hexagon:2611.01086v1

Face posets, local weight matroids and Betti numbers of torus actions

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November 2026

Abstract

Let a compact torus act on a closed manifold X with isolated fixed points. The face poset $S(X)$ of the action is covered by the lattices of flats of the linear matroids of tangent weights at the fixed points. We study how much of the cohomology of X these combinatorial data see.

- *Local equivariant cohomology.* The group $H^*_T(X, X \setminus \mathrm{pt})$, embedded in $H^*_T(\mathrm{pt})$ by restriction, is the principal ideal generated by the product of the tangent weights. It determines the weights up to scalars, with multiplicities, and hence the local lattice $S(X)_{\geqslant \mathrm{pt}}$.
- *Local matroids do not suffice.* Two connected regular semisimple Hessenberg varieties in $\mathrm{Fl}(\mathbb{C}^5)$, with torus actions of complexity one, have isomorphic weight matroids at all fixed points, and hence the same Chow rings, Orlik–Solomon algebras and Kazhdan–Lusztig polynomials there, but different Betti numbers.
- *Type A.* The face poset determines the Betti numbers of every regular semisimple Hessenberg variety of type A, full or partial, and of the isospectral manifolds of Ayzenberg and Buchstaber. The reason is rigidity: the cluster-permutohedron Cl_Γ determines the graph Γ, by an elementary argument that recovers the vertices of Γ from stars of edges.
- *Simply-laced types.* The same holds for regular semisimple Hessenberg varieties of types A, D and E: the face poset determines the Weyl group together with the lines spanned by the roots of the Hessenberg ideal, up to isometry.
- *A formula.* The Poincaré polynomial is a sum over the face poset,

$$P_X(q) = \sum_{\sigma \in S(X)} (q-1)^{\mathrm{rk}\,\sigma} e_\sigma(q),$$

  in which e_σ is a sum of Tutte polynomials of the matroids of weights that are negative on a generic circle. For toric manifolds this is the h-vector formula.

An appendix collects the background on matroids, equivariant cohomology, GKM theory, root systems and Hessenberg varieties.

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*This paper was prepared with substantial assistance from Claude, an AI system developed by Anthropic; see the statement on the use of AI at the end of the paper.

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1 Introduction

Throughout, a *torus action* is an effective action of a compact torus $T = T^k$ on a closed connected smooth manifold $X = X^{2n}$ with a finite nonempty set of fixed points; its *complexity* is $n - k$. Toric topology studies actions of complexity zero by combinatorial means [18]. Buchstaber and Terzić extended these methods to positive complexity [19, 20], and Ayzenberg, Masuda and their coauthors made the *face poset* $S(X)$ of the action the central combinatorial object [7, 8, 6]. Its elements are the closures of the components of the orbit type strata of X/T that contain a fixed point, ordered by inclusion. By a theorem of Ayzenberg and Cherepanov [7], $S(X)$ is *locally geometric*: for a fixed point p, the faces containing p form the lattice of flats of the linear matroid of the tangent weights at p.

This paper asks what these combinatorial data know about the cohomology of X. We prove five results: one about the local data, one negative result, two rigidity theorems for the face poset, and a formula. Readers who are new to some of the notions involved will find the definitions and the facts we use in Appendix A.

Theorem A (Theorem 3.3)**.** *Let* pt *be a fixed point with tangent weights* $\alpha_1, \dots, \alpha_n \in H^2(BT; \mathbb{Q})$*. Restriction identifies* $H^*_T(X, X \setminus \mathrm{pt})$ *with the principal ideal* $(\alpha_1 \cdots \alpha_n)$ *of* $H^*_T(\mathrm{pt}) = H^*(BT; \mathbb{Q})$*. This ideal determines the multiset of weights up to nonzero scalars, hence the linear matroid of weights at* pt *with its parallel classes, and the lattice* $S(X)_{\geqslant \mathrm{pt}}$*.*

The point of Theorem A is that no independence assumption on the weights is needed. The familiar GKM argument computes the ideal as an intersection $\bigcap_j (\alpha_j)$, which equals the product only when the weights are pairwise independent and loses multiplicities otherwise. We use the Thom isomorphism and unique factorization instead.

The local data are far from determining the global cohomology. The Chow ring [2] and the Orlik–Solomon algebra [36] of a matroid depend only on the matroid, and so do the augmented Chow ring and the intersection cohomology module [12, 13] and the Kazhdan–Lusztig polynomial [25]. The following therefore shows that none of them, taken at all fixed points, determines $H^*_T(X)$.

Theorem B (Theorem 4.3)**.** *Let* $M = \mathrm{diag}(1, 2, 3, 4, 5)$*. The regular semisimple Hessenberg varieties* $X_1 = \mathrm{Hess}(M, (2, 3, 5, 5, 5))$ *and* $X_2 = \mathrm{Hess}(M, (2, 4, 4, 5, 5))$ *in* $\mathrm{Fl}(\mathbb{C}^5)$ *are connected GKM manifolds with the same* 120 *fixed points and torus actions of complexity one. At every fixed point of either variety the weight matroid is* $U_{2,3} \oplus U_{1,1} \oplus U_{1,1}$*. Their Poincaré polynomials are* $1 + 17q + 42q^2 + 42q^3 + 17q^4 + q^5$ *and* $1 + 12q + 47q^2 + 47q^3 + 12q^4 + q^5$*, where* q *has degree* 2*.*

The global face posets of X_1 and X_2 are different. This leaves the question, raised in [34], whether the face poset determines the Betti numbers of an equivariantly formal action (Question 1). In complexity zero it does: for a quasitoric manifold the Betti numbers are the h-numbers of the orbit polytope [18]. Our third result answers the question for the main family of examples in positive complexity.

Theorem C (Theorems 5.4 and 5.8, Corollary 5.6)**.** *(1) For simple graphs* Γ, Γ'*, an isomorphism of cluster-permutohedra* $\mathrm{Cl}_\Gamma \cong \mathrm{Cl}_{\Gamma'}$ *implies* $\Gamma \cong \Gamma'$*.*

(2) *Let* X *and* X' *each be an isospectral manifold* $M_{\Gamma,\lambda}$ *or a regular semisimple Hessenberg variety* $\mathrm{Hess}(M, h)$ *of type A, with* Γ*, resp.* Γ_h*, connected. If* $S(X) \cong S(X')$*, then* X *and* X' *have the same Betti numbers and isomorphic equivariant cohomology rings.*

(3) *The same conclusion about Betti numbers holds for partial regular semisimple Hessenberg varieties, including all partial flag varieties.*

The cluster-permutohedron Cl_Γ is the face poset of the isospectral manifold $M_{\Gamma,\lambda}$, and through the twin construction of Ayzenberg and Buchstaber also of $\mathrm{Hess}(M, h)$ when $\Gamma = \Gamma_h$ [5, 6]. Part (1) is purely combinatorial. Counting the minimal elements below joins of pairs of atoms tells which edges of Γ meet, i.e. recovers the line graph, and ranks of joins tell which triples of edges form triangles. These two data determine Γ by an elementary star argument (Proposition 5.3). Unlike Whitney's line graph theorem [42, 32], it has no exceptions. For partial flags we combine the same argument with a fibration over the partial Hessenberg variety.

Theorem C shows that in these families the face poset remembers the whole graph, and the graph remembers the manifold. A counterexample to the general question, if one exists, must therefore come from outside type A. Regular semisimple Hessenberg varieties exist in every Lie type, and our fourth result excludes half of the types.

Theorem D (Theorem 6.6, Corollary 6.7)**.** *Let* X *and* X' *be regular semisimple Hessenberg varieties of simply-laced types, given by Hessenberg ideals* $I \subset \Phi^+$ *and* $I' \subset \Phi'^+$*. If* $S(X) \cong S(X')$*, then there is an isometry between the spans of* I *and* I' *that maps the lines spanned by the roots of* I *onto those spanned by the roots of* I' *and conjugates the reflection groups they generate. Consequently* X *and* X' *have the same Betti numbers, and if they are connected, isomorphic equivariant cohomology rings.*

The proof reads the angles between roots from the numbers of fixed points on faces of rank 2, and the signs of the Gram matrix from the ranks of joins along induced cycles. For the types B, C, F_4 and G_2 the same conclusion holds for all Hessenberg ideals in rank at most 5 by computation (Remark 6.8), but the first step of the proof fails and we do not have an argument.

Theorems C and D show that in these families the face poset determines the Betti numbers, but they do not give a formula. A formula exists if one also uses the signs of the weights.

Theorem E (Theorem 7.1)**.** *Let* X *carry an invariant almost complex structure, and let* ξ*, be a generic one-parameter subgroup of* T *such that* $P_Y(q) = \sum_{p \in Y^T} q^{\mathrm{ind}_Y(p)}$ *for* X *and for each face submanifold* Y*, where* $\mathrm{ind}_Y(p)$ *is the number of weights of* Y *at* p *that are negative on* ξ*. For a face* σ *and a fixed point* $p \in \sigma$ *let* $N_p(\sigma)$ *be the set of weights of* σ *at* p *that are negative on* ξ*. Then*

$$P_X(q) = \sum_{\sigma \in S(X)} (q-1)^{\mathrm{rk}\,\sigma} e_\sigma(q), \qquad e_\sigma(q) = \sum_p T_{M(N_p(\sigma))}(1, q),$$

where the inner sum is over the fixed points p *of* σ *at which* $N_p(\sigma)$ *spans the weights of* σ*, and* T_M *is the Tutte polynomial. The polynomials* e_σ *do not depend on* ξ*, and* $e_\sigma = 1$ *when the weights of* σ *at its fixed points are linearly independent.*

The hypothesis holds for smooth projective varieties with an algebraic torus action, by the Białynicki-Birula decomposition, and for Hamiltonian actions. For a toric manifold Theorem E is the formula $h(q) = \sum_\sigma (q-1)^{\dim \sigma}$ for the h-polynomial of the moment polytope. In general e_σ measures how far the action on σ is from being toric, and it explains Theorem B: the two varieties there have the same e_σ on their top faces and differ only in the numbers of lower faces. Theorem E computes e_σ from the signs of the weights, but for Hessenberg varieties of types A, D and E the result depends only on the interval $S(X)_{\leqslant \sigma}$ (Proposition 7.3). A formula for e_σ in terms of the face poset alone would answer Question 1 with a formula; we do not know one.

Computations and formalization. All computations cited in this paper were done with programs that are available, with their output, at `https://github.com/davidmiheev/matroid_toric`. The Betti numbers in Theorem B were computed both from the Białynicki-Birula formula and, independently, from the GKM presentation of H^*_T in Macaulay2 [28]. Part (1) of Theorem C was confirmed by computing exact poset isomorphism classes, with nauty's Traces [33], for all 208 graphs with at most 6 vertices. Before the proofs were found, parts (2) and (3) of Theorem C were verified by direct computation for all full Hessenberg varieties with $4 \leqslant n \leqslant 7$ and all partial ones with $n \leqslant 7$ (617 and 2 940 cases). The conclusion of Theorem D was checked for all 1 230 Hessenberg ideals of the irreducible root systems of rank at most 5, and Theorem E against a recursive computation of the polynomials e_σ for all connected Hessenberg varieties of type A with $n \leqslant 5$. Part (1) of Theorem C (Theorem 5.4, including Proposition 5.3) is formally verified in Lean 4 with Mathlib, in the same repository, with the cluster-permutohedron encoded as maps from vertices to label sets ordered pointwise. So is the algebraic step of Theorem A, Lemma 3.2(2), together with the lattice of flats of a matroid, Whitney's Boolean expansion of its characteristic polynomial and Philip Hall's theorem.

Conventions. Cohomology has rational coefficients unless stated otherwise. We identify $H^*(BT) = \mathrm{Sym}(t^*_{\mathbb{Q}}) \cong \mathbb{Q}[t_1, \dots, t_k]$ with $\deg t_i = 2$ and $H^2(BT) = \mathrm{Hom}(T, S^1) \otimes \mathbb{Q}$. For a polynomial in q we write q for the class of degree 2, so $P_X(q) = \sum_j \beta_{2j}(X) q^j$ when $H^{\mathrm{odd}}(X) = 0$. Matroids are matroids on a finite set; the linear matroid of a finite family of vectors has the family as its ground set, so that repeated and proportional vectors are different elements. Theorem numbers of [5, 6, 7] refer to their arXiv versions.

Organization. Section 2 recalls face posets, GKM manifolds and Hessenberg varieties. Theorems A to E are proved in Sections 3 to 7, and Section 8 lists open questions. Appendix A is a short introduction to the background.

2 Preliminaries

2.1 Weights and face posets

Let $p \in X^T$. The tangent representation decomposes as $T_pX \cong \bigoplus_{j=1}^n V(\alpha_{p,j})$, where $V(\alpha) = \mathbb{C}$ with $t \cdot z = \alpha(t)z$. The $\alpha_{p,j} \in \mathrm{Hom}(T, S^1)$ are the *tangent weights* at p. Without an invariant complex structure they are defined up to sign, which does not affect linear dependence. The point p is isolated if and only if no weight vanishes, and since the action is effective, the weights at p span $\mathrm{Hom}(T, S^1) \otimes \mathbb{Q}$ [7, Constr. 2.3]. For an oriented T-vector bundle, the equivariant Euler class of T_pX at a fixed point is $e_T(T_pX) = \pm \prod_j \alpha_{p,j} \in H^{2n}(BT)$.

Let $X_j \subset X$ be the union of the orbits of dimension at most j, and $\pi\colon X \to X/T$ the projection. A *face* of rank j is the closure σ of a connected component of $\pi(X_j) \setminus \pi(X_{j-1})$ that contains a fixed point, and $X_\sigma = \pi^{-1}(\sigma)$ is its *face submanifold*. The *face poset* $S(X)$ is the set of faces ordered by inclusion; we identify a face with its face submanifold, and a fixed point with the face $\{p\}$ of rank 0. For $y \in S(X)$ we write $\nu(y)$ for the number of minimal elements below y, that is, the number of fixed points in X_y.

Theorem 2.1 (Ayzenberg–Cherepanov [7, Thm. 1])**.** *For every $p \in X^T$, the poset $S(X)_{\geqslant p}$ of faces containing p is isomorphic to the lattice of flats of the linear matroid of the weights $\alpha_{p,1}, \dots, \alpha_{p,n} \in \mathrm{Hom}(T, S^1) \otimes \mathbb{Q}$. In particular $S(X)$ is graded, and each $S(X)_{\geqslant p}$ is a geometric lattice.*

We shall use the following more precise form of this theorem. For a set F of indices let $H_F \subset T$ be the identity component of $\bigcap_{j \in F} \ker \alpha_{p,j}$, a subtorus.

