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THE ESPERET–HENDREY–LAGOUTTE–MARSELOO–NORIN–STEINER (2025) FLOW RECONFIGURATION PROBLEM HAS A POSITIVE ANSWER

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Abstract. For a finite multigraph G and an integer $k \geq 2$, let $M_k(G)$ be the graph on the nowhere-zero $\mathbb{Z}_k$-flows of G, and $I_k(G)$ the graph on the integer flows F of G with $1 \leq |F(e)| \leq k-1$ on every edge; two flows are adjacent if the support of their difference is a cycle. We prove that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G if $k = 40$, if $k = 48$, or if k is a multiple of 4 with $k \geq 52$. This gives a positive answer to Problem 1.2 of Esperet, Hendrey, Lagoutte, Marseloo, Norin and Steiner. The same holds for $k = 66$, for every prime power p^N with $N \geq 6$, and for a set of moduli of natural density one; these results do not cover every $k \geq 40$. For one fixed graph G it holds for every k above a bound that depends on G, primes included. The main tool is a master theorem for a finite abelian group Γ with a subgroup B and a set of steps. It gives the connectivity of the flow graph $F(G, \Gamma)$ for every finite abelian group Γ that is not cyclic of prime-power order and has a subgroup of order at least 4 and index at least 10, for example every abelian group of order 40.

0. Introduction

The table compares the main results with the best results known before this paper; the notation is fixed below.

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Key words and phrases. nowhere-zero flows, flow reconfiguration, abelian groups, integer flows.

A preprint of this paper is Part I of the paper *Flow reconfiguration with one modulus for all graphs, Cereceda's conjecture for bipartite 2-degenerate graphs, and other miscellanea*; its section, theorem and table numbers are the same as here. It is available at https://hexagonmath.org/2611.00520

New result	Best result before this paper
One modulus for all graphs: $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G if $k = 40$, $k = 48$, or k is a multiple of 4 with $k \geq 52$ (Theorem 9.2). This answers Problem 1.2 of [EHLMNS].	$I_k(G)$ is connected for a modulus k that depends on G [EHLMNS]; $M_k(G)$ is connected for every k above a bound of the order of 10^{694} [EHLMNS].
Groups: $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G if Γ is not cyclic of prime-power order and $\lvert\Gamma\rvert$ has a divisor d with $4 \leq d \leq \lvert\Gamma\rvert/10$; for example for every abelian group of order 40 (Theorem 4.5).	$\Gamma = \mathbb{F}_2^8$, of order 256, and every abelian group Γ of order above a bound of the order of 10^{694} [EHLMNS].
One graph: for a fixed G, $M_k(G)$ and $I_k(G)$ are connected for every $k \geq 40$ if G has at most 149 vertices, or at most 189 edges, or no cycle with more than 39 edges (Section 10).	$I_k(G)$ is connected for a modulus k that depends on G [EHLMNS].

The new results are the following.

- *One modulus for all graphs*: $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G if $k = 40$, $k = 48$, or k is a multiple of 4 with $k \geq 52$ (Theorem 9.2, proved in Section 9 from the cases proved in Sections 5, 6 and 7). Before, $I_k(G)$ was known to be connected only for a modulus that depends on G, and Problem 1.2 of [EHLMNS] asked for one modulus for all graphs.
- *Further moduli*: $k = 66$ (Corollary 5.5, Section 5); every prime power p^N with $N \geq 6$ (Theorem 7.4, Section 7); every k that is not a prime power and has at least four prime factors, counted with multiplicity, except 24, 36 and 54 (Theorem 9.5, Section 9); a set of moduli of natural density one (Corollary 9.9). Before, no modulus was known for $I_k(G)$ for all graphs, and for $M_k(G)$ only those above a bound of the order of 10^{694} [EHLMNS].
- *Groups*: $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G, loops allowed, if Γ is not cyclic of prime-power order and $\lvert\Gamma\rvert$ has a divisor d with $4 \leq d \leq \lvert\Gamma\rvert/10$ (Theorem 4.5, proved in Section 4). Before, this was known for $\mathbb{F}_2^8$ and for orders above a bound of the order of 10^{694} [EHLMNS]; whether the order 2^8 can be improved was one of the two main questions left open there. Every abelian group of order 40 works.
- *One graph*: for a fixed G, the results go from a base k_0 to every $k \geq k_0$, prime k included; for example every $k \geq 40$ if G has at most 149 vertices, or at most 189 edges, or no cycle with more than 39 edges, or if G is connected with diameter at most 3 (Section 10, Table 2).

The problem. Let G be a finite multigraph and $k \geq 2$ an integer. A *nowhere-zero $\mathbb{Z}_k$-flow* of G gives every edge a non-zero element of $\mathbb{Z}_k$, with conservation at every vertex; an *integer k-flow* gives every edge an integer $F(e)$ with $1 \leq \lvert F(e)\rvert \leq k-1$, with conservation at every vertex. The reconfiguration graphs $M_k(G)$ and $I_k(G)$ have these flows as their vertices; two flows are adjacent if their difference is $\lambda\chi_C$ for a cycle C of G and a non-zero λ, that is, if the support of the difference is a cycle. For a finite abelian group Γ the graph $F(G, \Gamma)$ is defined in the same way, and $M_k(G) = F(G, \mathbb{Z}_k)$. The exact definitions are in Section 1. A graph with a bridge has no nowhere-zero flow, so the graphs G of interest are the bridgeless ones.

These graphs are studied by Esperet, Hendrey, Lagoutte, Marseloo, Norin and Steiner [EHLMNS]. Their Problem 1.2 asks whether there is *one* integer k such that $I_k(G)$ is connected for every 2-edge-connected graph G; they prove that $I_k(G)$ is connected for a modulus k that depends on G. For groups they prove that $F(G, \mathbb{F}_2^8)$ is connected for every 2-edge-connected graph G ($\mathbb{F}_2^m$ is the elementary abelian group of order 2^m), and that $F(G, \Gamma)$ is connected for every abelian group Γ whose order is at least an explicit bound of the order of 10^{694}; they also show that the modulus 5 fails. (In [EHLMNS] the graphs $I_k(G)$ and $M_k(G)$ are $\mathcal{F}(G, k)$ and $\mathcal{F}(G, \mathbb{Z}_k)$.) Their Problem 2.6 asks whether the connectivity of $M_k(G)$ implies the connectivity of $I_k(G)$; Cranston, Li, Su, Wang and Xu [CLSWX] conjecture that $I_k(G)$ is connected if and only if $M_k(G)$ is connected.

The results. All results below are for every finite bridgeless loopless multigraph G; G need not be connected. "k is reached" means: one of the results of this paper applies to k (in Section 9: k is *covered*, Definition 9.6); then $M_k(G)$ and $I_k(G)$ are connected for every such G.

(1) *The bases* (Theorem 9.2). $k = 40$, $k = 48$ and every multiple k of 4 with $k \geq 52$ are reached. In particular $k = 40$ gives a positive answer to Problem 1.2 of [EHLMNS].

(2) *Further moduli.* $k = 66$ (Corollary 5.5); every prime power $k = p^N$ with $N \geq 6$ (Theorem 7.4); every product $k = ab$ of two coprime *admitted factors*, that is integers a, b each of which is at least 43 or a product de with $d \geq 4$, $e \geq 7$ (Corollary 8.9): so every multiple of 1120, every pq with different primes $p, q \geq 43$, and every p^2q with different primes $p \geq 7$, $q \geq 43$; every k that is not a prime power and has at least four prime factors, counted with multiplicity, except 24, 36 and 54 (Theorem 9.5); every $k = 2p^3$ with a prime $p \geq 5$ (Proposition 9.4); and every k to which Criterion 5.11 applies.

(3) *Exactly which k these results reach* (Corollary 9.8, Proposition 9.7), by the number $\Omega(k)$ of prime factors with multiplicity: see Table 1. The reached moduli have natural density 1 (Corollary 9.9). Up to 100 they are 40, 48, 52, 56, 60, 64, 66, 68, 72, 76, 78, 80, 84, 88, 90, 92, 96, 100.

(4) *One graph* (Section 10). For a fixed G the results go from a base k_0 to every $k \geq k_0$, prime k included: see Table 2. For example, $M_k(G)$ and $I_k(G)$ are connected for every $k \geq 40$ if G has at most 149 vertices, or at most 189 edges, or no cycle with more than 39 edges, or if G is connected with diameter at most 3.

(5) *Groups* (Theorem 4.5). If Γ is not cyclic of prime-power order and $|\Gamma|$ has a divisor d with $4 \leq d \leq |\Gamma|/10$, then $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G, loops allowed; the least such orders are 40, 44, 48, 50. The final remarks of [EHLMNS] name two main questions left open there: Problem 1.2, and whether the order $2^8 = 256$ of the group $\mathbb{F}_2^8$ in the positive answer to their Problem 1.4 can be improved. Item (1) answers the first, and this theorem answers the second: every abelian group of order 40 works. By [Go1, Corollary 13.2], so does every abelian group of order 32 whose exponent divides 4.

(6) *Loops.* Every k of (1) and (2) is reached through Theorem 3.19 or through Theorem 8.8, and part (a) of these two theorems gives the connectivity of $M_k(G)$ also for every finite bridgeless multigraph G with loops.

Open cases. For all graphs, the moduli that are not reached include those from 6 to 39, those from 41 to 47, and the primes from 7 on (Remark 9.10); the modulus 5 fails [EHLMNS].

Table 1 gives exactly which moduli are reached. For the modular graph alone more is known: by the theorem of [EHLMNS] for large groups, $M_k(G)$ is connected for every k above a bound of the order of 10^{694}, primes included; by [Go2], for every $k \geq 3024$. For $k = 44$ the group theorem gives the connectivity of $M_{44}(G)$ for every bridgeless G; the connectivity of $I_{44}(G)$ is a case of Problem 2.6 of [EHLMNS]. Section 11 explains where the method stops, and Section 12 lists open problems.

TABLE 1. The moduli k for which this paper proves that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G (Corollary 9.8; "covered" in Definition 9.6).

k	reached exactly when
prime power p^e	$e \geq 6$ (no prime)
pq, $p \neq q$ primes	$p \geq 43$ and $q \geq 43$
$\Omega(k) = 3$, not a prime power	Proposition 9.7: p^2q by the rules (α), (β), (γ) there; every odd pqr except 105; every $2pq$ except 30, 42, 70, 110, 182, 286, 506, 646, 1406, 1798
$\Omega(k) \geq 4$, not a prime power	$k \notin \{24, 36, 54\}$

TABLE 2. One graph G (finite, bridgeless, loopless): the moduli k for which this paper proves that $M_k(G)$ and $I_k(G)$ are connected. $c(G)$ is the number of edges of a longest cycle, n the number of vertices.

condition on G	every k with	where
none	$k \geq \max(48, c(G) + 4)$	Cor. 10.6
$c(G) \leq 39$	$k \geq 40$	Cor. 10.6
at most 189 edges (at most 235 edges)	$k \geq 40$ ($k \geq 48$)	Cor. 10.12
at most 149 vertices (at most 187 vertices)	$k \geq 40$ ($k \geq 48$)	Cor. 10.16
$n \geq 150$ vertices	$k \geq 2n/\ln n + 4$	Cor. 10.16
every edge on a cycle with at most 8 edges	$k \geq 40$	Cor. 10.22
connected, diameter at most 3	$k \geq 40$	Cor. 10.23
all edges but at most 40 on cycles with at most 8 edges	$k \geq 40$	Cor. 10.24

The method. *From groups to integers.* A non-zero element z of $\mathbb{Z}_k$ is *liftable* if, for every loopless G and every move $f \to f + z\chi_C$ of $M_k(G)$, all integer lifts of the two flows lie in one component of $I_k(G)$. Section 1 shows that ± 1, ± 2 and $\pm q$ for every divisor $q < k$ of k are liftable, and that cz is liftable with z if the unit c is 1 modulo both factors of a factorization of k (so $\pm(1 + 2p)$ is liftable for $k = 4p$). Integer lifts exist, and the lifts of one flow are joined in $I_k(G)$. So if any two nowhere-zero $\mathbb{Z}_k$-flows are joined by moves whose steps are liftable, then $I_k(G)$ is connected (Theorem 3.19).

The master theorem (Section 3). Let Γ be a finite abelian group, B a subgroup, $A = \Gamma/B$, and $\Sigma = -\Sigma$ a set of non-zero elements of Γ, the *steps*. Theorem 3.11 says: if the steps join the elements of A after the deletion of any 6 elements (R1), join the elements of B after the deletion of any 2 elements (R2), join the non-zero elements of Γ (R3*), if an averaging condition (AV″) holds, and if $|A| \geq 10$ and $|B| \geq 4$ (Corollary 3.15; more precisely, $|A| \geq 8$ and certain counts), then any two nowhere-zero Γ-flows of a bridgeless multigraph are joined

by moves with steps in Σ. The proof is an induction on the number of vertices. Loops are handled by (R3*), and a cut with two edges by a contraction. In a 3-edge-connected graph, (AV″) leads to a flow whose A-coordinate has no zero on a *good subgraph* H: a cycle with at most 5 edges, or a subdivision of a 3-edge-connected graph H_0 in which every edge becomes a path with at most 3 edges. Such an H exists in every loopless graph with $n \geq 3$ vertices and more than $(4/3)\,n - 2$ edges (Section 2). The graph G/H is smaller, and each of its moves is lifted to moves of G that keep the A-coordinate non-zero on H (Lemma 3.10). For this, every block of H_0 is expanded to a truncated cubic graph; their union is Q, and the well-colouring lemma of the appendix (Lemma A.5, one of the two facts of Jaeger, Linial, Payan and Tarsi [JLPT] that the appendix proves) gives a *frame* of H: a spanning forest of Q, with a tree in every component, whose edges come in pairs, and one *official cycle* for every pair. The A-flows of Q are combinations of the official cycles; the coefficients are chosen in reverse order to avoid lists of forbidden values (the reverse choice), and they are changed one step at a time inside the allowed flows (the walk lemma). A thread has at most 3 edges, so a list has at most 3 values; the largest count is $9 < |A|$, and in B it is $3 < |B|$. This is where $|A| \geq 10$ and $|B| \geq 4$ come from, hence the bound $k = ab \geq 40$.

This scheme of extraction and contraction follows [EHLMNS, Section 6, in particular Lemmas 6.1 and 6.14]: the averaging that finds the subgraph, and the preparation of the flow on it before a move of the contracted graph is lifted; their Section 6.2 also brings in the closure theory of [JLPT]. What we add is the quotient Γ/B with a prescribed set of steps, the counts for the avoidance and for the averaging, and the double step of Section 6; together with the liftable steps of Section 1 this gives the results on integer flows.

