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WHEN D-COMPANION MATRIX MEETS SENDOV'S CONJECTURE

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Abstract. Sendov's conjecture is a longstanding problem in the geometry of polynomials. It was recently proved by Lech Mazur. Terence Tao subsequently presented a simplified proof on his blog. Both proofs, however, rely on computer-assisted verification for intermediate degrees. Following Tao's framework, we give a new proof based on the D-companion matrix that requires no computer-assisted verification. The key difference lies in treating the ranges $0 < \alpha \leq 3$ and $\alpha \geq 3$ separately, where $\alpha = (n-1)(1-|a|^2)/2$ and a is the zero under consideration. This enables us to replace the finite verification with uniform analytic estimates valid for all degrees.

1. Introduction

In 1958, Sendov proposed the following conjecture. As Marden [Mar83] recalled, Sendov told him of the conjecture at the International Congress of Mathematicians in Stockholm in 1962. The conjecture first appeared in Hayman's collection of problems [Hay67, Problem 4.5], where it was attributed to Ilieff. For a historical account, see Sendov [Sen02].

Conjecture 1.1 (Sendov)**.** *Let $p(z)$ be a complex polynomial of degree $n \geq 2$ whose zeros lie in the closed unit disk. For every zero a of $p(z)$, there is a critical point ζ such that $|a-\zeta| \leq 1$.*

Remark 1.2. *Phelps and Rodriguez [PR72] conjectured that equality in Conjecture 1.1 holds if and only if $|a| = 1$ and $p(z) = c(z^n - a^n)$ for some nonzero constant c.*

Progress on Conjecture 1.1 is summarized in Table 1. The conjecture was recently resolved by Mazur [Maz26]. In a subsequent blog post, Tao [Tao26] simplified the proof and observed that the argument also yields the characterization of the equality case conjectured by Phelps and Rodriguez in Remark 1.2.

Upper bound	Hypotheses	Reference
2		Gauss–Lucas theorem
$2^{1/n}$		Bojanov–Rahman–Szynal [BRS85]
1.0753828		Rahman–Schmeisser [RS02]
$n^{-1/(n-1)}$	$a = 0$	Elementary
1	$\lvert a\rvert = 1$	Rubinstein [Rub68]
1	$n = 2$	Trivial
1	$n = 3$	Brannan [Bra68]
1	$n = 4$	Rubinstein [Rub68]
1	$n = 5$	Meir–Sharma [MS69]
1	$n = 6$	Borcea [Bor96a]
1	$n = 7$	Borcea [Bor96b], Brown [Bro97]
1	$n = 8$	Brown–Xiang [BX99]
1	At most eight distinct zeros	Brown–Xiang [BX99]
1	$n \geq N(\lvert a\rvert)$, $0 < \lvert a\rvert < 1$	Dégot [Deg14]
1	n sufficiently large	Tao [Tao22]
1	$n \geq 10^{200000}$	Zhang [Zha26b]
1	$n \geq 2$	Mazur [Maz26]; Tao's exposition [Tao26]

Table 1. Progress on Conjecture 1.1.

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However, the only drawback of the proof in [Tao26] is its reliance on computer-assisted verification for the intermediate degrees $6 \le n \le 200$. This is precisely the starting point of the present paper. We also mention that three further strengthenings of Sendov's conjecture have also been established: the quadratic Tang–Zhang inequality [Zha26b], Borcea's 2-variance conjecture [BF26, Zha26c], and Schmeisser's conjecture [Zha26d]. For the formulations of Borcea's variance conjectures and Schmeisser's conjecture, see Khavinson, Pereira, Putinar, Saff, and Shimorin [KPPSS11] and Schmeisser [Sch77], respectively.

In this paper, we give a new proof based on the D-companion matrix that requires no computer-assisted verification.

1.1. **High-level description of the proof.** Our proof follows the general framework of Tao [Tao26], but differs in two respects. We derive the reflection and origin inequalities by means of the D-companion matrix and avoid computer-assisted verification by establishing uniform estimates for the integrals appearing in these inequalities. We outline the main steps below.

