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DIFFERENTIAL TRANSCENDENCE OF CUBIC AND QUARTIC POLYNOMIAL SERIES

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Abstract. We prove that the characteristic generating function of the values of any integer-valued quartic polynomial with positive leading coefficient satisfies no algebraic differential equation, even after restricting its nonnegative integer inputs to a nonempty union of residue classes. In particular, this proves the fourth-power case of the conjecture of Chowla and Chowla. The proof extends Helminen's theorem for the cubic series $H(q) = \sum_{n\geq 0} q^{n^3}$. We give a self-contained exposition of his argument and establish the additional analytic estimates required for quartics. Exact Fourier decompositions separate exponential contributions whose arithmetic labels lead to an invertible Vandermonde matrix.

1. Introduction

1.1. **Main results.** A formal power series $F \in \mathbb{C}[[q]]$ is called *differentially algebraic* if it satisfies an *algebraic differential equation* (ADE) of the form $Q(q, F, F', \ldots, F^{(k)}) = 0$ for some $k \geq 0$ and some nonzero polynomial $Q \in \mathbb{C}[q, X_0, \ldots, X_k]$. Derivatives are taken term by term. Otherwise, F is called *differentially transcendental*. Equivalently, the family $q, F, F', F'', \ldots$ is algebraically independent over $\mathbb{C}$ in $\mathbb{C}[[q]]$.

Consider the following four generating functions, which exhibit different behavior:

$$\sum_{n\geq 0} q^n \ = \ \frac{1}{1-q}\,, \quad \sum_{n\geq 0} q^{n^2}\,, \quad H(q) \ := \sum_{n\geq 0} q^{n^3} \quad \text{and} \qquad L(q) \ := \sum_{n\geq 0} q^{n^4}.$$

Here the first is rational, while the second satisfies an ADE by Jacobi's classical work [Jac48]. The corresponding question for the last two series remained open until now.

Theorem 1.1 (Helminen [Hel26a])**.** *The series $H(q)$ is differentially transcendental over $\mathbb{C}(q)$.*

Theorem 1.2. *The series $L(q)$ is differentially transcendental over $\mathbb{C}(q)$.*

In Theorem 2.1, we extend the quartic theorem to allow lower-degree terms, rational coefficients giving integer values, and periodic restrictions on the input indices.

We present Helminen's cubic proof in detail and extend its method to quartics. The argument rewrites each generating function as a sum of exponential contributions and distinguishes their products by arithmetic properties of the exponents. Exact integral formulas are needed because the contributions that obstruct an ADE can be exponentially small. See also §12.2 for further work on the subject.

1.2. **Background.** Chowla and Chowla [CC84] conjectured in 1984 that $\sum_{n\geq 1} q^{n^d}$ satisfies no ADE for every integer $d > 2$. Lipshitz and Rubel later recorded the cubic case explicitly [LR91, p. 212]; it appears as Conjecture 2.6 in [Pak18]. Their earlier gap theorem [LR86] proves differential transcendence for a power series with infinitely many nonzero coefficients and *Hadamard gaps*: its successive nonzero exponents m_j satisfy $m_{j+1}/m_j \geq 1+\delta$ for some $\delta > 0$. This criterion does not apply to cubes or fourth powers, since $(n+1)^d/n^d \to 1$. Rubel's survey [Rub89] gives broader historical background.

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Costin and Huang [CH09, Thm. 4] obtained analytic expansions near the circle of convergence for series with power exponents, including the exponential scales proportional to $n^{3/2}$ for cubes. Helminen [Hel26a] uses these scales to exclude differential relations. His release contains two proofs, one using analytic continuation and one on the positive real ray formalized in Lean. We develop the second argument here. The separate prime-counting result [Hel26b] resolves the corresponding part of Conjecture 2.5 in [Pak18].

The linear case is much easier. A series $F \in \mathbb{C}[[q]]$ is called *D-finite* if

$$\sum_{j=0}^{r} a_j(q) F^{(j)}(q) = 0 \quad \text{for some} \quad a_j \in \mathbb{C}[q],$$

where not all a_j are zero. The square, cubic, and quartic series above have radius of convergence one and gaps between successive nonzero exponents tending to infinity. Fabry's gap theorem therefore makes the unit circle a *natural boundary*, meaning that analytic continuation is impossible across any point of that circle [Fab96, p. 382]. Now, a D-finite function analytic at the origin has only finitely many finite singularities [FGS05, Thm. 1], so these series are not D-finite.

Similarly, recall that the coefficient sequences of D-finite series are *P-recursive*, i.e. satisfy the fixed-order linear recurrences with polynomial coefficients [Sta80, Ch. 6]. Now, beyond the finitely many zeros of the leading recurrence coefficient, a sufficiently long block of zeros forces all later coefficients to vanish. This excludes a linear equation but is compatible with Jacobi's ADE for the theta function.

AI disclaimer. Except for mild editing by the author (mostly restricted to Introduction and References), this paper is fully written by ChatGPT-6 Pro in response to a single prompt. The author made no mathematical contribution to the paper, made no effort to check the validity of the results, and takes no credit for obtaining them. The goal was to clarify Helminen's AI assisted cubic proof in [Hel26a] and check whether this approach can be generalized to higher degrees.

2. Definitions and notation

We write $\mathbb{N} = \{0, 1, 2, \ldots\}$. The ring $\mathbb{C}[[q]]$ consists of formal series $F = \sum_{m \geq 0} f_m q^m$. Write $[q^m]F = f_m$ for coefficient extraction. Addition is coefficientwise, and multiplication is the *Cauchy product*: $[q^m](FG) = \sum_{i=0}^{m} f_i g_{m-i}$ for $G = \sum_{m \geq 0} g_m q^m$. The formal derivative is

$$\partial_q F = \sum_{m \geq 0} (m+1) f_{m+1} q^m, \qquad F^{(j)} = \partial_q^j F, \quad F^{(0)} = F.$$

The field $\mathbb{C}((q))$ of formal Laurent series allows finitely many negative powers and contains the rational function field $\mathbb{C}(q)$. Thus differential algebraicity is equivalent to the existence of a nonzero $R \in \mathbb{C}(q)[X_0, \ldots, X_r]$ with $R(F, F', \ldots, F^{(r)}) = 0$ in $\mathbb{C}((q))$: multiplying by a common nonzero denominator gives an ADE with polynomial coefficients.

An indexed family $(f_i)_{i \in I}$ in a commutative $\mathbb{C}$-algebra is *algebraically independent over* $\mathbb{C}$ if no nonzero polynomial over $\mathbb{C}$ in finitely many entries with distinct indices evaluates to zero.

The total degree of a nonzero polynomial is the largest sum of the exponents in a monomial with nonzero coefficient. Complex fractional powers use the principal logarithm, on $\mathbb{C} \setminus (-\infty, 0]$. Real powers p^s use $p > 0$. The relations $f(p) = O(p^s)$ and $f(p) = o(p^s)$ are understood as $p \to +\infty$; the first means $|f(p)| \leq C p^s$ for all sufficiently large p, and the second means $f(p)/p^s \to 0$. We say a series of holomorphic functions converges *normally* if it converges absolutely and uniformly on each compact subset of its domain. Constants in estimates may depend on a fixed derivative order, but not on a varying mode index.

For a set $S \subseteq \mathbb{N}$, its *characteristic generating function* is $F_S(q) = \sum_{m\in S} q^m$: each value has coefficient one, regardless of how many descriptions it has. A polynomial $P \in \mathbb{Q}[x]$ is *integer-valued on* $\mathbb{N}$ if $P(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Equivalently, it is an integer linear combination of the *binomial polynomials* $\binom{x}{i}$; see, e.g., [Nag19, Pol19].

Given an integer $M \geq 1$ and a nonempty set $R \subseteq \{0, \dots, M-1\}$, define

$$S_{P,R} = \{P(n) : n \in \mathbb{N},\ n \bmod M \in R,\ P(n) \geq 0\}.$$

Thus the restriction is periodic in the input index; the values of P need not form a periodic set.

Theorem 2.1. *Let $P \in \mathbb{Q}[x]$ be integer-valued on $\mathbb{N}$, have degree four, and have positive leading coefficient. For every M and R as above, the characteristic generating function $F_{S_{P,R}}$ is differentially transcendental over $\mathbb{C}(q)$.*

Theorem 2.1 is proved in §11. Its applications include fourth powers in an arithmetic progression, the values n^4+n^2, and the binomial values $\binom{n}{4}$. Repeated values and finitely many negative values cause no difficulty: the characteristic series counts each nonnegative value once.

3. From a formal equation to an autonomous equation

We first justify analytic evaluation. We then show that a putative formal equation places the function Y, defined below, in a finite polynomial differential system. A dimension argument produces an autonomous equation for Y. This reduction is independent of the Fourier analysis that follows.

3.1. **The analytic statement.** For $s > 0$ let $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}\,du$. For $p > 0$, let

$$q(p) \ := \ e^{-1/p^2}, \qquad Y(p) \ := \ H(q(p)) \ - \ \frac{1}{2} \ - \ \Gamma(4/3)\,p^{2/3}. \tag{3.1}$$

These functions are real analytic on $(0,\infty)$, that is, locally given by convergent power series in p with complex coefficients.

From now on, $Y^{(j)}$ and derivatives of other functions of p mean d^j/dp^j, while derivatives of H mean $\partial_q^j H$. A *tail* is an interval (p_0,∞) with $p_0 > 0$. An *autonomous polynomial differential relation* has constant complex coefficients and no explicit occurrence of the variable.

Theorem 3.1. *For every integer $J \geq 1$ and every $P \in \mathbb{C}[X_0, \dots, X_{J-1}]$, if $P(Y, Y', \dots, Y^{(J-1)}) = 0$ on a tail, then $P = 0$.*

The rest of this section reduces Theorem 1.1 to this statement. Sections 4–6 then give an exact decomposition of Y into a background and terms with exponential factors $e^{-an^{3/2}p}$ and $e^{-\bar{a}n^{3/2}p}$, where a is a fixed nonreal number with $\operatorname{Re} a > 0$. We call the constant λ in $e^{-\lambda p}$ its *exponential rate*. Section 7 proves that each distinct rate in a polynomial identity has zero coefficient. Writing $n = dk^2$, where no prime square divides d, assigns formal variables to these factors without losing their multiplicative relations. Sections 8–9 finish the proof with an invertible matrix of partial derivatives evaluated at distinct prime indices. Section 11 supplies the new analytic and arithmetic steps for quartic polynomials.

3.2. **Evaluation of the formal series.** For a condition E, write $\mathbf{1}_E = 1$ if E holds and $\mathbf{1}_E = 0$ otherwise. For $j, m \geq 0$, formal differentiation gives

$$[q^m]H^{(j)} = \frac{(m+j)!}{m!}\,\mathbf{1}_{\{m+j \text{ is a cube in } \mathbb{N}\}}, \qquad \left|[q^m]H^{(j)}\right| \leq j!(m+1)^j. \tag{3.2}$$

The inequality follows from $m + i \leq i(m+1)$ for $1 \leq i \leq j$; for $j = 0$ it is immediate.

Lemma 3.2. *Every formal derivative $H^{(j)}$ converges absolutely and locally uniformly in $|q| < 1$. It is the corresponding analytic derivative of H. Every polynomial identity among $q, H, H', \dots$ in $\mathbb{C}[[q]]$ therefore holds at every $q \in \mathbb{C}$ with $|q| < 1$.*

Proof. For $0 < r < 1$, the series $\sum_{m\geq 0}(m+1)^j r^m$ converges. Equation (3.2) gives uniform convergence on $|q| \leq r$ for every fixed derivative order. The usual power-series differentiation theorem applies. Absolute convergence also justifies each finite Cauchy product, so formal polynomial evaluation agrees with analytic evaluation. A formal zero series has zero analytic sum. □

In particular, the composition with $q(p)$ in (3.1) is real analytic. To see this without a boundary argument, fix $p_0 > 0$. The map $p \mapsto e^{-1/p^2}$ has a holomorphic extension to a small complex disk about p_0 whose image lies in $|q| < 1$. The same applies to the chosen branch of $p^{2/3}$ near p_0.

3.3. **Polynomial differential systems.** Let I be an open real interval. A *polynomial differential system* on I is a smooth map $u = (u_1, \ldots, u_N) : I \to \mathbb{C}^N$ satisfying $u_i' = \Phi_i(u)$, where each $\Phi_i \in \mathbb{C}[U_1, \ldots, U_N]$. For any polynomial g in the U_i, the chain rule gives

$$\frac{d}{dp} g(u(p)) = (\mathcal{D}g)(u(p)), \qquad \mathcal{D}g = \sum_{i=1}^{N} \Phi_i \frac{\partial g}{\partial U_i}.$$

Thus $\mathcal{D}$ sends polynomials to polynomials.