Lemma 2.2. *Let $p \in X^T$, and let M_p be the linear matroid of the weights at p.*

(1) *Every face submanifold X_σ is a connected component of the fixed point set X^H of a closed subgroup $H \subset T$, hence a closed T-invariant submanifold of X.*
(2) *The faces containing p correspond to the flats F of M_p: the face σ_F is the connected component of X^{H_F} containing p. Its tangent space at p is $\bigoplus_{j\in F} V(\alpha_{p,j})$, its rank is the rank of F, and $\sigma_F \subset \sigma_{F'}$ if and only if $F \subset F'$.*
(3) *For a face σ, the faces of the T-action on X_σ are the faces of X contained in X_σ, so that $S(X_\sigma) = S(X)_{\leqslant\sigma}$. The fixed points of X_σ are the fixed points of X in X_σ, and the tangent weights of X_σ at such a point p are the $\alpha_{p,j}$ with j in the flat of σ at p.*

Proof. Part (1) is [7, Lemma 2.8]. Part (2) is the content of [7, Lemma 2.10, Lemma 3.1 and Prop. 3.5]: near p, the slice theorem identifies X with T_pX, the faces through p with the invariant subspaces $(T_pX)^H$, and these with the subspaces $\bigoplus_{j\in F} V(\alpha_{p,j})$ for the flats F; the subgroup H_F has Lie algebra $\bigcap_{j\in F} \ker d\alpha_{p,j}$, whose annihilator is the span of the $\alpha_{p,j}$, $j \in F$.

(3) Let F_σ be the flat of σ at a fixed point $p \in X_\sigma$. By (2) the T-manifold X_σ has tangent weights $\alpha_{p,j}$, $j \in F_\sigma$, at p, and its fixed points are the fixed points of X that lie in X_σ. The flats of the matroid of these weights are the flats of M_p contained in F_σ. We apply (2) to X_σ, with the effective action of the quotient of T by the kernel of its action on X_σ; this does not change fixed point sets. The face of X_σ through p for such a flat F is the component of $(X_\sigma)^{H_F}$ through p. As $F \subset F_\sigma$ we have $H_F \supset H_{F_\sigma}$, so $X^{H_F} \subset X^{H_{F_\sigma}}$. The component of X^{H_F} through p is connected, lies in $X^{H_{F_\sigma}}$ and contains p, so it lies in X_σ, which is the component of $X^{H_{F_\sigma}}$ through p; hence it equals the component of $(X_\sigma)^{H_F}$ through p. Thus the faces of X_σ through p are the faces σ_F of X with $F \subset F_\sigma$, which by (2) are the faces of X through p contained in X_σ. Every face contains a fixed point, so the two sets of faces coincide. □

For example, a face of rank 1 through p corresponds to a parallel class of weights at p; if the weights at p are pairwise linearly independent, it is a T-invariant 2-sphere tangent to one $V(\alpha_{p,j})$.

2.2 Equivariant formality and GKM manifolds

The action is *equivariantly formal* if the Serre spectral sequence of the fibration $X_T = ET \times_T X \to BT$ degenerates. For X^T finite and nonempty this holds if and only if $H^{\mathrm{odd}}(X) = 0$, and then $H^*_T(X) \cong H^*(X) \otimes H^*(BT)$ as graded $H^*(BT)$-modules, so that the Hilbert series of $H^*_T(X)$ is $P_X(q)/(1-q)^k$ [27]. A smooth projective variety with an algebraic torus action with finitely many fixed points has $H^{\mathrm{odd}} = 0$, by the Białynicki-Birula decomposition [9].

A *GKM manifold* is an equivariantly formal X whose weights at each fixed point are pairwise linearly independent. By Lemma 2.2, each weight $\alpha_{p,j}$ is then tangent to exactly one T-invariant 2-sphere through p, the face of rank 1 for the flat $\{j\}$, and this sphere contains exactly one other fixed point. The fixed points and these spheres form the *GKM graph*, whose edge pq is labelled by the weight α_{pq} along the sphere, defined up to sign. By the GKM theorem [27, 23],

$$H^*_T(X) \cong \left\{(f_p) \in \textstyle\bigoplus_{p\in X^T} H^*(BT) \;\middle|\; f_p - f_q \in (\alpha_{pq}) \text{ for every edge } pq\right\}, \tag{2.1}$$

as $H^*(BT)$-algebras, with componentwise operations; the map is restriction to the fixed points. The same holds with real coefficients, with $H^*(BT;\mathbb{R}) = \mathrm{Sym}(t^*)$.

Lemma 2.3. *If X is a GKM manifold, possibly disconnected, then the connected components of X correspond to the connected components of its GKM graph: two fixed points lie in the same component of X if and only if they are joined by a path in the GKM graph.*

Proof. In degree 0, (2.1) identifies $H^0_T(X)$ with the functions $X^T \to \mathbb{Q}$ that are constant along the edges, since the ideal (α_{pq}) contains no nonzero constants. As BT is simply connected, $H^0_T(X) = H^0(X)$. Hence the GKM graph and X have the same number of connected components. Every component Y of X contains a fixed point, because $\chi(Y) = \chi(Y^T)$ and $\chi(Y) = \sum_i \beta_{2i}(Y) > 0$. The two endpoints of an edge lie on a connected sphere, hence in the same component of X.

So the partition of X^T into the vertex sets of the components of the graph refines its partition by the components of X, both partitions have the same number of blocks, and therefore they coincide. □

2.3 Hessenberg varieties

A *Hessenberg function* is a non-decreasing map $h\colon [n] \to [n]$ with $h(j) \geqslant j$. For $M = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with pairwise distinct real λ_i, the *regular semisimple Hessenberg variety* is

$$\mathrm{Hess}(M, h) = \{V_\bullet \in \mathrm{Fl}(\mathbb{C}^n) \mid MV_i \subset V_{h(i)} \text{ for all } i\}.$$

It is a smooth projective variety of complex dimension $\sum_j (h(j) - j)$ with $H^{\mathrm{odd}} = 0$ [24, 1], invariant under the diagonal torus $T = T^n$. The fixed points are the $n!$ coordinate flags, indexed by $w \in S_n$, and the tangent weights at w are $t_{w(i)} - t_{w(j)}$ for $j < i \leqslant h(j)$ [24, 41]. Let Γ_h be the graph on $[n]$ with edges $\{j, i\}$, $j < i \leqslant h(j)$. Up to isomorphism, the graphs Γ_h are exactly the indifference (unit interval) graphs [5]. By [24],

$$P_{\mathrm{Hess}(M,h)}(q) = \sum_{w \in S_n} q^{\ell_h(w)}, \qquad \ell_h(w) = \#\{(j, i) \mid j < i \leqslant h(j),\ w(j) > w(i)\}. \tag{2.2}$$

The variety is connected if and only if Γ_h is connected, that is, $h(j) > j$ for all $j < n$; then $w = \mathrm{id}$ is the only permutation with $\ell_h(w) = 0$.

Let $\bar{k} = (k_1, \dots, k_s)$ be a composition of n, $K_b = k_1 + \cdots + k_b$, and h a Hessenberg function on $[s]$. The *partial Hessenberg variety* $\mathrm{Hess}_{\bar{k}}(M, h)$ is the set of partial flags $V_{K_1} \subset \cdots \subset V_{K_s} = \mathbb{C}^n$ with $MV_{K_a} \subset V_{K_{h(a)}}$ for all a. It is smooth (Lemma 5.10). Its fixed points are the ordered set partitions $J = (J_1, \dots, J_s)$ of $[n]$ with $|J_a| = k_a$, and its tangent weights at J are $t_y - t_x$ for $x \in J_a$, $y \in J_b$, $a < b \leqslant h(a)$ [6, §5]. In [6] this variety is denoted $M'_{\bar{k},\Gamma_h,\lambda}$ and is defined by the conditions $ML_a \subset \bigoplus_b L_b$, the sum over $b = a$ and the neighbours b of a in Γ_h, on the orthogonal pieces $L_a = V_{K_a} \cap V_{K_{a-1}}^{\perp}$. Since M is real and diagonal, hence Hermitian, $ML_a \perp L_b$ if and only if $ML_b \perp L_a$, and the two definitions agree. For a generic one-parameter subgroup, the Białynicki-Birula decomposition gives

$$P_{\mathrm{Hess}_{\bar{k}}(M,h)}(q) = \sum_J q^{\mathrm{ind}(J)}, \qquad \mathrm{ind}(J) = \#\{(x, y) \mid x \in J_a,\ y \in J_b,\ a < b \leqslant h(a),\ x > y\}, \tag{2.3}$$

which reduces to (2.2) for $\bar{k} = (1, \dots, 1)$. The *inflation* $\Gamma_h(\bar{k})$ is the graph obtained from Γ_h by replacing each vertex a by k_a pairwise non-adjacent copies, copies of a and b being adjacent if and only if ab is an edge of Γ_h. At the fixed point J, identifying $x \in J_a$ with a copy of a, the tangent weights are $t_y - t_x$ over the edges xy of $\Gamma_h(\bar{k})$, so the local weight matroid is the graphic matroid $M(\Gamma_h(\bar{k}))$. If Γ_h is connected, the only J with $\mathrm{ind}(J) = 0$ is $J_1 < J_2 < \cdots < J_s$ elementwise, so $\mathrm{Hess}_{\bar{k}}(M, h)$ is connected.

2.4 Isospectral manifolds and cluster-permutohedra

Let Γ be a connected graph on $[n]$ and λ a generic spectrum. The *isospectral manifold* $M_{\Gamma,\lambda}$ is the space of Hermitian matrices A with spectrum λ and $a_{ij} = 0$ whenever $\{i, j\} \notin E(\Gamma)$, with T^n acting by conjugation [6]. For $\Gamma = K_n$ it is $\mathrm{Fl}(\mathbb{C}^n)$.

Definition 2.4 ([6, Def. 2.14])**.** Let Γ be a simple graph on $[n]$. A *clustering* is a partition $C = \{V_1, \dots, V_m\}$ of $[n]$ into blocks inducing connected subgraphs; ordered by refinement, the clusterings form the lattice of flats $\mathrm{Flats}(\Gamma)$ of the graphic matroid. An *assignment* for C is a family of label sets $L_1, \dots, L_m$ partitioning $[n]$ with $|L_i| = |V_i|$. The *cluster-permutohedron* Cl_Γ is the set of pairs (clustering, assignment), where $(C', A') \leqslant (C, A)$ if C' refines C and the label set of each block of C is the union of the label sets of the blocks of C' it contains. It is graded by $\mathrm{rk}(C, A) = n - |C|$.

Theorem 2.5 (Ayzenberg–Buchstaber [6, Thm. 1, Cor. 2.43], [5])**.** *For connected* Γ *and generic* λ, $S(M_{\Gamma,\lambda}) \cong \mathrm{Cl}_\Gamma$. *If* Γ_h *is connected, then* $\mathrm{Hess}(M,h)$ *and* $M_{\Gamma_h,\lambda}$ *are twins: they have homeomorphic orbit spaces by a homeomorphism preserving orbit dimensions, equivariant cohomology rings that are isomorphic as graded rings, and equal Betti numbers [5, Thm. 3.11]. In particular* $S(\mathrm{Hess}(M,h)) \cong \mathrm{Cl}_{\Gamma_h}$.

The isomorphism of equivariant cohomology in Theorem 2.5 is not one of $H^*(BT)$-algebras in general: the weights of $M_{\Gamma_h,\lambda}$ at the fixed point w are $t_i - t_j$, those of $\mathrm{Hess}(M,h)$ are $t_{w(i)} - t_{w(j)}$. The face poset of a partial Hessenberg variety is described in Lemma 5.7, following [6, Thm. 2].

3 Local equivariant cohomology

Let $V \cong \bigoplus_{j=1}^n V(\alpha_j)$ be a representation of T with all $\alpha_j \neq 0$.

Proposition 3.1. *Restriction to* 0 *is an injective map* $H^*_T(V, V \setminus 0) \to H^*_T(0) = H^*(BT)$ *with image the principal ideal* $(\alpha_1 \cdots \alpha_n)$.

Proof. Choose an invariant complex structure on each $V(\alpha_j)$; it orients V. Let $V_T = ET \times_T V \to BT$ be the associated vector bundle, which is the direct sum of the complex line bundles $L_j = ET \times_T V(\alpha_j)$ with $c_1(L_j) = \alpha_j$. By definition $H^*_T(V, V \setminus 0) = H^*(V_T, V_T \setminus BT)$, where BT is the zero section. By the Thom isomorphism [35, Thm. 10.4], $H^*(V_T, V_T \setminus BT)$ is a free $H^*(BT)$-module of rank one on the Thom class τ, and the restriction of τ to the zero section is the Euler class $e(V_T) = \prod_j c_1(L_j) = \prod_j \alpha_j$. Restriction to the zero section is $H^*(BT)$-linear, so it maps $x\tau$ to $x \prod_j \alpha_j$: it is multiplication by $\prod_j \alpha_j$, which is injective because $H^*(BT)$ is a domain, with image $(\alpha_1 \cdots \alpha_n)$. □

Lemma 3.2. *Let* k *be a field and* $R = k[t_1, \ldots, t_m]$.

(1) Every nonzero linear form is irreducible, hence prime, in R.

(2) If $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_{n'}$ *are nonzero linear forms with* $(\alpha_1 \cdots \alpha_n) = (\beta_1 \cdots \beta_{n'})$, *then* $n = n'$ *and there are* $\sigma \in S_n$ *and* $c_j \in k^\times$ *with* $\beta_{\sigma(j)} = c_j \alpha_j$.

Proof. (1) If a linear form ℓ factors as $\ell = fg$, then $\deg f + \deg g = 1$, so one of the factors has degree 0 and is a unit. Since R is a unique factorization domain, irreducible elements are prime. (2) Equal principal ideals of a domain have associate generators: $\beta_1 \cdots \beta_{n'} = c\,\alpha_1 \cdots \alpha_n$ with $c \in R^\times = k^\times$. Both sides are factorizations into primes by (1), and by uniqueness of factorization in R the factors agree up to order and units. □

Theorem 3.3. *Let* $p \in X^T$ *be an isolated fixed point with tangent weights* $\alpha_1, \ldots, \alpha_n$.

(1) The composition $H^*_T(X, X \setminus p) \to H^*_T(X) \to H^*_T(p)$ *is injective, with image* $(\alpha_1 \cdots \alpha_n) = (e_T(T_pX))$.

(2) This ideal determines the multiset $\{\alpha_j\}$ *up to nonzero scalars. Hence it determines the linear matroid of weights at* p, *including parallel elements, and by Theorem 2.1 the lattice* $S(X)_{\geqslant p}$.