- **Step I. From critical points to eigenvalues.** The starting point is the D-companion matrix of Cheung and Ng [CN06]. Given a simple zero a of a monic polynomial $p(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$, we consider the matrix
$$R = (I + \mathbf{1}\mathbf{1}^*)(aI - Z)^{-1}, \qquad Z = \operatorname{diag}(z_1, \dots, z_{n-1}),$$
whose eigenvalues are $q_j = 1/(a-\zeta_j)$, where $\zeta_1, \dots, \zeta_{n-1}$ are the critical points of $p(z)$. Conjecture 1.1 is thus reduced to the spectral radius inequality $\rho(R) \ge 1$. Lemma 2.1 gives the identity
$$\int_0^1 \det(I + tvR)\, dt = \frac{\det(aI - Z + vI)}{\det(aI - Z)}, \qquad v \in \mathbb{C}.$$
The two choices $v = (1-a^2)/a$ and $v = -a$ lead to the reflection and origin inequalities, respectively. The cases $a = 0$ and $a = 1$ follow directly from the determinant and trace identities in Lemma 2.1. The trace argument also characterizes the equality case when $a = 1$. It remains to exclude the possibility that $\rho(R) \le 1$ for $0 < a < 1$.
- **Step II. The reflection inequality.** For $0 < a < 1$, the geometry of the unit disk gives $|1 - az_j| \ge |a - z_j|$. Taking $v = (1-a^2)/a$ in (2.2), we therefore obtain a lower bound for the modulus of the integral of a determinant. Following Tao [Tao26], we introduce
$$x + iy = \frac{1}{n-1}\sum_{j=1}^{n-1} q_j, \qquad \alpha = \frac{n-1}{2}(1-a^2), \qquad \beta(t) = 1 - 2axt + a^2t^2.$$
The arithmetic–geometric mean inequality for the squared singular values, together with the hyperbolic-function estimate in Lemma 3.1, yields
$$\beta(1) < \frac{\alpha}{3+\alpha}.$$
The latter inequality allows us to eliminate x from the quadratic function β by replacing it with
$$\widetilde{\beta}(t) = 1 - \frac{3t}{3+\alpha} - a^2t(1-t).$$
In particular, $0 < \beta(t) \le \widetilde{\beta}(t) \le 1$ for $0 \le t \le 1$. For $2 \le n \le 5$, the reflection identity alone suffices, as shown by the fourth-moment argument in Lemma 2.2.
- **Step III. The origin inequality.** Let $Q = \operatorname{diag}(q_1, \dots, q_{n-1})$ and $F(t) = \det(I - atQ)$. Jacobi's formula expresses $F'(t) + a(n-1)(x+iy)F(t)$ in terms of the adjugate of $I - atQ$. Since Q is a contraction, the trace norm estimate in Lemma 3.3 controls the resulting expression by $(n-1)a^2t\beta(t)^{(n-2)/2}$. To estimate the boundary terms arising from integration, we apply the Schwarz–Pick lemma to a holomorphic function defined by a quotient of determinants (Lemma 3.4). This provides an alternative to the defect lemma of Tao [Tao26, Lemma 13]. The resulting inequality is
$$2\alpha + ax \le \frac{1-x^2}{2(n-1)} + a^2n(n-1)\int_0^1 t\beta(t)^{(n-2)/2}\, dt.$$
For small α, the left-hand side is close to 1. It is therefore useful to separate the principal term $(1-axt)^{n-2}$ from $\beta(t)^{(n-2)/2}$, using the decomposition $\beta(t) = (1-axt)^2 + a^2t^2(1-x^2)$. This leads to the refined origin inequality (3.9), in which the remaining integral involves $t^3\widetilde{\beta}(t)^{(n-4)/2}$.
- **Step IV. Uniform estimates for the integrals.** This is where our proof departs from Tao's argument. Rather than first establishing an upper bound for α and then treating the intermediate degrees separately, we divide the proof into the two ranges $0 < \alpha \le 3$ and $\alpha \ge 3$.

The key observation is the factorization

$$1-\widetilde{\beta}(t)=t\left(\frac{3}{3+\alpha}+a^2(1-t)\right).$$

For $0<\alpha\le 3$, we use the maximum of $u^{(n-4)/2}(1-u)^3$ on $[0,1]$. Substituting $u=\widetilde{\beta}(t)$ allows us to cancel the factor t^3 in the integrand of (3.9) and replace the high power of $\widetilde{\beta}(t)$ by a rational function. For $n\ge 10$, we split the integral at $t=1/2$, using an affine majorant of $\widetilde{\beta}$ on the first subinterval and the maximum-value estimate on the second. The resulting upper bound is affine in $1/(n-1)$ when α is fixed, so it suffices to check its values at 0 and $1/9$. Lemma 4.1 thus excludes the entire range $0<\alpha\le 3$.

For $\alpha\ge 3$, we return to the original inequality (3.4) and use the maximum of $u^{(n-2)/2}(1-u)^2$ on $[0,1]$. Splitting the integral at $t=1/3$ and proceeding as above, we reduce the desired estimate to a comparison between rational functions. The comparison is established in Lemma 4.2 by monotonicity and elementary polynomial inequalities.

The two ranges exhaust all possible values of α. In particular, no a priori upper bound on α and no computer-assisted verification for intermediate degrees are required.

1.2. **Organization of this paper.** In Section 2, we reformulate Sendov's conjecture in terms of the spectral radius of the inverse of a shifted D-companion matrix and settle the cases $a=0$, $a=1$, and $2\le n\le 5$. In Section 3, we derive Tao's reflection and origin inequalities using matrix methods. Finally, in Section 4, we establish uniform integral estimates for the ranges $0<\alpha\le 3$ and $\alpha\ge 3$, thereby completing the proof without computer-assisted verification.

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2. The D-companion matrix

Throughout the paper, zeros and critical points are counted with multiplicity. Multiplying a polynomial by a nonzero constant and rotating its zeros do not affect the assertion of Conjecture 1.1. Moreover, if a is a multiple zero, then $p'(a)=0$. Thus we may assume that $p(z)$ is monic and that $a\in[0,1]$ is a simple zero. Write $p(z)=(z-a)\prod_{j=1}^{n-1}(z-z_j)$, where $|z_j|\le 1$, and denote the critical points by $\zeta_1,\dots,\zeta_{n-1}$. Set

$$Z=\operatorname{diag}(z_1,\dots,z_{n-1}),\qquad R=(I+\mathbf{1}\mathbf{1}^*)(aI-Z)^{-1},\qquad q_j=\frac{1}{a-\zeta_j},\tag{2.1}$$

where I is the identity matrix of order $n-1$ and $\mathbf{1}\in\mathbb{C}^{n-1}$ is the vector with every entry equal to 1. The inverses in (2.1) exist because a is a simple zero.

By the D-companion matrix representation of Cheung and Ng [CN06, Theorem 1.1], the matrix $(aI-Z)(I-\frac{1}{n}\mathbf{1}\mathbf{1}^*)$ has eigenvalues $a-\zeta_1,\dots,a-\zeta_{n-1}$. Its inverse R therefore has eigenvalues $q_1,\dots,q_{n-1}$. We write $Q=\operatorname{diag}(q_1,\dots,q_{n-1})$ and denote the spectral radius of a matrix by ρ.