Lemma 3.3. *If u satisfies such a system and $y = g(u)$ for a polynomial g, then there is a nonzero $P \in \mathbb{C}[T_0, \ldots, T_N]$ such that $P(y, y', \ldots, y^{(N)}) = 0$ on I.*

Proof. For $j \geq 0$, induction gives $y^{(j)} = (\mathcal{D}^j g)(u)$. The $N+1$ polynomials $g, \mathcal{D}g, \ldots, \mathcal{D}^N g$ in N variables are algebraically dependent. For completeness, this last fact follows by counting monomials. If their degrees are at most $M \geq 1$, monomials of total degree at most L in $N+1$ formal inputs number $\binom{L+N+1}{N+1}$. Their evaluations are polynomials in $U_1, \ldots, U_N$ of degree at most ML, a vector space of dimension $\binom{ML+N}{N}$. For large L the former number is larger. A nontrivial linear dependence gives the required nonzero polynomial P. Evaluating it along u proves the assertion. □

3.4. **The transfer argument.** Define actual functions on $(0, \infty)$ by

$$v_0(p) = q(p), \qquad v_{j+1}(p) = H^{(j)}(q(p)) \quad (j \geq 0).$$

A polynomial identity in these functions means an identity holding for every $p > 0$.

Proposition 3.4. *If a nonzero polynomial over $\mathbb{C}$ vanishes at a finite set of $q, H, H', \ldots$ in $\mathbb{C}[[q]]$, then Y satisfies a nonzero autonomous polynomial differential relation on all of $(0, \infty)$.*

Proof. Lemma 3.2 supplies at least one nonzero polynomial identity in the functions v_i. Among all such identities choose one of least total degree, say $Q(v_0, \ldots, v_n) = 0$, with v_n actually occurring. A polynomial in v_0 alone cannot vanish identically, since $q(p)$ ranges over $(0, 1)$. Hence $n \geq 1$.

Let $S = \partial Q/\partial X_n$. Characteristic zero and the actual occurrence of X_n imply $S \neq 0$. Its smaller total degree implies that $S(v_0, \ldots, v_n)$ is not identically zero. Choose $p_0 > 0$ where it is nonzero and an open interval I about p_0 where it remains nonzero. On I, consider the $N = n + 4$ state functions

$$(v_0, \ldots, v_n, r, b, w), \qquad r = p^{-1}, \quad b = p^{2/3}, \quad w = S(v_0, \ldots, v_n)^{-1}.$$

The derivatives of the first n coordinates and of r, b are

$$v_0' = 2r^3 v_0, \qquad v_i' = 2r^3 v_0 v_{i+1} \ (1 \leq i < n), \qquad r' = -r^2, \qquad b' = \tfrac{2}{3} r b.$$

Differentiating $Q(v) = 0$ determines the remaining derivative:

$$v_n' = -w \sum_{i=0}^{n-1} Q_{X_i}(v) v_i'. \tag{3.3}$$

Finally,

$$w' = -w^2 \sum_{i=0}^{n} S_{X_i}(v) v_i'.$$

After inserting the displayed formulas, every right-hand side is a polynomial in the state coordinates. Moreover, $Y = v_1 - \frac{1}{2} - \Gamma(4/3)b$ is a polynomial in the state. Lemma 3.3 therefore gives a nonzero polynomial P with $P(Y, Y', \dots, Y^{(N)}) = 0$ on I.

This last expression is real analytic on the connected interval $(0, \infty)$. The identity theorem for real-analytic functions extends its vanishing from I to all of $(0, \infty)$. Notice that w need only be defined locally; it is the resulting identity in Y, whose derivatives exist everywhere, that is continued. □

4. Exact Fourier and Airy formulas

The separation argument needs an exact integral representation, with all convergence estimates stated explicitly. We obtain one by first applying Poisson summation, then identifying the Airy term by its differential equation. All fractional powers of complex numbers in this section use the principal logarithm.

Put

$$h = e^{-i\pi/6}, \qquad k = e^{i\pi/6}, \qquad a = \frac{4\pi^{3/2}}{3\sqrt{3}}(1-i), \qquad \kappa = \left(\frac{\pi}{6}\right)^{1/4} e^{-i\pi/8}.$$

For $t > 0$ and $z \in \mathbb{C}$ define

$$J_t(z) = \int_0^\infty e^{-tr^3 + zr}\, dr.$$

The cubic factor makes this integral absolutely convergent for every z. We use the contour definition

$$\mathrm{Ai}(X) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{w^3/3 - Xw}\, dw, \tag{4.1}$$

where $\mathcal{C}$ follows the ray of argument $-\pi/3$ inward to zero, then the ray of argument $\pi/3$ outward. The integral and its derivatives in X converge locally uniformly. An integration by parts gives $\mathrm{Ai}''(X) = X\,\mathrm{Ai}(X)$.

For $n \geq 1$ let $b_n = 2\pi n$, and for $p > 0$ put $\alpha = \alpha(p) = 3^{1/3} p^{-2/3}$. Define

$$B(p) = 2 \sum_{n \geq 1} \mathrm{Re}\big(ikJ_{p^{-2}}(-kb_n)\big), \qquad W_n(p) = \frac{2\pi h}{\alpha}\, \mathrm{Ai}\left(\frac{b_n h}{\alpha}\right).$$

The following lemma includes convergence of these expressions.

Lemma 4.1 (Poisson–Airy identity). *For every $p > 0$, the function Y defined in (3.1) satisfies*

$$Y(p) = B(p) + \sum_{n \geq 1} \big(W_n(p) + \overline{W_n(p)}\big). \tag{4.2}$$

The real background series and the wave series converge absolutely.

Proof. Apply Poisson summation to $f(r) = e^{-p^{-2}|r|^3}$. This function is twice continuously differentiable, and f, f', f'' are integrable. Its periodization $\sum_{m \in \mathbb{Z}} f(x+m)$ and the series of its first two derivatives converge uniformly for $x \in [0,1]$. Two integrations by parts show absolute summability of its Fourier coefficients. Evaluating its Fourier series at zero therefore gives (see also [SS03, Ch. 5, §3])

$$\sum_{m \geq 0} e^{-m^3/p^2} = \frac{1}{2} + \int_0^\infty e^{-r^3/p^2}\, dr + 2 \sum_{n \geq 1} \mathrm{Re}\, J_{p^{-2}}(ib_n).$$

The zero-frequency integral is $\Gamma(4/3)p^{2/3}$, by the substitution $u = r^3/p^2$ and $\Gamma(4/3) = \Gamma(1/3)/3$. To evaluate a nonzero mode, parametrize the two rays in (4.1) and substitute $w = \alpha r e^{\pm i\pi/3}$. For $b > 0$ and $\alpha = (3t)^{1/3}$ this gives exactly

$$2\pi i \operatorname{Ai}(bh/\alpha) = \alpha\big(e^{i\pi/3}J_t(-kb) - e^{-i\pi/3}J_t(ib)\big),$$

hence

$$J_t(ib) = ikJ_t(-kb) + \frac{2\pi h}{\alpha}\operatorname{Ai}(bh/\alpha). \tag{4.3}$$

For summability of the background, subtract its purely imaginary leading term:

$$ikJ_t(-kb) - \frac{i}{b} = ik\int_0^\infty e^{-kbr}(e^{-tr^3} - 1)\,dr.$$

Since $|1 - e^{-tr^3}| \le tr^3$,

$$\big|\operatorname{Re}\big(ikJ_t(-kb)\big)\big| \le \frac{6t}{(b\operatorname{Re}k)^4}.$$

Thus the real background is absolutely summable. The wave summability also follows directly from Lemma 4.3 below (so the identity can first be established for finite partial sums). □

Remark 4.2. The real part is taken before summation. Integration by parts gives $J_t(ib_n) = i/b_n + O(b_n^{-2})$ as $n \to \infty$ with t fixed, so the complex series $\sum_n J_t(ib_n)$ diverges. Consequently $2\operatorname{Re}\sum_n J_t(ib_n)$ would be an undefined expression. The same distinction applies to the rotated background.

4.1. **The normalized Airy kernel.** For complex z with $\operatorname{Re} z > -2$ define

$$\mathcal{B}(z) = \frac{1}{\Gamma(5/6)\Gamma(1/6)}\int_0^1 s^{-1/6}(1-s)^{-5/6}(1+zs/2)^{-1/6}\,ds. \tag{4.4}$$

The beta identity $\int_0^1 s^{x-1}(1-s)^{y-1}\,ds = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ for $x, y > 0$ gives $\mathcal{B}(0) = 1$. The integral is holomorphic on this half-plane, by domination on compact subsets. If $\operatorname{Re} z \ge 0$, then $|\mathcal{B}(z)| \le 1$. For a function F on $[0,\infty)$ define its normalized Laplace transform, when the integral converges, by

$$\mathcal{L}F(p) = p\int_0^\infty e^{-pu}F(u)\,du.$$

For $n \ge 1$ let $c_n = an^{3/2}$ and $E_n(u) = \mathcal{B}(u/c_n)$. The number c_n is the complex exponential *rate* of W_n.

Lemma 4.3 (Exact wave kernel)**.** *The function E_n is analytic on a neighborhood of $[0,\infty)$, $E_n(0) = 1$, and $|E_n(u)| \le 1$ for $u \ge 0$. For every $p > 0$,*

$$W_n(p) = \kappa n^{-1/4}p^{1/2}e^{-c_n p}\mathcal{L}E_n(p). \tag{4.5}$$

Proof. We first prove, on the ray $X = hr$, $r > 0$, the exact formula

$$\operatorname{Ai}(X) = \frac{e^{-z}}{2\sqrt{\pi}X^{1/4}\Gamma(5/6)}\int_0^\infty e^{-v}v^{-1/6}(1+v/(2z))^{-1/6}\,dv, \qquad z = \frac{2}{3}X^{3/2}. \tag{4.6}$$

This is the classical Airy integral representation [NIST, 9.5.8]; the following argument checks its normalization and branch choices on the required ray. Set

$$T(z) = \int_0^\infty e^{-v}v^{-1/6}(v+2z)^{-1/6}\,dv, \qquad G(X) = C_0e^{-z}T(z), \quad C_0 = \frac{3^{-1/6}2^{-2/3}}{\sqrt{\pi}\Gamma(5/6)}.$$

On the required ray, $\arg(2z) = -\pi/4$ and $\arg(1+v/(2z)) \in [0, \pi/4)$ for $v \ge 0$. Thus $(v+2z)^{-1/6} = (2z)^{-1/6}(1+v/(2z))^{-1/6}$ with the stated principal powers. This identifies G with the right-hand

side of (4.6). On $\operatorname{Re} z > 0$, differentiation under the integral is valid on compact subsets. If $w = e^{-v}v^{-1/6}(v+2z)^{-1/6}$, direct differentiation gives

$$3zT'' + (1-6z)T' - T = \int_0^\infty \frac{d}{dv}\left(\frac{vw}{v+2z}\right) dv = 0.$$

The primitive tends to zero at both endpoints. Since $dz/dX = X^{1/2}$, the displayed equation is equivalent to $G''(X) = XG(X)$.

For $X = hr$, the parameter z has argument $-\pi/4$. The real part of $(v+2z)^{-1/6}$ is strictly positive, so $T(z) \neq 0$. Moreover, $G(hr)$ and its first radial derivative decay exponentially as $r \to \infty$. For example, $|v+2z| \geq v + 2\operatorname{Re} z$ gives $T(z) = O(r^{-1/4})$ and $T'(z) = O(r^{-7/4})$. In contrast, the contour representation (4.1) shows directly that $\mathrm{Ai}(hr)$ and its first radial derivative are bounded: on the two contour rays, $\operatorname{Re}(he^{\pm i\pi/3}) \geq 0$, and the cubic factor is $e^{-s^3/3}$. Write the two radial functions as $u(r) = \mathrm{Ai}(hr)$ and $v(r) = G(hr)$. Both solve the equation $f''(r) = h^3 r f(r)$. Their *Wronskian* $W(r) = u(r)v'(r) - u'(r)v(r)$ satisfies $W' = uv'' - u''v = 0$, and tends to zero at infinity by the bounds just proved. Thus $W = 0$. Since $v(r) \neq 0$, the derivative of u/v is zero, and their ratio is constant.

At zero, dominated convergence gives $G(0) = C_0\Gamma(2/3)$, whereas the contour gives

$$\mathrm{Ai}(0) = \frac{3^{-1/6}\Gamma(1/3)}{2\pi}.$$

The duplication formula $\Gamma(1/3)\Gamma(5/6) = 2^{1/3}\sqrt{\pi}\Gamma(2/3)$ shows that these values agree. This proves (4.6).

To identify its kernel, use the elementary beta–gamma identity: if V has density e^{-v} on $(0,\infty)$ and S has density $s^{-1/6}(1-s)^{-5/6}/(\Gamma(5/6)\Gamma(1/6))$ on $(0,1)$, independently, then VS has density $e^{-y}y^{-1/6}/\Gamma(5/6)$. Indeed, the change of variables $y = vs$ and $r = v(1-s)$ transforms their joint density into

$$\frac{e^{-y-r}y^{-1/6}r^{-5/6}}{\Gamma(5/6)\Gamma(1/6)}\, dy\, dr.$$

Applying this identity to $(1+y/(2z))^{-1/6}$ gives

$$\int_0^\infty e^{-v}\mathcal{B}(v/z)\, dv = \frac{1}{\Gamma(5/6)}\int_0^\infty e^{-y}y^{-1/6}(1+y/(2z))^{-1/6}\, dy.$$

All these integrals are absolutely convergent. With $X = b_n h/\alpha$ we have $z = c_n p$ and

$$\frac{2\pi h}{\alpha}\cdot\frac{1}{2\sqrt{\pi}X^{1/4}} = \kappa n^{-1/4}p^{1/2}.$$

Finally, substitute $v = pu$ in the normalized Laplace transform. This proves (4.5) and all the kernel assertions. □

4.2. **Ordinary derivatives and a common bound.** For nonnegative integers m define

$$M_{n,m}(p) = \int_0^\infty e^{-pu}u^m E_n(u)\, du.$$

Differentiation under the integral gives $M'_{n,m} = -M_{n,m+1}$ and

$$|M_{n,m}(p)| \leq m!p^{-m-1}, \qquad p > 0. \tag{4.7}$$

For a real number d and an integer $s \geq 0$, write $(d)_{\underline{s}} = d(d-1)\cdots(d-s+1)$, with value 1 if $s = 0$.