Averaging (Section 4). If $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial, and the steps generate both factors, then random changes of one coordinate on the support of the other make the value of every edge almost uniform. So some flow has at most $m/9$ edges where the A-coordinate is 0, if $(|B| - 1)/(|\Gamma| - 1) < 1/9$ (Lemma 4.1); with the count of the edges this gives a good subgraph.

The moduli. For $k = 40 = 10 \cdot 4$ the steps are ± 1, ± 2 and $\pm q$ for the divisors q; modulo 10 they miss only ± 3, which an elementary lemma allows; and $\mathbb{Z}_{40} = \mathbb{Z}_8 \times \mathbb{Z}_5$ (Section 5; 48 and 66 in the same way, and Criterion 5.11 in general). For $k = 4p$ the steps modulo p are only ± 1, ± 2, ± 4, and six deletions can isolate a value, so (R1) fails. The frame form of the master theorem (Theorem 3.12) needs (R1) only with five deletions, which holds by the theorem of Cauchy and Davenport (and Vosper's theorem shows that six deletions can only isolate one value), and replaces the walk lemma by a walk with *double steps* that pass an isolated value; one of four choices (two routes, and the two lifts ± 1, $\pm(1 + 2p)$ of the step ± 1) works, by a parity argument in $B = \mathbb{Z}_4$ (Section 6). A cyclic group of prime-power order is not a product, so for $k = p^N$ the averaging is replaced by a flattening of the p-adic valuations of the values (Section 7). For two coprime factors a, b, moves that change one residue on a cycle where the other residue has no zero lift to integers whatever their step (Section 8). Section 9 puts the moduli together.

One graph (Section 10). If $I_{k-1}(G)$ is connected and every flow of $I_k(G)$ is joined to a flow without the value $\pm(k-1)$, then $I_k(G)$ is connected. This step from $k-1$ to k is made under a bound on the longest cycle, on the number of edges, on the number of vertices, or on the lengths of short cycles, starting at a base.

Related results. Part II of the preprint [Go1], whose Part I is this paper, is mainly about groups of exponent 2 or 4, with the moves and the graphs $F(G, \Gamma)$ of this paper. Its main

theorem [Go1, Theorem 13.1] is a binary form of the master theorem: if a subgroup B of Γ and $A = \Gamma/B$ satisfy $2B = 0$, $2A = 0$, $|B| \geq 4$, $|A| \geq 8$ and $(|B|-1)/(|\Gamma|-1) \leq 1/9$, then $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G. Its proof uses the extraction and the frames of this paper (Section 2 and the appendix) and the estimate of Remark 4.2 (from the proof of Lemma 4.1), which also covers the ratio $1/9$. So $F(G,\Gamma)$ is connected for every finite abelian group Γ whose exponent divides 4 and whose order is at least 32 [Go1, Corollary 13.2]; at order 32 these groups are $\mathbb{F}_2^5$, $\mathbb{Z}_4 \times \mathbb{F}_2^3$ and $\mathbb{Z}_4^2 \times \mathbb{Z}_2$, which item (5) does not reach. Part II of [Go1] also proves, independently of this paper and with the tree-packing theorem of Nash-Williams and Tutte, that $F(G,\mathbb{F}_2^m)$ is connected for every 4-edge-connected multigraph G and every $m \geq 3$ [Go1, Theorem 13.3]; and that $F(G,\mathbb{F}_2^3)$ and $F(G,\mathbb{F}_2^4)$ are connected if every component of the finite bridgeless multigraph G has at most 44, respectively 19022, vertices of odd degree [Go1, Theorems 22.1 and 28.6]. The paper [Go2] proves that $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G and every finite abelian group Γ of order at least 3024. No proof in this paper uses these results.

Organization. Section 1: conventions, integer lifts, liftable steps. Section 2: good subgraphs, the truncated cubic reduction, threads, frames and official cycles. Section 3: step systems, the master theorem and its integer form. Section 4: averaging in a product, and the group theorem. Sections 5–9: the moduli. Section 10: one graph. Section 11: limits of the method. Section 12: open problems. The appendix proves the closure lemma of Jaeger, Linial, Payan and Tarsi and the other facts from their paper that this paper uses. The results for $k = 40$, 48 and 66, and so the answer to Problem 1.2 of [EHLMNS], use only Sections 1–5 and the appendix.

Theorems quoted from the literature. The proofs of Sections 1–10 use the following theorems from the literature, in the form stated here, and no other ones. Standard facts are used without a reference: the Chinese remainder theorem; a finite abelian group is a direct product of cyclic groups of prime-power order and has a subgroup of every order that divides its order; blocks of a graph. Some published results are proved again, in the form that is used: Tutte's lifting theorem (Corollary 1.3; [EHLMNS, Theorem 2.3]), the connected fibres (Lemma 1.4; [EHLMNS, Lemma 5.5]), the extraction lemma (Lemma 2.2; [EHLMNS, Lemma 6.1]), the closure lemma of [JLPT] (Lemma A.7), the signed cycle decomposition (Lemma 1.1 (b)), and the potential of a cost function without negative cycles (Lemma 10.13).

Theorem 0.1 (Seymour [Se]; Tutte [Tu1])**.** *Every bridgeless multigraph, loops allowed, has a nowhere-zero flow with values in any abelian group of order at least* 6*.*

Used in the proof of Theorem 3.11 (the target flow) and of Theorem 8.8.

Theorem 0.2 (Cauchy–Davenport [Ca, Da])**.** *Let p be a prime, and let U_1 and U_2 be non-empty subsets of $\mathbb{Z}_p$. Then*

$$|U_1 + U_2| \;\geq\; \min\{\, p,\; |U_1| + |U_2| - 1 \,\}, \qquad \text{where } U_1 + U_2 = \{\, u_1 + u_2 : u_1 \in U_1,\; u_2 \in U_2 \,\}.$$

Used in Lemma 5.8 and Lemma 6.1.

Theorem 0.3 (Vosper [Vo])**.** *Let p be a prime, and let U_1, U_2 be subsets of $\mathbb{Z}_p$ with $|U_1| \geq 2$, $|U_2| \geq 2$ and $|U_1 + U_2| = |U_1| + |U_2| - 1 \leq p - 2$. Then U_1 and U_2 are arithmetic progressions with the same difference.*

Used in Lemma 6.1.

Theorem 0.4 (Mader, Watkins [Ma, Wa]). *Let Δ be a connected vertex-transitive graph without loops and without multiple edges, in which every vertex has $h \geq 2$ neighbours. Then the vertex-connectivity of Δ is at least $2(h+1)/3$: the deletion of any set of fewer than $2(h+1)/3$ vertices leaves a connected graph.* (*See also* [GR, Theorem 3.4.2, p. 40].)

Used in Lemma 5.9, hence in the last sentence of Criterion 5.11 and in its uses, and in Theorem 7.4 for an odd prime p (the check of (R1)).

Theorem 0.5 (Hoffman's circulation theorem [Ho]). *Let G be a multigraph with an orientation, and let $l \leq u$ be integer functions on its arcs. There is an integer flow Y of G (in this orientation) with $l(e) \leq Y(e) \leq u(e)$ for every arc e if and only if, for every set X of vertices, the sum of l over the arcs that leave X is at most the sum of u over the arcs that enter X.*

Used three times in Section 10.

Theorem 0.6 (Euler; see [HW]). *The sum of $1/p$ over all primes p diverges.*

Theorem 0.7 (Chebyshev's inequality; see [Fe]). *Let X be a real random variable on a finite probability space, with mean μ and variance V. Then $\Pr[\,|X-\mu| \geq t\,] \leq V/t^2$ for every real $t > 0$.*

Theorems 0.6 and 0.7 are used only in Corollary 9.9.

1. Conventions; lifting to integers; liftable steps

Graphs, flows and the reconfiguration graphs. Graphs are finite multigraphs. Loops are allowed, unless the graph is called loopless. A *cycle* is a connected subgraph in which every vertex has degree 2; a loop is a cycle of length 1, and two parallel edges form a cycle of length 2. An edge that lies on no cycle is a *bridge*; a graph is *bridgeless* if it has no bridge, that is, if every edge lies on a cycle. A graph without edges is bridgeless. A multigraph is *j-edge-connected* if, for every partition of its vertex set into two non-empty sets, at least j edges have one end in each set.

Every graph has a fixed reference orientation: every edge has a tail and a head (for a loop they are the same vertex). Let x be a map from $E(G)$ to an abelian group. Its *boundary* at a vertex v is

$$(\partial x)(v) \;=\; \sum_{e\,:\,v \text{ is the head of } e} x(e) \;-\; \sum_{e\,:\,v \text{ is the tail of } e} x(e).$$

A loop at v is in both sums, so it adds nothing. The map x is a *flow* if $\partial x = 0$ (conservation at every vertex). An *integer flow* is a flow with values in $\mathbb{Z}$. Groups are finite abelian groups, if nothing else is said. For a group Γ, a *Γ-flow* of G is a flow $E(G) \to \Gamma$; it is *nowhere zero* if no value is 0. The *support* $\mathrm{supp}(x)$ is the set of the edges e with $x(e) \neq 0$.

For a cycle C with a direction, χ_C is its signed indicator: $\chi_C(e) = 1$ if C passes the edge e from the tail to the head (*forward*), $\chi_C(e) = -1$ if C passes e from the head to the tail, and $\chi_C(e) = 0$ if e is not on C. So χ_C is an integer flow, and $\lambda\chi_C$ is a Γ-flow for every λ in Γ.

If the reference orientation of one edge is reversed and the value of every flow on this edge is replaced by its negative, then flows stay flows, and nothing else changes. No statement of this paper depends on the reference orientation, and we change it freely inside proofs.

$F(G,\Gamma)$ is the graph on the nowhere-zero Γ-flows of G; two flows are adjacent if their difference is $\lambda\chi_C$ for a cycle C of G and $\lambda \neq 0$ (a *move*). $M_k(G) = F(G, \mathbb{Z}_k)$. $I_k(G)$ is the graph on the integer flows F of G with $1 \leq |F(e)| \leq k-1$ for every edge e; two of them are

adjacent if their difference is $\lambda\chi_C$ with an integer $\lambda \neq 0$. An *integer lift* of a $\mathbb{Z}_k$-flow f is a vertex F of $I_k(G)$ with $F = f$ modulo k. These are the three *reconfiguration graphs*.

In each of the three graphs, two different vertices are adjacent if and only if the support of their difference is the edge set of a cycle: a flow whose support is a cycle C has, by conservation at the vertices of C, the form $\lambda\chi_C$.

A graph G without edges has exactly one flow, the empty map; it is a vertex of each of the three reconfiguration graphs. A reconfiguration graph that has no vertex counts as connected.

In this section G is a finite multigraph, and loops are allowed. Only Definition 1.7, Lemma 1.12 and Corollary 1.13 speak of loopless graphs.

The cycle decomposition.

Lemma 1.1 (Cycle decomposition)**.** *Let G be a multigraph.*

(a) *Let A be an abelian group, finite or not, and let T_0 be a spanning forest of G (a spanning tree in every component of G). For an edge e outside T_0 let C_e be the fundamental cycle of e in T_0, directed along e. Then every A-flow h of G is*

$$h = \sum_{e \notin E(T_0)} h(e)\, \chi_{C_e},$$

and this is the only way to write h as a sum $\sum_{e\notin E(T_0)} x_e\chi_{C_e}$ with x_e in A. In particular every A-flow of G is a sum of flows $x\chi_C$ with x in A and C a cycle of G.

(b) (*The signed form.*) *Every integer flow h of G is a sum*

$$h = \chi_{O_1} + \cdots + \chi_{O_j} \qquad (j \geq 0)$$

of signed indicators of directed cycles $O_1, \ldots, O_j$ of G (a cycle can occur several times), such that every O_i passes each of its edges in the direction of the sign of h: $\chi_{O_i}(e)\, h(e) > 0$ for every edge e of O_i. Hence, for every edge e: exactly $|h(e)|$ of the cycles contain e; every sum of some of the numbers $\chi_{O_1}(e), \ldots, \chi_{O_j}(e)$ lies between 0 and $h(e)$; and all cycles lie in $\operatorname{supp}(h)$*. If all values of h are in $\{-1, 0, 1\}$, then $O_1, \ldots, O_j$ are pairwise edge-disjoint.*

Proof. (a) Put $h' = h - \sum_{e\notin E(T_0)} h(e)\chi_{C_e}$. This is an A-flow. The cycle C_e contains e and no other edge outside T_0, and $\chi_{C_e}(e) = 1$. (A loop is not in T_0, and it is its own fundamental cycle.) So h' is 0 on every edge outside T_0. Suppose that $h' \neq 0$. The edges with $h' \neq 0$ form a forest with at least one edge. It has a vertex that meets exactly one of these edges. Conservation at this vertex says that the value of h' on this edge is 0, a contradiction. So $h' = 0$. The sum $\sum x_e\chi_{C_e}$ has the value x_e on the edge e outside T_0; so the coefficients are determined by the flow.

(b) Induction on $\sum_e |h(e)|$. If this sum is 0, take $j = 0$. Otherwise reverse the reference orientation of every edge e with $h(e) < 0$; now $h \geq 0$. If a loop has a positive value, let O be this loop, passed forward. Otherwise use conservation: at every vertex, the sum of h over the edges that enter is equal to the sum over the edges that leave (loops are left out on both sides). So the head of an edge with a positive value is the tail of an edge with a positive value. Start with an edge with a positive value and go on in this way. The walk does not end, so it comes back to a vertex that it has seen. This gives a directed cycle O that passes all its edges forward, and $h > 0$ on all its edges. Now $h - \chi_O$ is an integer flow with $h - \chi_O \geq 0$ and with a smaller sum. By induction it is a sum of signed indicators of directed cycles

that pass their edges forward. Add χ_O. In the first reference orientation, "forward" is the direction of the sign of $h(e)$.

For a fixed edge e, all non-zero numbers $\chi_{O_i}(e)$ have the sign of $h(e)$, and their sum is $h(e)$. This gives the three consequences. If $|h(e)| \leq 1$ for all e, every edge is on at most one of the cycles. □

Rounding, integer lifts, and the fibres.

Lemma 1.2 (Rounding)**.** *Let x be a map from $E(G)$ to the real interval $[0,1]$ such that $(\partial x)(v)$ is an integer for every vertex v. Then there is a map $z\colon E(G) \to \{0,1\}$ with $\partial z = \partial x$ and with $z(e) = x(e)$ for every edge e with $x(e)$ in $\{0,1\}$.*

Proof. Call an edge e fractional if $0 < x(e) < 1$. If there is no fractional edge, take $z = x$. Otherwise the fractional edges contain a cycle C. Indeed, a fractional loop is such a cycle. If no loop is fractional, then no vertex v meets exactly one fractional edge: the other edges at v have integer values, $(\partial x)(v)$ is an integer, and so the one fractional value would be an integer. A loopless graph with at least one edge in which no vertex has degree 1 contains a cycle.