Proof. By the slice theorem [14], p has a closed invariant neighbourhood D that is equivariantly diffeomorphic to the unit disc of $V = T_pX$. The closed set $X \setminus \operatorname{int} D$ lies in the open set $X \setminus p$, so excision, which holds for equivariant cohomology because it holds for the pair of Borel constructions, gives $H^*_T(X, X \setminus p) \cong H^*_T(D, D \setminus p)$. Radial deformation gives $H^*_T(D, D \setminus p) \cong H^*_T(V, V \setminus 0)$. Both isomorphisms commute with restriction to p. Part (1) now follows from Proposition 3.1, and part (2) from Lemma 3.2(2), since rescaling the weights does not change linear dependence. □

Remark 3.4. As an abstract module, $H^*_T(X, X \setminus p)$ is free of rank one and carries no information; everything lies in its embedding into $H^*_T(p)$. For a GKM manifold the image can also be computed from the GKM theorem, as $\bigcap_j (\alpha_j)$. Without pairwise independence this intersection differs from the product: for $\alpha_1 = \alpha_2$ it is (α_1), not (α_1^2). The Thom isomorphism argument needs no such assumption, which is why Theorem 3.3 sees multiplicities.

4 Local weight matroids do not determine equivariant cohomology

Let $M = \mathrm{diag}(1,2,3,4,5)$, $T = T^5$, and

$$h_1 = (2,3,5,5,5), \quad h_2 = (2,4,4,5,5), \quad X_1 = \mathrm{Hess}(M, h_1), \quad X_2 = \mathrm{Hess}(M, h_2).$$

So $X_1 = \{MV_1 \subset V_2,\ MV_2 \subset V_3\}$ and $X_2 = \{MV_1 \subset V_2,\ MV_2 \subset V_4,\ MV_3 \subset V_4\}$. The graph Γ_{h_1} has edges $12, 23, 34, 35, 45$: a triangle $\{3,4,5\}$ with the path $1-2-3$ attached. The graph Γ_{h_2} has edges $12, 23, 24, 34, 45$: a triangle $\{2,3,4\}$ with two pendant edges. They are not isomorphic.

Lemma 4.1. *At every fixed point w of* $\mathrm{Hess}(M, h)$*, the weight matroid is isomorphic to the graphic matroid* $M(\Gamma_h)$*.*

Proof. The weights at w are $e_{w(i)} - e_{w(j)}$ over the edges $\{j, i\}$ of Γ_h. The linear automorphism $e_a \mapsto e_{w^{-1}(a)}$ carries them to $e_i - e_j$, which realize the graphic matroid $M(\Gamma_h)$ (Appendix A.2). □

Lemma 4.2. $M(\Gamma_{h_1}) \cong M(\Gamma_{h_2}) \cong U_{2,3} \oplus U_{1,1} \oplus U_{1,1}$.

Proof. In each graph the only cycle is the triangle and the other two edges are bridges. The circuits of a graphic matroid are the cycles of the graph, so in both cases the matroid has one circuit, of size three, and two coloops. □

Theorem 4.3. *(1)* X_1 *and* X_2 *are connected smooth projective varieties of complex dimension* 5 *with* $H^{\mathrm{odd}} = 0$*. The torus* T *acts on both with the same* 120 *fixed points, as GKM actions whose effective quotient* $T/\Delta(S^1) \cong T^4$ *acts with complexity one.*

(2) For all $p_1 \in X_1^T$ *and* $p_2 \in X_2^T$ *the weight matroids are isomorphic to* $U_{2,3} \oplus U_{1,1} \oplus U_{1,1}$*. Hence* $S(X_1)_{\geqslant p_1} \cong S(X_2)_{\geqslant p_2}$*, and the Chow rings, augmented Chow rings, intersection cohomology modules, Kazhdan–Lusztig polynomials and Orlik–Solomon algebras of the local matroids are isomorphic. Their Hilbert functions are* $(1, 14, 14, 1)$ *for the Chow ring,* $(1, 19, 44, 19, 1)$ *for the augmented Chow ring,* $(1, 5, 8, 5, 1)$ *for the intersection cohomology module and* $(1, 5, 9, 7, 2)$ *for the Orlik–Solomon algebra, and the Kazhdan–Lusztig polynomial is* 1*.*

(3) $P_{X_1}(q) = 1 + 17q + 42q^2 + 42q^3 + 17q^4 + q^5$ *and* $P_{X_2}(q) = 1 + 12q + 47q^2 + 47q^3 + 12q^4 + q^5$*. In particular* $H^*_T(X_1) \ncong H^*_T(X_2)$*, even as graded vector spaces.*

Proof. (3) Summing (2.2) over S_5 gives the polynomials. Independently, the GKM presentation (2.1) of $H^*_T(X_i)$, computed in Macaulay2 as a graded $\mathbb{Q}[t_1, \ldots, t_5]$-module, has Hilbert series $P_{X_i}(q)/(1-q)^5$ with the same numerators. Since the actions are equivariantly formal, $\dim H^2_T(X_i) = \beta_2 + 5\beta_0$, which is 22 for X_1 and 17 for X_2.

(1) Smoothness, dimension and $H^{\mathrm{odd}} = 0$ were recalled in Section 2; connectedness follows from $\beta_0 = 1$. The weights at a fixed point are distinct roots $e_a - e_b$, hence pairwise independent. An element of T acting trivially lies in the common kernel of the weights $e_a - e_b$ over the edges of a connected graph, which is the diagonal circle $\Delta(S^1)$, and $\Delta(S^1)$ acts trivially on $\mathrm{Fl}(\mathbb{C}^5)$. So the effective torus has dimension 4, and the complexity is $5 - 4 = 1$.

(2) The first statement follows from Lemmas 4.1 and 4.2, and the second from Theorem 2.1. The rings and polynomials listed are invariants of the matroid (Appendix A.2). Their Hilbert functions were computed from the presentations in Macaulay2 and checked against independent formulas in terms of the lattice of flats: the Feichtner–Yuzvinsky basis [26] for the two Chow rings, the Kazhdan–Lusztig recursion [25] for the Z-polynomial, which is the Hilbert series of the intersection cohomology module [13], and the Poincaré polynomial $(1 + 3t + 2t^2)(1 + t)^2$ of the matroid for the Orlik–Solomon algebra [36]. □

Remark 4.4. A computer search over all pairs of Hessenberg functions with $n \leqslant 6$ finds smaller pairs in $\mathrm{Fl}(\mathbb{C}^4)$, for example $h = (2,3,3,4)$ and $(2,2,4,4)$, but these varieties are disconnected and β_0 merely counts components. Connected pairs also exist in $\mathrm{Fl}(\mathbb{C}^6)$, such as $h = (2,3,4,6,6,6)$ and $(2,3,5,5,6,6)$, with local matroid $U_{2,3} \oplus U_{1,1}^{\oplus 3}$ and Betti numbers $(1, 43, 179, 274, 179, 43, 1)$ and $(1, 29, 175, 310, 175, 29, 1)$.

The global face posets of X_1 and X_2 are not isomorphic: $\mathrm{Cl}_{\Gamma_{h_1}}$ and $\mathrm{Cl}_{\Gamma_{h_2}}$ have 200 and 190 elements of rank 2. This is no accident, as the next section shows.

5 The face poset determines the Betti numbers in type A

Question 1 ([34])**.** Let X_1, X_2 carry torus actions with $H^{\mathrm{odd}}(X_1) = H^{\mathrm{odd}}(X_2) = 0$. Does $S(X_1) \cong S(X_2)$ imply $\beta_j(X_1) = \beta_j(X_2)$ for all j?

For quasitoric manifolds the answer is yes, since $\beta_{2i} = h_i$ of the orbit polytope. The converse fails already there: a pentagonal prism and a cube with one vertex cut off have the same h-vector but different face lattices.

5.1 Rigidity of cluster-permutohedra

For $y \in \mathrm{Cl}_\Gamma$ let $\nu(y)$ be the number of minimal elements below y. The minimal elements are the pairs (discrete clustering, assignment), that is, the bijections $p\colon [n] \to [n]$, where p assigns the label set $\{p(i)\}$ to the block $\{i\}$. An element $y = (C, A)$ lies above p exactly when p maps every block of C onto its label set, so

$$\nu(y) = \prod_{V_i \in C} |V_i|!.$$

In particular Cl_Γ has $n!$ minimal elements.

Lemma 5.1. *Let $x \in \mathrm{Cl}_\Gamma$ be minimal. Then $(\mathrm{Cl}_\Gamma)_{\geqslant x} \cong \mathrm{Flats}(\Gamma)$, and its atoms correspond to the edges of Γ. The join of the atoms of a set S of edges has rank $n - c(S)$, where $c(S)$ is the number of components of $([n], S)$. For distinct edges e, f, the join of their atoms has $\nu = 6$ if e and f share a vertex and $\nu = 4$ otherwise. Three pairwise adjacent edges either form a triangle, and then the join of their atoms has rank 2, or share a vertex, and then it has rank 3.*

Proof. Let x be given by the bijection p. An element $(C, A) \geqslant x$ has its assignment determined by C, namely $L_i = p(V_i)$, and every clustering C occurs; the order on these elements is refinement of clusterings. This identifies $(\mathrm{Cl}_\Gamma)_{\geqslant x}$ with $\mathrm{Flats}(\Gamma)$, and the atoms with the clusterings having one block of size two, which is an edge, and singletons elsewhere. In $\mathrm{Flats}(\Gamma)$ the join of the atoms of a set S of edges is the clustering into the vertex sets of the components of $([n], S)$, which has $c(S)$ blocks and rank $n - c(S)$. For two edges sharing a vertex this clustering has one block of size 3, so $\nu = 3! = 6$; for two disjoint edges it has two blocks of size 2, so $\nu = 2!\,2! = 4$. Three pairwise adjacent edges either form a triangle, spanning 3 vertices and one component, so the rank is $3 - 1 = 2$; or all contain a common vertex, spanning 4 vertices and one component, so the rank is $4 - 1 = 3$. □

Call a set F of edges of a simple graph a *star family* if any two edges of F share a vertex and no three form a triangle. Let $E(v)$ be the set of edges at a vertex v.

Lemma 5.2. *(1) All edges of a star family with at least two edges contain a common vertex.*
(2) The maximal star families with at least two edges are exactly the sets $E(v)$ with $\deg v \geqslant 2$, and $E(v) \neq E(v')$ for distinct such vertices.

Proof. (1) Let uv and vw be two edges of the family. Every further edge meets both. If it avoids v, it must be uw, and then the three edges form a triangle, which is excluded. So every edge contains v.

(2) If $\deg v \geqslant 2$, then $E(v)$ is a star family: its edges pairwise share v, and three edges through v span four vertices, so they do not form a triangle. Suppose that an edge $g \notin E(v)$ meets every edge of $E(v)$. It meets each of them at its endpoint other than v; as these endpoints are distinct and g has two endpoints, $E(v) = \{vu, vw\}$ and $g = uw$, so $E(v) \cup \{g\}$ contains a triangle. Hence

$E(v)$ is a maximal star family. Conversely, by (1) a star family with at least two edges is contained in some $E(v)$ with $\deg v \geqslant 2$, so a maximal one equals $E(v)$. Finally, two distinct vertices have at most one common edge, so $E(v) = E(v')$ with $v \neq v'$ forces $\deg v = 1$. □

Proposition 5.3. *Let $\varphi\colon E(\Gamma) \to E(\Gamma')$ be a bijection between the edge sets of simple graphs such that φ and φ^{-1} preserve adjacency of edges and send triangles to triangles. Then φ is induced by a bijection ψ between the non-isolated vertices: $\varphi(uv) = \psi(u)\psi(v)$ for every edge uv.*

Proof. The maps $\varphi^{\pm 1}$ preserve the property of being a star family, since this property is defined in terms of adjacency and triangles. By Lemma 5.2(2), there is a unique bijection ψ from the vertices of Γ of degree at least 2 to those of Γ' such that $\varphi(E(v)) = E(\psi(v))$. Let uv be an edge. We define ψ on the remaining non-isolated vertices and check $\varphi(uv) = \psi(u)\psi(v)$ in three cases.

(a) $\deg u, \deg v \geqslant 2$. Then $\varphi(uv) \in E(\psi(u)) \cap E(\psi(v))$, and as $\psi(u) \neq \psi(v)$, the only such edge is $\psi(u)\psi(v)$.

(b) $\deg u \geqslant 2$ and $\deg v = 1$. Then $\varphi(uv) \in E(\psi(u))$, so $\varphi(uv) = \psi(u)v'$ for a vertex v'. If $\deg v' \geqslant 2$, then $v' = \psi(w)$ for some w, and $\varphi(uv) \in E(\psi(w)) = \varphi(E(w))$ gives $uv \in E(w)$, so $w \in \{u, v\}$. But $w \neq u$ because $\psi(w) \neq \psi(u)$, and $w \neq v$ because $\deg v = 1$. Hence $\deg v' = 1$, and we put $\psi(v) = v'$. This is well defined, since v lies on one edge only.

(c) $\deg u = \deg v = 1$. Then uv is a connected component of Γ. If an endpoint of $\varphi(uv)$ had degree at least 2, then case (a) or (b) applied to φ^{-1} would show that an endpoint of uv has degree at least 2. So $\varphi(uv)$ is also a component with two vertices, and we map u, v to its endpoints in either order.

Applying the same construction to φ^{-1}, with the same choices in case (c), gives the inverse of ψ. □

Theorem 5.4. *Let Γ and Γ' be simple graphs on $[n]$ and $[n']$, $n, n' \geqslant 1$. If $\mathrm{Cl}_\Gamma \cong \mathrm{Cl}_{\Gamma'}$, then $\Gamma \cong \Gamma'$.*

Proof. Let Φ be an isomorphism. Counting minimal elements gives $n! = n'!$, so $n = n'$ (for $n = 1$ both graphs are a point). Fix a minimal x. Then Φ restricts to an isomorphism of lattices $(\mathrm{Cl}_\Gamma)_{\geqslant x} \to (\mathrm{Cl}_{\Gamma'})_{\geqslant \Phi(x)}$, and by Lemma 5.1 it induces a bijection $\varphi\colon E(\Gamma) \to E(\Gamma')$ on atoms. An isomorphism of lattices preserves joins and ranks, and Φ preserves ν because ν is defined by the order. By Lemma 5.1, two edges share a vertex if and only if the join of their atoms has $\nu = 6$, and three pairwise adjacent edges form a triangle if and only if the join of their atoms has rank 2. So φ and φ^{-1} preserve adjacency of edges and triangles, and by Proposition 5.3, Γ and Γ' are isomorphic after the removal of isolated vertices. Since both have n vertices, they have the same number of isolated vertices, and $\Gamma \cong \Gamma'$. □

Remark 5.5. Adjacency alone is the line graph. Whitney's theorem [42] together with the rank of three edges, which separates K_3 from $K_{1,3}$, would also prove Theorem 5.4. For the partial case, where a partition of the vertices must be tracked as well, one needs the edge isomorphism to be induced by a vertex bijection; by Jung's theorem [32] this holds for connected graphs with more than four vertices but fails for $K_{1,3} + e$, $K_4 - e$ and K_4. Proposition 5.3 uses the triangles that the poset sees, and has no exceptions.