For $j \geq 0$ define the *amplitude* $A_{j,n}$ by $W_n^{(j)}(p) = e^{-c_n p}A_{j,n}(p)$; thus an amplitude is the factor remaining after the specified exponential has been removed.

Lemma 4.4 (Derivative amplitudes). *For every $j \geq 0$,*

$$A_{j,n}(p) = \kappa n^{-1/4} \sum_{r+s+t=j} \frac{j!}{r!s!t!}(-c_n)^r (3/2)_{\underline{s}} (-1)^t p^{3/2-s} M_{n,t}(p). \tag{4.8}$$

For each fixed j there is a constant K_j such that

$$|A_{j,n}(p)| \leq K_j p^{1/2} n^{3j/2-1/4} \quad (n \geq 1,\ p \geq 1), \tag{4.9}$$

and for each fixed n,

$$p^{-1/2} A_{j,n}(p) \longrightarrow \kappa n^{-1/4} (-c_n)^j. \tag{4.10}$$

The series of the jth derivatives of the waves converges uniformly on every compact subinterval of $(0, \infty)$.

Proof. Since (4.5) is $W_n = \kappa n^{-1/4} e^{-c_n p} p^{3/2} M_{n,0}$, Leibniz's rule gives (4.8). Bound its terms using (4.7). For $p \geq 1$, $p^{3/2-s}|M_{n,t}(p)| \leq t! p^{1/2-s-t} \leq t! p^{1/2}$; also $|c_n|^r \leq \max(1, |a|)^j n^{3j/2}$. This proves (4.9).

The substitution $u = v/p$ and dominated convergence give $pM_{n,0}(p) = \int_0^\infty e^{-v} E_n(v/p)\, dv \to 1$. In (4.8), the term $r = j, s = t = 0$ therefore has the limit in (4.10); after division by $p^{1/2}$ all other terms are $O(p^{-s-t}) = o(1)$. On any interval $p_0 \leq p \leq p_1$ with $p_0 > 0$, the same termwise estimates give a constant times $n^{3j/2-1/4} e^{-p_0(\operatorname{Re} a) n^{3/2}}$, a summable majorant. Thus the series may be differentiated any fixed number of times. □

5. An analytic Laplace kernel for the background

We express the background B as a normalized Laplace transform with an analytic kernel. This will place its derivatives in the same amplitude algebra as the waves.

5.1. **The inverse of a cubic phase.** Fix $b > 0$ and retain $k = e^{i\pi/6}$. Define

$$f_b(y) = y^3 + kby, \qquad a_b = b \operatorname{Re} k > 0,$$

and the closed sector

$$S = \{y \in \mathbb{C} : \operatorname{Re} y \geq 0, -\operatorname{Re} y/\sqrt{3} \leq \operatorname{Im} y \leq 0\}.$$

Lemma 5.1. *For every $v \geq 0$ there is a unique $y_b(v) \in S$ with $f_b(y_b(v)) = v$. This inverse has a holomorphic extension near each point of $[0, \infty)$, and*

$$y_b(0) = 0, \qquad y_b'(v) = \frac{1}{3y_b(v)^2 + kb}, \qquad |y_b(v)| \leq \frac{v}{a_b}, \qquad |y_b'(v)| \leq \frac{1}{a_b}. \tag{5.1}$$

Proof. If $x, y \in S$, each nonzero product among x^2, xy, y^2 has argument in $[-\pi/3, 0]$, so each has nonnegative real part. Therefore

$$|f_b(x) - f_b(y)| = |x - y|\, |x^2 + xy + y^2 + kb| \geq a_b |x - y|. \tag{5.2}$$

This proves uniqueness. Also $\operatorname{Re}(3y^2 + kb) \geq a_b$ on S.

For existence, let $u \geq 0$ and define

$$\tau(u) = \frac{1}{\sqrt{3}(1 + u^2)}, \qquad s(u) = u \sqrt{\frac{a_b}{3 - \tau(u)^2}}, \qquad y(u) = s(u)(1 - i\tau(u)).$$

The denominator is positive and $y(u) \in S$. For $u > 0$, the identity $s(u)^2(3 - \tau(u)^2) = a_b u^2$ gives

$$\operatorname{Im} f_b(y(u)) = s(u)\{-s(u)^2 \tau(u)(3 - \tau(u)^2) + a_b(1/\sqrt{3} - \tau(u))\} = 0.$$

The same assertion holds at $u = 0$. Moreover,

$$\operatorname{Re} f_b(y(u)) \geq a_b s(u) \geq a_b \sqrt{a_b/3}\, u.$$

Thus the image is a continuous nonnegative real function starting at zero and tending to infinity. The intermediate value theorem proves existence for every $v \geq 0$.

Inequality (5.2) shows that the inverse on the ray is Lipschitz continuous. Since f_b' never vanishes at these roots, the complex inverse function theorem gives a local holomorphic inverse at each of them. Continuity and uniqueness identify it with y_b on the nearby real ray. The derivative formula follows by differentiation; its bound follows from $\operatorname{Re} f_b' \geq a_b$. Taking $y = 0$ in (5.2) gives the bound on $y_b(v)$. □

5.2. **Straightening the integration path.** On $[0, \infty)$ define

$$\psi_b(v) = \frac{ik}{3y_b(v)^2 + kb} - \frac{i}{b}, \qquad K_b(v) = 2 \operatorname{Re} \psi_b(v). \tag{5.3}$$

The subtraction is purely imaginary and changes no real part.

Lemma 5.2. *For every $p, b > 0$,*

$$2 \operatorname{Re}\big(ikJ_{p-2}(-kb)\big) = \mathcal{L}K_b(p). \tag{5.4}$$

The kernel K_b is holomorphic on a neighborhood of the positive ray after extension from its real values, and

$$|K_b(v)| \leq \frac{6v^2}{(\operatorname{Re} k)^3 b^4} \qquad (v \geq 0). \tag{5.5}$$

Proof. Scaling the integration variable by p gives

$$J_{p-2}(-kb) = p \int_0^\infty e^{-pf_b(y)}\, dy.$$

For a finite endpoint $y_b(V) = s + i\eta$, compare the real segment from 0 to s with the inverse curve $v \mapsto y_b(v)$, $0 \leq v \leq V$. Join their endpoints by the vertical segment at real part s. For $-s/\sqrt{3} \leq w \leq 0$,

$$\operatorname{Re} f_b(s + iw) = s^3 - 3sw^2 + b\big((\operatorname{Re} k)s - (\operatorname{Im} k)w\big) \geq a_b s.$$

The connector integral is consequently at most $(s/\sqrt{3})e^{-pa_b s}$ in absolute value. The integrand is entire, so the two integrals differ only by this connector. As $V \to \infty$, also $s \to \infty$: otherwise $y_b(V)$ would stay bounded in S, contradicting $f_b(y_b(V)) = V$. The connector therefore tends to zero. On the inverse curve, $dy = y_b'(v)\, dv$. The bound in (5.1) justifies the resulting improper integral, and gives

$$\int_0^\infty e^{-pf_b(y)}\, dy = \int_0^\infty \frac{e^{-pv}}{3y_b(v)^2 + kb}\, dv.$$

Taking the real part after multiplication by ikp proves (5.4).

The exact identity

$$\psi_b(v) = -\frac{3i\, y_b(v)^2}{b(3y_b(v)^2 + kb)}$$

together with (5.1) gives $|\psi_b(v)| \leq 3v^2/(ba_b^3)$, hence (5.5). Finally, near a real point extend ψ_b holomorphically by Lemma 5.1, and extend K_b by

$$\widetilde{K}_b(z) = \psi_b(z) + \overline{\psi_b(\overline{z})}. \tag{5.6}$$

This is a holomorphic function of z on the common reflected neighborhood. It agrees with K_b on the real axis. Local extensions agree on overlaps by the identity theorem, after restricting to disks centered on the real ray. Their union gives the required neighborhood. □

5.3. **Summing the kernels and differentiating.** Set

$$K(v) = \sum_{n\geq 1} K_{b_n}(v), \qquad b_n = 2\pi n. \tag{5.7}$$

The estimate in (5.5) proves absolute convergence and a bound $|K(v)| \leq C_B v^2$ on $v \geq 0$, where $C_B = 6(\operatorname{Re} k)^{-3} \sum_{n\geq 1} b_n^{-4} < \infty$.

Proposition 5.3. *The function K extends holomorphically to a neighborhood of $[0,\infty)$, and $B(p) = \mathcal{L}K(p)$ for every $p > 0$. For $j \geq 1$,*

$$B^{(j)}(p) = p \int_0^\infty e^{-pv}(-v)^j K(v)\, dv + j \int_0^\infty e^{-pv}(-v)^{j-1} K(v)\, dv. \tag{5.8}$$

For every $j \geq 0$, $B^{(j)}(p) = O(p^{-j-2})$.

Proof. We first establish analyticity of the infinite sum. The inverse functions satisfy the scaling identity

$$y_b(v) = \sqrt{b}\, y_1(v/b^{3/2}),$$

because both sides lie in S and solve the same cubic equation. Near zero, the inverse function theorem then gives the holomorphic identity

$$\psi_b(z) = b^{-1}\psi_1(z/b^{3/2}).$$

Both $\psi_1(0)$ and $\psi_1'(0)$ vanish. Fix $R > 0$. The Taylor bound $|\psi_1(z)| \leq C|z|^2$ near zero implies, for all sufficiently large b,

$$|\psi_b(z)| \leq C|z|^2 b^{-4} \qquad (|z| \leq R).$$

These extensions agree with the inverse on the real ray on the relevant real segment by uniqueness and continuation. The same estimate holds for the reflected term in (5.6). Thus the tail of (5.7) converges normally on every fixed disk after finitely many modes have been omitted. Near a prescribed real $v_0 \geq 0$, choose local inverse disks for the remaining finitely many modes. Adding them to that holomorphic tail proves analyticity near v_0. The local sums agree on overlapping real-centered disks, giving one neighborhood of the entire ray.

The quadratic real-axis bound and $\sum_n b_n^{-4} < \infty$ permit interchanging the sum with the Laplace integral. Applying Lemma 5.2 termwise gives $B = \mathcal{L}K$. On a compact interval $[p_0, p_1] \subset (0,\infty)$, all terms obtained by differentiating $pe^{-pv}K(v)$ j times are dominated by a constant times $e^{-p_0 v}(1+v)^{j+2}$. Differentiation under the integral and the product rule give (5.8). Finally, using $\int_0^\infty e^{-pv} v^m\, dv = m! p^{-m-1}$ yields, for $j \geq 1$,

$$|B^{(j)}(p)| \leq C_B\{(j+2)! + j(j+1)!\}p^{-j-2}.$$

For $j = 0$ the bound is $2C_B p^{-2}$. □

Combining the last proposition with the wave bounds proves the exact derivative expansion

$$Y^{(j)}(p) = B^{(j)}(p) + \sum_{n\geq 1} \left(e^{-c_n p} A_{j,n}(p) + e^{-\overline{c}_n p}\overline{A_{j,n}(p)} \right). \tag{5.9}$$

Every series here is absolutely convergent and may be differentiated any fixed number of times on compact subintervals of $(0,\infty)$. Complex conjugation commutes with differentiation in the real variable p.

6. An algebra of Laplace amplitudes

An asymptotic expansion does not, by itself, distinguish a zero function from a nonzero exponentially small one. The argument needs a class of amplitudes in which such ambiguity cannot occur, and that class must be closed under all finite sums and products arising from a polynomial. We construct it here by convolution of analytic kernels.

