mumford, and Mordell–Lang*, preprint, 17 April 2005.

[46] G. Rémond, *Inégalité de Vojta généralisée*, Bull. Soc. Math. France 133 (2005), 459–495.

[47] G. Rémond, *Généralisations du problème de Lehmer et applications à la conjecture de Zilber–Pink*, Panor. Synthèses 52 (2017), 243–284.

[48] A. Schinzel, *On the reducibility of polynomials and in particular of trinomials*, Acta Arith. 11 (1965), 1–34.

[49] A. Schinzel, *Reducibility of lacunary polynomials. X*, Acta Arith. 53 (1989), 47–97.

[50] X. Yuan and S.-W. Zhang, *Adelic line bundles on quasi-projective varieties*, Theorems 5.2.2(2), 5.3.3 and Appendix A. arXiv:2105.13587.

[51] U. Zannier, *Proof of Conjecture 1*, appendix to A. Schinzel, *Polynomials with Special Regard to Reducibility*, Encyclopedia Math. Appl. 77, Cambridge University Press, Cambridge, 2000, 517–539.

[52] S.-W. Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. 8 (1995), 187–221.

[53] B. Zilber, *Intersecting varieties with tori*, manuscript, 20 December 2001.

[54] B. Zilber, *Exponential sums equations and the Schanuel conjecture*, J. London Math. Soc. (2) 65 (2002), 27–44.