Interlacing triangles with an alternating bottom row

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Abstract. We enumerate interlacing triangles whose bottom row begins with an ascent and then alternates. At size eight there are exactly 22 726 530 931 960 444 539 200 such triangles. We give the counts for sizes one through eight and the accompanying table of all bottom-row patterns. At each tested size from two through eight, the two alternating patterns are the only ones attaining the largest count.

1. The enumeration

A triangle of size n has rows of lengths $1, 2, \ldots, n$, filled with the integers $1, \ldots, N$, each used once, where $N = n(n+1)/2$. It is *interlacing* if each entry above the bottom row lies strictly between its two children. Kimberling's Problem 18 asks for the unrestricted count $T(n)$ [1], tabulated as OEIS A347608 [2].

Write $b_1, \ldots, b_n$ for the bottom row, read from left to right. Each adjacent comparison gives one letter, U for an ascent and D for a descent. We count triangles by this word, keeping reflected fillings distinct. Let $a(n)$ denote the count for $b_1 < b_2 > b_3 < b_4 > \cdots$. For $n = 1$ the word is empty and the unique filling contributes one.

Table 1. Alternating bottom rows with a prescribed first ascent.

n	$a(n)$	$T(n)$, all patterns
1	1	1
2	1	2
3	8	20
4	524	1 744
5	425 816	2 002 568
6	6 143 567 544	42 263 042 752
7	2 129 327 925 003 968	21 686 691 099 024 768
8	22 726 530 931 960 444 539 200	344 069 541 824 691 045 987 328

At size three, the eight triangles counted by $a(3)$ are

$$\begin{smallmatrix} & & 3 & & \\ & 2 & & 5 & \\ 1 & & 6 & & 4 \end{smallmatrix} \quad \begin{smallmatrix} & & 3 & & \\ & 5 & & 2 & \\ 4 & & 6 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & & 4 & & \\ & 2 & & 5 & \\ 1 & & 6 & & 3 \end{smallmatrix} \quad \begin{smallmatrix} & & 4 & & \\ & 3 & & 5 & \\ 1 & & 6 & & 2 \end{smallmatrix} \quad \begin{smallmatrix} & & 4 & & \\ & 3 & & 5 & \\ 2 & & 6 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & & 4 & & \\ & 5 & & 2 & \\ 3 & & 6 & & 1 \end{smallmatrix} \quad \begin{smallmatrix} & & 4 & & \\ & 5 & & 3 & \\ 1 & & 6 & & 2 \end{smallmatrix} \quad \begin{smallmatrix} & & 4 & & \\ & 5 & & 3 & \\ 2 & & 6 & & 1 \end{smallmatrix}.$$

Of the twenty size-three triangles, eight have pattern UD, eight have pattern DU, and two each have patterns UU and DD.

Complement symmetry. Replacing each label x by $N + 1 - x$ reverses every comparison and preserves strict betweenness. This involution pairs each bottom word with its letterwise complement. Thus either alternating orientation gives $a(n)$, and together they give $2a(n)$ for $n \geq 2$. At size one both orientations describe the same filling. Reflection across the vertical axis also preserves the count, sending a word to its reversed complement.

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2. Counting methods and related enumerations

Three algorithms gave the alternating counts in Table 1 for $2 \le n \le 8$. *Rank insertion* builds a total order on the cells, working up from the bottom. A parent enters at a rank between its children. The algorithm keeps the ranks of cells needed by later comparisons. *Value placement* assigns labels in increasing order and counts ways to reach each set of occupied cells. A parent can receive the next label when exactly one child has been filled. The bottom comparisons impose additional precedence conditions. *Orientation summation* uses Sidoli's decomposition [3] and sums the numbers of linear extensions of the compatible partial orders. The latter two implementations use closely related hash-table routines.

A separate counter enumerates chains of occupied sets. At every threshold, two occupied children require an occupied parent, and two empty children require an empty parent. Each valid chain of single-cell additions gives one filling, with labels equal to the steps at which the cells enter. This reproduces each of the 128 counts at size eight. Summing the counts at each size gives the corresponding value of A347608 [2].

Howe, Huffman, Kohlmeyer and Kruer give a transfer-matrix formula for $T(n)$ and refine it by the bottom positions of the two extreme labels [4, Theorem 1.1]. Their formula specializes to $a(n)$ by restricting its bottom-row indicator to rows that also satisfy $b_1 < b_2 > b_3 < \cdots$. The order-preserving relabeling in their inclusion–exclusion argument preserves all these inequalities. This specialization gives $1, 8, 524$ at sizes two, three and four.

Increasing bottom row. When consecutive bottom entries satisfy $b_j < b_{j+1}$, their parent p_j satisfies $b_j < p_j < b_{j+1}$. If the whole bottom row increases, then $p_j < b_{j+1} < p_{j+1}$ for every adjacent pair of parents. Iterating proves that all rows increase. This is the increasing-row class counted by A003121 [2]. An increasing row can have a nonmonotone row below it, as in the triangle with rows 4, $(2, 5)$, $(3, 1, 6)$.

Two rows. Take m top entries t_i and $m+1$ bottom entries b_i, filled bijectively with $1, \ldots, 2m+1$, with the bottom pattern unrestricted. Reading $b_1, t_1, b_2, t_2, \ldots, t_m, b_{m+1}$ gives a permutation. Interlacing means exactly that no even position is a local extremum. The even and odd positions recover the array, proving that its count is A122647$(m + 1)$ [2].

Sequence search. Eight OEIS queries for prefixes and large terms, in one orientation and in both orientations, returned no matching entry on 26 November 2026. A control query for the unrestricted counts returned A347608. Priority for the alternating sequence remains undetermined.

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References

[1] C. Kimberling, *Unsolved problems and rewards*, Problem 18, Triangles with Interlacing Rows.

[2] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, entries A347608, A003121 and A122647.

[3] J. B. Sidoli, *On the Number of Interlacing Triangles of Size n*, 2021.

[4] W. Howe, G. Huffman, J. Kohlmeyer and L. Kruer, *Counting triangles with interlacing rows*, Hexagon 2611.00688v1, November 2026.