6.1. **Kernels, convolution, and first terms.** Call a function $F : [0,\infty) \to \mathbb{C}$ an *admissible kernel* if it extends holomorphically to an open connected neighborhood of $[0,\infty)$ and there are constants $C, D \geq 0$ such that $|F(u)| \leq Ce^{Du}$ for $u \geq 0$. For $p > D$ its normalized Laplace transform is $\mathcal{L}F(p) = p\int_0^\infty e^{-pu}F(u)\,du$, as above. For two kernels define their *convolution* on the real ray by

$$(F * G)(u) = \int_0^u F(v)G(u-v)\,dv.$$

Lemma 6.1. *Admissible kernels are closed under finite sums, complex scalar multiplication, multiplication by a polynomial in u, complex conjugation on the real ray, and convolution. On a sufficiently large tail,*

$$\mathcal{L}(F+G) = \mathcal{L}F + \mathcal{L}G, \qquad (\mathcal{L}F)(\mathcal{L}G) = p\mathcal{L}(F*G), \qquad \mathcal{L}1 = 1. \tag{6.1}$$

Proof. The first four closure assertions follow directly from the definition; for conjugation use the reflected extension $z \mapsto \overline{F(\overline{z})}$. For convolution, take a common open neighborhood U on which both kernels are holomorphic. The set

$$V = \{z \in U : tz \in U \text{ for every } 0 \leq t \leq 1\}$$

is open by compactness of each segment, contains $[0,\infty)$, and is star-shaped about zero. On V the formula

$$z\int_0^1 F(tz)G((1-t)z)\,dt$$

is holomorphic, as follows by differentiation on compact subsets. It extends $F * G$ from the real ray. If $|F(u)| \leq C_1e^{D_1u}$ and $|G(u)| \leq C_2e^{D_2u}$, then

$$|(F*G)(u)| \leq C_1C_2u\,e^{\max(D_1,D_2)u},$$

which is again of exponential growth. For sufficiently large p all double integrals are absolutely convergent. Fubini's theorem and the substitution $u = v + w$ prove (6.1). The last identity is the elementary integral $p\int_0^\infty e^{-pu}\,du = 1$. □

Lemma 6.2 (First nonzero Laplace term)**.** *Let F be an admissible kernel. Either F vanishes on the whole positive ray, or there is a least integer $r \geq 0$ with $F^{(r)}(0) \neq 0$. In the latter case,*

$$\mathcal{L}F(p) = F^{(r)}(0)p^{-r} + O(p^{-r-1}). \tag{6.2}$$

Proof. If all derivatives at zero vanish, the convergent Taylor series vanishes near zero. The holomorphic identity theorem on the connected neighborhood then gives vanishing on the whole ray. Otherwise write $F(u) = F^{(r)}(0)u^r/r! + O(u^{r+1})$ near zero and split the Laplace integral at a fixed small $\varepsilon > 0$. The Taylor remainder contributes $O(p^{-r-1})$ after the outside factor p is included. Extending the leading monomial integral to infinity changes it by an exponentially small term, and

$$p\int_0^\infty e^{-pu}u^r\,du = r!p^{-r}.$$

The growth bound on F makes the remaining tail at most a constant times $p(p-D)^{-1}e^{-(p-D)\varepsilon}$. This proves the formula. It is the elementary form of Watson's lemma [NIST, §2.3(ii)]. □

6.2. **Germs and the two power classes.** Two functions defined on tails are equivalent if they agree on some smaller tail. Their equivalence classes are called *germs at $+\infty$*. Pointwise addition and multiplication define a commutative ring $\mathcal{R}$ of such complex-valued germs. We write $[f]$ for the germ of f. Constants embed injectively in $\mathcal{R}$. A germ $[f]$ is a *unit*, meaning it has a multiplicative inverse, exactly when $f(p) \neq 0$ everywhere on some tail: its inverse is then the germ of $1/f$. We will not assume that $\mathcal{R}$ is a field or that it has no zero divisors.

For $\epsilon \in \{0, \frac{1}{2}\}$, let $\mathcal{A}_\epsilon$ consist of germs represented on a tail as

$$p^{m+\epsilon}\mathcal{L}F(p), \qquad m \in \mathbb{Z}, \quad F \text{ an admissible kernel.} \tag{6.3}$$

The zero germ belongs to both classes. Put $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_{1/2}$.

Proposition 6.3. *The set $\mathcal{A}$ is a $\mathbb{C}$-subalgebra of $\mathcal{R}$. Every nonzero $f \in \mathcal{A}$ has a first nonzero power: there are $s \in \frac{1}{2}\mathbb{Z}$ and $\gamma \in \mathbb{C} \setminus \{0\}$ such that any representative of f satisfies*

$$f(p) = p^s(\gamma + o(1)). \tag{6.4}$$

Proof. To add expressions with different outside powers in the same class, we first align those powers. Convolving a kernel with the constant kernel 1 gives, by (6.1),

$$p^{m+\epsilon}\mathcal{L}F = p^{m+1+\epsilon}\mathcal{L}(F * 1).$$

Repeated convolution therefore raises the outside exponent by any nonnegative integer without changing the represented function. For finitely many expressions choose the largest outside integer exponent, raise each to that exponent, and add the kernels. This proves closure under finite addition within each class.

The product identity in (6.1) adds the outside powers and one more integer. Consequently

$$\mathcal{A}_0\mathcal{A}_0 \subseteq \mathcal{A}_0, \qquad \mathcal{A}_0\mathcal{A}_{1/2} \subseteq \mathcal{A}_{1/2}, \qquad \mathcal{A}_{1/2}\mathcal{A}_{1/2} \subseteq \mathcal{A}_0.$$

Constants belong to $\mathcal{A}_0$ because $\mathcal{L}c = c$. All closure assertions follow.

By Lemma 6.2, a nonzero expression of the form (6.3) has leading exponent $m + \epsilon - r$ for an integer $r \geq 0$. Write $f = f_0 + f_{1/2}$ with $f_\epsilon \in \mathcal{A}_\epsilon$. If only one summand is nonzero, the conclusion follows immediately. If both are nonzero, their leading exponents lie in the disjoint sets $\mathbb{Z}$ and $\frac{1}{2}+\mathbb{Z}$. The larger exponent dominates the sum and its coefficient is nonzero. This proves (6.4). □

Corollary 6.4. *For every $j \geq 0$ and $n \geq 1$, the germs of $B^{(j)}$ lie in $\mathcal{A}_0$, and those of $A_{j,n}$ and $\overline{A_{j,n}}$ lie in $\mathcal{A}_{1/2}$. Every finite sum of finite products of these functions therefore has its germ in $\mathcal{A}$.*

Proof. The kernels $u^m K(u)$ from Proposition 5.3 are admissible. Formula (5.8) expresses $B^{(j)}$ as a sum of normalized Laplace transforms with integer outside powers, and $B = \mathcal{L}K$ handles $j = 0$. The kernels $u^m E_n(u)$ are admissible by Lemma 4.3. Since $M_{n,m} = p^{-1}\mathcal{L}(u^m E_n)$, formula (4.8) expresses each $A_{j,n}$ as a finite sum with outside powers $p^{1/2-s}$, $s \in \mathbb{N}$. Conjugation preserves this form. Apply Proposition 6.3. □

Remark 6.5. Products of two half-integer amplitudes belong to the integer class, so cancellations must be collected within each class before applying the first-term argument. Convolution handles these cancellations exactly. For more general Borel–Laplace methods, see [Sau14].

7. Separation of the complex exponential rates

We now prove that a polynomial identity in the actual functions $Y^{(j)}$ gives a separate identity at each complex rate. Two issues must be addressed: different imaginary parts can have the same real decay, and there are infinitely many modes. A finite-frequency lemma handles the first; a summable uniform estimate handles the second.

For distinct real numbers $b_1, \ldots, b_s$ and complex numbers $\gamma_1, \ldots, \gamma_s$, call $T(p) = \sum_{i=1}^{s} \gamma_i e^{ib_i p}$ a *finite trigonometric sum*.

Lemma 7.1. *If a finite trigonometric sum $T(p)$ tends to zero as $p \to \infty$, then all its coefficients are zero.*

Proof. Fix k and a lower endpoint p_0. The average

$$\frac{1}{R - p_0} \int_{p_0}^{R} T(p) e^{-ib_k p}\, dp$$

tends to zero because $T(p) \to 0$. Its kth summand is exactly γ_k. Each other summand tends to zero, since the integral of $e^{i(b_i - b_k)p}$ is bounded independently of R. Thus $\gamma_k = 0$. □

7.1. **Expanding a polynomial and bounding its tail.** Fix $J \geq 1$ and a polynomial $P \in \mathbb{C}[X_0, \dots, X_{J-1}]$ of total degree at most d. In each monomial substitute (5.9) and distribute the finite product. Label the resulting terms by a countable set I_P. A label records the original monomial and, for each factor, the choice of a background or of a signed mode. Thus the expanded expression has the form

$$P(Y, Y', \dots, Y^{(J-1)}) = \sum_{\nu \in I_P} e^{-\lambda_\nu p} a_\nu(p). \tag{7.1}$$

Here λ_ν is a sum of at most d numbers chosen from c_n and $\bar{c}_n$, and a_ν is a constant times a product of background derivatives and wave amplitudes. Background factors contribute zero to λ_ν. The empty product has rate zero. Let Λ_P be the set of these complex rates and put

$$C_\lambda(p) = \sum_{\nu : \lambda_\nu = \lambda} a_\nu(p) \qquad (\lambda \in \Lambda_P). \tag{7.2}$$

The next lemma proves, in particular, that each such sum is finite.

Lemma 7.2. *The set $\{\lambda \in \Lambda_P : \operatorname{Re}\lambda \leq R\}$ is finite for every real R. Every fiber in (7.2) is finite. There are nonnegative numbers W_λ such that*

$$|C_\lambda(p)| \leq p^{d/2} W_\lambda \quad (p \geq 1), \qquad \sum_{\lambda \in \Lambda_P} W_\lambda e^{-\operatorname{Re}\lambda} < \infty. \tag{7.3}$$

The expansion (7.1) is absolutely convergent, and may be grouped as $\sum_{\lambda \in \Lambda_P} e^{-\lambda p} C_\lambda(p)$. Every $[C_\lambda]$ belongs to $\mathcal{A}$.

Proof. Let $\rho = \operatorname{Re} a > 0$. Each selected wave with index n contributes $\rho n^{3/2}$ to the real part of the rate. For $R < 0$ there are no such labels. For $R \geq 0$, if that real part is at most R, every selected index satisfies $n \leq (R/\rho)^{2/3}$. There are at most d selected waves, finitely many monomials, and finitely many choices of their signs and positions. This proves finiteness of all labels below a real-rate ceiling, hence both finiteness assertions.

For $p \geq 1$, Proposition 5.3 bounds every background derivative occurring here by a constant. Lemma 4.4 bounds every selected wave amplitude by a constant times $p^{1/2} n^{3J/2}$. Thus for each label there is a nonnegative weight w_ν with $|a_\nu(p)| \leq p^{d/2} w_\nu$, and

$$\sum_{\nu \in I_P} w_\nu e^{-\operatorname{Re}\lambda_\nu} < \infty.$$

Indeed this sum is bounded by a finite sum of finite powers of $1 + \sum_{n \geq 1} n^{3J/2} e^{-\rho n^{3/2}}$, which converges. Taking $W_\lambda = \sum_{\lambda_\nu = \lambda} w_\nu$ proves (7.3). For arbitrary fixed $p > 0$, the compact-interval versions of the same derivative estimates prove absolute convergence of (7.1). Absolute convergence justifies both product expansion and regrouping. Finally each C_λ is a finite sum of products from Corollary 6.4, so belongs to $\mathcal{A}$ as a germ. □

Proposition 7.3. *If $P(Y, Y', \dots, Y^{(J-1)}) = 0$ on a tail, then $[C_\lambda] = 0$ in $\mathcal{R}$ for every $\lambda \in \Lambda_P$.*

Here, each coefficient vanishes on a tail that may depend on its rate. The ring of germs records precisely this conclusion.

Proof. Suppose some $[C_\lambda]$ is nonzero. By local finiteness in Lemma 7.2, the real parts of such rates have a least value, say r. There are only finitely many rates of real part at most r. Every coefficient with real part less than r vanishes on a common tail, since it is a zero germ and this is a finite collection. The zero germs within the cluster of real part r can also be discarded on a common tail, since that cluster is finite.

The rates with real part larger than r, if any, have real part at least $r+\delta$ for some $\delta>0$, again by local finiteness. Equation (7.3) gives the explicit estimate

$$\sum_{\operatorname{Re}\lambda>r} e^{(r-\operatorname{Re}\lambda)p}|C_\lambda(p)| \le p^{d/2}e^r e^{-\delta(p-1)} \sum_{\lambda\in\Lambda_P} W_\lambda e^{-\operatorname{Re}\lambda} \qquad (p\ge 1). \tag{7.4}$$

If there are no higher rates, this tail is zero.

Among the nonzero germs with real part r, use Proposition 6.3 to write $C_\lambda(p)=p^{s_\lambda}(\gamma_\lambda+o(1))$ with $\gamma_\lambda\neq 0$. This is a nonempty finite collection. Let s be its largest exponent. Multiply the grouped zero identity by $e^{rp}p^{-s}$. The lower rates vanish exactly on a tail, the higher rates tend to zero by (7.4), and terms at real part r with a smaller power tend to zero. We obtain

$$\sum_{\operatorname{Re}\lambda=r,\ s_\lambda=s} \gamma_\lambda e^{-i(\operatorname{Im}\lambda)p} \longrightarrow 0.$$

Distinct complex rates with the same real part have distinct imaginary parts. Lemma 7.1 forces every displayed γ_λ to vanish, a contradiction. □

8. Squarefree coordinates and marked formal series

The separation theorem distinguishes complete complex rates. To use formal differentiation we must identify those rates with monomials without either losing a genuine collision or inventing a new one. The squarefree decomposition of the mode index supplies these coordinates.