Give C a direction. Let $\delta > 0$ be the least number such that $x + \delta\chi_C$ has the value 0 or 1 on some edge of C; it exists, because all values of x on C are strictly between 0 and 1. The map $x + \delta\chi_C$ has values in $[0,1]$ and the same boundary as x, because $\partial\chi_C = 0$. The values on the edges that are not fractional have not changed, and the number of fractional edges is smaller. Repeat this until no edge is fractional. □

Corollary 1.3 (Integer lifts)**.** *Let $k \geq 2$ and let f be a $\mathbb{Z}_k$-flow of G; zeros are allowed. Then there is an integer flow F of G with $F = f$ modulo k, with $|F(e)| \leq k-1$ for every edge e, and with $F(e) = 0$ exactly for the edges e with $f(e) = 0$. In particular every nowhere-zero $\mathbb{Z}_k$-flow of G has an integer lift.*

Proof. For every edge e let $r(e)$ in $\{0,\dots,k-1\}$ be the residue of $f(e)$. Then $\partial r = \partial f = 0$ modulo k, so the map $x = r/k$ has values in $[0,1)$ and an integer boundary. Lemma 1.2 gives $z\colon E(G) \to \{0,1\}$ with $\partial z = \partial x$ and with $z(e) = 0$ if $r(e) = 0$. Put $F = r - kz$. Then $\partial F = \partial r - k\,\partial x = 0$, and $F = f$ modulo k. If $r(e) = 0$, then $F(e) = 0$. If $r(e) \neq 0$, then $F(e)$ is $r(e)$ or $r(e) - k$; the first number is in $[1, k-1]$ and the second one is in $[-(k-1), -1]$. □

Corollary 1.3 is Tutte's lifting theorem in the form of [EHLMNS, Theorem 2.3].

Lemma 1.4 (Connected fibres)**.** *Let $k \geq 2$ and let f be a nowhere-zero $\mathbb{Z}_k$-flow of G. Then the integer lifts of f induce a connected subgraph of $I_k(G)$.*

Proof. Let F and F' be integer lifts of f. On every edge e, both $F(e)$ and $F'(e)$ are in $\{r(e),\, r(e) - k\}$, where $r(e)$ in $\{1,\dots,k-1\}$ is the residue of $f(e)$. So $h = (F' - F)/k$ is an integer flow with values in $\{-1, 0, 1\}$. By Lemma 1.1 (b), $h = \chi_{O_1} + \cdots + \chi_{O_j}$ with pairwise edge-disjoint directed cycles. Put $F_i = F + k\,(\chi_{O_1} + \cdots + \chi_{O_i})$. On every edge, F_i has the value of F or the value of F'. So every F_i is an integer lift of f, $F_0 = F$, $F_j = F'$, and $F_i - F_{i-1} = k\chi_{O_i}$. This is a path from F to F' in $I_k(G)$ through integer lifts of f. □

Lemma 1.4 is [EHLMNS, Lemma 5.5].

The divisor extension. Let $k = mq$ with integers $m, q \geq 2$. Reduction modulo q is a homomorphism from $\mathbb{Z}_k$ onto $\mathbb{Z}_q$; so the *reduction modulo q* of a $\mathbb{Z}_k$-flow, taken edge by edge, is a $\mathbb{Z}_q$-flow. The multiples of q in $\mathbb{Z}_k$ form the subgroup $q\mathbb{Z}_k$, and $qj \mapsto j$ modulo m is an isomorphism from $q\mathbb{Z}_k$ onto $\mathbb{Z}_m$. For a $\mathbb{Z}_k$-flow y with all values in $q\mathbb{Z}_k$ we write y/q for its image, a $\mathbb{Z}_m$-flow.

The next two proofs use the same setting.

Setting. w is a $\mathbb{Z}_q$-flow of G, and E_0 is the set of the edges e with $w(e) = 0$. By Corollary 1.3 there is an integer flow W of G with $W = w$ modulo q, with $|W(e)| \le q - 1$, and with $W(e) = 0$ exactly on E_0. Reverse the reference orientation of every edge with $W(e) < 0$. So from now on

$$1 \le W(e) \le q - 1 \quad \text{for every edge } e \notin E_0, \qquad W(e) = 0 \quad \text{for every edge } e \in E_0.$$

A *state* is a nowhere-zero $\mathbb{Z}_k$-flow of G whose reduction modulo q is w.

Claim 1 (the standard lift of a state). *For every state h there is an integer flow L of G with $|L(e)| \le m - 1$ for every edge e, such that $F = W + qL$ is an integer lift of h. For every edge e in E_0: $L(e) \ne 0$, and $h(e) = qL(e)$ modulo k.*

Proof. The $\mathbb{Z}_k$-flow $h - W$ has the reduction 0 modulo q, so all its values are in $q\mathbb{Z}_k$, and $y = (h - W)/q$ is a $\mathbb{Z}_m$-flow. By Corollary 1.3 there is an integer flow L with $L = y$ modulo m, with $|L(e)| \le m-1$, and with $L(e) = 0$ exactly where $y(e) = 0$. Put $F = W + qL$. This is an integer flow. The residue of $qL(e)$ modulo k depends only on the residue of $L(e)$ modulo m; so $qL = h - W$ modulo k, and $F = h$ modulo k.

Let $e \notin E_0$. Then $F(e) = W(e)$ modulo q, and this is not 0; so $F(e) \ne 0$. And $1 - q(m - 1) \le F(e) \le q - 1 + q(m - 1)$, so $|F(e)| \le k - 1$.

Let $e \in E_0$. Then $W(e) = 0$ and $h(e) \ne 0$, so $y(e) \ne 0$ and $L(e) \ne 0$. So $F(e) = qL(e)$ is not 0, and $|F(e)| \le q(m - 1) < k$. □

Claim 2 (one step of size q). *Let O be a cycle of G with a direction that passes each of its edges outside E_0 forward. Let h and $h - q\chi_O$ be states. Then some integer lift of h is adjacent in $I_k(G)$ to some integer lift of $h - q\chi_O$.*

Proof. Take L and $F = W + qL$ from Claim 1. We show that $F - q\chi_O$ is a vertex of $I_k(G)$. It is an integer flow, it is equal to $h - q\chi_O$ modulo k, and outside O it is equal to F.

Let e be an edge of O outside E_0. Then $\chi_O(e) = 1$, and the new value is $W(e) + q\,(L(e) - 1)$. It is equal to $W(e)$ modulo q, so it is not 0. It is at least $W(e) - qm \ge 1 - k$ and at most $W(e) + q(m - 2) \le k - q - 1$.

Let e be an edge of O in E_0. Choose its reference orientation such that O passes e forward (on E_0 the orientation was free). By Claim 1, $h(e) = qi$ modulo k with $1 \le i \le m - 1$, and $L(e) = i$ modulo m; so $L(e)$ is i or $i - m$. The state $h - q\chi_O$ has the value $q(i - 1)$ on e, which is not 0; so $i \ge 2$. The new integer value on e is $q(i - 1)$ or $q(i - m - 1)$. Here $1 \le i - 1 \le m - 2$ and $1 - m \le i - m - 1 \le -2$. So the new value is not 0, and its absolute value is at most $q(m - 1) < k$.

So $F - q\chi_O$ is an integer lift of $h - q\chi_O$. It is adjacent to F, because $q \ne 0$. □

Theorem 1.5 (Divisor extension)**.** *Let $k = mq$ with $m, q \ge 2$. Let f and g be nowhere-zero $\mathbb{Z}_k$-flows of G that have the same reduction w modulo q, and let $f(e) = g(e)$ for every edge e with $w(e) = 0$. Then all integer lifts of f and all integer lifts of g lie in one component of $I_k(G)$.*

Proof. Use the setting above for this w. Then f and g are states. Let G' be the graph with the vertex set $V(G)$ and the edge set $E(G) \setminus E_0$. The $\mathbb{Z}_k$-flow $g - f$ has all values in $q\mathbb{Z}_k$, and it is 0 on E_0. So $y_0 = (g - f)/q$ is a $\mathbb{Z}_m$-flow of G'. By Corollary 1.3, used for G', there is an integer flow L_0 of G' with $L_0 = y_0$ modulo m and $|L_0(e)| \le m - 1$.

The integer flow $mW + L_0$ of G' is positive on every edge of G': $|L_0(e)| \le m - 1 < m \le mW(e)$. By Lemma 1.1 (b),

$$mW + L_0 = \chi_{O_1} + \cdots + \chi_{O_j}$$

with directed cycles $O_1, \ldots, O_j$ of G' that pass all their edges forward. Modulo m this gives $y_0 = \chi_{O_1} + \cdots + \chi_{O_j}$, and so $g - f = q\,(\chi_{O_1} + \cdots + \chi_{O_j})$ modulo k.

Put $h_0 = g$ and $h_i = h_{i-1} - q\chi_{O_i}$ for $i = 1, \ldots, j$. Then $h_j = f$. Every h_i is a state: its reduction modulo q is w, so it is not 0 outside E_0; and on E_0 it is equal to g, because the cycles O_i have no edge in E_0. By Claim 2, some integer lift of h_{i-1} is adjacent to some integer lift of h_i. By Lemma 1.4 all integer lifts of one h_i lie in one component of $I_k(G)$. So all integer lifts of $g = h_0$ and of $f = h_j$ lie in one component. (If $j = 0$, then $f = g$, and this is Lemma 1.4.) □

Remark 1.6. For $m = 2$ the two flows of Theorem 1.5 are always equal on the edges with $w(e) = 0$: the only element of $q\mathbb{Z}_k$ other than 0 is q. For $m \ge 3$ the theorem says nothing about two flows that are different on such an edge. Lemma 1.10 below treats one such case, a single move of size q.

Liftable steps.

Definition 1.7 (Liftable step)**.** Let $k \ge 3$. An element $z \ne 0$ of $\mathbb{Z}_k$ is *liftable* if for every finite loopless multigraph G and every move $f \to f + z\chi_C$ of $M_k(G)$, all integer lifts of f and all integer lifts of $f + z\chi_C$ lie in one component of $I_k(G)$.

In the next three lemmas C is a cycle of G with a direction. In the first two of them we choose the reference orientation such that C passes all its edges forward. So χ_C is 1 on the edges of C and 0 on all other edges.

Lemma 1.8 (A step of size 1)**.** *Let $k \ge 2$, and let f and $f + \chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then for every integer lift F of f, the flow $F + \chi_C$ is an integer lift of $f + \chi_C$.*

Proof. $F + \chi_C$ is an integer flow, and it is equal to $f + \chi_C$ modulo k. Let e be an edge of C and $r(e)$ in $\{1, \ldots, k-1\}$ the residue of $f(e)$. Then $r(e) \ne k - 1$, because $f + \chi_C$ is not 0 on e. The value $F(e)$ is $r(e)$ or $r(e) - k$. So $F(e) + 1$ is in $[2, k-1]$ or in $[2-k, -1]$. □

Lemma 1.9 (A step of size 2)**.** *Let $k \ge 3$, and let f and $f + 2\chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then there is an integer lift F of f such that $F + 2\chi_C$ is an integer lift of $f + 2\chi_C$.*

The flow $f + \chi_C$ can have zeros, so Lemma 1.8 cannot be used twice.

Proof. For every edge e let $r(e)$ in $\{1, \ldots, k-1\}$ be the residue of $f(e)$. No edge of C has $r(e) = k - 2$, because $f + 2\chi_C$ is nowhere zero. Let E_1 be the set of the edges e of C with $r(e) = k - 1$. Put

$$x = (r + \chi_C)/k.$$

All values of x are in $[0, 1]$, and $x(e) = 1$ exactly for the edges e in E_1. The boundary $\partial x = (\partial r)/k$ is an integer at every vertex, because $\partial r = \partial f = 0$ modulo k. By Lemma 1.2 there is $z\colon E(G) \to \{0, 1\}$ with $\partial z = \partial x$ and with $z(e) = 1$ for every e in E_1. Put $F = r - kz$. As in the proof of Corollary 1.3, F is an integer lift of f.

$F + 2\chi_C$ is an integer flow, it is equal to $f + 2\chi_C$ modulo k, and outside C it is equal to F. For e in E_1: $F(e) = -1$, and the new value is 1. For an edge e of C outside E_1: $r(e) \le k - 3$. If $F(e) = r(e)$, then $3 \le F(e) + 2 \le k - 1$. If $F(e) = r(e) - k$, then $3 - k \le F(e) + 2 \le -1$. So $F + 2\chi_C$ is a vertex of $I_k(G)$. □

Lemma 1.10 (Divisor move)**.** *Let q be a divisor of k with $1 \le q < k$, and let f and $f + q\chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then there is a path in $I_k(G)$ from an integer lift of f to an integer lift of $f + q\chi_C$.*

Proof. For $q = 1$ this is Lemma 1.8. Let $q \ge 2$ and $k = mq$; then $m \ge 2$. Let w be the reduction of f modulo q, and use the setting of this section for w, with E_0, W and the states. The reference orientation is now the one of the setting, and χ_C is read in it. The flows f and $f + q\chi_C$ are states.

The integer flow $mW - \chi_C$ has a value of at least $m - 1 \ge 1$ on every edge outside E_0, and the value $-\chi_C(e)$ on every edge e in E_0. By Lemma 1.1 (b),

$$mW - \chi_C = \chi_{O_1} + \cdots + \chi_{O_j}$$

with directed cycles that pass their edges in the direction of the sign of $mW - \chi_C$. So every O_i passes its edges outside E_0 forward. An edge of E_0 that is not on C is on no O_i. An edge e of E_0 on C is on exactly one O_i, and $\chi_{O_i}(e) = -\chi_C(e)$.

Put $h_0 = f$ and $h_i = h_{i-1} - q\chi_{O_i}$ for $i = 1, \ldots, j$. Then $h_j = f - q\,(mW - \chi_C) = f - kW + q\chi_C = f + q\chi_C$ modulo k. Every h_i is a state. Indeed, its reduction modulo q is w, so it is not 0 outside E_0. On an edge of E_0 outside C it is equal to f. On an edge e of E_0 on C it has the value $f(e)$, or the value $f(e) + q\chi_C(e)$ of the state $f + q\chi_C$; both are not 0.