The next lemma is the determinant form of the reciprocal representation in Tang and Zhang [TZ25, §6]. Its integral identity is a common form of the identities used by Tao [Tao26, Lemma 6].

Lemma 2.1. *The eigenvalues of R are $q_1,\dots,q_{n-1}$, counted with algebraic multiplicity. For every $v\in\mathbb{C}$,*

$$\det(I+vR)=\frac{p'(a+v)}{p'(a)},\qquad \int_0^1\det(I+tvR)\,dt=\frac{\det(aI-Z+vI)}{\det(aI-Z)}.\tag{2.2}$$

Moreover,

$$\operatorname{tr}R=2\operatorname{tr}(aI-Z)^{-1},\qquad \operatorname{tr}R^{-1}=\frac{n-1}{n}\big((n-1)a-\operatorname{tr}Z\big),\qquad \det R=\frac{n}{p'(a)}.\tag{2.3}$$

Proof. As noted above, the eigenvalues of R are $q_1,\dots,q_{n-1}$. Since $p'(z)=n\prod_{j=1}^{n-1}(z-\zeta_j)$, we have

$$\det(I+vR)=\prod_{j=1}^{n-1}(1+vq_j)=\prod_{j=1}^{n-1}\frac{a+v-\zeta_j}{a-\zeta_j}=\frac{p'(a+v)}{p'(a)}.$$

For $v \neq 0$, integrating the first identity in (2.2) and using $p(a) = 0$, we obtain

$$\int_0^1 \det(I + tvR)\, dt = \frac{1}{p'(a)} \int_0^1 p'(a + tv)\, dt = \frac{p(a+v)}{vp'(a)}.$$

Since $p(a+v) = v \det(aI - Z + vI)$ and $p'(a) = \det(aI - Z)$, the second identity in (2.2) follows for $v \neq 0$. For $v = 0$, both sides are equal to 1.

By (2.1), the diagonal entries of R are $2/(a - z_j)$, $1 \leq j \leq n-1$. Hence $\operatorname{tr} R = 2 \operatorname{tr}(aI - Z)^{-1}$. As noted before,

$$\begin{aligned} \operatorname{tr} R^{-1} &= \operatorname{tr}\left((aI - Z)\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^*\right)\right) \\ &= \operatorname{tr}(aI - Z) - \frac{1}{n}\mathbf{1}^*(aI - Z)\mathbf{1} \\ &= \frac{n-1}{n}\sum_{j=1}^{n-1}(a - z_j) = \frac{n-1}{n}\big((n-1)a - \operatorname{tr} Z\big). \end{aligned}$$

Finally, comparing the coefficients of v^{n-1} on both sides of the first identity in (2.2) gives $\det R = n/p'(a)$, since $p(z)$ is monic of degree n. □

By Lemma 2.1, $\det R = \det Q$, $\operatorname{tr} R = \operatorname{tr} Q$, and $\operatorname{tr} R^{-1} = \operatorname{tr} Q^{-1}$. We will also use $\det(I + vR) = \det(I + vQ)$. In particular, the minimum in Conjecture 1.1 equals $\rho(R)^{-1}$.

For $a = 0$, (2.3) gives $\rho(R)^{n-1} \geq |\det R| \geq n > 1$. For $a = 1$, the elementary inequality $\operatorname{Re}(1/(1 - z_j)) \geq 1/2$ follows from $1 - |z_j|^2 \geq 0$. Thus $\operatorname{Re} \operatorname{tr} R \geq n - 1$ by (2.3), and $\rho(R) \geq 1$. If $\rho(R) = 1$, all q_j have modulus at most 1, and $\operatorname{Re} \sum_{j=1}^{n-1} q_j \geq n - 1$ forces every q_j to equal 1. Hence every $\zeta_j = 0$, so $p(z) = z^n - 1$. This recovers Rubinstein's boundary result [Rub68], including its equality case.

It remains to prove that $\rho(R) > 1$ when $0 < a < 1$. We henceforth suppose, to the contrary, that

$$|q_j| \leq 1 \qquad (1 \leq j \leq n-1). \tag{2.4}$$

Substituting $v = (1 - a^2)/a$ into (2.2) and multiplying by a^{n-1} gives

$$\int_0^1 \det\big(aI + t(1 - a^2)Q\big)\, dt = \prod_{j=1}^{n-1} \frac{1 - az_j}{a - z_j}. \tag{2.5}$$

The right-hand side of (2.5) has modulus at least 1, since $|1 - az_j|^2 - |a - z_j|^2 = (1 - a^2)(1 - |z_j|^2) \geq 0$.

For the convenience of the reader, we give the short fourth-moment argument that handles degrees up to five.