8.1. **Independence of squarefree radicals.** A positive integer is *squarefree* if it is not divisible by the square of a prime. Let $\mathcal{S}$ denote the set of positive squarefree integers, including 1. Unique factorization gives a unique decomposition

$$n=dk^2, \qquad d\in\mathcal{S}, \quad k\ge 1, \qquad n^{3/2}=d\sqrt{d}\,k^3. \tag{8.1}$$

Lemma 8.1. *The numbers $d\sqrt{d}$, $d\in\mathcal{S}$, are linearly independent over $\mathbb{Q}$.*

Proof. It suffices to prove independence of the square roots in any finite collection. Let $p_1,\dots,p_s$ be all primes appearing in that collection and let $L_j=\mathbb{Q}(\sqrt{p_1},\dots,\sqrt{p_j})$. Two nonzero rational numbers have the same *square class* if their quotient is a square in $\mathbb{Q}$. We recall why adjoining each square root doubles the degree. Inductively, if a nonzero rational number x has a square root in L_j, its square class is a product of the classes of $p_1,\dots,p_j$. This is clear for $j=0$. Assuming it for $j-1$, p_j cannot be a square in L_{j-1}, by unique prime factorization in $\mathbb{Q}$. Hence every element of L_j has a unique form $u+v\sqrt{p_j}$, $u,v\in L_{j-1}$. If its square is rational, the coefficient $2uv$ of $\sqrt{p_j}$ is zero. Either $v=0$, reducing to $j-1$, or $u=0$, reducing to the rational number x/p_j. This proves the induction claim and the degree assertion.

It follows that the 2^s products $\prod_{i\in T}\sqrt{p_i}$, $T\subseteq\{1,\dots,s\}$, form a $\mathbb{Q}$-basis of L_s. Distinct squarefree integers give distinct such basis elements, including the empty product for $d=1$. Their square roots are therefore linearly independent. Multiplying each by its nonzero rational number d preserves independence. □

8.2. **The marking monoid.** Introduce one variable $z_{d,+}$ and one variable $z_{d,-}$ for each $d\in\mathcal{S}$. Let $I=\mathcal{S}\times\{+,-\}$. A *multi-index* is a function $\alpha: I\to\mathbb{N}$ whose *support* $\operatorname{supp}\alpha=\{i\in I:\alpha_i\neq 0\}$ is finite. Its total degree is $|\alpha|=\sum_{i\in I}\alpha_i$, and its monomial is $z^\alpha=\prod_{i\in I} z_i^{\alpha_i}$. The multi-index e_i has value 1 at i and zero elsewhere. The multi-indices form a commutative monoid under coordinatewise addition, meaning an associative addition with identity 0.

For a multi-index define its rate by

$$\lambda(\alpha)=a\sum_{d\in\mathcal{S}}\alpha_{d,+}d\sqrt{d}+\overline{a}\sum_{d\in\mathcal{S}}\alpha_{d,-}d\sqrt{d}. \tag{8.2}$$

Both sums are finite. Give the positive wave at $n = dk^2$ the mark $z_{d,+}^{k^3}$, and its conjugate the mark $z_{d,-}^{k^3}$. Equation (8.1) says that its actual rate equals the rate of its mark. Rates add when marks multiply.

Lemma 8.2. *The map $\alpha \mapsto \lambda(\alpha)$ is injective. Thus two products of signed waves have equal complex rates if and only if their marking monomials are equal.*

Proof. Write the two real totals in (8.2) as U and V. Since $a = \rho(1 - i)$ with $\rho > 0$,

$$\operatorname{Re}\lambda = \rho(U + V), \qquad \operatorname{Im}\lambda = \rho(V - U).$$

Equality of complex rates therefore gives equality of U and of V separately. Lemma 8.1 identifies every rational coefficient in each total, hence every coordinate of α. The converse follows at once from (8.2). □

For example, the mode $4d$ has rate $8c_d$ and mark $z_{d,+}^{8}$, the same mark as eight copies of the fundamental mode d. This real collision is retained. Distinct squarefree parts supply the independent directions; distinct mode indices need not do so.

8.3. **Formal series in the marks.** Let $\mathcal{R}[[z_i : i \in I]]$ be the ring of coefficient assignments $F = \sum_\alpha f_\alpha z^\alpha$, with $f_\alpha \in \mathcal{R}$, and write $[z^\alpha]F = f_\alpha$. No convergence in the variables z_i is required. Multiplication is defined by

$$[z^\alpha](FG) = \sum_{\beta+\gamma=\alpha} f_\beta g_\gamma.$$

This sum is finite, with $\prod_{i\in\operatorname{supp}\alpha}(\alpha_i+1)$ possible choices of β. The coefficient $[z^0]F$ is the constant coefficient. For each $i \in I$ formal differentiation is defined by

$$[z^\alpha]\frac{\partial F}{\partial z_i} = (\alpha_i + 1)f_{\alpha+e_i}.$$

Finite coefficient sums prove the product rule and the chain rule for substitution into a polynomial.

Define the marked jth derivative, for $j \geq 0$, by

$$F_j = [B^{(j)}] + \sum_{d\in\mathcal{S},\ k\geq 1} \left([A_{j,dk^2}] z_{d,+}^{k^3} + [\overline{A_{j,dk^2}}] z_{d,-}^{k^3} \right). \tag{8.3}$$

This is a well-defined formal series: a specified nonconstant monomial receives at most one term in this display.

Proposition 8.3. *Let $J \geq 1$ and $P \in \mathbb{C}[X_0, \ldots, X_{J-1}]$. If $P(Y, Y', \ldots, Y^{(J-1)}) = 0$ on a tail, then*

$$P(F_0, \ldots, F_{J-1}) = 0 \quad \text{in } \mathcal{R}[[z_i : i \in I]].$$

Proof. Fix a multi-index α. Expanding the polynomial formally collects exactly those labeled products whose marks multiply to z^α. By Lemma 8.2, these are exactly the products of complex rate $\lambda(\alpha)$ in the analytic expansion (7.1). Both collections are finite by Lemma 7.2. Consequently

$$[z^\alpha]P(F_0, \ldots, F_{J-1}) = [C_{\lambda(\alpha)}],$$

where an absent rate means coefficient zero. Proposition 7.3 sets this germ equal to zero. This holds for every α, proving the formal identity. □

9. The Vandermonde Jacobian

Only finitely many marking variables are needed for a fixed finite family of derivatives. We choose them at distinct primes. The linear coefficients form an invertible matrix over the germ ring, and this forces every polynomial relation to have vanishing partial derivatives.

Fix $J \geq 1$ and distinct primes $\ell_1, \ldots, \ell_J$. For $1 \leq i \leq J$ and $0 \leq j < J$ define

$$M_{ij}(p) = A_{j,\ell_i}(p).$$

Since each prime is squarefree, equation (8.3) gives $[z_{\ell_i,+}]F_j = [M_{ij}]$. The exponent of this mark is 1, so only the fundamental mode $n = \ell_i$ contributes.

Lemma 9.1. *The determinant of the matrix $([M_{ij}])$ is a unit in $\mathcal{R}$.*

Proof. By (4.10),

$$p^{-1/2} M_{ij}(p) \longrightarrow \kappa \ell_i^{-1/4} (-c_{\ell_i})^j.$$

The determinant of the limiting matrix is

$$\left(\prod_{i=1}^{J} \kappa \ell_i^{-1/4} \right) \prod_{1 \leq i < k \leq J} \big((-c_{\ell_k}) - (-c_{\ell_i}) \big), \tag{9.1}$$

by the Vandermonde determinant formula. Indeed, the determinant of (x_i^j) is alternating in the x_i, vanishes whenever two of them coincide, and has total degree $J(J-1)/2$. It is therefore a constant multiple of $\prod_{i<k}(x_k - x_i)$; the coefficient of $x_1^0 x_2^1 \cdots x_J^{J-1}$ is 1 on both sides. Every factor is nonzero: $\kappa \neq 0$ and the numbers $c_{\ell_i} = a\ell_i^{3/2}$ are distinct. Continuity of the determinant therefore gives a nonzero limit of $p^{-J/2} \det M(p)$. Thus $\det M(p) \neq 0$ for every sufficiently large p. Its reciprocal on that tail represents its inverse in $\mathcal{R}$. □

The *Jacobian matrix* for these choices is the $J \times J$ matrix of formal partial derivatives

$$\mathcal{J}_{ij} = \frac{\partial F_j}{\partial z_{\ell_i,+}}, \qquad 1 \leq i \leq J, \quad 0 \leq j < J.$$

Its entries lie in $\mathcal{R}[[z_i : i \in I]]$ and its constant coefficient matrix is $([M_{ij}])$.

Lemma 9.2. *A formal series in $\mathcal{R}[[z_i : i \in I]]$ with unit constant coefficient is a unit. Consequently the matrix $\mathcal{J}$ is invertible over this ring.*

Proof. For $F = f_0 + \sum_{\alpha \neq 0} f_\alpha z^\alpha$ with f_0 a unit, construct G recursively by $g_0 = f_0^{-1}$ and

$$g_\alpha = -f_0^{-1} \sum_{\beta + \gamma = \alpha,\ \beta \neq 0} f_\beta g_\gamma \qquad (\alpha \neq 0).$$

The sums are finite and $|\gamma| < |\alpha|$, so induction on total degree defines every coefficient and gives $FG = 1$. The constant coefficient of $\det \mathcal{J}$ is $\det([M_{ij}])$, a unit by Lemma 9.1. Hence $\det \mathcal{J}$ is a unit, and the adjugate formula gives an inverse for $\mathcal{J}$. This argument is valid over a ring with zero divisors. □

Proposition 9.3. *For every $J \geq 1$, the marked series $F_0, \ldots, F_{J-1}$ are algebraically independent over $\mathbb{C}$.*

Proof. If a nonzero relation existed, choose one of least total degree, $P(F_0, \ldots, F_{J-1}) = 0$. Differentiate with respect to the J chosen marking variables. Writing $P_{X_j} = \partial P / \partial X_j$, the chain rule gives

$$\mathcal{J} \begin{pmatrix} P_{X_0}(F_0, \ldots, F_{J-1}) \\ \vdots \\ P_{X_{J-1}}(F_0, \ldots, F_{J-1}) \end{pmatrix} = 0.$$

Lemma 9.2 implies that all entries of this vector vanish. A nonzero partial derivative P_{X_j} would have smaller total degree than P, contradicting the choice of P. Thus all partial derivatives vanish as polynomials. In characteristic zero this makes P constant: any nonconstant monomial has a positive exponent in some variable and contributes a nonzero derivative there. A nonzero complex constant remains nonzero in $\mathcal{R}$ and in the formal series ring, giving the final contradiction. □

Proof of Theorem 3.1. A polynomial identity among the derivatives of Y gives, by Proposition 8.3, the same polynomial identity on their marked series. Proposition 9.3 then says that its polynomial is zero. □

Proof of Theorem 1.1. A nonzero formal relation among $q, H, H', \dots$ would give a nonzero autonomous relation on Y by Proposition 3.4. Theorem 3.1 excludes it. Clearing denominators then excludes every algebraic differential equation over $\mathbb{C}(q)$. □

Remark 9.4 (Why the square series behaves differently)**.** In Gaussian Poisson summation for the square-exponent theta series, the dual rates are proportional to the integers n^2. They therefore do not provide the infinitely many independent squarefree directions used in (8.2). For example, a mode with rate 4 and four copies of a mode with rate 1 have the same rate. Assigning an independent variable to every mode would destroy that equality. The cubic marking preserves its own collisions while retaining infinitely many independent directions. Thus the argument is consistent with Jacobi's differential equation for the square series.

10. Generating functions for sums of cubes

For an integer $k \geq 1$, let $r_k(m)$ be the number of ordered tuples $(n_1, \dots, n_k) \in \mathbb{N}^k$ satisfying $n_1^3 + \cdots + n_k^3 = m$. The Cauchy product gives

$$G_k(q) := \sum_{m \geq 0} r_k(m) q^m = H(q)^k.$$

Corollary 10.1. *For every $k \geq 1$, the generating function G_k is differentially transcendental over $\mathbb{C}(q)$.*

Proof. We use only elementary facts about differential fields. A subfield of $\mathbb{C}((q))$ is a *differential field* if it is closed under ∂_q. The *transcendence degree* of a field extension is the largest size of an algebraically independent subset, or the corresponding cardinal when that size is infinite. A field generated by finitely many elements has finite transcendence degree, and adjoining an algebraic element does not change it. See [Rit50] for general differential-algebraic background.

Suppose $G = H^k$ is differentially algebraic. Choose a nonzero relation $Q(G, G', \dots, G^{(r)}) = 0$ over $\mathbb{C}(q)$ of least total degree, with the highest variable actually occurring. Its partial derivative in that variable does not vanish on the derivatives of G, by the same minimal-degree argument used in Proposition 3.4. Differentiating $Q = 0$, including its rational-function coefficients, expresses $G^{(r+1)}$ as a rational function of $q, G, \dots, G^{(r)}$. Hence

$$E = \mathbb{C}(q)(G, G', \dots, G^{(r)})$$

is a differential field of finite transcendence degree over $\mathbb{C}(q)$. The field $E(H)$ is algebraic over E, since H satisfies $T^k - G = 0$. It is also closed under differentiation: $H \neq 0$ and

$$H' = \frac{G'}{kH^{k-1}} \in E(H),$$

so the product and quotient rules differentiate every rational expression in H with coefficients in E inside the same field. All derivatives of H therefore lie in this field of finite transcendence degree. They are algebraically dependent over $\mathbb{C}(q)$, contradicting Theorem 1.1. □

11. Quartic polynomial values and periodic restrictions

We now extend the same proof to characteristic generating functions of quartic polynomial values. The new analytic step is an exact quartic contour formula, including the lower-degree terms of the polynomial. The separation theorem of §7 and the Vandermonde argument of §9 then apply with cube-free indices.