Corollary 5.6. *Let each of X and X' be either $M_{\Gamma,\lambda}$ with Γ connected and λ generic, or $\mathrm{Hess}(M, h)$ with Γ_h connected. If $S(X) \cong S(X')$, then $\beta_j(X) = \beta_j(X')$ for all j, and $H^*_T(X) \cong H^*_T(X')$ as graded rings. If X and X' are both Hessenberg varieties, this is an isomorphism of $H^*(BT)$-algebras; if both are isospectral manifolds, it is one up to an automorphism of T.*

Proof. By Theorem 2.5 the face posets are Cl_Γ and $\mathrm{Cl}_{\Gamma'}$ for the corresponding graphs, so $\Gamma \cong \Gamma'$ by Theorem 5.4. Let π be a permutation with $\pi(\Gamma) = \Gamma'$ and P_π its permutation matrix. Conjugation $A \mapsto P_\pi A P_\pi^{-1}$ preserves the spectrum and the zero pattern up to relabelling, $(P_\pi A P_\pi^{-1})_{\pi(i)\pi(j)} = a_{ij}$. It is therefore a diffeomorphism $M_{\Gamma,\lambda} \to M_{\Gamma',\lambda}$, equivariant along the automorphism $t \mapsto P_\pi t P_\pi^{-1}$ of T. This proves the corollary for two isospectral manifolds. A

Hessenberg variety and its twin $M_{\Gamma_h,\lambda}$ have equal Betti numbers and equivariant cohomology rings isomorphic as graded rings (Theorem 2.5), which gives the general case. Finally, let $X = \mathrm{Hess}(M, h)$ and $X' = \mathrm{Hess}(M, h')$ with $\pi(\Gamma_h) = \Gamma_{h'}$. By (2.1), $H^*_T(X)$ consists of the families $(f_w)_{w\in S_n}$ with $f_w - f_{w\circ(i\,j)} \in (t_{w(i)} - t_{w(j)})$ for every edge ij of Γ_h, and similarly for X'. Put $f'_{w'} = f_{w'\circ\pi}$. If ij is an edge of Γ_h and $w' = w \circ \pi^{-1}$, then $\pi(i)\pi(j)$ is an edge of $\Gamma_{h'}$, $w' \circ (\pi(i)\,\pi(j)) = w \circ (i\,j) \circ \pi^{-1}$, and $t_{w'(\pi(i))} - t_{w'(\pi(j))} = t_{w(i)} - t_{w(j)}$. So $(f_w) \mapsto (f'_{w'})$ maps $H^*_T(X)$ onto $H^*_T(X')$, and it is an $H^*(BT)$-algebra isomorphism. □

The corollary does not use $H^{\mathrm{odd}} = 0$. In particular it covers $M_{K_{1,3},\lambda}$ and the manifolds $M_{C_n,\lambda}$, $n \geqslant 4$, which are not equivariantly formal [6, 8].

5.2 Partial Hessenberg varieties

Let $X = \mathrm{Hess}_{\bar{k}}(M, h)$ and $G = \Gamma_h(\bar{k})$. For a partition $W = \{W_1, \ldots, W_l\}$ of $[n]$ let $T_W \subset T$ be the subtorus of the elements whose coordinates are equal within every block. The following description of the faces is [6, Thm. 2], in the form in which it appears in the proof given there. As Theorem 5.8 depends on its details, we include a proof.

Lemma 5.7. *The faces of* X *correspond to the families* $\{(W_j, c_j)\}_{j=1}^{l}$*, where* $W_1, \ldots, W_l$ *are the blocks of a partition of* $[n]$*, each* $c_j \in \mathbb{Z}^s_{\geqslant 0}$ *is a vector of part counts with* $\sum_a c_j(a) = |W_j|$*,* $\sum_j c_j = \bar{k}$*, and for every* j *either* $|W_j| = 1$*, or the support* $\{a \mid c_j(a) > 0\}$ *has at least two elements and induces a connected subgraph of* Γ_h*. The fixed points of the face are the* J *with* $|J_a \cap W_j| = c_j(a)$ *for all* a *and* j*, so their number is*

$$\nu\big(\{(W_j, c_j)\}\big) = \prod_j \frac{|W_j|!}{\prod_a c_j(a)!}.$$

The rank of the face is $n - l$*, and* $\{(W'_i, c'_i)\} \leqslant \{(W_j, c_j)\}$ *if and only if every* W'_i *lies in some* W_j *and* $c_j = \sum_{W'_i \subset W_j} c'_i$ *for all* j*. For every fixed point* J*,* $S(X)_{\geqslant J} \cong \mathrm{Flats}(G)$*.*

Proof. Fix a fixed point J and label each vertex of G by the element $x \in J_a$ it stands for, so that the weights at J are the $t_y - t_x$ over the edges xy of G. The flats of the graphic matroid $M(G)$ are the partitions W of $[n]$ whose blocks induce connected subgraphs of G. For such a flat, the common kernel of the weights $t_y - t_x$ with x, y in one block is T_W, which is connected. By Lemma 2.2(2) the face through J for this flat is the component of X^{T_W} containing J.

We describe X^{T_W}. A T_W-invariant subspace of $\mathbb{C}^n$ is the direct sum of its intersections with the coordinate subspaces $\mathbb{C}^{W_j}$, so a partial flag in X^{T_W} is a direct sum of partial flags $V^{(j)}_\bullet$ in the $\mathbb{C}^{W_j}$. Since M is diagonal, it preserves each $\mathbb{C}^{W_j}$, so the sum satisfies the Hessenberg conditions if and only if every summand does. The jumps $c_j(a) = \dim V^{(j)}_{K_a} - \dim V^{(j)}_{K_{a-1}}$ are locally constant on X^{T_W}, and the flags with given jumps $c = (c_j)$ form the set

$$Y(W, c) \cong \prod_{j=1}^{l} \mathrm{Hess}_{c_j}(M|_{\mathbb{C}^{W_j}}, h),$$

where in the jth factor the parts a with $c_j(a) = 0$ are omitted and h is restricted accordingly. (If $c_j(a) = 0$, then $V^{(j)}_{K_a} = V^{(j)}_{K_{a'}}$ for the last part $a' < a$ in the support, and the condition at a follows from the condition at a'.) The jth factor is the partial Hessenberg variety of the subgraph of Γ_h induced on the support of c_j, and its fixed points are the distributions of the elements of W_j to the parts with counts c_j.

The face through J is the component of X^{T_W} containing J. It lies in $Y(W, c)$, where $c_j(a) = |J_a \cap W_j|$. A block W_j induces a connected subgraph of G if and only if $|W_j| = 1$, or the support of c_j has at least two parts and induces a connected subgraph of Γ_h: copies of one part are pairwise non-adjacent, and every copy of a is adjacent to every copy of each neighbour of a.

In that case each factor of $Y(W, c)$ is connected (Section 2), so $Y(W, c)$ is connected and is the face. This condition depends only on (W, c), every $Y(W, c)$ contains fixed points, and $Y(W, c)$ determines W (its flat at any fixed point) and c (its fixed points). Hence the faces are the $Y(W, c)$ with (W, c) as in the statement, with the stated fixed points; their number ν counts, for each block W_j, the ways of distributing its elements to the parts with counts c_j. The rank is the rank of the flat, $n - l$.

By Lemma 2.2(2), applied at a common fixed point, $Y(W', c') \subset Y(W, c)$ holds if and only if the two faces have a common fixed point and W' refines W. A common fixed point exists if and only if the counts of the blocks of W' inside each W_j add up to c_j. Finally, $S(X)_{\geqslant J} \cong \mathrm{Flats}(G)$ by Theorem 2.1 and Section 2. □

Theorem 5.8. *If X and X' are partial regular semisimple Hessenberg varieties with $S(X) \cong S(X')$, then $\beta_j(X) = \beta_j(X')$ for all j.*

The proof rests on three lemmas.

Lemma 5.9 (Splitting)**.** *Let $Q_1, \dots, Q_r$ be the connected components of Γ_h and P_{Q_i} the Poincaré polynomial of the partial Hessenberg variety of the restriction of $(\bar{k}, h)$ to Q_i. Then*

$$P_X(q) = \#X^T \cdot \prod_{i=1}^{r} \frac{P_{Q_i}(q)}{P_{Q_i}(1)},$$

and a component consisting of a single part contributes the factor 1.

Proof. The components of Γ_h are intervals of consecutive parts, since the edges of Γ_h join a to the parts $a + 1, \dots, h(a)$. Every pair counted in (2.3) joins two adjacent parts, which lie in one component. For a fixed point J put $U_i = \bigcup_{a \in Q_i} J_a$. The pairs counted inside Q_i depend only on the relative order of the elements of U_i, that is, on the fixed point of the variety of Q_i obtained by renumbering U_i increasingly. Conversely, a choice of the sets $U_1, \dots, U_r$ of the right sizes and of a fixed point of each variety of Q_i determines J. Summing $q^{\mathrm{ind}(J)}$ over these choices gives $P_X = \binom{n}{|U_1|,\dots,|U_r|} \prod_i P_{Q_i}$, and setting $q = 1$ gives $\#X^T = \binom{n}{|U_1|,\dots,|U_r|} \prod_i P_{Q_i}(1)$. A single part gives a point. □

Lemma 5.10 (Fibration)**.** *Put $\tilde{h}(i) = k_{h(a)}$ for $k_{a-1} < i \leqslant k_a$, and let $\pi\colon \mathrm{Fl}(\mathbb{C}^n) \to \mathrm{Fl}_{\bar{k}}$ be the projection to the partial flag variety. Then $\tilde{h}$ is a Hessenberg function, $\pi^{-1}(X) = \mathrm{Hess}(M, \tilde{h})$, the restriction of π is a fibre bundle with fibre $\prod_a \mathrm{Fl}(\mathbb{C}^{k_a})$, and*

$$P_{\mathrm{Hess}(M,\tilde{h})}(q) = P_X(q) \cdot \prod_{a=1}^{s} [k_a]_q!, \qquad [k]_q! = \prod_{m=1}^{k} (1 + q + \cdots + q^{m-1}).$$

Moreover, $\Gamma_{\tilde{h}}$ is the inflation $\Gamma_h(\bar{k})$ with every part made a clique.

Proof. The function $\tilde{h}$ is non-decreasing and $\tilde{h}(i) = k_{h(a)} \geqslant k_a \geqslant i$. A full flag $F_\bullet$ lies in $\mathrm{Hess}(M, \tilde{h})$ if and only if $MF_i \subset F_{\tilde{h}(i)}$ for all i. At $i = k_a$ this is the condition $MF_{k_a} \subset F_{k_{h(a)}}$ that defines X. For i in part a we have $F_i \subset F_{k_a}$ and $\tilde{h}(i) = \tilde{h}(k_a)$, so the condition at i follows from the condition at k_a. Hence $F_\bullet \in \mathrm{Hess}(M, \tilde{h})$ if and only if $\pi(F_\bullet) \in X$. The projection π is a holomorphic fibre bundle whose fibre over $V_\bullet$ is $\prod_a \mathrm{Fl}(V_{k_a}/V_{k_{a-1}})$, so it restricts to a bundle over X. In particular X is smooth, since $\mathrm{Hess}(M, \tilde{h})$ is smooth and π is a submersion. The cohomology of the fibre is generated by the Chern classes of the tautological bundles, which are defined on the total space, and all spaces involved have only even cohomology, so the Leray–Hirsch theorem gives $P_{\mathrm{Hess}(M,\tilde{h})} = P_X \cdot P_{\mathrm{fibre}}$, with $P_{\mathrm{Fl}(\mathbb{C}^k)} = [k]_q!$. Finally, $i < j$ is an edge of $\Gamma_{\tilde{h}}$ if and only if $j \leqslant k_{h(a)}$ for the part a of i, that is, if and only if j lies in part a or in a part b with $a < b \leqslant h(a)$. □

Lemma 5.11 (Reconstruction)**.** *Let* $G = \Gamma_h(\bar{k})$ *with its partition into parts. The poset* $S(X)$ *determines* $\#X^T$ *and the multiset of isomorphism classes, as graphs with a vertex partition, of the components of* G *that have an edge.*

Proof. The number $\#X^T$ is the number of minimal elements. Fix a fixed point J and identify $S(X)_{\geqslant J}$ with $\mathrm{Flats}(G)$ as in Lemma 5.7; the atoms are the edges of G. For edges uv and vw the join of their atoms has the single nontrivial block $\{u, v, w\}$, so by the formula for ν in Lemma 5.7, $\nu = 3!/2! = 3$ if u and w lie in one part and $\nu = 3! = 6$ otherwise; here v lies in a part different from those of u and w, since adjacent vertices lie in different parts. Two disjoint edges give two blocks of size 2, each with its elements in different parts, so $\nu = 2 \cdot 2 = 4$. Let $\Phi\colon S(X) \to S(X')$ be an isomorphism. As in the proof of Theorem 5.4, Φ induces a bijection φ between the edge sets of the inflations that preserves the rank and ν of the join of every set of edges. By the values of ν, $\varphi^{\pm 1}$ preserve adjacency, and since the rank statement of Lemma 5.1 holds verbatim for $\mathrm{Flats}(G)$, they preserve triangles. By Proposition 5.3, φ is induced by a bijection ψ between the non-isolated vertices.