By Claim 2, for every i some integer lift of h_{i-1} is adjacent to some integer lift of h_i. By Lemma 1.4, all integer lifts of one h_i are joined in $I_k(G)$. Together this is a path from an integer lift of f to an integer lift of $f + q\chi_C$. □

Lemma 1.11 (Multiplier)**.** *Let $k = mq$ with $m, q \ge 2$, and let c be an integer that is invertible modulo k, with $c = 1$ modulo m and modulo q. Then for every nowhere-zero $\mathbb{Z}_k$-flow f of G, the map cf is a nowhere-zero $\mathbb{Z}_k$-flow of G, and all integer lifts of f and all integer lifts of cf lie in one component of $I_k(G)$.*

Proof. Multiplication by c is an automorphism of $\mathbb{Z}_k$, so cf is a nowhere-zero $\mathbb{Z}_k$-flow. Because q divides $c - 1$, the flows f and cf have the same reduction w modulo q. Let $w(e) = 0$. Then $f(e) = qi$ for an integer i, and $cf(e) - f(e) = (c - 1)\,qi$ is a multiple of $mq = k$, because m divides $c - 1$. So $cf(e) = f(e)$. Theorem 1.5 gives the result. □

Lemma 1.12 (Liftable steps)**.** *Let $k \ge 3$.*

(a) *Every nowhere-zero $\mathbb{Z}_k$-flow of a finite loopless multigraph G has an integer lift, and all its integer lifts lie in one component of $I_k(G)$.*

(b) *1 and -1 are liftable.*

(c) *2 and -2 are liftable.*

(d) *q and $-q$ are liftable for every divisor q of k with $1 \le q < k$.*

(e) *Let $k = mq$ with $m, q \ge 2$, and let c be invertible modulo k with $c = 1$ modulo m and modulo q. If z is liftable, then cz is liftable. In particular, for $k = 4p$ with an odd prime p, the elements $1 + 2p$ and $-(1 + 2p)$ are liftable.*

Proof. (a) Corollary 1.3 and Lemma 1.4.

(b), (c), (d) Let z be 1, or 2, or a divisor q of k with $1 \le q < k$, and let $f \to f + z\chi_C$ be a move of $M_k(G)$. By Lemma 1.8, Lemma 1.9 or Lemma 1.10, some integer lift of f is joined in $I_k(G)$ to some integer lift of $f + z\chi_C$. With (a) for f and for $f + z\chi_C$, all integer lifts of the two flows lie in one component. A move $f \to f - z\chi_C$ is the move $f \to f + z\chi_{C'}$ for the cycle C' that is C with the other direction. So $-z$ is liftable, too.

(e) Let c' be an integer with $cc' = 1$ modulo k. Then c' is invertible modulo k, and $c' = 1$ modulo m and modulo q. Let $f \to g = f + cz\chi_C$ be a move of $M_k(G)$. Then $c'f$ and $c'g = c'f + z\chi_C$ are nowhere-zero $\mathbb{Z}_k$-flows, and $c'f \to c'g$ is a move of $M_k(G)$. Because z is liftable, all integer lifts of $c'f$ and of $c'g$ lie in one component of $I_k(G)$. By Lemma 1.11, used for the flow $c'f$ and for the flow $c'g$, all integer lifts of $c'f$ and of $f = c\,(c'f)$ lie in one component, and all integer lifts of $c'g$ and of $g = c\,(c'g)$ lie in one component. So all integer lifts of f and of g lie in one component.

Let $k = 4p$ with an odd prime p. Take $m = 2$, $q = 2p$ and $c = 1 + 2p$. Then $c = 1$ modulo 2 and modulo $2p$. The number c is odd, and it is 1 modulo p; so it is invertible modulo $4p$. By (b), $z = 1$ and $z = -1$ are liftable. So $1 + 2p$ and $-(1 + 2p)$ are liftable. □

Corollary 1.13 (The steps for $k = 4p$)**.** *Let $k = 4p$ with an odd prime p. Then $1 + 2p$ and $-(1 + 2p)$ are liftable, and every element of the set*

$$\{\pm 1,\ \pm(1 + 2p),\ \pm 2,\ \pm 4,\ \pm p,\ 2p\}$$

of elements of $\mathbb{Z}_{4p}$ is liftable.

Proof. $\pm(1 + 2p)$: Lemma 1.12 (e). ± 1: (b). ± 2: (c). The numbers 4, p and $2p$ are divisors of $4p$ that are smaller than $4p$, and $-2p = 2p$ in $\mathbb{Z}_{4p}$; so (d) gives ± 4, $\pm p$ and $2p$. □

2. Extraction; the truncated cubic reduction; threads; official cycles

Good subgraphs and the extraction lemma.

Definition 2.1 (Good subgraph)**.** A *good subgraph* of a multigraph is a subgraph H of one of two kinds:

(CYC) a cycle with at most 5 edges and at least 2 vertices;

(SUB) a subdivision of a loopless 3-edge-connected multigraph H_0 on at least two vertices, in which every edge of H_0 is replaced by a path with at most 3 edges.

In (SUB) the paths are internally disjoint, and their interior vertices are the vertices of H that are not vertices of H_0; these have degree 2 in H.

Lemma 2.2 (Extraction)**.** *Let G be a loopless multigraph with $n \geq 3$ vertices and more than $(4/3)\,n - 2$ edges. Call a subgraph of G* dense *if it has $n' \geq 3$ vertices and more than $(4/3)\,n' - 2$ edges.*

(a) *G contains a good subgraph.*

(b) *Let G contain no cycle with at most 5 edges, and let H be a dense subgraph of G with the least number of vertices (a* vertex-minimal *dense subgraph). Then H is a good subgraph of the kind* (SUB)*, and:*

- (X1) *The vertices of H_0 are the vertices of H of degree at least 3. H_0 is loopless and 3-edge-connected, it has at least two vertices, and every edge of H_0 is replaced by a path with at most 3 edges.*
- (X2) *H contains no cycle with at most 5 edges. So H is a simple graph, and the paths that replace the edges of a cycle of H_0 have together at least 6 edges. In particular, if two edges of H_0 are parallel, both are replaced by paths with 3 edges.*
- (X3) *$3\,|E(H)| \geq 4\,|V(H)| - 5$. So H has at most $3m_0 - 4n_0 + 5$ vertices of degree 2, where n_0 and m_0 are the numbers of vertices and of edges of H_0.*

(X4) *For every set V' of vertices of H with $1 \le |V'| \le |V(H)| - 3$, more than $(4/3)\,|V'|$ edges of H have an end in V'.*

Proof. G is a dense subgraph of G, so there is a dense subgraph H with the least number of vertices. If G contains a cycle with at most 5 edges, this cycle is a good subgraph of the kind (CYC): it has at least 2 vertices, because G is loopless. So (a) follows from (b). For the proof of (b) let G contain no cycle with at most 5 edges. Then H contains no such cycle; in particular H has no parallel edges. Put $n_H = |V(H)|$ and $m_H = |E(H)|$.

(X3), *first part.* H is dense, so $3m_H > 4n_H - 6$, that is $3m_H \ge 4n_H - 5$.

(X4). The graph $H - V'$ has $n_H - |V'| \ge 3$ vertices, fewer than H. So it is not dense: $3\,|E(H - V')| \le 4\,(n_H - |V'|) - 6$. With (X3), the number of the edges of H with an end in V' is $m_H - |E(H - V')| \ge (4\,|V'| + 1)/3$.

Step 1: $n_H \ge 6$, $m_H > n_H$, and every vertex of H has degree at least 2. We have $m_H > (4/3)\,n_H - 2 \ge n_H - 1$, so $m_H \ge n_H$, and H contains a cycle. It has at least 6 edges, so $n_H \ge 6$. Then $(4/3)\,n_H - 2 \ge n_H$, so $m_H > n_H$. By (X4) for a set V' with one vertex, more than 4/3 edges meet this vertex.

Step 2: H is connected. Suppose that H is the disjoint union of two graphs H_1, H_2 with n_1, n_2 vertices and m_1, m_2 edges. By Step 1 every vertex has degree at least 2, so both graphs contain a cycle, and $n_1, n_2 \ge 6$. Both have fewer vertices than H, so they are not dense: $m_i \le (4/3)\,n_i - 2$. Then $m_H \le (4/3)\,n_H - 4$, a contradiction.

Step 3: the partitions with at most two edges. Let $V(H)$ be the disjoint union of two non-empty sets V_1, V_2 such that at most two edges of H join V_1 and V_2. Let H_i be the subgraph of H induced on V_i, with n_i vertices and m_i edges. If $m_i \le (4/3)\,n_i - 2$ for $i = 1$ and for $i = 2$, then $m_H \le (4/3)\,n_H - 4 + 2$, a contradiction. So $m_i > (4/3)\,n_i - 2$ for one i. This H_i has fewer vertices than H, so it is not dense; hence $n_i \le 2$. A graph with at most 2 vertices and without parallel edges has no cycle. So $n_i = 1$ and $m_i = 0$, or $n_i = 2$ and $m_i = 1$. Every vertex of V_i has degree at least 2 in H, and at most two edges leave V_i. So exactly two edges leave V_i, and every vertex of V_i has degree exactly 2 in H. Result: *exactly two edges join V_1 and V_2, and one of the two sets has at most 2 vertices, all of degree 2 in H.*

Step 4: the threads. Let V_0 be the set of the vertices of H of degree at least 3. It is not empty: H is connected, every degree is at least 2, and $m_H > n_H$. Because H is connected and V_0 is not empty, H is the union of paths and cycles of the following kind (we call them *threads* here): the two ends of a thread are in V_0 (they are equal if the thread is a cycle), its interior vertices have degree 2 in H, every edge of H is on exactly one thread, and every vertex of degree 2 is an interior vertex of exactly one thread. Let I be the set of the interior vertices of a thread, and let I be not empty. The first and the last edge of the thread are the only edges between I and $V(H) \setminus I$. The set $V(H) \setminus I$ contains a vertex of degree at least 3. So by Step 3, $|I| \le 2$. Hence every thread has at most 3 edges. A thread that is a cycle would be a cycle with at most 3 edges; there is none. So every thread is a path with two different ends in V_0. Let H_0 be the multigraph with the vertex set V_0 in which every thread is replaced by one edge between its ends. H_0 is loopless, and H is a subdivision of H_0 in which every edge of H_0 is replaced by a path with at most 3 edges.

Step 5: H_0 is 3-edge-connected and has at least two vertices. A thread has two different ends, so $|V_0| \ge 2$. Let V_0 be the disjoint union of two non-empty sets V_0', V_0'' such that at most two edges of H_0 join V_0' and V_0'' (possibly none). Put the interior vertices of every thread on the side of the first end of the thread. This is a partition of $V(H)$ into two sets; at

most two edges of H join them (for every thread between V_0' and V_0'' its last edge); and both sets contain a vertex of degree at least 3. This contradicts Step 3.

Steps 4 and 5 give (X1). (X2): a cycle of H_0 becomes a cycle of H when its edges are replaced by their paths; it has at least 6 edges; two parallel edges of H_0 are a cycle of H_0, and each of the two paths has at most 3 edges. (X3), second part: let j be the number of the vertices of degree 2 of H. Then $n_H = n_0 + j$ and $m_H = m_0 + j$, and $3\,(m_0 + j) \geq 4\,(n_0 + j) - 5$ gives $j \leq 3m_0 - 4n_0 + 5$. □

Part (a) of Lemma 2.2 is the extraction lemma of [EHLMNS, Lemma 6.1]. The proof above treats the case of a cycle that is a thread explicitly.

Remark 2.3 (What the later sections use)**.** The proofs of the later sections use of a good subgraph H only the two lines of Definition 2.1. In the case (SUB) this is (X1): H_0 is loopless and 3-edge-connected, it has at least two vertices, and every edge of H_0 is replaced by a path with at most 3 edges. So every good subgraph serves, and Lemma 2.2 (a) is all that the proofs need. The equivalent form of (a): a loopless multigraph with $n \geq 3$ vertices that contains no good subgraph has at most $(4/3)\,n - 2$ edges.

Part (b) says more. Whenever the lemma is used for a graph G without a cycle with at most 5 edges, the good subgraph can be chosen as a dense subgraph with the least number of vertices, and then it has the properties (X2), (X3) and (X4) in addition. These three properties are not used in the proofs. They limit the edges of H_0 that are replaced by paths with 3 edges, and the conditions (S) and (NR) of Definition 2.15 speak of these edges. The bound in (X3) is attained. Let G consist of two vertices joined by three paths with 3 edges each. It has 8 vertices and $9 > (4/3) \cdot 8 - 2$ edges, and no cycle with fewer than 6 edges. A subgraph with $j \leq 7$ vertices has at most j edges if $j \geq 6$, and at most $j - 1$ edges if $j \leq 5$; so it is not dense. Hence $H = G$ in Lemma 2.2 (b), with $n_0 = 2$, $m_0 = 3$ and $6 = 3m_0 - 4n_0 + 5$ vertices of degree 2.

The truncated cubic reduction.

Definition 2.4 (Truncated cubic graph)**.** A *truncated cubic* 3*-edge-connected graph* is a graph $Q = Q^+ - v$, where Q^+ is a cubic 3-edge-connected graph on at least four vertices and v is a vertex of Q^+.

Such a graph is simple and connected; three of its vertices have degree 2 and all others have degree 3 (Lemma A.2 (a)).

A *cut vertex* of a connected graph is a vertex whose deletion leaves a graph that is not connected. A *block* of a graph is a maximal connected subgraph without a cut vertex. A graph with two vertices and one or more edges between them has no cut vertex. We use the standard facts: every edge of a graph lies in exactly one block, and every cycle lies in one block.

Lemma 2.5 (Truncated cubic reduction)**.** (a) *Let H_0 be a loopless* 3*-edge-connected multigraph on at least two vertices. Then every block K of H_0 is loopless and* 3*-edge-connected, has at least two vertices, and has no cut vertex.*

(b) *Let K be a loopless* 3*-edge-connected multigraph on at least two vertices without a cut vertex. Then there is a truncated cubic* 3*-edge-connected graph Q_K with these properties: $E(K)$ is a subset of $E(Q_K)$; the edges of Q_K that are not in $E(K)$ form pairwise disjoint cycles; and K is obtained from Q_K by the contraction of each of these cycles to one vertex, that is, by the contraction of the edges of Q_K that are not in $E(K)$.*

Proof. (a) H_0 is connected and loopless and has an edge at every vertex; so a block has an edge, it has at least two vertices, and it is loopless. It has no cut vertex by definition. Let $V(K)$ be the disjoint union of two non-empty sets V_1, V_2. For a vertex x of K let $V[x]$ be the set of the vertices y of H_0 that are joined to x by a path without an edge of K (so x is in $V[x]$). Every vertex of H_0 is in some $V[x]$: take a path to a vertex of K and stop at the first vertex of K. No vertex is in $V[x]$ and in $V[x']$ with $x \neq x'$: otherwise there is a path without an edge of K between two different vertices of K, and with no other vertex in K; the graph K together with this path is connected, has no cut vertex, and is larger than K. An edge yy' of H_0 that is not in K has both ends in one set $V[x]$: a path from x to y without an edge of K, together with this edge, shows that y' is in $V[x]$. So the edges of H_0 between the union of the $V[x]$ with x in V_1 and the union of the $V[x]$ with x in V_2 are exactly the edges of K between V_1 and V_2. There are at least three of them, because H_0 is 3-edge-connected.