11.1. **Normalization and the statement to be proved.** Retain the polynomial P, modulus M, residue set R, and value set $S_{P,R}$ of Theorem 2.1. Write

$$P(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E, \qquad A > 0, \qquad a = B/A, \quad b = C/A, \quad c = D/A.$$

In particular, $E = P(0) \in \mathbb{Z}$. The coefficients a, b, c are fixed real numbers. Put

$$Q(x) = x^4 + ax^3 + bx^2 + cx, \qquad T(q) = \sum_{\substack{n \geq 0 \\ n \bmod M \in R}} q^{P(n)-E}, \qquad q(p) = e^{-1/(Ap^3)}.$$

Here T is a formal Laurent series: only finitely many of its exponents can be negative. Since P is eventually positive and strictly increasing, and eventually exceeds all its values at the remaining finitely many indices,

$$F_{S_{P,R}}(q) - q^E T(q) \quad \text{is a Laurent polynomial.} \tag{11.1}$$

In particular, replacing multiplicities by characteristic coefficients affects only finitely many terms.

Adding a rational function, or multiplying by a nonzero rational function, preserves differential algebraicity in both directions. For example, substituting the derivatives of $g(q)X_0 + h(q)$ into a differential polynomial is an invertible triangular substitution over $\mathbb{C}(q)$ when $g \neq 0$. It therefore preserves nonzero relations. It suffices to prove that T is differentially transcendental.

The analytic function to be studied is

$$Y(p) = T(q(p)) = \sum_{\substack{n \geq 0 \\ n \bmod M \in R}} e^{-Q(n)/p^3}, \qquad p > 0. \tag{11.2}$$

This sum and all its derivatives converge normally locally in p. Indeed, the positive quartic leading term dominates every lower-degree term, uniformly on a sufficiently small complex neighborhood of any fixed positive p. We will prove that $Y, Y', Y'', \ldots$ are algebraically independent over $\mathbb{C}$.

11.2. **Four classes of Laplace amplitudes.** We use the same admissible kernels and normalized Laplace transform as in §6. For $\epsilon \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, define $\mathcal{A}_\epsilon$ by (6.3), and put

$$\mathcal{A}^{(4)} = \mathcal{A}_0 + \mathcal{A}_{1/4} + \mathcal{A}_{1/2} + \mathcal{A}_{3/4}.$$

The superscript distinguishes this algebra from the two-class algebra used for cubes. For a kernel F, write $\mathcal{I}F(u) = \int_0^u F(v)\,dv$ and $\mathcal{I}^0 F = F$. Thus $\mathcal{I}F = F * 1$, and $\mathcal{L}(\mathcal{I}^h F) = p^{-h}\mathcal{L}F$ for every integer $h \geq 0$.

Lemma 11.1. *The set $\mathcal{A}^{(4)}$ is a $\mathbb{C}$-algebra, is closed under differentiation, and contains the original amplitude algebra $\mathcal{A}$. Every nonzero member has the form $p^s(\gamma + o(1))$, where $s \in \frac{1}{4}\mathbb{Z}$ and $\gamma \neq 0$. If G is holomorphic near zero and $s \in \frac{1}{4}\mathbb{Z}$, then $p^s G(p^{-1})$ belongs to $\mathcal{A}^{(4)}$ as a germ.*

Proof. The convolution argument in Proposition 6.3 aligns integer outside powers within each class and multiplies the four classes by adding their fractional parts modulo 1. Watson's lemma gives a first nonzero power within each nonzero class. These four possible fractional parts are distinct, so the largest leading power in a nonzero sum cannot cancel. Differentiation preserves each class because

$$\frac{d}{dp}\big(p^s \mathcal{L}F(p)\big) = (s+1)p^{s-1}\mathcal{L}F(p) - p^s \mathcal{L}(uF(u))(p).$$

Both kernels on the right are admissible.

For the last assertion write $G(z)=\sum_{h\ge0} g_h z^h$. A Cauchy estimate gives $|g_h|\le C_0R_0^{-h}$ for some $C_0,R_0>0$. The entire kernel $\widehat G(u)=\sum_{h\ge0} g_h u^h/h!$ satisfies $|\widehat G(u)|\le C_0e^{u/R_0}$ on the positive ray. Termwise integration for $p>R_0^{-1}$ gives $\mathcal L\widehat G(p)=G(p^{-1})$. □

Define the zero Fourier integral by

$$Z(p)=\int_0^\infty e^{-Q(x)/p^3}\,dx.$$

Its membership in the amplitude algebra must be checked, since it will occur in products with the nonzero modes.

Lemma 11.2. *The germs of Z and all its derivatives belong to $\mathcal A^{(4)}$, and $Z^{(j)}(p)=O(p^{3/4-j})$ for each fixed $j\ge0$.*

Proof. The function

$$\Phi(v)=\int_0^\infty e^{-z^4-avz^3-bv^2z^2-cv^3z}\,dz$$

is entire in v. On $|v|\le R_0$, its integrand and each fixed v-derivative are bounded by an integrable polynomial multiple of $e^{-z^4+|a|R_0z^3+|b|R_0^2z^2+|c|R_0^3z}$. This justifies holomorphic differentiation under the integral. Write $\Phi(v)=\sum_{h\ge0}\phi_h v^h$, and for $r=0,1,2,3$ put $\Phi_r(w)=\sum_{h\ge0}\phi_{4h+r}w^h$. Each Φ_r is entire, by Cauchy estimates for Φ. The substitution $x=p^{3/4}z$ gives the exact identity

$$Z(p)=p^{3/4}\Phi(p^{-3/4})=\sum_{r=0}^{3}p^{(3-3r)/4}\Phi_r(p^{-3}). \tag{11.3}$$

Apply Lemma 11.1 to $G_r(w)=\Phi_r(w^3)$, and then differentiate. The convergent expansion at $p^{-3/4}=0$ also gives the stated growth estimates. □

11.3. **An exact quartic saddle contour.** For a positive Fourier frequency λ, define

$$f_\lambda(y)=y^4-i\lambda y,\qquad r_\lambda=(\lambda/4)^{1/3},\qquad s_\lambda=r_\lambda e^{i\pi/6},\qquad \chi_\lambda=f_\lambda(s_\lambda)=3r_\lambda^4e^{-i\pi/3}.$$

A *critical point* of f_λ is a zero of its derivative; its image is a *critical value*. The three critical values here are χ_λ, $\overline{\chi_\lambda}$, and $-3r_\lambda^4$. The ray $\chi_\lambda+[0,\infty)$ therefore contains no other critical value.

Lemma 11.3. *There are two branches $y_{\lambda,+}(u),y_{\lambda,-}(u)$, $u>0$, satisfying*

$$f_\lambda(y_{\lambda,\pm}(u))=\chi_\lambda+u,\qquad y_{\lambda,\pm}(u)=s_\lambda\pm\frac{1}{\sqrt6\,s_\lambda}\sqrt u+O(u).$$

They lie in the open first quadrant. As $u\to\infty$, the plus branch tends to infinity in the positive real direction and the minus branch in the positive imaginary direction. For an absolute constant C_0, they satisfy

$$|y_{\lambda,\pm}(u)|\le C_0(r_\lambda+u^{1/4}),\qquad \sqrt u\,|y'_{\lambda,\pm}(u)|\le C_0r_\lambda^{-1}. \tag{11.4}$$

The contour from $+i\infty$ along the minus branch to s_λ and then along the plus branch to $+\infty$ may replace the broken path $+i\infty\to0\to+\infty$ in the integral

$$\int e^{-pf_\lambda(y)}e^{-ay^3-by^2/p-cy/p^2}\,dy,\qquad p>0.$$

Proof. The local expansion follows from $f''_\lambda(s_\lambda)=12s_\lambda^2\ne0$ and the analytic inverse function theorem after taking a square-root coordinate. The branches continue along $u>0$: they meet no critical point there, and a root of a polynomial with bounded coefficients cannot escape to infinity while u stays bounded.

Scaling $y=r_\lambda z$ reduces the geometric assertions to $\lambda=4$. In this normalization $s=e^{i\pi/6}$ and $\chi=3e^{-i\pi/3}$. On the imaginary axis f_4 is real, whereas $\operatorname{Im}\chi=-3\sqrt3/2$. On the positive real axis,

the equation $\operatorname{Im} f_4(x) = \operatorname{Im}\chi$ forces $x = 3\sqrt{3}/8$; there $\operatorname{Re} f_4(x) = 729/4096 < 3/2 = \operatorname{Re}\chi$. Thus neither branch can leave the first quadrant for $u > 0$.

The saddle ray separates the two branches. Indeed,

$$\operatorname{Im} f_4(te^{i\pi/6}) = \frac{\sqrt{3}}{2}(t^4 - 4t) \qquad (t \geq 0)$$

has its unique minimum at $t = 1$, with value $\operatorname{Im}\chi$. The lifted level curve cannot cross this ray away from the saddle. The initial plus branch lies below it and the minus branch above it. For large u, the four roots are asymptotic to the four fourth roots of u. Consequently the plus and minus branches approach the positive real and positive imaginary directions, respectively. Their derivatives equal $(4y^3 - i\lambda)^{-1}$. The local expansion, the large-u root asymptotics, and compactness of the intervening portion give (11.4) when $\lambda = 4$. Scaling proves the general bounds.

For the contour deformation, truncate each branch at modulus L and join its endpoint to L or iL by a short circular arc. For all large L, these arcs lie in sectors where $\operatorname{Re} f_\lambda(y) \geq L^4/2 - \lambda L$. The modulus of the integrand is at most

$$\exp\{-p(L^4/2 - \lambda L) + |a|L^3 + |b|L^2/p + |c|L/p^2\}.$$

Its product with the arc length tends to zero for each fixed $p > 0$. The integrand is entire, so Cauchy's theorem on the truncated contours proves the assertion upon taking the limit. □

Define

$$J_p(i\lambda) = \int_0^\infty e^{-Q(x)/p^3 + i\lambda x}\, dx, \qquad V_\lambda(p) = p\int_0^\infty e^{-p(y^4+\lambda y)} e^{iay^3 + by^2/p - icy/p^2}\, dy.$$

The substitution $x = py$ followed by the contour deformation gives the exact identity

$$J_p(i\lambda) = iV_\lambda(p) + W_\lambda(p), \tag{11.5}$$

where the saddle contribution is

$$\begin{aligned} W_\lambda(p) = pe^{-\chi_\lambda p}\int_0^\infty e^{-pu}\big\{ & y'_{\lambda,+}(u) e^{-ay_{\lambda,+}(u)^3 - by_{\lambda,+}(u)^2/p - cy_{\lambda,+}(u)/p^2} \\ & - y'_{\lambda,-}(u) e^{-ay_{\lambda,-}(u)^3 - by_{\lambda,-}(u)^2/p - cy_{\lambda,-}(u)/p^2}\big\}\, du. \end{aligned} \tag{11.6}$$

All these integrals converge absolutely. In particular, (11.5) is an exact replacement for the Airy identity, with no omitted exponentially small remainder.

11.4. **The quartic wave kernels.** For integers $j, k \geq 0$ put

$$\begin{aligned} D_{\lambda,j,k}(u) &= \sqrt{u}\big\{ y'_{\lambda,+}(u) y_{\lambda,+}(u)^{2j+k} e^{-ay_{\lambda,+}(u)^3} \\ &\qquad - y'_{\lambda,-}(u) y_{\lambda,-}(u)^{2j+k} e^{-ay_{\lambda,-}(u)^3}\big\}, \\ E_{\lambda,j,k}(v) &= \frac{1}{\pi}\int_0^1 \frac{D_{\lambda,j,k}(tv)}{\sqrt{t(1-t)}}\, dt. \end{aligned}$$

The first expression has a removable singularity at zero. To see this, write $y_{\lambda,\pm}(u) = U_\lambda(\pm\sqrt{u})$ locally and let $g(z) = z^{2j+k}e^{-az^3}$. Then $D_{\lambda,j,k}(u)$ is

$$\frac{1}{2}\{g(U_\lambda(w))U'_\lambda(w) + g(U_\lambda(-w))U'_\lambda(-w)\}, \qquad w = \sqrt{u}.$$

This is an even holomorphic function of w, hence holomorphic in u. Continuation of the inverse branches gives a common holomorphic neighborhood of the positive ray for all these kernels. Restricting to a star-shaped subneighborhood permits the beta integral and all primitives below to be taken on straight segments.