Lemma 2.2. *If $0 < a < 1$, then (2.4) is impossible for $2 \leq n \leq 5$.*

Proof. By (2.5) and (2.4), $1 \leq \int_0^1 (a + t(1 - a^2))^{n-1}\, dt$. Let $h = (1 - a^2)/2$ and $b = a + h$. Then

$$\int_0^1 (a + t(1 - a^2))^4\, dt = \frac{1}{2}\int_{-1}^1 (b + hs)^4\, ds = (b^2 + h^2)^2 - \frac{4}{5}h^4 < 1,$$

since $b^2 + h^2 = 1 - (1 - a)^3(1 + a)/2 < 1$. As $n - 1 \leq 4$, Hölder's inequality gives $\int_0^1 (a + t(1 - a^2))^{n-1}\, dt < 1$, a contradiction. □

3. The reflection and origin inequalities

For the rest of the proof we assume $n \geq 6$, $0 < a < 1$, and (2.4). Following Tao [Tao26], put

$$x + iy = \frac{1}{n-1}\sum_{j=1}^{n-1} q_j, \qquad \alpha = \frac{n-1}{2}(1 - a^2), \qquad \beta(t) = 1 - 2axt + a^2t^2 \quad (0 \leq t \leq 1). \tag{3.1}$$

In particular, $x^2 + y^2 \leq 1$, $0 < \alpha < (n-1)/2$, and $\beta(t) \geq (1 - at)^2 > 0$.

3.1. **The reflection inequality.** We first state the hyperbolic-function estimate used in Tao [Tao26, (27)]. A differential comparison gives the following short proof.

Lemma 3.1. *For $h > 0$,*

$$\log \frac{\sinh h}{h} \le \sqrt{h^2+9} - 3.$$

Proof. Let $L(h) = \coth h - 1/h$ and $V(h) = h/\sqrt{h^2+9}$. Then $L' = 1 - L^2 - 2L/h$, and direct differentiation gives

$$V' - \left(1 - V^2 - \frac{2V}{h}\right) = \frac{(2\sqrt{h^2+9}-3)(\sqrt{h^2+9}-3)}{(h^2+9)^{3/2}} \ge 0.$$

Consequently, $(V-L)' + (V+L+2/h)(V-L) \ge 0$. Both V and L tend to 0 as $h \downarrow 0$. Multiplying by the integrating factor $h^2 \exp(\int_0^h (V(u)+L(u))\,du)$ and integrating from 0 proves $L \le V$. Integration once more gives the assertion. □

The following is the part of Tao's Proposition 10 [Tao26] that is needed here.

Lemma 3.2. *Under (2.4) and (3.1),*

$$0 < \beta(1) < \frac{\alpha}{3+\alpha}.$$

Proof. For an $N \times N$ matrix X, the arithmetic–geometric mean inequality for the squared singular values gives $|\det X| \le (\operatorname{tr}(X^*X)/N)^{N/2}$. Apply this inequality to $aI + t(1-a^2)Q$. By (2.5) and $|q_j| \le 1$,

$$1 \le \int_0^1 \bigl(a^2 + 2atx(1-a^2) + t^2(1-a^2)^2\bigr)^{(n-1)/2}\,dt.$$

Using $t^2 < t$ for $0 < t < 1$, $2ax = 1 + a^2 - \beta(1)$ and $1 + u \le e^u$, we obtain

$$1 < \int_0^1 \exp\bigl(\alpha[-1 + (2-\beta(1))t]\bigr)\,dt. \tag{3.2}$$

If $\beta(1) \ge 1$, then the right-hand side of (3.2) is at most $\int_0^1 e^{\alpha(t-1)}\,dt < 1$. Hence $0 < \beta(1) < 1$. Set $b = \alpha\beta(1)/2$ and $h = \alpha - b$. The integral in (3.2) is $e^{-b}\sinh h/h$, so Lemma 3.1 gives $b + 3 < \sqrt{(\alpha-b)^2+9}$. Squaring yields $2b(3+\alpha) < \alpha^2$, as required. □

It will be useful to replace β by a function depending only on a and α. Define

$$\widetilde{\beta}(t) = 1 - \frac{3t}{3+\alpha} - a^2 t(1-t), \qquad 0 \le t \le 1. \tag{3.3}$$

Lemma 3.2 gives $2ax > a^2 + 3/(3+\alpha)$, so $0 < \beta(t) \le \widetilde{\beta}(t) \le 1$ on $[0,1]$.

3.2. **The origin inequality.** We will use the following form of Maclaurin's inequality (see Hardy, Littlewood, and Pólya [HLP52, p. 52, Theorem 52]). We state it for matrices to make the application transparent. Here $\operatorname{adj} X$ denotes the classical adjugate, $\|X\|$ the operator norm, and $\|X\|_1$ the trace norm, that is, the sum of the singular values of X.

Lemma 3.3. *If X is an $N \times N$ complex matrix with $N \ge 2$, then*

$$\|\operatorname{adj} X\|_1 \le N \left(\frac{\operatorname{tr}(X^*X)}{N}\right)^{(N-1)/2}.$$

Proof. For an invertible X with singular values $s_1, \dots, s_N$, the singular values of $\operatorname{adj} X = (\det X)X^{-1}$ are the products $\prod_{k \ne j} s_k$, $1 \le j \le N$. By Maclaurin's inequality, their sum is at most $N((s_1 + \cdots + s_N)/N)^{N-1}$. The Cauchy–Schwarz inequality gives the stated bound. Continuity of the adjugate and of the trace norm extends the result to singular matrices. □

The next lemma gives an alternative to the product estimate in Tao [Tao26, Lemma 13]. It is an immediate application of the Schwarz–Pick lemma to a product of holomorphic maps of the disk into itself. If a diagonal entry has modulus one, the corresponding factor is constant.