The map ψ preserves parts. Indeed, two distinct non-isolated vertices u, w lie in one part if and only if they have a common neighbour v with $\nu(a_{uv} \vee a_{vw}) = 3$, where a_e is the atom of the edge e. If u, w are copies of a part a, then a lies in a component Q of Γ_h with at least two parts, because u is not isolated; so a has a neighbour b in Γ_h, every copy v of b is adjacent to u and w, and $\nu = 3$. The converse holds by the computation above. Since ψ preserves adjacency and φ preserves ν, ψ preserves this relation. Finally, the components of G with an edge are the inflations G_Q of the components Q of Γ_h with at least two parts, and the copies of the remaining parts are isolated, so ψ matches these components as graphs with parts. □

Proof of Theorem 5.8. By Lemma 5.9 it suffices to show that $S(X)$ determines $\#X^T$ and the product of the $P_Q(q)/P_Q(1)$ over the components Q of Γ_h with at least two parts. For such Q, Lemma 5.10 gives $P_Q = P_{\mathrm{Hess}(M_Q, \tilde{h}_Q)}/\prod_{a \in Q}[k_a]_q!$. The graph $\Gamma_{\tilde{h}_Q}$ is connected and is obtained from G_Q by making the parts cliques, so an isomorphism of graphs with parts $G_Q \cong G_{Q'}$ induces an isomorphism $\Gamma_{\tilde{h}_Q} \cong \Gamma_{\tilde{h}_{Q'}}$. Isomorphic connected graphs $\Gamma_{\tilde{h}}$ give Hessenberg varieties with equal Betti numbers: their twins are equivariantly diffeomorphic, as in the proof of Corollary 5.6, and twins have equal Betti numbers. Hence $P_{\mathrm{Hess}(M_Q, \tilde{h}_Q)}$ depends only on the isomorphism class of G_Q with its parts, and so do the part sizes. Lemma 5.11 completes the proof. □

5.3 Remarks

Theorem 5.8 includes partial flag varieties ($\Gamma_h = K_s$) and Grassmannians, where it is classical, and the full case $\bar{k} = (1, \ldots, 1)$. For disconnected Γ_h the variety $\mathrm{Hess}(M, h)$ is a disjoint union of products of smaller Hessenberg varieties, and Theorem 5.8 covers it as well.

The statements of this subsection were rechecked by a program written independently of the proofs: Lemma 5.7 against the faces computed from the GKM graph for all 186 pairs $(\bar{k}, h)$ with $n \leqslant 5$; formula (2.3) against the number of $\mathbb{F}_q$-points of $\mathrm{Hess}_{\bar{k}}(M, h)$ for $n = 4$ and $q = 5, 7$; and the statement that the data recovered in Lemma 5.11 determine the Poincaré polynomial, for all pairs with $n \leqslant 7$. Lemma 5.10 and Theorem 5.4 were also checked by computer: the identity of Lemma 5.10 for all 2 949 pairs $(\bar{k}, h)$ with $n \leqslant 7$, and Theorem 5.4 by exact poset isomorphism for all 208 graphs with $n \leqslant 6$.

6 Other Lie types

6.1 Hessenberg varieties of a root system

Let G be a connected semisimple complex algebraic group with a Borel subgroup B and a maximal torus $T_{\mathbb{C}} \subset B$, and let $T \subset T_{\mathbb{C}}$ be the compact torus. Let $V = \mathfrak{t}^*_{\mathbb{R}} = \mathrm{Hom}(T, S^1) \otimes \mathbb{R}$ with a

W-invariant inner product, let $\Phi \subset V$ be the root system, W the Weyl group, Φ^+ the positive roots (those of B) and Δ the simple roots. For $\alpha \in \Phi$ let s_α be the reflection in α. Order Φ^+ by $\beta \leqslant \alpha$ if $\alpha - \beta$ is a nonnegative integer combination of simple roots. A *Hessenberg ideal* is a lower order ideal $I \subset \Phi^+$: if $\alpha \in I$, $\beta \in \Phi^+$ and $\beta \leqslant \alpha$, then $\beta \in I$. Then $\mathfrak{b} \oplus \bigoplus_{\alpha \in I} \mathfrak{g}_{-\alpha}$ is stable under B. Let $s \in \mathrm{Lie}(T_{\mathbb{C}})$ be regular. The *regular semisimple Hessenberg variety* is

$$\mathrm{Hess}(I) = \left\{ gB \in G/B \;\middle|\; \mathrm{Ad}(g^{-1})s \in \mathfrak{b} \oplus \bigoplus\nolimits_{\alpha \in I} \mathfrak{g}_{-\alpha} \right\}.$$

It is a smooth projective $T_{\mathbb{C}}$-invariant subvariety of G/B of dimension $|I|$ [24], with finitely many fixed points and hence $H^{\mathrm{odd}} = 0$ (Section 2). Its fixed points are the points wB, $w \in W$, and its tangent weights at wB are the $-w(\alpha)$, $\alpha \in I$; they are pairwise linearly independent, because distinct positive roots are not proportional. So $\mathrm{Hess}(I)$ is a GKM manifold. The invariant 2-sphere through wB with weight $-w(\alpha)$ is the closure of $wU_{-\alpha}B/B$, where $U_{-\alpha}$ is the root subgroup, and it joins w to ws_α: this is the unique invariant sphere of G/B with this tangent direction, and $\mathrm{Hess}(I)$ contains an invariant sphere with this tangent direction by Lemma 2.2. For $G = \mathrm{SL}_n$ and $I = \{e_j - e_i \mid j < i \leqslant h(j)\}$ we recover $\mathrm{Hess}(M, h)$, with the maximal torus of SL_n in place of T^n.

For a set R of roots, no two of them proportional, let $W_R \subset W$ be the subgroup generated by the s_α, $\alpha \in R$, let $\Gamma(R)$ be the graph with vertex set W_R and edges $\{u, us_\alpha\}$, $\alpha \in R$, labelled by $u(\alpha)$, and let

$$A(R) = \left\{ f \colon W_R \to \mathrm{Sym}(\mathrm{span}\, R) \;\middle|\; f(u) - f(us_\alpha) \in (u(\alpha)) \text{ for all } u \in W_R,\ \alpha \in R \right\},$$

a graded ring with deg of linear forms equal to 2. Here $\mathrm{Sym}(\mathrm{span}\, R)$ is the polynomial ring on the real span of R, and $u(\alpha) \in \mathrm{span}\, R$ because W_R preserves $\mathrm{span}\, R$.

Lemma 6.1. *Let I be a Hessenberg ideal and $J = \Delta \cap I$.*

(1) I lies in the root subsystem Φ_J spanned by J, and W_I is the parabolic subgroup W_J.

(2) The connected components of $\mathrm{Hess}(I)$ correspond to the cosets wW_J: the component X_w has the fixed points wW_J, and its GKM graph is the image of $\Gamma(I)$ under $u \mapsto wu$, with labels transformed by w.

(3) Every component has the Poincaré polynomial $P_{X_w}(q) = \mathrm{Hilb}(A(I))(q) \cdot (1-q)^{\dim \mathrm{span}\, I}$, and $P_{\mathrm{Hess}(I)} = \frac{|W|}{|W_J|} P_{X_w}$. In particular $\mathrm{Hess}(I)$ is connected if and only if $\Delta \subset I$.

Proof. (1) A positive root α satisfies $\gamma \leqslant \alpha$ for every simple root γ in its support. As I is a lower ideal, the support of every $\alpha \in I$ lies in J, so $I \subset \Phi_J$. Hence $W_I \subset W_J$, and $W_J \subset W_I$ because $J \subset I$.

(2) The edges of the GKM graph of $\mathrm{Hess}(I)$ join w to ws_α, $\alpha \in I$, so its components are the cosets $wW_I = wW_J$. By Lemma 2.3 these are the fixed point sets of the components of $\mathrm{Hess}(I)$.

(3) Each component X_w is a smooth projective GKM manifold, so (2.1) computes $H^*_T(X_w; \mathbb{R})$ from its GKM graph, which by (2) consists of the vertices wu, $u \in W_J$, and edges $\{wu, wus_\alpha\}$ labelled by $\pm wu(\alpha)$. The map $f \mapsto \big(u \mapsto w^{-1} f(wu)\big)$, where w^{-1} acts on $\mathrm{Sym}(V)$, identifies $H^*_T(X_w; \mathbb{R})$ with the ring of functions $f \colon W_J \to \mathrm{Sym}(V)$ satisfying the conditions of $A(I)$. Write $V = U \oplus U^\perp$ with $U = \mathrm{span}\, I$. Since $\mathrm{Sym}(V) = \mathrm{Sym}(U) \otimes \mathrm{Sym}(U^\perp)$ is free over $\mathrm{Sym}(U)$ and all labels lie in U, a function with values in $\mathrm{Sym}(V)$ satisfies the conditions if and only if each of its coefficients with respect to a basis of $\mathrm{Sym}(U^\perp)$ does; so this ring is $A(I) \otimes \mathrm{Sym}(U^\perp)$. Its Hilbert series is $\mathrm{Hilb}(A(I))(q)/(1-q)^{\dim U^\perp}$, and by equivariant formality it equals $P_{X_w}(q)/(1-q)^{\dim V}$. The last statement follows from (2), since $|W|/|W_J| = 1$ if and only if $J = \Delta$. The same argument shows that $P_Y(q) = \mathrm{Hilb}(A(R))(q) \cdot (1-q)^{\dim \mathrm{span}\, R}$ for every connected GKM manifold Y whose GKM graph is the image of $\Gamma(R)$ under $u \mapsto wu$, with labels transformed by w, for some $w \in W$. □

Proposition 6.2. *The face poset of $X = \mathrm{Hess}(I)$ is the set of pairs (F, wW_F) of a flat F of the linear matroid $M(I)$ and a left coset $wW_F \subset W$, where $(F, wW_F) \leqslant (F', w'W_{F'})$ if and only if $F \subset F'$*

and $wW_F \subset w'W_{F'}$. *The face* (F, wW_F) *has rank* $\operatorname{rk} F$, *its fixed points are the* $w' \in wW_F$, *and* $\nu(F, wW_F) = |W_F|$. *Its face submanifold* $X_{(F,wW_F)}$ *is a smooth projective GKM manifold whose GKM graph is the image of* $\Gamma(F)$ *under* $u \mapsto wu$, *with labels transformed by* w. *In particular* $S(X)_{\geqslant w} \cong \mathcal{L}(I)$, *the lattice of flats of* $M(I)$, *for every* w.

Proof. The weights at w are $-w(\alpha)$, $\alpha \in I$, and w is linear, so the matroid of weights at w is $M(I)$. By Lemma 2.2(2) the faces through w correspond to the flats F of $M(I)$; the face Y is the component through w of X^H, where H is the identity component of the common kernel of the characters $w(\alpha)$, $\alpha \in F$. The Lie algebra of H is annihilated exactly by the span $w(\operatorname{span} F)$ of these characters.

The manifold Y is a component of the fixed set of a torus in a smooth projective variety, hence a smooth projective variety with finitely many fixed points, and it is a GKM manifold: its weights at its fixed points are among those of X. Its invariant 2-spheres are the invariant spheres of X contained in Y. An invariant sphere C of X through a fixed point $v \in Y$, with weight χ at v, lies in Y if and only if χ is trivial on H, that is, $\chi \in w(\operatorname{span} F)$. Indeed, if $\chi|_H = 1$, then H fixes C pointwise, so C is a connected subset of X^H meeting Y and lies in Y; conversely, if $C \subset Y \subset X^H$, then H acts trivially on $T_v C = V(\chi)$.

Write a fixed point of Y as wu. The spheres at wu have weights $-wu(\beta)$, $\beta \in I$, and lead to wus_β. Suppose $u \in W_F$. Then u preserves $\operatorname{span} F$, because each s_α, $\alpha \in F$, does. So $wu(\beta) \in w(\operatorname{span} F)$ if and only if $\beta \in \operatorname{span} F$, that is, $\beta \in I \cap \operatorname{span} F = F$, since F is a flat; and then $us_\beta \in W_F$. Starting from $u = 1$ we conclude that the fixed points of Y reachable from w in its GKM graph are the wu with $u \in W_F$, all of which are reachable since W_F is generated by the s_β, $\beta \in F$. By Lemma 2.3, applied to the connected manifold Y, $Y^T = wW_F$, and the GKM graph of Y is as stated.

The computation also shows that the weights of Y at any fixed point wu are $-wu(\beta)$, $\beta \in F$, so the flat of Y at wu is again F. Hence the face Y is determined by F and any of its fixed points, which proves that $(F, wW_F) \mapsto Y$ is well defined; it is injective because Y determines its fixed points and its flats, and surjective because every face contains a fixed point. By Lemma 2.2(2), applied at a common fixed point, the order is as stated, and the rank of Y is $\operatorname{rk} F$. □

For type A the flats of $M(I)$ are the clusterings of Γ_h, W_F is the product of the symmetric groups of the blocks, and Proposition 6.2 recovers $S(\mathrm{Hess}(M, h)) \cong \mathrm{Cl}_{\Gamma_h}$ without the twin construction.

Proposition 6.3. *Let* $R \subset \Phi$ *and* $R' \subset \Phi'$ *be sets of roots of two root systems, no two proportional. Suppose that there is a linear isomorphism* $g\colon \operatorname{span} R \to \operatorname{span} R'$ *with* $gW_R g^{-1} = W_{R'}$, *as groups of linear maps of* $\operatorname{span} R$ *and* $\operatorname{span} R'$, *that maps the set of lines* $\{\mathbb{R}\alpha \mid \alpha \in R\}$ *onto* $\{\mathbb{R}\alpha' \mid \alpha' \in R'\}$. *Then* $A(R) \cong A(R')$ *as graded rings, compatibly with* $\mathrm{Sym}(g)$. *Consequently:*

(1) if $R = I$ *and* $R' = I'$ *are Hessenberg ideals and* $|W| = |W'|$, *then* $\mathrm{Hess}(I)$ *and* $\mathrm{Hess}(I')$ *have the same Betti numbers;*

(2) if moreover $\Delta \subset I$ *and* $\Delta' \subset I'$, *then* $H^*_T(\mathrm{Hess}(I); \mathbb{R}) \cong H^*_T,(\mathrm{Hess}(I'); \mathbb{R})$ *as graded rings.*

Proof. Let $\alpha \in R$. The element $gs_\alpha g^{-1}$ of $W_{R'}$ has order two, fixes the hyperplane $g(\alpha^\perp \cap \operatorname{span} R)$ pointwise and acts by -1 on the line $g(\mathbb{R}\alpha)$. The elements of W' that act on $\operatorname{span} R'$ in this way and trivially on its orthogonal complement are reflections, and every reflection in a Weyl group is s_γ for a root γ [31, §1.14]; its -1-eigenline is $\mathbb{R}\gamma$. By hypothesis $g(\mathbb{R}\alpha)$ is the line of some $\alpha' \in R'$, so $gs_\alpha g^{-1} = s_{\alpha'}$. Hence $u \mapsto gug^{-1}$ maps the edge $\{u, us_\alpha\}$ of $\Gamma(R)$ to the edge $\{gug^{-1}, gug^{-1}s_{\alpha'}\}$ of $\Gamma(R')$, and its label $u(\alpha)$ to $g(u(\alpha)) = (gug^{-1})(g\alpha)$, a nonzero multiple of the label $(gug^{-1})(\alpha')$. Therefore $f \mapsto \big(gug^{-1} \mapsto \mathrm{Sym}(g)(f(u))\big)$ is an isomorphism $A(R) \to A(R')$ of graded rings.