Lemma 11.4. *The series*

$$K_\lambda = \sum_{j,k\ge 0} \frac{(-b)^j(-c)^k}{j!\,k!}\mathcal{I}^{j+2k}E_{\lambda,j,k} \tag{11.7}$$

defines an admissible kernel. For every $p>0$,

$$W_\lambda(p) = \sqrt{\pi}\,p^{1/2}e^{-\chi_\lambda p}\mathcal{L}K_\lambda(p), \qquad K_\lambda(0) = \frac{e^{-ia\lambda/4}}{\sqrt{6}\,s_\lambda} \neq 0. \tag{11.8}$$

For each fixed integer $h\ge 0$, there is a constant C_h, depending on a,b,c,h, such that

$$\int_0^\infty e^{-pu}u^h|K_\lambda(u)|\,du \le C_h r_\lambda^{-1}e^{C_h(r_\lambda^3+r_\lambda^2+r_\lambda+1)}p^{-h-1} \qquad (p\ge 1). \tag{11.9}$$

Proof. Write $r=r_\lambda$ temporarily. The root bounds (11.4) give

$$|E_{\lambda,j,k}(u)| \le C_0 r^{-1}e^{C_0|a|(r^3+u^{3/4})}[C_0(r+u^{1/4})]^{2j+k}.$$

An N-fold primitive is bounded by the maximum of this expression on $[0,u]$ times $u^N/N!$. Set $X = C_1|b|u(r+u^{1/4})^2$ and $V = C_1|c|u^2(r+u^{1/4})$, with a sufficiently large absolute C_1. Since $(j+2k)!\ge j!(2k)!\ge j!(k!)^2$,

$$\sum_{j,k\ge 0}\frac{X^jV^k}{j!\,k!\,(j+2k)!} \le \left(\sum_{j\ge 0}\frac{X^j}{(j!)^2}\right)\left(\sum_{k\ge 0}\frac{V^k}{(k!)^3}\right) \le e^{2\sqrt{X}+3V^{1/3}}.$$

The last inequality follows by taking diagonal subseries in products of two or three exponential series. Thus, with constants depending only on a,b,c,

$$|K_\lambda(u)| \le C_2 r^{-1}\exp\{C_2r^3 + C_2r\sqrt{u} + C_2r^{1/3}u^{2/3} + C_2u^{3/4}\}. \tag{11.10}$$

For fixed λ this grows more slowly than $e^{\delta u}$ for every $\delta>0$, up to a constant factor. On a compact subset of the common star-shaped complex neighborhood, the inverse branches and their regularized derivatives have uniform bounds. The same factorial majorant proves normal convergence of (11.7). This proves admissibility.

The beta–gamma identity with parameters $(1/2,1/2)$ gives, for each of the kernels in the definition,

$$p\int_0^\infty e^{-pu}u^{-1/2}D_{\lambda,j,k}(u)\,du = \sqrt{\pi}\,p^{1/2}\mathcal{L}E_{\lambda,j,k}(p).$$

For example, apply the substitutions $x=vt$, $y=v(1-t)$ to

$$\frac{1}{\pi}\int_0^\infty e^{-v}\int_0^1 \frac{D_{\lambda,j,k}(vt/p)}{\sqrt{t(1-t)}}\,dt\,dv.$$

Using $\mathcal{L}(\mathcal{I}^NE) = p^{-N}\mathcal{L}E$ and expanding the two exponentials containing p^{-1} and p^{-2} in (11.6) proves (11.8). All interchanges are absolute: in the saddle integral a majorant is a constant multiple of

$$e^{-pu}u^{-1/2}\exp\{C_3|a|(r^3+u^{3/4}) + C_3|b|(r^2+\sqrt{u})/p + C_3|c|(r+u^{1/4})/p^2\},$$

which is integrable for every $p>0$; the kernel side is controlled by (11.10). At zero, every positive-order primitive vanishes. Hence $K_\lambda(0) = e^{-as_\lambda^3}/(\sqrt{6}s_\lambda)$, and $s_\lambda^3 = i\lambda/4$ proves the stated value.

Finally, each of $C_2r\sqrt{u}$, $C_2r^{1/3}u^{2/3}$ and $C_2u^{3/4}$ is at most $u/6$ plus, respectively, a constant multiple of r^2, r, or 1. Thus (11.10) is at most $C_4r^{-1}e^{C_4(r^3+r^2+r+1)}e^{u/2}$. For $p\ge 1$, use $p-\frac12\ge p/2$ and $\int_0^\infty e^{-(p/2)u}u^h\,du = h!(2/p)^{h+1}$. This proves (11.9). □

For later use, put

$$\lambda_m = 2\pi m/M, \qquad \alpha_M = 3\left(\frac{\pi}{2M}\right)^{4/3} e^{-i\pi/3}, \qquad \chi_m = \chi_{\lambda_m} = \alpha_M m^{4/3} \qquad (m \geq 1).$$

Define $A_{j,m}(p)$ by $W^{(j)}_{\lambda_m}(p) = e^{-\chi_m p} A_{j,m}(p)$.

Corollary 11.5. *The germs of $A_{j,m}$ and their conjugates lie in $\mathcal{A}_{1/2}$. With constants allowed to depend on the fixed coefficients a, b, c and modulus M, for every fixed j one has*

$$|A_{j,m}(p)| \leq C_j p^{1/2} m^{4j/3-1/3} e^{C_j m} \qquad (m \geq 1,\ p \geq 1), \tag{11.11}$$

$$p^{-1/2} A_{j,m}(p) \longrightarrow \eta_m(-\chi_m)^j, \qquad \eta_m = \frac{\sqrt{\pi}\, e^{-ia\lambda_m/4}}{\sqrt{6}\, s_{\lambda_m}} \neq 0. \tag{11.12}$$

The sums of the waves and of any fixed number of their derivatives converge absolutely and uniformly on compact positive p-intervals.

Proof. Write $W_{\lambda_m} = \sqrt{\pi} e^{-\chi_m p} p^{3/2} \int_0^\infty e^{-pu} K_{\lambda_m}(u)\, du$. Leibniz's rule gives exactly the derivative formula of Lemma 4.4, with the new kernel and rate. The moments in (11.9) prove (11.11), since r_{λ_m} and $|\chi_m|$ are fixed positive multiples of $m^{1/3}$ and $m^{4/3}$, respectively. The smaller powers $m^{2/3}$ and $m^{1/3}$ in the exponential bound can be absorbed into $C_j m$. The kernel value at zero and Watson's lemma give (11.12); every other derivative term loses at least one power of p. Polynomial multiples of the kernels are admissible, which proves the class assertion. On any compact positive p-interval the same estimates give a majorant of the form $Cm^N e^{Cm-\rho m^{4/3}}$ with $\rho > 0$. It is summable and justifies termwise differentiation. □

11.5. **The rotated endpoint contribution.** The term V_λ in (11.5) has an analytic Laplace kernel as well. For $u \geq 0$, let $v_\lambda(u)$ be the unique nonnegative solution of $v^4 + \lambda v = u$. The polynomial is strictly increasing on the nonnegative ray, and

$$v'_\lambda(u) = \frac{1}{4v_\lambda(u)^3 + \lambda}, \qquad v_\lambda(u) \leq \min(u/\lambda, u^{1/4}), \qquad v'_\lambda(u) \leq \lambda^{-1}.$$

The inverse function theorem extends v_λ holomorphically near that ray. Define

$$Q_\lambda(u) = \sum_{j,k\geq 0} \frac{b^j(-ic)^k}{j!\,k!} \mathcal{I}^{j+2k}\big(e^{iav_\lambda^3} v_\lambda^{2j+k} v'_\lambda\big)(u), \qquad L_\lambda(u) = Q_\lambda(u) - \lambda^{-1}. \tag{11.13}$$

Lemma 11.6. *These series define admissible kernels and $V_\lambda = \mathcal{L}Q_\lambda$ for every $p > 0$. With $D_0 = 2\sqrt{|b|} + 3|c|^{1/3}$, one has*

$$|L_\lambda(u)| \leq u^3 \left[4\lambda^{-5} + |a|\lambda^{-4} + \big(|b|\lambda^{-3} + |c|\lambda^{-2}\big) e^{D_0 u^{3/4}}\right] \qquad (u \geq 0). \tag{11.14}$$

On every fixed complex disk, for all sufficiently large λ, L_λ has a holomorphic extension satisfying $L_\lambda = O(\lambda^{-2})$ uniformly on that disk.

Proof. On compact subsets of a star-shaped neighborhood of the ray, the (j,k) summand in (11.13) is bounded by $C_0 C_1^j C_2^k/(j!\,k!\,(j+2k)!)$. This proves normal convergence. On the positive ray a is real and $|e^{iav^3} - 1| \leq |a|v^3$. The zeroth term minus λ^{-1} is therefore bounded by $u^3(4\lambda^{-5} + |a|\lambda^{-4})$. For $j \geq 1$, use $v_\lambda^2 v'_\lambda \leq u^2\lambda^{-3}$ once, and $v_\lambda \leq u^{1/4}$ for the remaining powers. For $j = 0, k \geq 1$, use $v_\lambda v'_\lambda \leq u\lambda^{-2}$ once. The respective primitive bounds are

$$\frac{\lambda^{-3} u^{3j/2+9k/4+3/2}}{(j+2k)!}, \qquad \frac{\lambda^{-2} u^{9k/4+3/4}}{(2k)!}.$$

After removing one factor $|b|u^3\lambda^{-3}$ or $|c|u^3\lambda^{-2}$, sum by the factorial inequalities used in Lemma 11.4. This gives (11.14).

Substitution $u = y^4 + \lambda y$ in the integral for V_λ, expansion of the two exponentials involving negative powers of p, and $\mathcal{L}\mathcal{I}^h = p^{-h}\mathcal{L}$ give $V_\lambda = \mathcal{L}Q_\lambda$. Before expansion an absolute majorant is

$$\lambda^{-1} e^{-pu+|b|u^{1/2}/p+|c|u^{1/4}/p^2},$$

which is integrable for every $p > 0$. After expansion, (11.14) supplies the Laplace majorant. Thus the identities are exact.

For the complex uniform estimate, the scaling identity $v_\lambda(z) = \lambda^{1/3} v_1(z/\lambda^{4/3})$ and the inverse function theorem at zero give $v_\lambda(z) = O(z/\lambda)$ and $v'_\lambda(z) = O(\lambda^{-1})$ on any prescribed disk, once λ is large enough. The zeroth summand minus λ^{-1} is $O(\lambda^{-4})$; the sum with $j \geq 1$ is $O(\lambda^{-3})$; the terms with $j = 0, k \geq 1$ are $O(\lambda^{-2})$. The compact factorial majorants justify these estimates for the entire sums. □

11.6. **One-sided Poisson summation with periodic weights.** Define the bounded periodic Fourier coefficients and a boundary constant by

$$\gamma_m = \sum_{r\in R} e^{-2\pi i m r/M}, \qquad C_R = \sum_{r\in R} \left(\frac{1}{2} - \frac{r}{M}\right).$$

Then $|\gamma_m| \leq |R|$, $\gamma_{-m} = \overline{\gamma_m}$, and $\gamma_{M\ell} = |R|$ for every integer ℓ.

Proposition 11.7. *There is an admissible kernel K_R such that*

$$Y(p) = C_R + \frac{|R|}{M} Z(p) + \mathcal{L}K_R(p) + \frac{1}{M} \sum_{m\geq 1} \left(\gamma_m W_{\lambda_m}(p) + \overline{\gamma_m W_{\lambda_m}(p)}\right). \tag{11.15}$$

The displayed wave series is absolutely convergent and may be differentiated any fixed number of times. In addition, $(\mathcal{L}K_R)^{(j)}(p) = O(p^{-j-3})$ for $p \geq 1$.

Proof. Periodize, with period M, the function which is $e^{-Q(x)/p^3}$ for $x > 0$, zero for $x < 0$, and $1/2$ at zero. It is smooth away from the multiples of M, with one jump per period, and has integrable derivatives on each smooth piece. Its Fourier series converges to the midpoint of the one-sided limits. Evaluating at the residues in R gives

$$Y(p) = \frac{\mathbf{1}_{\{0\in R\}}}{2} + \frac{|R|}{M} Z(p) + \frac{2}{M} \sum_{m\geq 1} \operatorname{Re}\big(\gamma_m J_p(i\lambda_m)\big).$$

At this stage the sum means the limit of symmetric Fourier partial sums. The term i/λ_m from (11.5) has the boundary contribution

$$\frac{\mathbf{1}_{\{0\in R\}}}{2} + \frac{2}{M} \sum_{m\geq 1} \operatorname{Re}(i\gamma_m/\lambda_m) = C_R.$$

Indeed, for $0 < r < M$, $\sum_{m\geq 1} \sin(2\pi m r/M)/m = \pi(1/2 - r/M)$; at $r = 0$ the sine series is zero. This familiar sawtooth Fourier identity also follows by integrating the geometric series with radius $0 < t < 1$ and taking its radial limit, or by the same midpoint Fourier theorem.

After this subtraction put, on the real ray,

$$K_R(u) = \frac{2}{M} \sum_{m\geq 1} \operatorname{Re}\big(i\gamma_m L_{\lambda_m}(u)\big).$$

For holomorphic extension take each real part by reflection, as in (5.6). The estimate (11.14) and $\sum_m \lambda_m^{-2} < \infty$ give $|K_R(u)| \leq C_0 u^3 e^{D_0 u^{3/4}}$. Lemma 11.6 proves normal convergence of the tail on any fixed disk. The remaining finitely many summands have local holomorphic extensions near every point of the ray. Thus K_R is admissible. Its real-axis bound justifies summing the Laplace integrals absolutely and gives (11.15). Differentiating these integrals and substituting $v = pu$ gives the bound $O(p^{-j-3})$, because $e^{-v+D_0 v^{3/4}}$ has every polynomial moment finite. The wave assertions follow from Corollary 11.5. □

11.7. **Separation and cube-free coordinates.** The background $B_*(p) = C_R + (|R|/M)Z(p) + \mathcal{L}K_R(p)$ and all its derivatives belong to $\mathcal{A}^{(4)}$ by Lemmas 11.1 and 11.2. They are $O(p^{3/4})$ for $p \geq 1$. For a polynomial relation of total degree at most D among Y and finitely many of its derivatives, expansion of (11.15) and its derivatives therefore has the same properties as Lemma 7.2, with $p^{3D/4}$ in place of $p^{D/2}$. In detail, each selected wave has rate $\alpha_M m^{4/3}$ or $\overline{\alpha_M} m^{4/3}$, with positive real part, and amplitude bounded by a polynomial in m times $p^{1/2} e^{C_0 m}$. The weights are summable since

$$\sum_{m \geq 1} m^N e^{C_0 m - \rho m^{4/3}} < \infty \qquad (\rho > 0).$$

There are only finitely many labels below any real-rate ceiling, as before. Every grouped coefficient lies in $\mathcal{A}^{(4)}$ and hence has a first nonzero power unless it vanishes as a germ. The proof of Proposition 7.3, including its finite-frequency averaging and exponentially small tail estimate, now applies verbatim. Every coefficient at a full complex rate must vanish separately.