(b) *Step 1.* Add to K a new vertex v and three new edges from v to vertices of K, not all three to the same vertex. Call the result K^+. It is loopless. It has no cut vertex: for a vertex u of K the graph $K - u$ is connected, and v has a neighbour other than u; so $K^+ - u$ is connected; and $K^+ - v = K$ is connected. It is 3-edge-connected: let $V(K^+)$ be the disjoint union of two non-empty sets; if both contain vertices of K, at least three edges of K join them; if not, one of them is $\{v\}$, and three edges join them. So every vertex of K^+ has degree at least 3. A neighbour of v has degree at least 3 in K, so it has degree at least 4 in K^+.

Step 2. For every vertex $u \neq v$ of K^+ of degree $j \geq 4$ take a cycle O_u of length j with new vertices and new edges. Replace u by O_u: the j edges at u end now at the j vertices of O_u, one edge at each vertex. Call the result K^c. It is cubic and loopless, the cycles O_u are pairwise disjoint, and the contraction of every O_u to one vertex gives K^+ back. Every vertex of O_u has exactly one edge that is not on O_u, and the other end of this edge is not on O_u, because K^+ is loopless.

Step 3: K^c is 3-edge-connected. Let $V(K^c)$ be the disjoint union of two non-empty sets V_1, V_2 such that at most two edges join them. First case: every cycle O_u is in one of the two sets. Then the contraction of the cycles gives two non-empty sets of vertices of K^+ with the same edges between them; this contradicts Step 1. Second case: some cycle O_u has vertices in both sets. Then at least two edges of O_u join the two sets. So exactly two edges join them, both are on O_u, and every other cycle $O_{u'}$ is in one of the two sets. A vertex of O_u in V_1 has a neighbour outside O_u, and this neighbour is in V_1; the same for V_2. Now contract all cycles other than O_u and delete O_u. The result is $K^+ - u$. Its vertex set is split into two non-empty sets, which come from V_1 and from V_2, with no edge between them. So $K^+ - u$ is not connected; this contradicts Step 1.

Step 4. By Step 1 the ends in K of the three edges at v have degree at least 4 in K^+. So in K^c the three edges at v end at three different vertices, and K^c has at least four vertices. Put $Q_K = K^c - v$. This is a truncated cubic 3-edge-connected graph. Its edges are the edges of K and the edges of the cycles O_u. The contraction of every O_u to one vertex gives $K^+ - v = K$. □

Threads and the auxiliary graph. In the rest of this section H is a subdivision as in Definition 2.1 (SUB), and A is an abelian group. (In the later sections A is the quotient Γ/B of a group Γ by a subgroup B.)

Definition 2.6 (Threads and the auxiliary graph)**.** For an edge t of H_0, the path P_t of H that replaces t is the *thread* of t, and $\ell(t) \in \{1, 2, 3\}$ is its number of edges. Every thread is directed as a path; its edges and the edge t have this direction as reference orientation.

Let $K_1, \dots, K_r$ be the blocks of H_0. For every block K_b let Q_b be a truncated cubic 3-edge-connected graph (a cubic 3-edge-connected graph on at least four vertices minus one vertex) such that $E(K_b)$ is a subset of $E(Q_b)$ and K_b is obtained from Q_b by the contraction of the other edges of Q_b. Q is the disjoint union of the Q_b. The edges of H_0 are the *original* edges of Q; the other edges of Q are *expansion* edges, and $\ell(t) = 0$ for an expansion edge t. For an A-flow Y of Q, $\pi(Y)$ is the A-flow of H that has the value $Y(t)$ on every edge of P_t, for every original edge t.

The graphs Q_b exist by Lemma 2.5. We fix them, one for every block, with disjoint vertex sets. A vertex x of K_b is a set V_x of vertices of Q_b: a component of the graph formed by the expansion edges of Q_b. The map $\pi(Y)$ is defined in the same way for every abelian group in the place of A, also for $\mathbb{Z}$; Lemma 2.7 (c) says that it is a flow.

Lemma 2.7 (The flows of H, of H_0 and of Q)**.** *Let A be an abelian group, finite or not.*

(a) *Every cycle of H is a union of threads. If O is a cycle of H with a direction, then χ_O is constant, 1 or -1, on the edges of each thread of O. Every A-flow of H is constant on every thread. So the A-flows of H are the maps that have the value $h_0(t)$ on every edge of P_t, for an A-flow h_0 of H_0.*

(b) *The restriction of an A-flow of H_0 to a block K_b is an A-flow of K_b.*

(c) *For every A-flow Y of Q, the restriction of Y to the original edges is an A-flow of H_0, and $\pi(Y)$ is an A-flow of H. Every A-flow of H is $\pi(Y)$ for some A-flow Y of Q.*

(d) *Let C be a cycle of Q with a direction. Then there are pairwise edge-disjoint directed cycles $O_1, \dots, O_j$ of H ($j \geq 0$) such that $\pi(x\chi_C) = x\,(\chi_{O_1} + \dots + \chi_{O_j})$ for every x in A.*

Proof. (a) An interior vertex of a thread has degree 2 in H, and the thread is directed as a path. So a cycle that contains one edge of a thread contains the whole thread and passes it in one direction; this gives the first two statements. And conservation at an interior vertex says that its two edges have the same value. For a map that is constant on every thread, conservation at a vertex of H_0 is the same condition in H and in H_0.

(b) By Lemma 1.1 (a) an A-flow of H_0 is a sum of flows $x\chi_C$ with cycles C of H_0. Every cycle lies in one block. So the restriction to K_b is the sum of the terms with C in K_b.

(c) Let x be a vertex of K_b. Add the conservation conditions of Y at the vertices of V_x. Every edge with both ends in V_x is in this sum with both signs. What is left is conservation at x for the restriction of Y to $E(K_b)$. So this restriction is an A-flow of K_b. At a vertex of H_0, conservation is the sum of the conservation conditions in the blocks that contain this vertex. So the restriction of Y to the original edges is an A-flow of H_0, and by (a) $\pi(Y)$ is an A-flow of H.

Now let an A-flow of H be given. By (a) and (b) it comes from an A-flow h_0 of H_0 whose restriction to every block is a flow of the block. Let x be a vertex of K_b. For a vertex y in V_x let $\nu(y)$ be the boundary at y of the map that is h_0 on the original edges and 0 on the expansion edges. The sum of $\nu(y)$ over y in V_x is the boundary of h_0 at x in K_b, which is 0. The expansion edges connect V_x; choose a spanning tree of V_x that consists of expansion edges. There are values on the edges of this tree whose boundary at every y in V_x is $-\nu(y)$. (Induction on $|V_x|$: for one vertex, $\nu = 0$. Otherwise let y be a vertex that meets only one edge yy' of the tree; the value on this edge is fixed by the condition at y; delete y and add $\nu(y)$ to $\nu(y')$.) Give all other expansion edges the value 0. Do this for every x and every b. The result is an A-flow Y of Q that is equal to h_0 on the original edges; so $\pi(Y)$ is the given flow.

(d) By (c), used for the group $\mathbb{Z}$, the map $\pi(\chi_C)$ is an integer flow of H. Its values are in $\{-1, 0, 1\}$. By Lemma 1.1 (b) it is $\chi_{O_1} + \cdots + \chi_{O_j}$ with pairwise edge-disjoint directed cycles of H. And $\pi(x\chi_C) = x\,\pi(\chi_C)$ edge by edge. □

Frames.

Definition 2.8 (2-joins and 2-constructions)**.** A 2-*join* of two graphs U' and U'' with disjoint vertex sets is a graph that consists of U', of U'' and of exactly two new edges, each with one end in U' and one end in U'' (the two new edges can be parallel). A graph is 2-*constructible* if it has one vertex and no edge, or if it is a 2-join of two 2-constructible graphs.

A 2-*construction* of a 2-constructible graph U_0 fixes one way to obtain U_0. If U_0 has more than one vertex, it names two graphs U_0', U_0'' of which U_0 is a 2-join, and it contains a 2-construction of U_0' and a 2-construction of U_0''. The *stages* of the 2-construction are U_0 and, if U_0 has more than one vertex, the stages of the 2-constructions of U_0' and of U_0''. So the stages are subgraphs of U_0, and the stages with one vertex are the vertices of U_0. A stage U with more than one vertex is a *node*; it stands for the 2-join that makes it. The two stages U', U'' that it joins are its *operands*, and its two new edges are the *edges of the node*.

Definition 2.9 (Frame)**.** A *frame* of H consists of the following, for every component Q_b of Q:

(F1) a *well colouring* of Q_b: every edge is blue or red, the blue edges form a spanning tree T_b of Q_b, and the graph Q_b/red (its vertices are the components of the red edges, its edges are the blue edges) is 2-constructible;

(F2) a 2-construction of Q_b/red from single vertices; its 2-joins are the *nodes*, and the two graphs that a node joins are its *operands*;

(F3) for every node i, names e_i and f_i for its two new edges. The set $N_i = \{e_i, f_i\}$ is the *pair* of the node i.

The nodes of all components are numbered $1, \ldots, s$: component after component, and in every component every node after all nodes that are subgraphs of its two operands. T is the union of the trees T_b; its edge set is the disjoint union of the pairs $N_1, \ldots, N_s$. Every truncated cubic 3-edge-connected graph has a well colouring (Lemma A.5), so every H has a frame. The names in (F3) are free: at every node the two names can be exchanged.

In (F1), "the components of the red edges" are the *red components*: the components of the graph with the vertex set $V(Q_b)$ and the red edges. The ends of a blue edge in Q_b/red are the red components of its two ends.

Notation for a frame. Let a frame of H be given. For a stage U of the 2-construction of Q_b/red, V_U is the union of the red components that are the vertices of U, and Q_U is the subgraph of Q_b induced on V_U, the *graph of* U. For a node i, U_i is its stage, and U_i', U_i'' are its operands. A node h is *below* the node i if $h \neq i$ and U_h is a subgraph of U_i; then U_h lies in U_i' or in U_i'' and is a stage of its 2-construction (two stages are disjoint or nested, Lemma A.2 (b), and $U_h \neq U_i$), and $h < i$ by the numbering of Definition 2.9. The following facts are proved in the appendix (Lemma A.2 (b), (c) and Lemma A.3).

(G1) The red edges of Q_b form a forest. No blue edge has both ends in one red component.

(G2) Every blue edge is an edge of exactly one node. So $E(T)$ is the disjoint union of the pairs.

(G3) The edges of Q_U are the red edges with both ends in V_U and the edges of the pairs N_h of the nodes h with U_h a subgraph of U. Q_U is connected.

(G4) For a node i: V_{U_i} is the disjoint union of $V_{U_i'}$ and $V_{U_i''}$; each of e_i, f_i has one end in $V_{U_i'}$ and one end in $V_{U_i''}$; and $E(Q_{U_i}) = E(Q_{U_i'}) \cup E(Q_{U_i''}) \cup \{e_i, f_i\}$.

Protected and free threads; the official cycles.

Definition 2.10 (Protected and free threads; official cycles)**.** Let a frame of H be given. The thread P_t is *protected* if t is an original blue edge, and *free* if t is an original red edge; then t is a *free edge*. H_P is the union of the protected threads. For a free edge ε, D_ε is a fixed cycle of H that contains P_ε and no other free thread, directed along P_ε. T^e is the subgraph of Q formed by all red edges and the edges $e_1, \dots, e_s$. The *official cycle* C_i is the fundamental cycle of f_i in T^e, directed along f_i.

Lemma 2.11 (Official cycles)**.** *For every frame of H:*

(O1) *The red edges of Q form a forest. For every choice of one edge g_i in every pair N_i, the red edges and the edges $g_1, \dots, g_s$ form a spanning tree T^g of every component Q_b, and of the graph of every stage. In particular T^e is such a tree.*

(O2) *C_i contains e_i and f_i.*

(O3) *If $l < i$, then C_l contains no edge of N_i.*

(O4) *If $l > i$, then C_l does not contain f_i. (It can contain e_i.)*

(O5) *For every abelian group A, every A-flow of Q is a sum $x_1\chi_{C_1} + \dots + x_s\chi_{C_s}$ with x_i in A, in exactly one way.*

(O6) *H_P is connected and contains every vertex of H that is not an interior vertex of a free thread; and the cycle D_ε exists for every free edge ε.*

In (O1), "T^g is a spanning tree of the graph Q_U of a stage U" means: the edges of T^g in Q_U form a spanning tree of Q_U. This holds for every stage, also for the stages with one vertex.

Proof. (O1) The red edges form a forest by (G1). Let one edge g_i in every pair N_i be chosen. For a stage U let T^g_U be the graph with the vertex set V_U whose edges are the red edges with both ends in V_U and the edges g_h of the nodes h with U_h a subgraph of U. We show by induction on the 2-construction that T^g_U is a spanning tree of Q_U. If U has one vertex, then V_U is one red component; it is connected by its red edges, and they form a forest; so it is a tree. Let $U = U_i$ be a node. By induction $T^g_{U_i'}$ and $T^g_{U_i''}$ are trees; by (G4) their vertex sets are disjoint with the union V_{U_i}, and g_i has one end in each of them. A red edge with both ends in V_{U_i} has both ends in one red component, so in $V_{U_i'}$ or in $V_{U_i''}$. A node h with U_h a subgraph of U_i is i, or U_h is a subgraph of U_i' or of U_i''. So $T^g_{U_i}$ is the union of the two trees and the edge g_i: a spanning tree of Q_{U_i}. For $U = Q_b/\text{red}$ this is the statement on Q_b.

Consequence. For every stage U and two vertices of V_U, the path of T^g between them lies in Q_U: it is the path of the tree T^g_U.