Lemma 3.4. *Let $T = \operatorname{diag}(t_1, \dots, t_N)$ and $H = \operatorname{diag}(h_1, \dots, h_N)$, where $t_j \in \mathbb{C}$, $|t_j| \le 1$ and $h_j \in [-1, 1]$. Then*

$$\bigl|\operatorname{tr}(H \operatorname{adj} T) - \det T \operatorname{tr}(HT^*)\bigr| \le 1 - |\det T|^2.$$

Proof. The function $f(v) = \det((T + vH)(I + vHT^*)^{-1})$ is holomorphic for $|v| < 1$ and has modulus at most 1, as is seen from the product $\prod_{j=1}^{N}(t_j + vh_j)/(1 + vh_j\overline{t_j})$. The Schwarz–Pick lemma (see [Gar07, Lemma 1.2]) gives $|f'(0)| \le 1 - |f(0)|^2$ (the assertion is trivial if f is a constant of modulus 1). Differentiating the two determinants gives $f'(0) = \operatorname{tr}(H \operatorname{adj} T) - \det T \operatorname{tr}(HT^*)$, which proves the inequality even when T is singular. □

The following inequality is Tao's Proposition 11(i) [Tao26]. We give its derivation using Lemmas 2.1, 3.3, and 3.4, in particular retaining the precise coefficient in the inverse-trace identity.

Lemma 3.5. *Under (2.4) and (3.1),*

$$2\alpha + ax \le \frac{1 - x^2}{2(n-1)} + a^2 n(n-1) \int_0^1 t\beta(t)^{(n-2)/2}\, dt. \tag{3.4}$$

Proof. Let $F(t) = \det(I - atQ)$ and $J = \det Z \det Q$. The diagonal matrix Q is a contraction by (2.4); we do not require that R be a contraction.

Step 1. A differential identity. Jacobi's formula and the resolvent identity $Q(I - atQ)^{-1} = Q + atQ^2(I - atQ)^{-1}$ give

$$F'(t) + a(n-1)(x+iy)F(t) = -a^2 t \operatorname{tr}\bigl(Q^2 \operatorname{adj}(I - atQ)\bigr).$$

By Lemma 3.3, $\|\operatorname{adj}(I - atQ)\|_1 \le (n-1)\beta(t)^{(n-2)/2}$, since $\operatorname{tr}((I - atQ)^*(I - atQ))/(n-1) \le \beta(t)$. Using $\|Q^2\| \le 1$ and the duality of the operator and trace norms, we bound the modulus of the trace on the right by $(n-1)\beta(t)^{(n-2)/2}$. Integration, using $F(0) = 1$, yields

$$1 \le \left| F(1) + a(n-1)(x+iy) \int_0^1 F(t)\, dt \right| + a^2(n-1) \int_0^1 t\beta(t)^{(n-2)/2}\, dt. \tag{3.5}$$

Step 2. Estimation at the origin. By Lemma 2.1 at $v = -a$,

$$\int_0^1 F(t)\, dt = \frac{(-1)^{n-1} J}{n}, \qquad F(1) = \frac{(-1)^{n-1} J}{n} \left(1 + a \sum_{j=1}^{n-1} \frac{1}{z_j} \right). \tag{3.6}$$

For the first identity, use $\det Q = n/p'(a)$ and $\det(-Z) = (-1)^{n-1} \det Z$. For the second, use $F(1) = p'(0)/p'(a)$ and differentiate $p(z) = (z - a)\prod_{j=1}^{n-1}(z - z_j)$ at 0. Products such as $J\sum_{j=1}^{n-1} z_j^{-1}$ are understood throughout as $\det Q \operatorname{tr}(\operatorname{adj} Z)$; in particular, (3.6) is meaningful when some $z_j = 0$.

Apply Lemma 3.4 to $T = \operatorname{diag}(Z, \overline{Q})$ and $H = \operatorname{diag}((n-1)I/n, -I)$, both of order $2(n-1)$. For invertible Z, the trace in the lemma, after factoring out $\det T$, is

$$\begin{aligned} \operatorname{tr}\bigl(H(T^{-1} - T^*)\bigr) &= \frac{n-1}{n} \operatorname{tr} Z^{-1} - \frac{n-1}{n} \operatorname{tr} \overline{Z} - \operatorname{tr} \overline{Q}^{-1} + \operatorname{tr} Q \\ &= \frac{n-1}{n} \sum_{j=1}^{n-1} \frac{1}{z_j} + (n-1)(x+iy) - \frac{a(n-1)^2}{n}, \end{aligned}$$

where the last equality follows from the conjugate of the inverse-trace identity in (2.3). Since $|\det T| = |J|$, Lemma 3.4 implies

$$\left| J \left(\frac{n-1}{n} \sum_{j=1}^{n-1} \frac{1}{z_j} + (n-1)(x+iy) - \frac{a(n-1)^2}{n} \right) \right| \le 1 - |J|^2. \tag{3.7}$$

For singular Z, the same conclusion follows by expanding $\operatorname{tr}(H \operatorname{adj} T)$ in Lemma 3.4 before applying (2.3); the term $J\sum_{j=1}^{n-1} z_j^{-1}$ is precisely the adjugate expression described after (3.6). No limiting assumption on the original polynomial is required.

Set $c = 1 + (n-1)a^2 - a(x+iy)$. By (3.6) and (3.7),

$$\left| F(1) + a(n-1)(x+iy) \int_0^1 F(t)\, dt \right| \le \frac{|J||c|}{n} + \frac{a}{n-1}(1 - |J|^2). \tag{3.8}$$

Indeed, the expression in parentheses in (3.6) after adding $a(n-1)(x+iy)$ equals c plus $an/(n-1)$ times the expression in parentheses in (3.7). This equality is also valid after multiplying by J if Z is singular.