(1) By Lemma 6.1(1), $W_I = W_J$ and $W_{I'} = W_{J'}$, and $|W_J| = |W_{I'}|$ since g conjugates these groups. Since $\dim \operatorname{span} I = \dim \operatorname{span} I'$, Lemma 6.1(3) gives the equality of Poincaré polynomials. (2) Now $\operatorname{span} I = V$, $\operatorname{span} I' = V'$ and $A(I) = H^*_T(\mathrm{Hess}(I); \mathbb{R})$ by the proof of Lemma 6.1(3). □

Automorphisms of the Dynkin diagram and the duality $\alpha \leftrightarrow \alpha^\vee$, which identifies the lines of B_n with those of C_n, are instances of Proposition 6.3.

6.2 Simply-laced types

Lemma 6.4. *Let $\alpha_1, \dots, \alpha_k$, $k \geqslant 3$, be vectors in a Euclidean space with $(\alpha_i, \alpha_i) = 2$ such that $(\alpha_i, \alpha_{i+1}) = \pm 1$ for all i, indices modulo k, and all other pairs are orthogonal. Let $\varepsilon = \prod_{i=1}^{k} (\alpha_i, \alpha_{i+1})$. Then the α_i span a space of dimension $k-1$ if $\varepsilon = (-1)^k$, and of dimension k otherwise.*

Proof. The dimension of the span is the rank of the Gram matrix $2 \cdot \mathrm{Id} + A$, where A is a signed adjacency matrix of the k-cycle, with entries $A_{i,i+1} = A_{i+1,i} = (\alpha_i, \alpha_{i+1})$. Replacing α_i by $-\alpha_i$ changes the signs of the two entries in row and column i, so it preserves ε and the rank. Changing signs successively at $\alpha_2, \dots, \alpha_k$, we may assume that $(\alpha_i, \alpha_{i+1}) = -1$ for $i = 1, \dots, k-1$; then $(\alpha_k, \alpha_1) = -1$ if and only if $\varepsilon = (-1)^k$. In that case $2 \cdot \mathrm{Id} + A$ is the Cartan matrix of the affine Dynkin diagram $\tilde{A}_{k-1}$, which is positive semidefinite of corank one, with kernel spanned by $(1, \dots, 1)$. Otherwise $-A$ is the adjacency matrix of a k-cycle with one negative edge; the vectors $(\zeta^j, \zeta^{2j}, \dots, \zeta^{kj})$ with $\zeta = e^{\pi i(2m+1)/k}$ show that its eigenvalues are $2\cos((2m+1)\pi/k)$, $m = 0, \dots, k-1$, none of which equals 2. So $2 \cdot \mathrm{Id} + A$ is nonsingular. □

A *signing* of a graph is a map ε from its edges to $\{\pm 1\}$. For a cycle C write $\varepsilon(C)$ for the product of the signs of its edges.

Lemma 6.5. *Let $\varepsilon, \varepsilon'$ be signings of a graph N with $\varepsilon(C) = \varepsilon'(C)$ for every induced cycle C of N. Then there is a map d from the vertices to $\{\pm 1\}$ with $\varepsilon'(uv) = d_u d_v \varepsilon(uv)$ for every edge uv.*

Proof. First, $\varepsilon(C) = \varepsilon'(C)$ for every cycle C, by induction on its length: if C has a chord e, then $C \cup \{e\}$ is the union of two shorter cycles C_1, C_2 sharing only e, and $\varepsilon(C) = \varepsilon(C_1)\varepsilon(C_2)$, because the sign of e appears twice; similarly for ε'. Put $\eta = \varepsilon\varepsilon'$, so that $\eta(C) = 1$ for every cycle. Choose a spanning forest of N and a root in each tree, put $d = 1$ at the roots, and define $d_v = d_u \eta(uv)$ along the tree edges. For an edge uv outside the forest, the cycle formed by uv and the tree path from u to v has $\eta = 1$, which gives $\eta(uv) = d_u d_v$. This is the statement. (This is Harary's balance theorem [30]; see also [44].) □

Theorem 6.6. *Let Φ and Φ' be simply-laced root systems and $R \subset \Phi$, $R' \subset \Phi'$ sets of roots, no two proportional. For a flat F of $M(R)$ put $\nu(F) = |W_F|$. If there is an isomorphism of lattices of flats $\mathcal{L}(R) \to \mathcal{L}(R')$ that preserves ν on the flats of rank 2, then there is an isometry $g\colon \operatorname{span} R \to \operatorname{span} R'$ with $gW_R g^{-1} = W_{R'}$ that maps the lines of R onto the lines of R'.*

Proof. Normalize the roots to $(\alpha, \alpha) = 2$. In a simply-laced root system, two non-proportional roots α, β have $(\alpha, \beta) \in \{0, \pm 1\}$. Let $\varphi\colon R \to R'$ be the bijection of atoms induced by the isomorphism.

Step 1: orthogonality. Let $\alpha, \beta \in R$ be distinct, and let F be the flat they span, so that $F = R \cap \operatorname{span}\{\alpha, \beta\}$. If $(\alpha, \beta) = 0$, the only roots in the plane $\operatorname{span}\{\alpha, \beta\}$ are $\pm\alpha, \pm\beta$: a root $a\alpha + b\beta$ has $(a\alpha + b\beta, \alpha) = 2a \in \{0, \pm 1, \pm 2\}$ and similarly for b, and $2a^2 + 2b^2 = 2$ leaves only these four. So $F = \{\alpha, \beta\}$ and $W_F = \{1, s_\alpha, s_\beta, s_\alpha s_\beta\}$, of order 4. If $(\alpha, \beta) = \pm 1$, the plane is a root system of type A_2 and s_α, s_β generate its Weyl group, of order 6, which contains the reflections in all roots of the plane; so $|W_F| = 6$. Hence φ maps orthogonal pairs to orthogonal pairs and non-orthogonal pairs to non-orthogonal pairs: it is an isomorphism of the graphs $N(R)$ and $N(R')$ with vertex sets R and R' whose edges are the pairs of non-orthogonal roots.

Step 2: signs. Sign each edge $\alpha\beta$ of $N(R)$ by $\varepsilon(\alpha\beta) = (\alpha, \beta)$, and each edge of $N(R')$ likewise, and pull back the second signing along φ to a signing ε' of $N(R)$. Let $C = (\alpha_1, \dots, \alpha_k)$ be an induced cycle of $N(R)$. The join of the atoms of C in $\mathcal{L}(R)$ is the flat spanned by C, whose rank is $\dim \operatorname{span} C$, and Lemma 6.4 expresses $\varepsilon(C)$ through this dimension. Since φ preserves induced cycles and the ranks of joins, $\varepsilon(C) = \varepsilon'(C)$. By Lemma 6.5 there are signs d_α with

$(\varphi\alpha, \varphi\beta) = d_\alpha d_\beta(\alpha, \beta)$ for all edges $\alpha\beta$ of $N(R)$, and this holds for all pairs $\alpha, \beta \in R$, since both sides vanish for non-edges and equal 2 for $\alpha = \beta$.

Step 3: the isometry. The families $(\alpha)_{\alpha\in R}$ and $(d_\alpha\varphi(\alpha))_{\alpha\in R}$ have the same Gram matrix. Hence a linear relation $\sum c_\alpha\alpha = 0$ implies $\left|\sum c_\alpha d_\alpha\varphi(\alpha)\right|^2 = \left|\sum c_\alpha\alpha\right|^2 = 0$, so $g(\alpha) = d_\alpha\varphi(\alpha)$ defines a linear map $g\colon \operatorname{span} R \to \operatorname{span} R'$. It preserves inner products and its image contains R' up to signs, so it is an isometry onto $\operatorname{span} R'$. It maps the line of α to the line of $\varphi(\alpha)$, and $gs_\alpha g^{-1} = s_{g\alpha} = s_{\varphi(\alpha)}$. Therefore $gW_R g^{-1}$ is generated by the $s_{\varphi(\alpha)}$, $\alpha \in R$, that is, $gW_R g^{-1} = W_{R'}$. □

Corollary 6.7. *Let* $X = \mathrm{Hess}(I)$ *and* $X' = \mathrm{Hess}(I')$ *be regular semisimple Hessenberg varieties of simply-laced types. If* $S(X) \cong S(X')$, *then* $\beta_j(X) = \beta_j(X')$ *for all* j. *If moreover* X *and* X' *are connected, then* $H^*_T(X) \cong H^*_{T'}(X')$ *as graded rings.*

Proof. An isomorphism $S(X) \to S(X')$ maps minimal elements to minimal elements, so $|W| = |W'|$, and it restricts to an isomorphism of the intervals above a minimal element w and its image. By Proposition 6.2 these intervals are $\mathcal{L}(I)$ and $\mathcal{L}(I')$, and ν, which is defined by the order, is $F \mapsto |W_F|$ on both. Theorem 6.6 with $R = I$ and $R' = I'$ gives an isometry g as in Proposition 6.3, which proves the first statement. In the connected case $\operatorname{span} I = V$, and g maps the roots of I to roots up to sign; as I spans the rational structure $t^*_{\mathbb{Q}}$, the map g and the isomorphism of Proposition 6.3(2) are defined over $\mathbb{Q}$. □

Remark 6.8. For Hessenberg varieties of type A, Corollary 6.7 gives a proof of Corollary 5.6 that does not use twins. Propositions 6.2 and 6.3 hold in every type. The first step of the proof of Theorem 6.6 does not: in a plane of type B_2 or G_2 that contains at least three lines of R, ν is the order of the whole dihedral group and no longer tells which pairs of lines are orthogonal. We computed the lattice $\mathcal{L}(I)$ with the function ν, up to isomorphism, for all 1 230 Hessenberg ideals of the irreducible root systems of rank at most 5: A_1–A_5, B_2–B_5, C_3–C_5, D_4, D_5, F_4 and G_2. Two ideals with isomorphic data and Weyl groups of the same order, of the same or of different types, always have equal Poincaré polynomials, and for connected varieties they always differ by a diagram automorphism or by duality. So the conclusion of Corollary 6.7 holds for all types in these ranks.

7 A formula over the face poset

7.1 The formula

In this section the torus action on X preserves an almost complex structure, so that the tangent weights have well-defined signs, and $\xi \in \mathrm{Lie}(T)$ is a vector on which no weight vanishes and which satisfies

$$P_Y(q) = \sum_{p\in Y^T} q^{\mathrm{ind}_Y(p)} \qquad \text{for } Y = X \text{ and for every face submanifold } Y \text{ of } X, \tag{7.1}$$

where $\mathrm{ind}_Y(p)$ is the number of weights of Y at p that are negative on ξ. The face submanifolds are almost complex submanifolds, since $T_pX_\sigma = (T_pX)^H$ is invariant under the almost complex structure. Condition (7.1) holds for smooth projective varieties with an algebraic torus action, by the Białynicki-Birula decomposition [9, 22] for a one-parameter subgroup with the same signs as ξ on all weights, because face submanifolds are components of fixed point sets of subtori, hence again smooth projective. It also holds for Hamiltonian actions on compact symplectic manifolds, where $\langle\mu, \xi\rangle$ is a perfect Morse function and face submanifolds are symplectic [3, 29].

For a face σ and a fixed point $p \in X_\sigma$, let $N_p(\sigma)$ be the set of weights of X_σ at p that are negative on ξ, and $M(N_p(\sigma))$ its linear matroid. Recall that the Tutte polynomial of a matroid M

of rank r on a set E is

$$T_M(x,y) = \sum_{S\subset E} (x-1)^{r-r(S)}(y-1)^{|S|-r(S)},$$

so that $T_M(1,y) = \sum_S (y-1)^{|S|-r}$, the sum over the spanning subsets $S \subset E$, that is, those with $r(S) = r$ (Appendix A.2).

Theorem 7.1. *Under assumption* (7.1),

$$P_X(q) = \sum_{\sigma\in S(X)} (q-1)^{\operatorname{rk}\sigma} e_\sigma(q), \qquad e_\sigma(q) = \sum_p T_{M(N_p(\sigma))}(1,q),$$

where the second sum is over the fixed points p of X_σ such that $N_p(\sigma)$ spans the space spanned by the weights of X_σ at p. The same formula holds for every face submanifold, $P_{X_\tau} = \sum_{\sigma\leqslant\tau}(q-1)^{\operatorname{rk}\sigma} e_\sigma$, with the same e_σ. Hence the e_σ are determined recursively by the Poincaré polynomials of the face submanifolds,

$$e_\tau = (q-1)^{-\operatorname{rk}\tau}\Big(P_{X_\tau} - \sum_{\sigma<\tau}(q-1)^{\operatorname{rk}\sigma} e_\sigma\Big), \tag{7.2}$$

and they do not depend on ξ.

Proof. Let p be a fixed point and N_p the set of weights at p that are negative on ξ, so that $\operatorname{ind}_X(p) = |N_p|$. Expanding $q^{|N_p|} = (1+(q-1))^{|N_p|}$ gives

$$q^{\operatorname{ind}_X(p)} = \sum_{S\subset N_p} (q-1)^{|S|}.$$

Sort the subsets S according to the flat F of M_p that they span. By Lemma 2.2(2), the flats F of M_p correspond to the faces σ containing p, and the weights of X_σ at p are those in F. For a given face $\sigma \ni p$ with flat F, the subsets $S \subset N_p$ that span F are the subsets of $N_p \cap F = N_p(\sigma)$ of rank $\operatorname{rk} F = \operatorname{rk}\sigma$. If $N_p(\sigma)$ has rank smaller than $\operatorname{rk}\sigma$ there are none. Otherwise they are the spanning subsets of the matroid $M(N_p(\sigma))$, which has rank $\operatorname{rk}\sigma$, and

$$\sum_{\substack{S\subset N_p(\sigma)\\ r(S)=\operatorname{rk}\sigma}} (q-1)^{|S|} = (q-1)^{\operatorname{rk}\sigma}\, T_{M(N_p(\sigma))}(1,q).$$

Summing over p and using (7.1) for $Y = X$,

$$P_X(q) = \sum_{p\in X^T}\sum_{\sigma\ni p}(q-1)^{\operatorname{rk}\sigma}\,[\,N_p(\sigma)\text{ spans }]\,T_{M(N_p(\sigma))}(1,q) = \sum_\sigma (q-1)^{\operatorname{rk}\sigma} e_\sigma(q).$$

For a face τ, Lemma 2.2(3) shows that the faces of X_τ are the faces $\sigma \leqslant \tau$ of X, with the same fixed points and the same weights; so the sets $N_p(\sigma)$ are the same for X_τ as for X, and the computation, now using (7.1) for $Y = X_\tau$, gives $P_{X_\tau} = \sum_{\sigma\leqslant\tau}(q-1)^{\operatorname{rk}\sigma} e_\sigma$. Solving for the term $\sigma = \tau$ gives (7.2), and induction on the rank shows that the e_σ are determined by the polynomials P_{X_τ}, which do not depend on ξ. □

Example 7.2. (1) Suppose that the weights of X_σ at each of its fixed points are linearly independent, as for every face of a toric manifold. Then $N_p(\sigma)$ spans only if it contains all the weights of X_σ at p, that is, if $\operatorname{ind}_{X_\sigma}(p) = \dim_{\mathbb{C}} X_\sigma$. By (7.1) for $Y = X_\sigma$ the number of such p is the top Betti number of the closed connected almost complex manifold X_σ, which is 1. At that point $M(N_p(\sigma))$ is a free matroid, with $T = 1$. So $e_\sigma = 1$. For a smooth projective toric variety, whose face poset is the face poset of its moment polytope with the rank of a face equal to its dimension, Theorem 7.1 becomes the formula $P_X(q) = h(q) = \sum_\sigma (q-1)^{\dim\sigma}$ for the h-polynomial of a simple polytope.