(O2) By (G2) the edges of Q outside T^e are $f_1, \dots, f_s$. So C_i consists of f_i and of the path of T^e between the ends of f_i. Both ends are in V_{U_i}, so by the consequence of (O1) this path lies in Q_{U_i}. By (G4) one end of f_i is in $V_{U_i'}$ and the other one is in $V_{U_i''}$, and the only edges of Q_{U_i} between these two sets are e_i and f_i. The path does not contain f_i. So it

contains e_i. We note the result as $(*)$: *C_i is a cycle of Q_{U_i}; it consists of f_i, of e_i, of a path of T^e in $Q_{U_i'}$ and of a path of T^e in $Q_{U_i''}$.*

(O3) Let $l < i$. By $(*)$, C_l is a cycle of Q_{U_l}. By (G3) the blue edges of Q_{U_l} are the edges of the pairs N_h with $h = l$ or h below l; so $h \leq l < i$. The pairs are disjoint. So no edge of N_i is an edge of Q_{U_l}.

(O4) All edges of C_l other than f_l are in T^e. The edge f_i is not in T^e, and $f_i \neq f_l$.

(O5) By (O1), T^e is a spanning tree in every component of Q. The edges outside T^e are $f_1, \ldots, f_s$, and C_i is the fundamental cycle of f_i, directed along f_i. This is Lemma 1.1 (a); the coefficients are $x_i = Y(f_i)$ for the flow Y.

(O6) T_b is a spanning tree of Q_b. The contraction of the expansion edges maps it onto a connected spanning subgraph of K_b; its edges are the original edges of T_b, that is the original blue edges of Q_b. The graph H_0 is connected, and every edge of H_0 is in a block. So the original blue edges of Q form a connected spanning subgraph of H_0: two vertices of H_0 are joined by a path of H_0, and the two ends of every edge of this path are joined by original blue edges in the block of this edge. Replace the edges by their threads: H_P is connected, and it contains all vertices of H_0 and the interior vertices of the protected threads. The other vertices of H are interior vertices of free threads.

Let ε be a free edge, in the block K_b. It is an original edge of Q_b that is not in T_b. Its fundamental cycle in T_b is mapped by the contraction of the expansion edges to a closed trail of K_b that contains ε once and, apart from ε, original blue edges only. The edges of a closed trail form a graph in which every vertex has an even degree; so every edge of it lies on a cycle of this graph. Take such a cycle through ε. K_b is loopless, so this cycle has more edges than ε, and they are original blue edges. Replace the edges of this cycle by their threads. This gives a cycle of H that contains P_ε and, apart from it, protected threads only. □

By (O2), (O3) and by $(*)$ in the proof above, the official cycles are a closure sequence in the sense of Lemma A.6: every edge of C_i other than e_i and f_i is a red edge or an edge of a pair N_h with $h < i$, and no edge of N_i is on a cycle C_l with $l < i$.

Definition 2.12 (The two cycles of a node)**.** Let a frame of H be given. T^f is the tree T^g of (O1) for the choice $g_i = f_i$ for every i: all red edges and the edges $f_1, \ldots, f_s$. For a node i put $C_i^e = C_i$, and let C_i^f be the fundamental cycle of e_i in T^f, with the direction in which it passes f_i forward (by Lemma 2.13 (a) it contains f_i).

Lemma 2.13 (The two cycles of a node)**.** *Let a frame of H be given and let i be a node.*

(a) *C_i^f is a cycle of Q_{U_i}. It consists of e_i, of f_i, of a path of T^f in $Q_{U_i'}$ and of a path of T^f in $Q_{U_i''}$. The cycles C_i^e and C_i^f pass f_i forward, and they pass e_i in the same direction. So $\chi_{C_i^e}$ and $\chi_{C_i^f}$ are equal on N_i.*

(b) *Let $h \neq i$ be a node. If h is below i, then C_i^e does not contain f_h, and C_i^f does not contain e_h. If h is not below i, then no edge of N_h is on C_i^e or on C_i^f; this is so for every $h > i$.*

(c) *$\Delta_i = \chi_{C_i^f} - \chi_{C_i^e}$ is an integer flow of Q. It is 0 on N_i and outside Q_{U_i}. So it is an integer flow of the graph $Q_{U_i'} \cup Q_{U_i''}$, whose edges are red edges and edges of pairs N_h with h below i. In particular Δ_i is 0 on $N_i \cup N_{i+1} \cup \cdots \cup N_s$. For every abelian group A and every x in A, the A-flow $x\Delta_i$ is a sum of flows $x'\chi_C$ with x' in A and C a cycle of $Q_{U_i'} \cup Q_{U_i''}$.*

Proof. (a) The proof of (O2), with the tree T^f in the place of T^e and with the names e_i and f_i exchanged, gives the first two sentences. By $(*)$ in the proof of Lemma 2.11, the cycle C_i^e has the same form, with paths of T^e. Let both cycles pass f_i forward, say from its end in $V_{U_i'}$ to its end in $V_{U_i''}$ (the other case is the same). After f_i each of the two cycles stays in $Q_{U_i''}$ until it passes e_i, from the end of e_i in $V_{U_i''}$ to its end in $V_{U_i'}$; then it goes back in $Q_{U_i'}$. So both pass e_i in the same direction.

(b) Let h be below i. All edges of C_i^e other than f_i are in T^e, and f_h is not in T^e. All edges of C_i^f other than e_i are in T^f, and e_h is not in T^f. Let $h \neq i$ be not below i. Both cycles are cycles of Q_{U_i}, and by (G3) the pair N_h has no edge in Q_{U_i}. A node $h > i$ is not below i.

(c) Δ_i is a difference of two integer flows. It is 0 on N_i by (a), and outside Q_{U_i} because both cycles are in Q_{U_i}. By (G4) it is therefore 0 outside $Q_{U_i'} \cup Q_{U_i''}$; so it is an integer flow of this graph. By (G3) the edges of this graph are red edges and edges of pairs N_h with U_h a subgraph of U_i' or of U_i'', that is with h below i. The last sentence is Lemma 1.1 (a) for the A-flow $x\Delta_i$ of the graph $Q_{U_i'} \cup Q_{U_i''}$. □

Lists and counts. In this subsection Γ is a group, B is a subgroup, $A = \Gamma/B$, and $x \mapsto \overline{x}$ is the quotient map. The *A-coordinate* of a Γ-flow f is the A-flow u with the values $u(e) = \overline{f(e)}$. The *A-part* of an element x of Γ is $\overline{x}$.

Definition 2.14 (Lists; H-good flows)**.** Let a frame of H be given. A *list assignment* Λ gives every blue edge t of Q a set $\Lambda(t)$ of elements of A, with $\Lambda(t)$ empty for every expansion edge. $P(\Lambda)$ is the set of the A-flows Y of Q with $Y(t) \notin \Lambda(t)$ for every blue edge t.

Let G be a multigraph that contains H, and f a nowhere-zero Γ-flow of G with A-coordinate u. In the case (SUB), f is *H-good* if $u(e) \neq 0$ for every edge e of every protected thread; then

$$\Lambda_f(t) = \{\, -u(e) : e \in E(P_t) \,\} \qquad \text{for every original blue edge } t.$$

So $|\Lambda_f(t)| \leq \ell(t)$, 0 is in $P(\Lambda_f)$, and for an A-flow Y of Q: Y is in $P(\Lambda_f)$ if and only if $u + \pi(Y)$ is non-zero on all protected threads. ($u + \pi(Y)$ means: add $\pi(Y)$ on $E(H)$ and nothing outside $E(H)$.) In the case (CYC), f is *H-good* if $u(e) \neq 0$ for every edge e of H.

The three claims after the display are clear: the thread P_t has $\ell(t)$ edges; no element $-u(e)$ of a list is 0; and $Y(t) \neq -u(e)$ for all edges e of P_t says that $u + \pi(Y)$ is not 0 on P_t.

Definition 2.15 (The counts; counted frame)**.** Let a frame of H be given. The *counts* for A and B are the following inequalities.

(C1) $2\,\ell(e_i) + \ell(f_i) < |A|$ for every node i (the walk lemma).

(C2) $\ell(e_i) + \ell(f_i) + \max\{\ell(e_i), \ell(f_i)\} < |A|$ for every node i (the reverse choice for a step with non-zero A-part).

(C3) $\ell(e_i) + \ell(f_i) + 2 < |A|$ for every node i (the reverse choice for a step in B).

(C4) $\ell(\varepsilon) < |B|$ for every free edge ε (the preparation in B).

The frame is *counted* (for A and B) if (C1)–(C4) hold. Two conditions on a frame that do not use the groups:

(S) no pair N_i consists of two original edges whose threads both have 3 edges;

(NR) no free thread has 3 edges (every original edge whose thread has 3 edges is blue).

Remark 2.16. The left sides of (C1), (C2), (C3), (C4) are at most 9, 9, 8, 3. So every frame is counted if $|A| \geq 10$ and $|B| \geq 4$. If the names are chosen with $\ell(e_i) \leq \ell(f_i)$ for every

i, then (C2) implies (C1). With these names: a frame with (S) is counted if $|A| \geq 9$ and $|B| \geq 4$ (left sides at most 7, 8, 7, 3); a frame with (NR) is counted if $|A| \geq 10$ and $|B| \geq 3$ (the left side of (C4) is at most 2).

3. Step systems and the master theorem

This section proves the master theorem in its two forms (Theorems 3.11 and 3.12), three corollaries, and the integer form (Theorem 3.19). All proofs are complete here. What is taken from other sections is cited at the place of use, with the property that is used: liftable steps (Definition 1.7, Lemma 1.12); good subgraphs (Definition 2.1) and the extraction lemma (Lemma 2.2); threads, the auxiliary graph, frames, official cycles, lists and counts (Definitions 2.6, 2.9, 2.10, 2.14, 2.15 and Lemma 2.11). The averaging conditions (AV) and (AV″) are hypotheses in this section; Lemma 4.1 proves them for products.

Two rules hold in every proof of this section.

(a) Every use of $|A|$ or of $|B|$, other than in the hypotheses of Lemmas 3.4 and 3.5, names the count that it uses: (C0) of Theorem 3.11, or one of (C1)–(C4) of Definition 2.15; every use of the bound $\ell(t) \leq 3$ names (i). Remark 3.14 lists the places.

(b) (R1) with six deleted elements, the count (C1) and the walk lemma (Lemma 3.5) are used only in the proof of Lemma 3.8. Every later proof uses Lemma 3.8 only through its conclusion. This is why the frame form of the theorem holds.

Step systems. Throughout, Γ is a finite abelian group, B a subgroup, $A = \Gamma/B$, and $x \mapsto \overline{x}$ is the quotient map. For a Γ-flow f of a multigraph the A-flow $u = \overline{f}$ (edge by edge) is the *A-coordinate* of f.

Definition 3.1 (Step system)**.** A *step system* is a set Σ of non-zero elements of Γ with $\Sigma = -\Sigma$. Put

$$\Sigma_B = \Sigma \cap B, \qquad \Sigma_A = \{\, \overline{\lambda} : \lambda \in \Sigma \,\} \setminus \{0\},$$

and for every s in Σ_A fix one $\gamma(s)$ in Σ with $\overline{\gamma(s)} = s$. A *Σ-move* of a multigraph G is a move $f \to f + \lambda\chi_C$ of $F(G, \Gamma)$ with λ in Σ. Two flows are *Σ-joined* if a sequence of Σ-moves leads from one to the other.

So a Σ-move leads from a nowhere-zero flow to a nowhere-zero flow. The reverse of a Σ-move is a Σ-move, because $\Sigma = -\Sigma$. A flow is Σ-joined to itself (the empty sequence).

Definition 3.2 (The conditions on a step system)**.** (R1) For every set X of at most 6 elements of A, any two elements of $A \setminus X$ are joined by a walk in $A \setminus X$ all of whose steps are in Σ_A.

(R1$^-$) The same for every set X of at most 5 elements of A.

(R2) For every set X of at most 2 elements of B, any two elements of $B \setminus X$ are joined by a walk in $B \setminus X$ all of whose steps are in Σ_B.

(R3*) Any two non-zero elements of Γ are joined by a walk in $\Gamma \setminus \{0\}$ all of whose steps are in Σ.

(AV) For every loopless 3-edge-connected multigraph G with $n \geq 2$ vertices and m edges, every nowhere-zero Γ-flow of G is Σ-joined to a nowhere-zero Γ-flow whose A-coordinate has at most $m/9$ zeros.

(AV″) For every loopless 3-edge-connected multigraph G with $n \geq 2$ vertices, every nowhere-zero Γ-flow of G is Σ-joined to a nowhere-zero Γ-flow f' such that the multigraph $(V(G), \mathrm{supp}(\overline{f'}))$ contains a good subgraph.

A *step* of a walk is the difference of two consecutive terms. (R1) implies (R1$^-$). (R1) or (R1$^-$) with X empty says: every element of A is a sum of elements of Σ_A. (R2) with X empty says: every element of B is a sum of elements of Σ_B.

Five elementary facts. Here G is a multigraph, f a Γ-flow of G and $u = \overline{f}$. An *integer flow* of G is a map $E(G) \to \mathbb{Z}$ with conservation at every vertex.

(E1) $f(e) \neq 0$ if and only if $u(e) \neq 0$, or $u(e) = 0$ and $f(e)$ is an element of B other than 0. (If $u(e) = 0$, then $f(e)$ is in B.)

(E2) If y is in B and C is a cycle of G, then $f + y\chi_C$ has the A-coordinate u.

(E3) *Cycle decomposition* (Lemma 1.1). Every flow of G with values in an abelian group is a sum of flows $x\chi_C$, where C is a cycle of G and x is in the group. Every integer flow of G with values in $\{0, 1, -1\}$ is a sum $\chi_{O_1} + \cdots + \chi_{O_\nu}$ with pairwise edge-disjoint cycles $O_1, \ldots, O_\nu$ of G ($\nu = 0$ for the zero flow).

(E4) Every A-flow of G is the A-coordinate of some Γ-flow of G.

(E5) *Contraction.* Let G' be obtained from G in this way: the vertices of each class of a partition of $V(G)$ are identified to one vertex, and some edges that have both ends in one class are deleted. Then the restriction of a flow of G to $E(G')$ is a flow of G' (values in any abelian group).

Proofs. (E1) and (E2) hold because B is the kernel of the quotient map.

(E3) The first statement is Lemma 1.1 (a). The second one is Lemma 1.1 (b): for an integer flow with values in $\{0, 1, -1\}$ the cycles of the signed form are pairwise edge-disjoint.

(E4) By (E3) the A-flow is a sum $\sum_r a_r \chi_{C_r}$. Choose x_r in Γ with $\overline{x_r} = a_r$. The Γ-flow $\sum_r x_r \chi_{C_r}$ has this A-coordinate.