The right-hand side of (3.8) is increasing as a function of $|J| \in [0, 1]$, because

$$|c| \ge 1 + (n-1)a^2 - a \ge (2\sqrt{n-1} - 1)a > \frac{2na}{n-1}.$$

The last inequality holds for $n \geq 6$: $2\sqrt{n-1} - 1 > 3$ and $2n/(n-1) \leq 12/5$. Thus (3.8) is at most $|c|/n$. Since $1 + (n-1)a^2 - ax > (n-1)a^2 > 0$ and $y^2 \leq 1 - x^2$, the inequality $\sqrt{u^2+v^2} \leq u + v^2/(2u)$ for $u > 0$ gives

$$|c| \leq 1 + (n-1)a^2 - ax + \frac{1-x^2}{2(n-1)}.$$

Substituting this bound into (3.5), multiplying by n, and using $2\alpha = (n-1)(1-a^2)$ gives (3.4). □

For small α, the following refinement of Lemma 3.5 is useful. It is obtained from Tao [Tao26, Proposition 11(iii)] by substituting the upper bound in Lemma 3.2; we include the short deduction to fix the constants.

Lemma 3.6. *Under (2.4) and (3.1),*

$$1 \leq \frac{1}{6} + \frac{1}{4(3+\alpha)} + \frac{1}{2(n-1)} + \frac{1}{4(n-1)(3+\alpha)} + \frac{a^4 n(n-1)(n-2)}{4(3+\alpha)} \int_0^1 t^3 \widetilde{\beta}(t)^{(n-4)/2}\, dt. \tag{3.9}$$

Proof. By Lemma 3.2, $0 < ax < 1$. Write $\beta(t) = (1-axt)^2 + a^2t^2(1-x^2)$. The inequality $(u+v)^r \leq u^r + rv(u+v)^{r-1}$ for $u, v \geq 0$ and $r \geq 1$ follows by integration of the derivative of $s \mapsto s^r$. Applying it with $r = (n-2)/2$ and using $\int_0^{1/(ax)} t(1-axt)^{n-2}\, dt = 1/(a^2x^2n(n-1))$, we find

$$a^2 n(n-1) \int_0^1 t\beta(t)^{(n-2)/2}\, dt \leq \frac{1}{x^2} + \frac{a^4 n(n-1)(n-2)(1-x^2)}{2} \int_0^1 t^3 \beta(t)^{(n-4)/2}\, dt.$$

Now $ax = 1 - \alpha/(n-1) - \beta(1)/2$ and $1 - x^2 \leq \beta(1)$, since $\beta(1) = 1 - x^2 + (x-a)^2$. In particular $x^{-2} - 1 \leq \beta(1)/(1-\beta(1))$. Inserting the integral bound into (3.4) and dividing the rearranged inequality by 2α gives

$$1 \leq \frac{\beta(1)}{2\alpha(1-\beta(1))} + \frac{\beta(1)}{4\alpha} + \frac{1}{2(n-1)} + \frac{\beta(1)}{4\alpha(n-1)} + \frac{a^4 n(n-1)(n-2)\beta(1)}{4\alpha} \int_0^1 t^3 \beta(t)^{(n-4)/2}\, dt.$$

By Lemma 3.2, $\beta(1)/\alpha < 1/(3+\alpha)$ and $\beta(1)/(\alpha(1-\beta(1))) < 1/3$. Using (3.3) to bound $\beta(t)$ now proves (3.9). □

4. Uniform estimates for the integrals

In this section, $n \geq 6$ is an arbitrary integer, $0 < a < 1$, $\alpha = (n-1)(1-a^2)/2$, and $\widetilde{\beta}$ is defined by (3.3). Write $d = 3/(3+\alpha)$, so $0 < d < 1$ and $\widetilde{\beta}(t) = (1-t)(1-a^2t) + (1-d)t > 0$ on $[0,1]$. The key observation is the factorization

$$1 - \widetilde{\beta}(t) = t\big(d + a^2(1-t)\big). \tag{4.1}$$

Thus a bound for $u^r(1-u)^k$ on $[0,1]$, applied with $u = \widetilde{\beta}(t)$, yields an upper bound for $t^k\widetilde{\beta}(t)^r$ by a rational function. The power k will be 3 for the integral in (3.9) and 2 for the integral in (3.4). We will use the elementary integral identities

$$\int_0^1 u(1-u)^r\, du = \frac{1}{(r+1)(r+2)}, \qquad \int_0^1 u^3(1-u)^r\, du = \frac{6}{(r+1)(r+2)(r+3)(r+4)} \tag{4.2}$$

for $r > -1$, which follow by integration by parts.

4.1. The range $0 < \alpha \leq 3$.

Lemma 4.1. *If $0 < \alpha \leq 3$, then the right-hand side of (3.9) is strictly less than 1.*

Proof. In this range $1/2 \leq d < 1$. Differentiating the function $u \mapsto u^{(n-4)/2}(1-u)^3$ on $[0,1]$ gives

$$C_n := n(n-1)(n-2) \max_{0 \leq u \leq 1} u^{(n-4)/2}(1-u)^3 = \frac{216n(n-1)(n-2)}{(n+2)^3} \left(\frac{n-4}{n+2}\right)^{(n-4)/2}.$$

Regarded as a function of a real variable $n > 4$, C_n is strictly decreasing. Indeed,

$$\frac{d}{dn} \log C_n = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} - \frac{1}{2} \int_{n-4}^{n+2} \frac{du}{u} < 0.$$

To see the sign, divide the interval of integration into three intervals of length 2. By strict convexity of $u \mapsto 1/u$, half of the integral exceeds $1/(n-3) + 1/(n-1) + 1/(n+1)$. The latter sum exceeds $1/(n-2) + 1/(n-1) + 1/n$, again by strict convexity. In particular, $C_n \leq C_6 = 405/32 < 13$ for $n \geq 6$ and $C_n \leq C_{10} = 45/4$ for $n \geq 10$.