(2) For $\mathrm{Fl}(\mathbb{C}^3)$ there are 6 faces of rank 0 and 9 of rank 1, all with $e = 1$, and the top face. Take ξ with $\xi_1 < \xi_2 < \xi_3$; then N_w is the set of inversions of w. The two permutations with two inversions have N_w free of rank two and contribute 1 each, and the longest permutation contributes $T_{U_{2,3}}(1, q) = q + 2$. So $e = q + 4$ and

$$P = 6 + 9(q-1) + (q-1)^2(q+4) = 1 + 2q + 2q^2 + q^3.$$

(3) More generally, for a connected Hessenberg graph Γ_h on $[n]$ and $\xi_1 < \cdots < \xi_n$, the set N_w consists of the edges $\{j < i\}$ with $w(j) > w(i)$, and exchanging the two sums in the definition gives the top face

$$e(q) = \sum_S (q-1)^{|S|-n+1} L(S),$$

the sum over the connected spanning sets S of edges of Γ_h, where $L(S)$ is the number of $w \in S_n$ with $w(j) > w(i)$ for all edges $\{j < i\}$ in S. With $L \equiv 1$ this would be $T_{\Gamma_h}(1, q)$, an invariant of the graphic matroid. The values are 1 for a path, $q + 4$ for K_3, $q^3 + 6q^2 + 20q + 33$ for K_4, and $q + 6$ for *both* graphs of Theorem 4.3. The two varieties of that theorem are thus distinguished only by their faces of lower rank.

(4) e_σ is not a function of the local matroids: $h = (2, 3, 5, 6, 6, 6)$ and $h = (2, 4, 5, 5, 6, 6)$ have isomorphic graphic matroids, but $e = q^2 + 8q + 28$ and $e = q^2 + 8q + 33$ for the top faces.

The polynomial e_σ has nonnegative coefficients, since $T_M(1, y)$ does, and $e_\sigma(1)$ is the total number of bases of the matroids $M(N_p(\sigma))$. For a toric variety each face is the closure of a single orbit of the complex torus and $e_\sigma = 1$; in general one may think of e_σ as a measure of the orbits of the complex torus in the open stratum of σ, but we do not use or prove such an interpretation.

7.2 Dependence on the face poset

Theorem 7.1 computes e_σ from the signs of the weights on ξ. For Hessenberg varieties of simply-laced type the result depends only on the face poset.

Proposition 7.3. *Let* $X = \mathrm{Hess}(I)$ *and* $X' = \mathrm{Hess}(I')$ *be regular semisimple Hessenberg varieties of simply-laced types, and let* $\sigma \in S(X)$, $\sigma' \in S(X')$. *If* $S(X)_{\leqslant \sigma} \cong S(X')_{\leqslant \sigma'}$, *then* $e_\sigma = e_{\sigma'}$.

Proof. By (7.2), e_σ is determined by the poset $S(X)_{\leqslant \sigma}$ together with the function $\tau \mapsto P_{X_\tau}$ on it. An isomorphism $\Psi\colon S(X)_{\leqslant \sigma} \to S(X')_{\leqslant \sigma'}$ restricts to isomorphisms $S(X)_{\leqslant \tau} \cong S(X')_{\leqslant \Psi(\tau)}$ and preserves ranks. So it suffices to show that $S(X)_{\leqslant \tau} \cong S(X')_{\leqslant \tau'}$ implies $P_{X_\tau} = P_{X'_{\tau'}}$.

Write $\tau = (F, wW_F)$ and $\tau' = (F', w'W_{F'})$ as in Proposition 6.2. An isomorphism $S(X)_{\leqslant \tau} \cong S(X')_{\leqslant \tau'}$ maps the minimal element w to a minimal element w'' and restricts to an isomorphism of the intervals $[w, \tau]$ and $[w'', \tau']$. By Proposition 6.2 the first interval is the set of $(F'', wW_{F''})$ with $F'' \subset F$ a flat of $M(I)$, that is, the lattice $\mathcal{L}(F)$ of flats of $M(F)$, with $\nu(F'') = |W_{F''}|$; similarly for the second. Theorem 6.6 with $R = F$ and $R' = F'$ gives an isometry as in Proposition 6.3, so $A(F) \cong A(F')$ and $\dim \operatorname{span} F = \dim \operatorname{span} F'$. By Proposition 6.2, X_τ is a connected GKM manifold whose GKM graph is a translate of $\Gamma(F)$, and by the last statement of Lemma 6.1, $P_{X_\tau} = \mathrm{Hilb}(A(F)) \cdot (1-q)^{\dim \operatorname{span} F}$. The same holds for $X'_{\tau'}$, and the two polynomials are equal. □

Remark 7.4. The argument shows more generally: if in a class of torus actions that is closed under passing to face submanifolds the face poset determines the Betti numbers, then e_σ depends only on $S(X)_{\leqslant \sigma}$. Conversely, if each e_σ depends only on $S(X)_{\leqslant \sigma}$, then by Theorem 7.1 so does P_X. So for a class closed under faces, a formula for e_σ in terms of the face poset is the same thing as a formula answering Question 1. For connected Hessenberg varieties of simply-laced type, Theorem 6.6 shows that e of the top face even depends only on the lattice $\mathcal{L}(I)$ above one fixed point with the function ν on its flats of rank 2, together with $|W|$. For the non-simply-laced types we checked by computer that e of the face (I, W_I) depends only on $|W|$ and on $\mathcal{L}(I)$ with ν, for

all 427 nonempty Hessenberg ideals of A_1–A_5, B_2–B_4, C_3, C_4, D_4 and G_2. The same computation shows that e is finer than the Betti numbers: two Hessenberg ideals of C_3 with four roots give varieties with the same Poincaré polynomial $1+12q+22q^2+12q^3+q^4$ but $e = q+9$ and $e = q+7$. Example 7.2(4) shows that the lattice $\mathcal{L}(I)$ alone does not suffice.

7.3 Oriented matroids and moment polytopes

Theorem 7.1 uses three kinds of data: the faces with their fixed points, the matroids of weights, and the signs of the weights on a common ξ. The last is an oriented structure: at each fixed point, the oriented matroid of the weights together with the tope cut out by ξ (Appendix A.3). The oriented matroids alone carry much less. For Hess(I) the signed weights at w are the image under the linear map w of those at the identity, so the oriented matroids at all fixed points are isomorphic; and in type A they are graphic, hence regular, so that even their reorientation class is determined by the underlying matroid [11]. At the other extreme, for a generic circle in T the weights at p are nonzero integers, and the oriented matroid at p is determined by the unordered pair $\{\mathrm{ind}(p), n - \mathrm{ind}(p)\}$ of the numbers of negative and positive weights. By (7.1) and Poincaré duality, $\beta_{2i} = \beta_{2n-2i} = \#\{p \mid \mathrm{ind}(p) = i\}$, so these pairs determine the Betti numbers.

For Hamiltonian actions one property of the oriented matroids has a direct geometric meaning. A vector configuration is *acyclic* if it lies in an open half-space, and *totally cyclic* if it positively spans its linear span.

Proposition 7.5. *Let a torus H act effectively on a compact connected symplectic manifold X in a Hamiltonian fashion with isolated fixed points, with moment map μ and moment polytope $\Delta = \mu(X) \subset \mathfrak{h}^*$. Take the weights with respect to an invariant compatible almost complex structure, with the signs for which $\mu(z) = \mu(p) + \frac{1}{2}\sum_j |z_j|^2 \alpha_{p,j}$ in equivariant Darboux coordinates z near a fixed point p. Then:*

(1) the weights at p are acyclic if and only if $\mu(p)$ is a vertex of Δ, and μ maps the fixed points with acyclic weights bijectively onto the vertices of Δ;

(2) the weights at p are totally cyclic if and only if $\mu(p)$ is an interior point of Δ.

Proof. We use three facts: Δ is the convex hull of $\mu(X^H)$ [3, 29]; the fibres of μ are connected [3]; and for $\zeta \in \mathfrak{h}$ every local maximum of $\langle \mu, \zeta \rangle$ is a global maximum, since $\langle \mu, \zeta \rangle$ is a Morse–Bott function with even indices [3]. Since the action is effective, the weights at p span $\mathfrak{h}^*$. Near p we have $\langle \mu, \zeta \rangle = \langle \mu(p), \zeta \rangle + \frac{1}{2}\sum_j |z_j|^2 \langle \alpha_{p,j}, \zeta \rangle$.

(1) If the weights at p lie in an open half-space, the set K of ζ with $\langle \alpha_{p,j}, \zeta \rangle < 0$ for all j is a nonempty open cone. For $\zeta \in K$, p is a strict local maximum of $\langle \mu, \zeta \rangle$, hence a global maximum, so $\mu(p)$ maximizes $\langle \cdot, \zeta \rangle$ on Δ. A point of a polytope that maximizes an open set of linear functionals is a vertex. Conversely, let $\mu(p)$ be a vertex. The set K' of ζ for which $\mu(p)$ is the unique maximizer of $\langle \cdot, \zeta \rangle$ on Δ is a nonempty open cone. For $\zeta \in K'$, p is a maximum of $\langle \mu, \zeta \rangle$, so $\langle \alpha_{p,j}, \zeta \rangle \leqslant 0$ for all j. Choosing $\zeta \in K'$ outside the finitely many hyperplanes $\alpha_{p,j}^{\perp}$ gives $\langle \alpha_{p,j}, \zeta \rangle < 0$ for all j. For the bijection, let $x \in \mu^{-1}(\mu(p))$ with $\mu(p)$ a vertex. For $\zeta \in K'$, x is a maximum of $\langle \mu, \zeta \rangle$, so the vector field ζ_X vanishes at x, because $d\langle \mu, \zeta \rangle = \iota_{\zeta_X}\omega$ and ω is nondegenerate. As K' spans $\mathfrak{h}$, x is fixed by H. So the fibre over a vertex consists of fixed points; it is connected and the fixed points are isolated, so it is a single point. Every vertex is the image of a fixed point, because Δ is the convex hull of $\mu(X^H)$.

(2) If the weights positively span $\mathfrak{h}^*$, then μ maps a neighbourhood of p onto a neighbourhood of $\mu(p)$, by the formula above, so $\mu(p)$ is an interior point. If they do not, their cone lies in a closed half-space, so there is $\zeta \neq 0$ with $\langle \alpha_{p,j}, \zeta \rangle \leqslant 0$ for all j. Then p is a local maximum of $\langle \mu, \zeta \rangle$, hence a global one, and $\mu(p)$ lies on the boundary of Δ. □

So the number of fixed points with acyclic weights is not a new invariant: it is the number of vertices of the moment polytope. For example, if a 2-torus acts on $\mathbb{C}P^3$ through four characters, three of which are the vertices of a triangle containing the fourth, the weight matroid is $U_{2,3}$ at

all four fixed points, and the oriented matroid distinguishes the interior fixed point from the other three.

8 Open questions

(1) *Non-simply-laced types.* Extend Theorem 6.6 to the types B, C, F_4, G_2: does the lattice $\mathcal{L}(I)$ with the function ν determine the lines of I and the group W_I up to isometry? By Remark 6.8 the conclusion of Corollary 6.7 holds in rank at most 5. The missing step is to recover the angles between the lines of I inside planes of type B_2 and G_2. The same question may be asked for partial flag varieties G/P and partial Hessenberg varieties outside type A.

(2) *A formula over the poset.* Theorem 7.1 computes e_σ from the signs of the weights, and by Proposition 7.3 the result depends only on $S(X)_{\leqslant\sigma}$ for Hessenberg varieties of types A, D and E. Is there a formula for e_σ in terms of $S(X)_{\leqslant\sigma}$, for instance in terms of the numbers ν and the Möbius function? In complexity zero $e_\sigma = 1$.

(3) *Beyond algebraic actions.* Question 1 remains open for general equivariantly formal torus manifolds of positive complexity.

(4) *Subtori of rank two.* For a generic 2-torus in T, the oriented matroid of the weights at a fixed point is a cyclic word. For all connected Hessenberg varieties $\mathrm{Hess}(M, h)$ with $n \leqslant 6$ and several generic choices of the 2-torus, the multiset of these words over the fixed points determines the Betti numbers, and indeed h up to reversal. Is there a reason, as there is for a circle?

Statement on the use of AI

This paper was prepared with substantial assistance from Claude, an AI system developed by Anthropic. Claude contributed to the computer programs, the computations and the Lean formalization cited in the text; it found and wrote the proofs of the results of Sections 6 and 7, carried out an independent check of Section 5.2 against the cited sources, and drafted much of the text, including Appendix A. The programs, their output and the formalization are in the repository cited in the introduction. In accordance with the policy of the repository in which this paper is deposited, AI systems are not listed as authors.

Acknowledgements. The author is grateful to V. M. Buchstaber, the advisor of the thesis [34] from which this work grew.

A Background

This appendix collects the definitions and standard facts used in the paper, with pointers to textbooks and surveys. It is meant for readers who know basic algebraic topology and linear algebra but not all of the areas involved.

A.1 Posets and lattices

A *poset* is a set with a partial order. For $x \leqslant y$ the *interval* is $[x, y] = \{z \mid x \leqslant z \leqslant y\}$, and $S_{\geqslant x}$, $S_{\leqslant x}$ denote the elements above and below x. A finite poset is *graded* if it has a rank function rk with $\mathrm{rk}\, y = \mathrm{rk}\, x + 1$ whenever y covers x, and minimal elements of rank 0. A *lattice* is a poset in which any two elements x, y have a least upper bound, the *join* $x \vee y$, and a greatest lower bound, the *meet* $x \wedge y$. The *atoms* of a lattice with least element $\hat{0}$ are the elements covering $\hat{0}$. A finite lattice

is *geometric* if it is graded, every element is a join of atoms, and $\mathrm{rk}(x \vee y) + \mathrm{rk}(x \wedge y) \leqslant \mathrm{rk}\, x + \mathrm{rk}\, y$. The *Möbius function* of a finite poset is defined by $\mu(x,x) = 1$ and $\mu(x,y) = -\sum_{x \leqslant z < y} \mu(x,z)$ for $x < y$. Examples: the Boolean lattice of subsets of a finite set, the lattice of set partitions ordered by refinement, and the face lattice of a polytope. See [39, Ch. 3].