(E5) Add the conservation equations of G over the vertices of one class. An edge with both ends in the class enters the class once and leaves it once, so it drops out. What is left is the conservation equation of G' at the vertex made from the class. □

Lemma 3.3. (AV) *implies* (AV″). (*A loopless multigraph with $n \geq 3$ vertices and more than $(4/3)\, n - 2$ edges contains a good subgraph; for $n = 2$ the support has two parallel edges.*)

Proof. Let G be loopless and 3-edge-connected with $n \geq 2$ vertices and m edges. Every vertex has degree at least 3, so $m \geq 3n/2$. Let f' be the flow that (AV) gives, with A-coordinate u'. Then

$$|\,\mathrm{supp}(u')| \;\geq\; m - m/9 \;\geq\; (8/9)(3/2)\, n \;=\; (4/3)\, n \;>\; (4/3)\, n - 2.$$

Let $n \geq 3$. The multigraph $(V(G), \mathrm{supp}(u'))$ is loopless and has n vertices and more than $(4/3)\, n - 2$ edges. By Lemma 2.2 (a) it contains a cycle with at most 5 edges or a subdivision of a loopless 3-edge-connected multigraph on at least two vertices in which every edge is replaced by a path with at most 3 edges. The cycle has at least 2 vertices, because the multigraph is loopless. This is a good subgraph (Definition 2.1). Let $n = 2$. Then $m \geq 3$, so $\mathrm{supp}(u')$ has at least $8m/9 > 2$ edges. G is loopless, so two of them are parallel. They form a cycle with 2 edges and 2 vertices, a good subgraph of the kind (CYC). □

The subdivision: what is used. From here to the end of the proof of Lemma 3.9, H is a subdivision as in Definition 2.1 (SUB), with the graphs H_0, K_b, Q_b, Q of Definition 2.6,

and a frame of H is given (Definition 2.9). The proofs use the following properties of these objects, and no other property.

(i) *Threads* (Definition 2.6). Every edge of H lies on exactly one thread P_t, where t is an edge of H_0, and $1 \leq \ell(t) \leq 3$. The interior vertices of a thread have degree 2 in H; the other vertices of H are the vertices of H_0. H_0 is connected and loopless, every edge of H_0 lies in exactly one block K_b, and every cycle of H_0 lies in one block.

(ii) *The auxiliary graph* (Definition 2.6). $E(K_b)$ is a subset of $E(Q_b)$, and K_b is obtained from Q_b by the contraction of the expansion edges: the vertices of K_b are the components of the subgraph of Q_b formed by all vertices and the expansion edges, and an original edge joins the components of its two ends. $\ell(t) = 0$ for an expansion edge t.

(iii) *The frame* (Definition 2.9). The blue edges of Q_b form a spanning tree T_b of Q_b. Every blue edge of Q lies in exactly one pair N_i. A *stage* is a graph U of the 2-construction (Definition 2.8); its vertices are components of the red edges. V_U is the union of these components, and Q_U is the subgraph of Q induced on V_U. If U' and U'' are the two operands of the node i, then $V_{U'}$ and $V_{U''}$ are disjoint, and each of the edges e_i, f_i has one end in $V_{U'}$ and the other end in $V_{U''}$ (a 2-join adds two edges, each with one end in each of the two graphs; Lemma A.3 (c)). The names e_i, f_i can be exchanged at every node; the result is a frame again.

(iv) *Official cycles* (Definition 2.10, Lemma 2.11). The statements (O1)–(O6). Of (O1) we use, with Q_U as in the notation for a frame after Definition 2.9 and the paragraph after Lemma 2.11: for every choice of one edge g_i in every pair, T^g is a spanning tree of every component Q_b; and for every stage U, the edges of T^e in Q_U form a spanning tree of Q_U (the proof of (O1) shows this for every stage, by induction on the 2-construction, from Lemmas A.2 and A.3). The blue edges of T^g are the edges g_i. The cycle D_ε contains P_ε and no other free thread, and it is directed along P_ε.

(v) *Lists and counts* (Definitions 2.14, 2.15). A list assignment is empty on every expansion edge. An H-good flow is nowhere zero. For an H-good flow f with A-coordinate u: $|\Lambda_f(t)| \leq \ell(t)$; 0 is in $P(\Lambda_f)$; and an A-flow Y of Q is in $P(\Lambda_f)$ if and only if $u + \pi(Y)$ is non-zero on every edge of every protected thread. Here and below, $u + \pi(Y)$ means: add $\pi(Y)$ on $E(H)$ and nothing outside $E(H)$.

Whenever H is a subgraph of a multigraph G, the reference orientation of G on $E(H)$ is the direction of the threads. This is no restriction: a change of the reference orientation of an edge changes the sign of all flow values on this edge and nothing else. For a flow Y of Q, with values in A or in $\mathbb{Z}$, $\pi(Y)$ has the value $Y(t)$ on every edge of P_t, for every original edge t.

Lemma 2.7 describes the flows of H and of Q. Below we use its parts (a), (c) and (d): every cycle of H is a union of threads, and for a cycle O of H with a direction, χ_O is constant, 1 or -1, on the edges of each thread of O; every flow of H is constant on the edges of each thread; $\pi(Y)$ is an A-flow of H for every A-flow Y of Q, and every A-flow of H is $\pi(Y)$ for some A-flow Y of Q; and for every cycle C of Q with a direction there are pairwise edge-disjoint directed cycles $O_1, \ldots, O_\nu$ of H ($\nu = 0$ is possible) with $\pi(x\chi_C) = x\,(\chi_{O_1} + \cdots + \chi_{O_\nu})$ for every x in A.

The reverse choice and the walk lemma.

Lemma 3.4 (Reverse choice)**.** *Let Λ be a list assignment.*

(a) *If* $|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$ *for every node* i*, then for every* A*-flow* Y_0 *of* Q *there are* $x_1, \dots, x_s$ *in* A *with* $Y_0 + x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ *in* $P(\Lambda)$.

(b) *Let* E' *be a set of red edges of* Q*, and for every* ε *in* E' *let* $\Phi(\varepsilon)$ *be a set of elements of* A*, not all of* A*. Let* $i(\varepsilon)$ *be the least* i *with* ε *on* C_i*, if there is one. If for every node* i

$$|\Lambda(e_i)| + |\Lambda(f_i)| + \sum_{\varepsilon\in E',\ i(\varepsilon)=i} |\Phi(\varepsilon)| \ < \ |A|,$$

then there is Y *in* $P(\Lambda)$ *with* $Y(\varepsilon) \notin \Phi(\varepsilon)$ *for every* ε *in* E'.

Proof. Every blue edge of Q lies in exactly one pair (iii). By (O2) the cycle C_i contains e_i and f_i; so χ_{C_i} has the value 1 or -1 on each of them. By (O3) no cycle C_l with $l < i$ contains an edge of N_i.

(a) Choose $x_s, x_{s-1}, \dots, x_1$ in this order. Let $x_s, \dots, x_{i+1}$ be chosen, and put $Z_i = Y_0 + \sum_{l>i} x_l\chi_{C_l}$. For an edge t of N_i the flow $Y = Y_0 + x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ has the value

$$Y(t) = Z_i(t) + x_i\,\chi_{C_i}(t),$$

whatever $x_{i-1}, \dots, x_1$ are (O3). Since $\chi_{C_i}(t)$ is 1 or -1, the condition $Y(t) \notin \Lambda(t)$ forbids exactly $|\Lambda(t)|$ values of x_i. The two edges of N_i together forbid at most $|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$ values. So x_i can be chosen. In the end $Y(t) \notin \Lambda(t)$ for every blue edge t.

(b) Let E'_0 be the set of the edges of E' that lie on no official cycle. For ε in E'_0 choose y_ε in A outside $\Phi(\varepsilon)$. The blue edges form a spanning tree of the component of Q that contains ε (iii). Let F_ε be the cycle formed by ε and the path of this tree between the ends of ε, directed along ε. Its only red edge is ε. Put $Y_0 = \sum_{\varepsilon\in E'_0} y_\varepsilon\chi_{F_\varepsilon}$; so $Y_0(\varepsilon) = y_\varepsilon$ for every ε in E'_0. Choose $x_s, \dots, x_1$ as in (a), for the flow $Y = Y_0 + x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$, with one more condition on x_i: $Y(\varepsilon) \notin \Phi(\varepsilon)$ for every ε in E' with $i(\varepsilon) = i$. For such an ε the value $Y(\varepsilon) = Z_i(\varepsilon) + x_i\,\chi_{C_i}(\varepsilon)$ does not depend on $x_{i-1}, \dots, x_1$, because no cycle C_l with $l < i(\varepsilon)$ contains ε. And $\chi_{C_i}(\varepsilon)$ is 1 or -1, so the condition forbids $|\Phi(\varepsilon)|$ values of x_i. By the hypothesis fewer than $|A|$ values of x_i are forbidden in all. So x_i can be chosen. The flow Y is in $P(\Lambda)$; $Y(\varepsilon) \notin \Phi(\varepsilon)$ for every ε in E' that lies on an official cycle; and $Y(\varepsilon) = Y_0(\varepsilon) = y_\varepsilon \notin \Phi(\varepsilon)$ for ε in E'_0, because no official cycle contains ε. □

Lemma 3.5 (Walk lemma)**.** *Assume* (R1)*. Let* Λ *be a list assignment such that for every node* i

$$|\Lambda(e_i)| + |\Lambda(f_i)| \le 6 \qquad \textit{and} \qquad 2\,|\Lambda(e_i)| + |\Lambda(f_i)| < |A|.$$

Then any two Y*,* Y' *in* $P(\Lambda)$ *are joined by a sequence* $Y = Y_0, Y_1, \dots, Y_n = Y'$ *in* $P(\Lambda)$ *in which every difference* $Y_r - Y_{r-1}$ *is* $\sigma\chi_{C_l}$ *with* σ *in* Σ_A *and an official cycle* C_l*. In particular this holds for* $\Lambda = \Lambda_f$*, for every* H*-good flow* f *(Definition 2.14), if the frame has* (C1).

Proof. A *good sequence* is a sequence of A-flows of Q in which every difference of two consecutive terms is $\sigma\chi_{C_l}$ with σ in Σ_A and an official cycle C_l. For $i = 1, \dots, s+1$ let Λ_i be the list assignment that is equal to Λ on the edges of $N_i \cup \cdots \cup N_s$ and empty on all other blue edges. So $\Lambda_1 = \Lambda$ (iii), and $P(\Lambda_{s+1})$ is the set of all A-flows of Q. Claim(i): any two members of $P(\Lambda_i)$ are joined by a good sequence in $P(\Lambda_i)$. Claim(1) is the lemma.

(1) *Claim*$(s+1)$. Let Y, Y' be A-flows of Q. By (O5), $Y' - Y = x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ with x_i in A. By (R1) with X empty every x_i is a sum of elements of Σ_A. Add these terms one by one, cycle after cycle. This is a good sequence from Y to Y'.

(2) *Notation for the step from Claim$(i+1)$ to Claim(i)*, $1 \le i \le s$. Write $\chi = \chi_{C_i}$. By (O2), $\chi(e_i)$ and $\chi(f_i)$ are 1 or -1. For an A-flow Z of Q put

$$X_i^e(Z) = \{\, c \in A : Z(e_i) + c\,\chi(e_i) \in \Lambda(e_i) \,\}, \qquad X_i^f(Z) = \{\, c \in A : Z(f_i) + c\,\chi(f_i) \in \Lambda(f_i) \,\},$$

and $X_i(Z) = X_i^e(Z) \cup X_i^f(Z)$. Then $|X_i^e(Z)| = |\Lambda(e_i)|$ and $|X_i^f(Z)| = |\Lambda(f_i)|$; $X_i^e(Z)$ depends only on $Z(e_i)$, and $X_i^f(Z)$ only on $Z(f_i)$; and

$$|X_i(Z)| \;\le\; |\Lambda(e_i)| + |\Lambda(f_i)| \;\le\; 6.$$

By (O3) the cycle C_i contains no edge of $N_{i+1} \cup \cdots \cup N_s$. So for Z in $P(\Lambda_{i+1})$ and c in A:

$$(*) \qquad Z + c\chi \in P(\Lambda_{i+1}); \qquad \text{and} \qquad Z + c\chi \in P(\Lambda_i) \;\text{ if and only if }\; c \notin X_i(Z).$$

(3) *The step.* Assume Claim$(i+1)$, and let Y, Y' be in $P(\Lambda_i)$. They are in $P(\Lambda_{i+1})$. Take a good sequence $Y = Y_0, \dots, Y_n = Y'$ in $P(\Lambda_{i+1})$, with $Y_r - Y_{r-1} = \sigma_r \chi_{C_{l_r}}$. We build a good sequence in $P(\Lambda_i)$ that starts with Y and passes through flows $Y_r + c_r\chi$, $r = 0, 1, \dots, n$, with $c_0 = 0$ and $c_r \notin X_i(Y_r)$. For $r = 0$: $0 \notin X_i(Y_0)$ by $(*)$, because Y is in $P(\Lambda_i)$. Let $r \ge 1$, and let the sequence be built up to $Y_{r-1} + c_{r-1}\chi$.

(3a) $l_r = i$. Then $Y_{r-1} + c_{r-1}\chi = Y_r + (c_{r-1} - \sigma_r)\chi$. Put $c_r = c_{r-1} - \sigma_r$ and add no term. This flow is in $P(\Lambda_i)$, so $c_r \notin X_i(Y_r)$ by $(*)$. No count is used.

(3b) $l_r < i$. By (O3), C_{l_r} contains no edge of N_i. So $Y_r = Y_{r-1}$ on N_i and $X_i(Y_r) = X_i(Y_{r-1})$. Choose c_r outside this set. This is possible: $|\Lambda(e_i)| + |\Lambda(f_i)| \le 2\,|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$, by the second hypothesis of the lemma.

(3c) $l_r > i$. By (O4), C_{l_r} does not contain f_i. So $X_i^f(Y_r) = X_i^f(Y_{r-1})$, and

$$|X_i(Y_{r-1}) \cup X_i(Y_r)| \;\le\; |X_i^e(Y_{r-1})| + |X_i^e(Y_r)| + |X_i^f(Y_{r-1})| \;=\; 2\,|\Lambda(e_i)| + |\Lambda(f_i)| \;<\; |A|.$$

The last inequality is the second hypothesis of the lemma; for $\Lambda = \Lambda_f$ it is the count (C1), see (5). Choose c_r outside $X_i(Y_{r-1}) \cup X_i(Y_r)$.