By (4.1), for $0 < t \le 1$,

$$n(n-1)(n-2)t^3\widetilde{\beta}(t)^{(n-4)/2} \le \frac{C_n}{(d+a^2(1-t))^3}. \tag{4.3}$$

Step 1. The range $6 \le n \le 10$. Integrating (4.3), using $a^2/d \le 2$ and $d = 3/(3+\alpha)$, gives

$$\frac{a^4 n(n-1)(n-2)}{4(3+\alpha)} \int_0^1 t^3\widetilde{\beta}(t)^{(n-4)/2}\,dt \le \frac{C_n(3+\alpha)a^2}{72}\left(1 - \frac{1}{(1+a^2/d)^2}\right) \le \frac{C_n(3+\alpha)a^2}{81}.$$

Also, $a^2 = 1 - 2\alpha/(n-1)$, so

$$(3+\alpha)\left(1-\frac{2\alpha}{9}\right) = \frac{25}{8} - \frac{2}{9}\left(\alpha - \frac{3}{4}\right)^2 \le \frac{25}{8}, \tag{4.4}$$

and $(3+\alpha)a^2$ is bounded by the left-hand side of (4.4) for $n \le 10$. The first four terms on the right of (3.9) are at most $1/4 + 7/(12(n-1)) < 3/8$ for $n \ge 6$. Therefore that right-hand side is less than $3/8 + 13 \cdot 25/(81 \cdot 8) = 71/81 < 1$.

Step 2. The range $n \ge 10$. Split the integral on the right of (3.9) at $t = 1/2$. Denote the contributions of the two subintervals, including the common prefactor $a^4 n(n-1)(n-2)/(4(3+\alpha))$, by T_0 and T_1, respectively.

For $0 \le t \le 1/2$, (4.1) gives $\widetilde{\beta}(t) \le 1 - (d + a^2/2)t$. The affine function is positive on this subinterval because it majorizes the positive function $\widetilde{\beta}$. Extending the integral of its nonnegative power to its zero at $1/(d+a^2/2)$, and using (4.2), we obtain

$$\begin{aligned} T_0 &\le \frac{24(n-1)a^4}{(3+\alpha)(n+2)(n+4)(d+a^2/2)^4} \\ &\le \frac{2(3+\alpha)(n-1)}{3(n+2)(n+4)} \le \frac{3+\alpha}{192} + \frac{3(3+\alpha)}{8(n-1)}. \end{aligned}$$

The second inequality uses $(d+a^2/2)^2 \ge 2da^2$. For the third, use $(n-1)/((n+2)(n+4)) \le 1/(n+7)$ and

$$\frac{1}{192} + \frac{3}{8(n-1)} - \frac{2}{3(n+7)} = \frac{(n-25)^2}{192(n-1)(n+7)} \ge 0.$$

On $1/2 \le t \le 1$, (4.3) gives

$$T_1 \le \frac{C_n(3+\alpha)a^2}{72}\left(1 - \frac{1}{(1+a^2/(2d))^2}\right) \le \frac{15(3+\alpha)a^2}{128},$$

because $a^2/(2d) \le 1$ and $C_n \le 45/4$.

Writing $v = 1/(n-1)$ and $a^2 = 1 - 2\alpha v$, the right-hand side of (3.9) is at most

$$\Psi_\alpha(v) = \frac{1}{6} + \frac{1}{4(3+\alpha)} + \left(\frac{1}{2} + \frac{1}{4(3+\alpha)}\right)v + \frac{3+\alpha}{192} + \frac{3(3+\alpha)}{8}v + \frac{15(3+\alpha)}{128}(1-2\alpha v).$$

For fixed α, this is affine in v. As $0 < v \le 1/9$, it suffices to bound the two endpoint values. Since $3 < 3+\alpha \le 6$,

$$\Psi_\alpha(0) \le \frac{1}{4} + \frac{1}{32} + \frac{90}{128} = \frac{63}{64}.$$

At $v = 1/9$, the sum of the first three terms in Ψ_α is at most $17/54 < 1/3$. Using (4.4) for the last term gives

$$\Psi_\alpha(1/9) < \frac{1}{3} + \frac{1}{32} + \frac{1}{4} + \frac{3}{8} = \frac{95}{96},$$

where $15 \cdot 25/(128 \cdot 8) < 3/8$. Both endpoint values are less than 1, completing the proof. □

4.2. **The range** $\alpha \ge 3$. In this range we use the unrefined origin inequality. Lemma 3.2 implies that the left-hand side of (3.4) exceeds $2\alpha + (a^2+d)/2$. We therefore need the following estimate, which makes no reference to the parameter x.

Lemma 4.2. *If* $\alpha \ge 3$, *then*

$$\frac{1}{2(n-1)} + a^2 n(n-1)\int_0^1 t\widetilde{\beta}(t)^{(n-2)/2}\,dt < 2\alpha + \frac{a^2+d}{2}.$$

Proof. As $\alpha < (n-1)/2$, we have $n \geq 8$ and $0 < d \leq 1/2$. Let $s = a^2/d$, so $0 < ds < 1$.