A positive integer is *cube-free* if no prime cube divides it. Let $\mathcal{T}$ be the set of positive cube-free integers. Unique factorization gives

$$m = dk^3, \qquad d \in \mathcal{T}, \quad k \geq 1, \qquad m^{4/3} = d^{4/3} k^4.$$

Lemma 11.8. *The numbers $d^{1/3}$, $d \in \mathcal{T}$, are linearly independent over $\mathbb{Q}$.*

Proof. This is a special case of the classical independence theorem for radicals; see [Mor53]. We give the required argument. Let $\zeta = e^{2\pi i/3}$ and $K = \mathbb{Q}(\zeta)$. For distinct primes $\ell_1, \dots, \ell_s$, put $\theta_i = \ell_i^{1/3} > 0$. Inductively, $K(\theta_1, \dots, \theta_r)$ has the basis

$$\prod_{i=1}^{r} \theta_i^{e_i}, \qquad 0 \leq e_i < 3,$$

over K, and has automorphisms independently multiplying each θ_i by ζ while fixing K. Here an automorphism is an invertible field map; fixing K means that every element of K is unchanged.

Suppose $\theta^3 = \ell_{r+1}$ for an element θ of the field already constructed. Under the ith automorphism, θ is multiplied by some ζ^{t_i}, because its cube is fixed. Expand θ in the displayed basis. Comparing coefficients under each automorphism shows that every nonzero basis coefficient has $e_i = t_i$ for all i. Thus $\theta = h \prod_i \theta_i^{e_i}$ with $h \in K$. Cubing and multiplying the resulting identity by its complex conjugate gives

$$\ell_{r+1}^2 = (h\overline{h})^3 \prod_{i=1}^{r} \ell_i^{2e_i}.$$

But $h\overline{h} \in \mathbb{Q}$, so the exponent of ℓ_{r+1} on the right is divisible by 3, whereas it is 2 on the left. This contradicts unique prime factorization for rational numbers. Hence $X^3 - \ell_{r+1}$ has no root in the old field, so is irreducible. Adjoining its positive root extends the basis. Each old automorphism extends while fixing this root, and the new automorphism fixes the old field and multiplies the root by ζ; these maps are well defined by the irreducible polynomial. The induction follows, including the prime 3 without an exception. Distinct cube-free radicals in any finite collection are distinct basis monomials for a sufficiently large collection of primes. They are therefore independent even over K, and in particular over $\mathbb{Q}$. □

Give the positive wave at $m = dk^3$ the mark $z_{d,+}^{k^4}$ and its conjugate the mark $z_{d,-}^{k^4}$. For a finite multi-index ν the corresponding rate is

$$\Lambda(\nu) = \alpha_M \sum_{d \in \mathcal{T}} \nu_{d,+} d^{4/3} + \overline{\alpha_M} \sum_{d \in \mathcal{T}} \nu_{d,-} d^{4/3}.$$

This rate map is injective. Its real and imaginary parts determine the two real sums separately, since both $\operatorname{Re} \alpha_M$ and $\operatorname{Im} \alpha_M$ are nonzero. Lemma 11.8 then identifies every rational coefficient

in each sum. Consequently the same coefficientwise argument as in Proposition 8.3 turns a polynomial identity among the actual functions into an identity of the marked series

$$F_j = [B_*^{(j)}] + \frac{1}{M} \sum_{\substack{d \in \mathcal{T} \\ k \geq 1}} \left(\gamma_{dk^3} [A_{j,dk^3}] z_{d,+}^{k^4} + \overline{\gamma_{dk^3}}\, \overline{[A_{j,dk^3}]} z_{d,-}^{k^4} \right).$$

These series use the formal-series construction over the germ ring $\mathcal{R}$ given in §8.

11.8. **Periodic zeros and the Vandermonde argument.** A cube-free direction d is called *active* if $\gamma_{dk^3} \neq 0$ for at least one $k \geq 1$. There are infinitely many such directions. Indeed, write $M = uv^3$ with u cube-free. For every prime ℓ not dividing M, the integer $u\ell$ is cube-free and $\gamma_{(u\ell)v^3} = \gamma_{M\ell} = |R| > 0$.

Choose J distinct active directions $d_1, \ldots, d_J$. For each i, let k_i be the least positive integer with $\gamma_{d_i k_i^3} \neq 0$, and set $m_i = d_i k_i^3$, $h_i = k_i^4$. Every entry in row i of the marked Jacobian $(\partial F_j / \partial z_{d_i,+})_{i,j}$ is divisible by $z_{d_i,+}^{h_i - 1}$. After this common monomial is removed, the constant coefficient matrix is

$$\left(\frac{h_i \gamma_{m_i}}{M} [A_{j,m_i}] \right)_{1 \leq i \leq J,\ 0 \leq j < J}.$$

By (11.12), its determinant divided by $p^{J/2}$ has the nonzero limit

$$\left(\prod_{i=1}^{J} \frac{h_i \gamma_{m_i} \eta_{m_i}}{M} \right) \prod_{1 \leq i < k \leq J} \left((-\chi_{m_k}) - (-\chi_{m_i}) \right).$$

The rates are distinct because the m_i are distinct. The divided Jacobian is therefore invertible over the formal-series ring, by Lemma 9.2.

Multiplication by any monomial is injective in that ring: it merely shifts the coefficient indices. This remains true even though the germ ring can have zero divisors. Thus the original Jacobian has trivial kernel. Differentiating a polynomial relation of least total degree with respect to the J selected marking variables now forces all its partial derivatives to vanish, exactly as in Proposition 9.3. There is no nonzero relation. Rate separation then proves that $Y, Y', \ldots, Y^{(J-1)}$ are algebraically independent over $\mathbb{C}$ for every J.

Proof of Theorem 2.1. If $F_{S_{P,R}}$ were differentially algebraic, so would be T, by (11.1). The coefficients of T are bounded by four, and T has only finitely many negative powers. Thus the formal-to-analytic evaluation argument of §3 applies to T and its derivatives on $0 < |q| < 1$. The polynomial differential-system proof in that section also applies, with

$$r = p^{-1}, \qquad r' = -r^2, \qquad q' = (3/A) r^4 q$$

in place of the cubic change of variable. Choose a least-degree relation on the actual functions, invert its nonzero partial derivative with respect to the highest-index variable locally, and differentiate as in (3.3). The resulting finite polynomial system contains $Y = T(q(p))$ as a coordinate; no subtraction of the background is needed. Lemma 3.3 gives a nonzero autonomous relation on Y, and real-analytic continuation extends it to the positive ray. The marked argument just proved excludes this relation. Theorem 1.2 follows by taking $P(x) = x^4$, $M = 1$, and $R = \{0\}$. □

11.9. **Examples.** The theorem gives characteristic generating functions of quartic value sets. In particular, each of the following is differentially transcendental:

$$\sum_{n \geq 0} q^{n^4}, \qquad \sum_{n \geq 0} q^{(3n+1)^4}, \qquad \sum_{\substack{n \geq 1 \\ (n,M)=1}} q^{n^4} \quad (M \geq 2), \qquad \sum_{n \geq 0} q^{n^4 + n^2}.$$

The second and third examples are proper subsets of the fourth powers. More generally, every nonempty union of input residue classes is allowed, without a symmetry condition. The polynomial need not be a translated or rescaled fourth power. For example,

$$\sum_{n\geq 0} q^{n^4+n^3+n^2+n} \qquad \text{and} \qquad 1+\sum_{n\geq 4} q^{\binom{n}{4}}$$

are characteristic series covered by the theorem. The sparse subset of squares $\{n^2(n+1)^2 : n \geq 0\}$ is another example, by taking $P(x) = x^2(x+1)^2$. For the binomial example $P(x) = x(x-1)(x-2)(x-3)/24$ is integer-valued, the repeated value at $n = 0, 1, 2, 3$ is counted once, and its values are strictly increasing for $n \geq 3$. Finite modifications of any of these sets preserve differential transcendence, by the rational-translation argument above. All these applications use the exact mode decomposition, Laplace amplitude algebra, rate separation, and Vandermonde determinant proved in this paper.

12. Final remarks

12.1. **Asymptotics for sums of cubes.** Fix $k \geq 8$ and let $e(u) = e^{2\pi i u}$. The *singular series* is

$$\Sigma_k(n) = \sum_{m\geq 1} m^{-k} \sum_{\substack{1\leq a\leq m\\(a,m)=1}} \left(\sum_{x=1}^{m} e(ax^3/m)\right)^k e(-an/m).$$

It converges absolutely and is bounded above and below by positive constants depending on k; see [Vau97, Ch. 4]. The classical formula is

$$r_k(n) = \frac{\Gamma(4/3)^k}{\Gamma(k/3)}\Sigma_k(n)n^{k/3-1} + o(n^{k/3-1}). \tag{12.1}$$

Hardy and Littlewood proved this for $k \geq 9$, and Vaughan extended it to $k = 8$ [Vau86]; see also [BW10, §1]. Allowing zero coordinates changes only the lower-order term, by the fourth- and eighth-moment estimates for cubic exponential sums [Vau97, Ch. 2]. Thus differential transcendence of the generating function is compatible with precise asymptotics for its coefficients.

12.2. **Prime-counting and prime-exponent series.** Let $\pi(x)$ be the number of primes at most x, for real $x \geq 0$, and put

$$I(q) = \sum_{p \text{ prime}} q^p, \qquad \Pi(q) = \sum_{n\geq 0} \pi(n) q^n.$$

Helminen [Hel26b] proved that both series are differentially transcendental over $\mathbb{C}(q)$, resolving the prime-counting part of [Pak18, Conj. 2.5]. Their differential transcendence is equivalent, since

$$(1-q)\Pi(q) = I(q), \tag{12.2}$$

and multiplication by a nonzero rational function preserves differential algebraicity in both directions by Leibniz's rule. Define $\mathrm{li}(x) = \int_2^x du/\log u$ for $x \geq 2$. Helminen's argument combines a separate analytic criterion with

$$\pi(x) - \mathrm{li}(x) = O_M\left(\frac{x}{(\log x)^M}\right) \qquad (x \to \infty)$$

for every $M \in \mathbb{N}$, where the implied constant may depend on M. In the Lean formalization, this estimate is proved from the quantitative prime number theorem of the *Prime Number Theorem and More* project (PNT+) [KT24]. The cubic and quartic arguments above use different analytic representations; their prime labels are Fourier indices. If p_n denotes the n-th prime, the series $\sum_{n\geq 1} p_n q^n$ is the other part of Conjecture 2.5 and is not settled by the cited release.

12.3. **Automatic sets.** Fix an integer $b \geq 2$. A set $S \subseteq \mathbb{N}$ is *b-automatic* if a finite automaton decides membership in S by reading base-b digits. It is *ultimately periodic* if, apart from finitely many integers, membership depends only on the residue modulo a fixed positive integer. A linear *Mahler equation* for $F_S(t) = \sum_{n\in S} t^n$ is a relation

$$\sum_{j=0}^{r} a_j(t) F_S(t^{b^j}) = 0, \qquad a_j(t) \in \mathbb{C}(t),$$

with coefficients not all zero. Automatic generating functions satisfy such equations [DHR18]. Adamczewski, Dreyfus and Hardouin proved that a formal power-series solution of such an equation is either rational or differentially transcendental [ADH21, Thm. 1.2, Case M]. For a series with coefficients in $\{0, 1\}$, rationality is equivalent to ultimate periodicity: a constant-coefficient recurrence has only finitely many states, and the converse follows by summing geometric series. Thus an automatic characteristic series satisfies an algebraic differential equation exactly when its set is ultimately periodic. These results use Mahler equations, independently of the method of this paper; earlier automatic examples appear in [DHR18].

12.4. **Higher degrees and the scope of the extension.** Theorem 2.1 shows that the method does extend beyond cubes. It covers every integer-valued quartic with positive leading coefficient and every nonempty periodic restriction on its inputs, including proper subsets of the fourth powers. Its proof uses exact Laplace representations throughout; the lower-degree polynomial terms have not been discarded in a saddle-point approximation.

For $d \geq 5$, write $H_d(q) = \sum_{n\geq 0} q^{n^d}$. With $q = e^{-p^{-(d-1)}}$ and $x = py$, the Fourier phase is $p(-y^d + 2\pi i m y)$, and its stationary values are proportional to $m^{d/(d-1)}$. To extend the proof one must identify the contributing contours, place every amplitude and endpoint term in a common algebra with the first-nonzero-power property, and resolve the additive relations among all contributing complex rates. The two quartic rates and the cube-free marking proved in §11 do not supply these steps in higher degree. The asymptotic expansions in [Par16] alone are insufficient: their remainders can contain precisely the exponentially small terms that must be separated. No assertion for general degree $d \geq 5$, or for arbitrary nonperiodic subsets of the power values, is made here.

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