A.2 Matroids

A *matroid* M on a finite set E can be given by its rank function $r\colon 2^E \to \mathbb{Z}_{\geqslant 0}$, which satisfies $r(S) \leqslant |S|$, is monotone, and is submodular: $r(S \cup S') + r(S \cap S') \leqslant r(S) + r(S')$. A set S is *independent* if $r(S) = |S|$, a *basis* is a maximal independent set, and a *circuit* is a minimal dependent set. The *closure* of S is the largest set containing S of the same rank, and a *flat* is a closed set. The flats ordered by inclusion form a geometric lattice $\mathcal{L}(M)$, the *lattice of flats*, whose rank is the rank function; the atoms are the closures of single elements, the *parallel classes*. A *loop* is an element of rank 0. See [37].

Linear matroids. A finite family of vectors $v_1, \dots, v_m$ in a vector space defines the matroid on $[m]$ with $r(S) = \dim \operatorname{span}\{v_i \mid i \in S\}$. Its flats are the sets $\{i \mid v_i \in U\}$ for linear subspaces U. Rescaling the vectors by nonzero scalars, or applying a linear isomorphism, does not change the matroid. The *uniform matroid* $U_{r,m}$ is realized by m vectors in general position in an r-dimensional space; for example $U_{2,3}$ by three pairwise independent vectors in a plane, and $U_{1,1}$ by one nonzero vector. The *direct sum* $M \oplus M'$ is realized by the union of realizations of M and M' in complementary subspaces.

Graphic matroids. The graphic matroid $M(\Gamma)$ of a graph Γ on $[n]$ has the edges as its ground set, the forests as its independent sets and the cycles as its circuits. It is realized by the vectors $e_i - e_j \in \mathbb{R}^n$ for the edges ij. Its flats correspond to the partitions of $[n]$ into blocks that induce connected subgraphs (the flat consists of the edges inside the blocks), and its rank is n minus the number of blocks. Whitney's 2-isomorphism theorem says that two graphs have isomorphic graphic matroids if and only if one can be obtained from the other by a sequence of vertex identifications, cleavings and twistings [43], [37, §5.3]; the two graphs of Theorem 4.3 are an example. Graphic matroids are *regular*, that is, representable over every field.

Invariants. All of the following depend only on the isomorphism class of the matroid, and except for the Tutte polynomial only on its lattice of flats. Let M be loopless of rank r.

- The *characteristic polynomial* $\chi_M(t) = \sum_F \mu(\hat{0}, F) t^{r - \mathrm{rk}\, F}$ and the *Poincaré polynomial* $\pi_M(t) = \sum_F |\mu(\hat{0}, F)| t^{\mathrm{rk}\, F}$, sums over the flats.

- The *Tutte polynomial* $T_M(x,y) = \sum_{S \subset E} (x-1)^{r - r(S)} (y-1)^{|S| - r(S)}$. It satisfies $T_M(1,1) = \#\{\text{bases}\}$, $\chi_M(t) = (-1)^r T_M(1-t, 0)$, and $T_M(1,y) = \sum_S (y-1)^{|S| - r}$ over the spanning sets S; for example $T_{U_{2,3}} = x^2 + x + y$ and $T_{U_{r,r}} = x^r$. See [17].

- The *Orlik–Solomon algebra* $\mathrm{OS}(M)$ is the quotient of the exterior algebra on generators e_i, $i \in E$, by the ideal generated by the elements ∂e_C for the circuits $C = \{i_1 < \cdots < i_k\}$, where $\partial(e_{i_1} \cdots e_{i_k}) = \sum_j (-1)^{j-1} e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_k}$. Its Hilbert series is $\pi_M(t)$. For a linear matroid over $\mathbb{C}$ it is the cohomology ring of the complement of the corresponding hyperplane arrangement [36].

- The *Chow ring* [2] is $A(M) = \mathbb{Q}[x_F]/(I + J)$, with a generator x_F of degree one for every nonempty proper flat F, where I is generated by the products $x_F x_G$ of incomparable flats and J by the differences $\sum_{F \ni i} x_F - \sum_{F \ni j} x_F$, $i, j \in E$. It vanishes above degree $r - 1$ and satisfies Poincaré duality; Feichtner and Yuzvinsky gave a monomial basis [26].

- The *augmented Chow ring* [12] has generators y_i, $i \in E$, and x_F for the proper flats F, and relations $y_i - \sum_{F \not\ni i} x_F$, $x_F x_G$ for incomparable F, G, and $y_i x_F$ for $i \notin F$. It vanishes above degree r.

- The *Kazhdan–Lusztig polynomial* [25] is defined by $P_M = 1$ if $r = 0$, and otherwise by $\deg P_M < r/2$ and $t^r P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t)$, where M_F and M^F are the matroids with lattices of flats $[\hat{0}, F]$ and $[F, \hat{1}]$. The *Z-polynomial* $Z_M(t) = \sum_F t^{rk\,F} P_{M^F}(t)$ [38] is the Hilbert series of the intersection cohomology module of M [13]. For example $P_{U_{2,3}} = 1$ and $Z_{U_{2,3}} = 1 + 3t + t^2$.

A.3 Oriented matroids and signed graphs

Real vectors $v_1, \dots, v_m$ carry more structure than their matroid: the *signed circuits* are the sign patterns of the minimal linear dependencies $\sum c_i v_i = 0$, and the *covectors* are the sign vectors $(\mathrm{sign}\langle \xi, v_i \rangle)_i$ of linear functionals ξ; the covectors without zeros are the *topes*. These data form the *oriented matroid* of the configuration [10]. The configuration is *acyclic* if some covector is everywhere positive, that is, if the vectors lie in an open half-space, and *totally cyclic* if every element lies in a positive circuit, that is, if the vectors positively span their linear span. Replacing some v_i by $-v_i$ is a *reorientation*; it does not change the underlying matroid. A regular matroid has only one orientation up to reorientation [11].

A *signing* of a graph assigns ± 1 to the edges; it is *balanced* if the product of the signs around every cycle is $+1$. Harary's theorem [30] says that a signing is balanced if and only if it is obtained from the all-positive signing by *switching*, that is, by changing the signs of all edges at some set of vertices; see also [44]. Lemma 6.5 is a form of this theorem.

A.4 Torus actions

Let $T = (S^1)^k$. Its characters $\mathrm{Hom}(T, S^1) \cong \mathbb{Z}^k$ form a lattice in t^*, and every complex representation of T is a direct sum of one-dimensional representations $V(\alpha)$, $\alpha \in \mathrm{Hom}(T, S^1)$. If T acts smoothly on a manifold X and p is a fixed point, the slice theorem [14] identifies an invariant neighbourhood of p with the tangent representation $T_p X$; so the fixed point is isolated if and only if no tangent weight vanishes, and the stabilizers of points near p are the subgroups $\bigcap_{j \in S} \ker \alpha_{p,j}$. The complexity of an effective action with isolated fixed points is $\frac{1}{2} \dim X - \dim T$.

Examples. (1) T^{n+1} acts on $\mathbb{C}P^n$ by $t \cdot [z_0 : \cdots : z_n] = [t_0 z_0 : \cdots : t_n z_n]$, with the diagonal circle acting trivially. The fixed points are the coordinate points p_i, the weights at p_i are $t_j - t_i$, $j \neq i$, and the faces are the coordinate subspaces, which correspond to the faces of an n-simplex. The complexity is $n - n = 0$. (2) The diagonal torus T^n acts on the flag variety $\mathrm{Fl}(\mathbb{C}^n)$; the fixed points are the $n!$ coordinate flags, the weights at the flag of $w \in S_n$ are the $t_{w(i)} - t_{w(j)}$, $j < i$, and the complexity is $\binom{n}{2} - (n-1) = \binom{n-1}{2}$. (3) *Toric manifolds* (smooth projective toric varieties) and *quasitoric manifolds* are the actions of complexity zero with nice orbit spaces: the orbit space is a simple polytope P, $S(X)$ is the face poset of P, and $\beta_{2i}(X) = h_i(P)$, where $h(q) = \sum_i h_i q^i = \sum_i f_i(P)(q-1)^i$ and $f_i(P)$ is the number of i-dimensional faces [18].

A.5 Equivariant cohomology

Let $ET \to BT$ be a universal principal T-bundle; one can take $BT = (\mathbb{C}P^\infty)^k$. The *equivariant cohomology* of a T-space X is $H^*_T(X) = H^*(ET \times_T X)$; it is an algebra over $H^*(BT) = H^*_T(pt) = \mathbb{Q}[t_1, \dots, t_k]$, with t_i of degree 2 corresponding to the coordinate characters. A character α gives a complex line bundle $ET \times_T V(\alpha) \to BT$ with first Chern class $\alpha \in H^2(BT)$. If T acts on a vector space $V = \bigoplus_j V(\alpha_j)$, the equivariant Euler class of V is $\prod_j \alpha_j$, and the Thom isomorphism [35] gives Proposition 3.1. The *localization theorem* says that the restriction $H^*_T(X) \to H^*_T(X^T)$ becomes an isomorphism after inverting the nonzero elements of $H^2(BT)$ [4]. The action is *equivariantly formal* if $H^*_T(X)$ is a free $H^*(BT)$-module; then $H^*_T(X) \cong H^*(X) \otimes H^*(BT)$ as modules and restriction to X^T is injective. Good introductions are [40] and [15].

A.6 GKM theory

Goresky, Kottwitz and MacPherson [27] computed the equivariant cohomology of an equivariantly formal action with isolated fixed points whose weights at each fixed point are pairwise independent: it is the ring (2.1) of functions on the fixed points, with values in $H^*(BT)$, whose values at the two ends of each invariant 2-sphere agree modulo the weight of that sphere. For $\mathbb{C}P^1$ with the action of T^2 this gives the pairs (f_0, f_1) of polynomials with $f_0 \equiv f_1 \bmod (t_1 - t_0)$, a free module over $\mathbb{Q}[t_0, t_1]$ on $(1, 1)$ and $(0, t_1 - t_0)$, of Hilbert series $(1+q)/(1-q)^2$, as equivariant formality predicts for $P_{\mathbb{C}P^1} = 1 + q$. For $\mathbb{C}P^n$ the GKM graph is the complete graph on the $n+1$ fixed points with labels $t_i - t_j$; for G/B it has vertex set W and edges $\{w, ws_\alpha\}$, $\alpha \in \Phi^+$, labelled by $w(\alpha)$. See [40].

A.7 Morse theory, moment maps and the Białynicki-Birula decomposition

If T acts on a compact symplectic manifold (X, ω) preserving ω, a *moment map* is a T-invariant map $\mu\colon X \to \mathfrak{t}^*$ with $d\langle\mu, \zeta\rangle = \iota_{\zeta_X}\omega$ for $\zeta \in \mathfrak{t}$, where ζ_X is the generating vector field. By the convexity theorem of Atiyah and Guillemin–Sternberg [3, 29], $\mu(X)$ is the convex hull of $\mu(X^T)$, the *moment polytope*. For generic ξ, the function $\langle\mu, \xi\rangle$ is a perfect Morse function whose critical points are the fixed points, the Morse index at p being twice the number of weights negative on ξ; so $P_X(q) = \sum_p q^{\mathrm{ind}(p)}$. See [21]. The algebraic counterpart is the Białynicki-Birula decomposition [9, 22]: if $\mathbb{C}^*$ acts algebraically on a smooth projective variety with finitely many fixed points, the points flowing to a given fixed point p form an affine space whose dimension is the number of weights at p that are positive (or, for the opposite flow, negative) on $\mathbb{C}^*$, and these cells give $\beta_{2i} = \#\{p \mid \mathrm{ind}(p) = i\}$.

A.8 Root systems and Weyl groups

A (reduced, crystallographic) *root system* is a finite set Φ of nonzero vectors spanning a Euclidean space V such that $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$, the reflections $s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$ preserve Φ, and $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. The *Weyl group* W is generated by the s_α; every reflection in W is some s_α [31, §1.14]. A choice of positive roots Φ^+ determines the simple roots Δ, a basis of V such that every positive root is a nonnegative integer combination of simple roots. For $J \subset \Delta$ the roots in the span of J form the parabolic subsystem Φ_J, with Weyl group W_J generated by the s_α, $\alpha \in J$. The irreducible root systems are A_n ($e_i - e_j$ in $\mathbb{R}^{n+1}$), B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2 [31]. The *simply-laced* ones are A_n, D_n, E_n: all roots have the same length, and after normalizing $(\alpha, \alpha) = 2$, two non-proportional roots have $(\alpha, \beta) \in \{0, \pm 1\}$. The dual root system consists of the coroots $\alpha^\vee = 2\alpha/(\alpha, \alpha)$; it exchanges B_n and C_n, and B_2, G_2, F_4 are isomorphic to their duals after a rotation and rescaling. In type A_{n-1}, the Hessenberg ideals of Section 6 correspond to the Hessenberg functions, and both are counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

A.9 Flag varieties and Hessenberg varieties

For a semisimple group G with Borel subgroup $B \supset T_{\mathbb{C}}$, the flag variety G/B is a smooth projective variety with an action of $T_{\mathbb{C}}$; its fixed points are the wB, $w \in W$, and its invariant curves are the closures of $wU_{-\alpha}B/B$, joining wB to $ws_\alpha B$ [16]. For $G = SL_n$ it is the variety $\mathrm{Fl}(\mathbb{C}^n)$ of complete flags. Regular semisimple Hessenberg varieties were introduced by De Mari, Procesi and Shayman [24]; see [1] for a survey. For $\mathrm{Hess}(M, h)$ the extreme cases are $h = (n, \ldots, n)$, which gives $\mathrm{Fl}(\mathbb{C}^n)$; $h = (1, 2, \ldots, n)$, which gives the $n!$ fixed points; and $h = (2, 3, \ldots, n, n)$, which gives the toric variety of the permutohedron, whose graph Γ_h is a path. The graph Γ_h is the *indifference graph* of h, and the unit interval graphs are exactly the graphs that arise in this way. The isospectral manifolds $M_{\Gamma,\lambda}$ of Ayzenberg and Buchstaber [5, 6] are defined for all

graphs; for indifference graphs they are *twins* of Hessenberg varieties, with the same orbit space and equivariant cohomology, but they are in general different manifolds.

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