Step 1. An upper bound for the integral. The maximum of $u^{(n-2)/2}(1-u)^2$ is attained at $u = (n-2)/(n+2)$. In particular,

$$D_n := n(n-1) \max_{0 \leq u \leq 1} u^{(n-2)/2}(1-u)^2 = \frac{16n(n-1)}{(n+2)^2} \left(\frac{n-2}{n+2}\right)^{(n-2)/2}.$$

Using $\log(1-u) \leq -u - u^2/2$ for $0 < u < 1$ on each of the three factors in $D_n/16$, namely $1 - 2/(n+2)$, $1 - 3/(n+2)$, and $(1 - 4/(n+2))^{(n-2)/2}$, gives

$$\log(D_n/16) \leq -2 - \frac{1}{n+2} + \frac{19}{2(n+2)^2} < -2,$$

since $n + 2 \geq 10$. As $e > 27/10$ (by summing the first five terms of its Taylor series), $D_n < 16/e^2 < 11/5$.

On $0 \leq t \leq 1/3$, (4.1) gives $\widetilde{\beta}(t) \leq 1 - (d + 2a^2/3)t$. Extending the integral of the affine power to its zero and applying (4.2), we obtain

$$da^2 n(n-1) \int_0^{1/3} t\widetilde{\beta}(t)^{(n-2)/2}\, dt \leq \frac{4(n-1)da^2}{(n+2)(d+2a^2/3)^2} < \frac{36s}{(3+2s)^2}.$$

On $1/3 \leq t \leq 1$, the definition of D_n and (4.1) give

$$\begin{aligned} da^2 n(n-1) \int_{1/3}^1 t\widetilde{\beta}(t)^{(n-2)/2}\, dt &\leq D_n s \int_{1/3}^1 \frac{dt}{t(1+s(1-t))^2} \\ &= D_n s \left(\frac{\log(3+2s)}{(1+s)^2} + \frac{2s}{(1+s)(3+2s)}\right), \end{aligned}$$

where the last equality follows by partial fractions. The tangent line to $\log u$ at $u = 7$, together with $\log 7 < 2$, yields $\log(3+2s) \leq \log 7 + (2s-4)/7 < (10+2s)/7$. Thus

$$\begin{aligned} da^2 n(n-1) \int_0^1 t\widetilde{\beta}(t)^{(n-2)/2}\, dt &\leq G(s), \\ G(s) &= \frac{36s}{(3+2s)^2} + \frac{11}{5}\left(\frac{s(10+2s)}{7(1+s)^2} + \frac{2s^2}{(1+s)(3+2s)}\right). \end{aligned} \tag{4.5}$$

Step 2. A rational comparison. Let $g(s) = (19s+12)/(5(s+1))$. Multiplication by the positive common denominator gives the identity

$$35(s+1)^2(2s+3)^2\big(g(s) - G(s)\big) = (s-2)^2(136s^2 + 274s + 189) + 371s.$$

Its right-hand side is strictly positive for all $s \geq 0$. Hence $G(s) < g(s)$.

On the other hand, using $\alpha = 3(1-d)/d$, $a^2 = ds$, and $(n-1)^{-1} = d(1-ds)/(6(1-d))$, we may write

$$d\left(2\alpha + \frac{a^2+d}{2} - \frac{1}{2(n-1)}\right) = U(s,d) := 6(1-d) + \frac{d^2(1+s)}{2} - \frac{d^2(1-ds)}{12(1-d)}.$$

We claim that $U(s,d) > g(s)$ whenever $0 < d \leq 1/2$ and $ds \leq 1$. For fixed $s \geq 0$, differentiation gives

$$\frac{\partial U}{\partial d} \leq -6 + d(1+s) + \frac{d^2 s}{12(1-d)} \leq -6 + \frac{3}{2} + \frac{1}{12} < 0.$$

For the first inequality, differentiating $-d^2(1-ds)/(12(1-d))$ gives the displayed positive term and two nonpositive terms, using $ds \leq 1$. For the second inequality, use $d \leq 1/2$ and $ds \leq 1$. It follows that the minimum over the allowed values of d is attained at the largest allowed value, $1/2$ if $0 \leq s \leq 2$, and $1/s$ if $s \geq 2$.

For $0 \leq s \leq 2$, we have

$$240(s+1)\big(U(s,1/2) - g(s)\big) = 35s^2 - 137s + 164 > 0,$$

since the quadratic has positive leading coefficient and discriminant -4191. For $s \geq 2$, we have

$$10s^2(s+1)\big(U(s,1/s) - g(s)\big) = 22(s-2)^3 + 113(s-2)^2 + 138(s-2) + 5 > 0.$$

Both identities follow by inserting the indicated value of d into $U(s,d)$ and collecting terms. This proves the claim. By (4.5), the integral multiplied by $da^2 n(n-1)$ is at most $G(s) < g(s) < U(s,d)$. Dividing by $d > 0$ and using the definition of U proves the lemma. □

Proof of Conjecture 1.1 and equality in Remark 1.2. By the reductions in Section 2 and Lemma 2.2, it suffices to exclude (2.4) for $n \geq 6$ and $0 < a < 1$. If $0 < \alpha \leq 3$, Lemmas 3.6 and 4.1 give a contradiction. If $\alpha \geq 3$, Lemma 4.2, together with $2ax > a^2 + d$ and $\beta \leq \widetilde{\beta}$ from Lemma 3.2 and (3.3), contradicts (3.4). Thus $\rho(R) > 1$, which yields the strict inequality in Conjecture 1.1 for all interior zeros. The boundary argument in Section 2 shows that equality forces $p(z) = z^n - 1$ after normalization. Undoing the rotation and monic normalization gives $p(z) = c(z^n - a^n)$ with $|a| = 1$ and $c \neq 0$. Conversely, this polynomial has all its critical points at 0, so equality holds. □

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