In (3b) and (3c): c_{r-1} and c_r are outside $X_i(Y_{r-1})$, a set of at most 6 elements by the first hypothesis of the lemma. By (R1) there is a walk from c_{r-1} to c_r in $A \setminus X_i(Y_{r-1})$ with steps in Σ_A. Add the flows $Y_{r-1} + c\chi$ for the terms c of this walk after the first term. They are in $P(\Lambda_i)$ by $(*)$, and two consecutive ones differ by $\sigma\chi_{C_i}$ with σ in Σ_A. Then add $Y_r + c_r\chi$. It is in $P(\Lambda_i)$ by $(*)$, because $c_r \notin X_i(Y_r)$; it differs from the term before it, $Y_{r-1} + c_r\chi$, by $\sigma_r\chi_{C_{l_r}}$.

(4) *The end of the step.* The sequence has reached $Y_n + c_n\chi$ with $c_n \notin X_i(Y_n)$. Also $0 \notin X_i(Y_n)$ by $(*)$, because $Y_n = Y'$ is in $P(\Lambda_i)$. By (R1) there is a walk from c_n to 0 in $A \setminus X_i(Y_n)$ with steps in Σ_A. Add the flows $Y_n + c\chi$ for its terms after the first term. The last one is Y'. This proves Claim(i).

(5) *The case $\Lambda = \Lambda_f$.* $|\Lambda_f(t)| \le \ell(t) \le 3$ for every blue edge t (v), (i). So $|\Lambda_f(e_i)| + |\Lambda_f(f_i)| \le 6$, and $2\,|\Lambda_f(e_i)| + |\Lambda_f(f_i)| \le 2\,\ell(e_i) + \ell(f_i) < |A|$ by (C1). □

A change of the A-coordinate inside H. Lemma 3.8 is put together from three parts: the walk lemma, which gives a sequence of steps $\sigma\chi_{C_l}$; Lemma 3.7, which makes one such step in G; and Lemma 3.6, which prepares the free threads for the step. Lemmas 3.6 and 3.7 do not use (R1), (C1) or the walk lemma, and Lemma 3.7 holds for every cycle C of Q.

Lemma 3.6 (Preparation in B)**.** *Assume* (R2)*, and let the frame have* (C4)*. Let G be a multigraph that contains H, and f an H-good Γ-flow of G with A-coordinate u. Let E° be a set of free edges. For every ε in E° put*

$$W_\varepsilon = \{\, -f(e) : e \in E(P_\varepsilon),\ u(e) = 0 \,\},$$

and let T_ε be a set of at most $\ell(\varepsilon)$ elements of B that contains W_ε. Then f is Σ-joined, by Σ-moves $y\chi_{D_\varepsilon}$ with y in Σ_B and ε in E°, to an H-good flow f° with A-coordinate u such that for every ε in E°

$$f^\circ = f + y_\varepsilon \ \text{ on } E(P_\varepsilon), \qquad \text{with an element } y_\varepsilon \text{ of } B \text{ outside } T_\varepsilon.$$

f° is equal to f outside $E(H)$, and on every free thread P_ε with ε not in E°.

Proof. W_ε is a subset of B by (E1), and 0 is not in W_ε, because f is nowhere zero.

(1) *One free edge ε.* If 0 is not in T_ε, put $y_\varepsilon = 0$ and make no move. Let 0 be in T_ε. By (C4), $|T_\varepsilon| \le \ell(\varepsilon) < |B|$. So there is y_ε in B outside T_ε. 0 is in T_ε and not in W_ε, so $|W_\varepsilon| \le |T_\varepsilon| - 1 \le \ell(\varepsilon) - 1 \le 2$; the last step holds because a thread has at most 3 edges (i). By (R2) there is a walk $0 = z_0, z_1, \dots, z_\nu = y_\varepsilon$ in $B \setminus W_\varepsilon$ with steps in Σ_B. Make the moves $(z_r - z_{r-1})\chi_{D_\varepsilon}$, $r = 1, \dots, \nu$. After the r-th move the flow is $f + z_r\chi_{D_\varepsilon}$. Its A-coordinate is u (E2). It is nowhere zero. Indeed, D_ε is a union of threads (Lemma 2.7 (a)); it contains P_ε and no other free thread (iv); so its edges outside P_ε lie on protected threads. An edge e of a protected thread has $u(e) \neq 0$, because f is H-good. An edge e of P_ε with $u(e) = 0$ has the value $f(e) + z_r$, because D_ε is directed along P_ε; this is not 0, because z_r is not in W_ε. The other edges of P_ε have $u(e) \neq 0$. The edges outside D_ε do not change. With (E1): every move is a Σ-move. The result $f + y_\varepsilon\chi_{D_\varepsilon}$ is H-good, it is equal to $f + y_\varepsilon$ on $E(P_\varepsilon)$, and it is equal to f outside $E(D_\varepsilon)$: on every other free thread and outside $E(H)$.

(2) *All edges of E°, one after the other.* Step (1) for ε does not change the A-coordinate, and it changes no value on a free thread other than P_ε. So when the turn of ε comes, the flow at hand is H-good with A-coordinate u, it is still equal to f on P_ε, its set W_ε is the set W_ε of the statement, and Step (1) can be made with the same T_ε. Later steps do not change the values on P_ε. □

Lemma 3.7 (One step inside H)**.** *Assume* (R2)*, and let the frame have* (C4)*. Let G be a multigraph that contains H, and f an H-good Γ-flow of G with A-coordinate u. Let C be a cycle of Q with a direction and σ an element of Σ_A such that $u' = u + \pi(\sigma\chi_C)$ is non-zero on every edge of every protected thread. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate u' that is equal to f outside $E(H)$.*

Proof. Put $\gamma = \gamma(\sigma)$; so γ is in Σ and $\overline{\gamma} = \sigma$. By Lemma 2.7 (d) and (a),

$$u' = u + \sigma\,(\chi_{O_1} + \cdots + \chi_{O_\nu}) \quad \text{on } E(H),$$

with pairwise edge-disjoint cycles $O_1, \dots, O_\nu$ of H; every O_j is a union of threads, and χ_{O_j} is constant, 1 or -1, on each of its threads. A thread lies on at most one O_j. For a free edge ε put

$$\delta_\varepsilon = \gamma \cdot (\text{the value of } \chi_{O_j} \text{ on } P_\varepsilon) \ \text{ if } P_\varepsilon \text{ lies on } O_j, \qquad \delta_\varepsilon = 0 \ \text{ if } P_\varepsilon \text{ lies on no } O_j.$$

So δ_ε is an element of Γ with

$$(**) \qquad \overline{\delta_\varepsilon} = u'(e) - u(e) \quad \text{for every edge } e \text{ of } P_\varepsilon.$$

(i) *Preparation in B.* For every free edge ε let W_ε be as in Lemma 3.6, and

$$T_\varepsilon = W_\varepsilon \cup \{\, -f(e) - \delta_\varepsilon : e \in E(P_\varepsilon),\ u'(e) = 0 \,\}.$$

T_ε is a subset of B: if $u'(e) = 0$, then $\overline{f(e) + \delta_\varepsilon} = u(e) + u'(e) - u(e) = 0$ by $(**)$. Every edge e of P_ε gives at most one element of T_ε. Indeed, let $u(e) = 0$ and $u'(e) = 0$. Then $\overline{\delta_\varepsilon} = 0$ by $(**)$. If P_ε were on some O_j, then $\overline{\delta_\varepsilon}$ would be σ or $-\sigma$, which is not 0. So P_ε lies on no O_j, $\delta_\varepsilon = 0$, and the two elements $-f(e)$ and $-f(e) - \delta_\varepsilon$ are equal. Hence $|T_\varepsilon| \le \ell(\varepsilon)$. Lemma 3.6, with E° the set of all free edges, gives an H-good flow f° with A-coordinate u, equal to f outside $E(H)$, with $f^\circ = f + y_\varepsilon$ on $E(P_\varepsilon)$ and $y_\varepsilon \notin T_\varepsilon$. So for every edge e of a free thread P_ε:

$$\text{(T)} \qquad \text{if } u'(e) = 0, \quad \text{then } f^\circ(e) + \delta_\varepsilon \neq 0.$$

($-f(e) - \delta_\varepsilon$ is in T_ε, so it is not y_ε.)

(ii) *The step.* Make the moves $\gamma\chi_{O_1}, \dots, \gamma\chi_{O_\nu}$, in this order. After the first j of them an edge e of H carries the value $f^\circ(e) + \gamma\chi_O(e)$, with the A-coordinate $u(e) + \sigma\chi_O(e) = u'(e)$, if e lies on a cycle O among $O_1, \dots, O_j$; and it carries the value $f^\circ(e)$, with the A-coordinate $u(e)$, if not. These values are not 0:

- on a protected thread, $u(e) \neq 0$ because f is H-good, and $u'(e) \neq 0$ by hypothesis; use (E1);
- on a free thread P_ε that lies on one of $O_1, \dots, O_j$, the value is $f^\circ(e) + \delta_\varepsilon$; it is not 0 by (E1) if $u'(e) \neq 0$, and by (T) if $u'(e) = 0$;
- on every other free thread the value is $f^\circ(e)$, and f° is nowhere zero.

The edges outside $E(H)$ do not change. So all these moves are Σ-moves on cycles of H. After the last one every edge e of H has the A-coordinate $u'(e)$; for an edge on no O_j note that $u'(e) = u(e)$. The result is nowhere zero and its A-coordinate u' is non-zero on the protected threads: it is H-good. It is equal to f outside $E(H)$. □

Lemma 3.8 (A change of the A-coordinate inside H)**.** *Assume* (R1) *and* (R2)*, and let the frame have* (C1) *and* (C4)*. Let G be a multigraph that contains H, f an H-good Γ-flow of G with A-coordinate u, and Y a flow in $P(\Lambda_f)$. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate $u + \pi(Y)$ that is equal to f outside $E(H)$.*

Proof. 0 and Y are in $P(\Lambda_f)$ (v). The frame has (C1), so the walk lemma (Lemma 3.5, its last sentence) gives a sequence $0 = Y_0, Y_1, \dots, Y_n = Y$ in $P(\Lambda_f)$ with $Y_r - Y_{r-1} = \sigma_r\chi_{C_{l_r}}$, σ_r in Σ_A. We build H-good flows $f = f_0, f_1, \dots, f_n$ of G, each Σ-joined to the one before it by Σ-moves on cycles of H and equal to f outside $E(H)$, such that f_r has the A-coordinate $u_r = u + \pi(Y_r)$. Let f_{r-1} be built. Then

$$u_{r-1} + \pi(\sigma_r\chi_{C_{l_r}}) = u + \pi(Y_{r-1}) + \pi(Y_r - Y_{r-1}) = u_r,$$

and u_r is non-zero on every edge of every protected thread, because Y_r is in $P(\Lambda_f)$ (v). So Lemma 3.7, for the flow f_{r-1}, the cycle C_{l_r} and σ_r, gives f_r; its hypotheses (R2) and (C4) are assumed. The flow f_n has the A-coordinate $u + \pi(Y)$. □

In the rest of this section we write (L1) for the conclusion of Lemma 3.8, for a given subdivision H with a given frame: *for every multigraph G that contains H, every H-good Γ-flow f of G with A-coordinate u, and every Y in $P(\Lambda_f)$, the flow f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate $u + \pi(Y)$ that is equal to f outside $E(H)$.* By Lemma 3.8, (L1) holds if (R1) and (R2) hold and the frame has (C1) and (C4).

The path.

Lemma 3.9 (The path)**.** *Let $p \neq p'$ be vertices of H. There is a path J of H from p to p' such that, for every node i, at most $\max\{\ell(e_i), \ell(f_i)\}$ edges of J lie on the threads of the edges of N_i (both threads together; so at most 3). J can contain any number of free threads.*

Proof. An expansion edge has no thread. So "the threads of the edges of N_i" are the threads of the original edges of N_i; there are two, one or none of them.

(1) *The subgraph H_g.* Let $g = (g_1, \dots, g_s)$ be a choice of one edge g_i in every pair N_i. Let H_g be the subgraph of H formed by the vertices of H_0 and the threads of all original edges of the trees T^g of all components Q_b. Then:

   (1a) H_g is connected. Reason: T^g is a spanning tree of Q_b (iv). Under the contraction of the expansion edges (ii) it becomes a connected subgraph of K_b that contains all vertices of K_b; its edges are the original edges of T^g. Any two vertices of H_0 are joined by a path of H_0; the two ends of an edge of this path lie in one block K_b (i), and they are joined by original edges of T^g. So the original edges of the trees T^g form a connected subgraph of H_0 on all vertices of H_0. Replace these edges by their threads.

   (1b) A vertex of H is outside H_g only if it is an interior vertex of the thread of an original edge that is in some pair N_i and is not g_i. Reason: T^g contains all red edges and all edges g_i, and every blue edge is in a pair (iii).

   (1c) A path of H_g has, on the threads of the edges of N_i, only edges of the thread of g_i: at most $\ell(g_i) \leq \max\{\ell(e_i), \ell(f_i)\}$ edges. Reason: g_i is the only edge of N_i in T^g (iv).

(2) *Case 1: p and p' are interior vertices of the same thread P_t.* Let J be the part of P_t between them. A thread has at most 3 edges (i), so at most 2 interior vertices; hence J has one edge. If t is in a pair N_i, then J has $1 \leq \ell(t) \leq \max\{\ell(e_i), \ell(f_i)\}$ edge on the threads of N_i. J has no edge on the threads of the other pairs.

(3) *Case 2: not Case 1, and there is no pair N_i such that p is an interior vertex of the thread of one edge of N_i and p' an interior vertex of the thread of the other edge.* A vertex is an interior vertex of at most one thread. So, by the case, every pair N_i has at most one edge whose thread has p or p' as an interior vertex. If there is such an edge, let it be g_i; if not, put $g_i = e_i$. By (1b), p and p' are in H_g. By (1a) there is a path J of H_g from p to p'. By (1c) it is as stated.

(4) *Case 3: p is an interior vertex of P_t and p' an interior vertex of $P_{t'}$, where $t \neq t'$ are the two edges of one pair N_i.* Let U', U'' be the two operands of the node i. By (iii), t has an end x_1 in $V_{U'}$ and an end in $V_{U''}$, and t' has an end x_2 in $V_{U'}$ and an end in $V_{U''}$. So each of the threads P_t, $P_{t'}$ has one end *on the side of* U' (the vertex of H_0 that comes from x_1, from x_2) and one end on the side of U''. Let c be the number of edges of P_t between p and the end on the side of U', and c' the same for $P_{t'}$ and p'. The numbers of edges to the ends on the side of U'' are $\ell(t) - c$ and $\ell(t') - c'$. The two sums $c + c'$ and $(\ell(t) - c) + (\ell(t') - c')$ add up to $\ell(t) + \ell(t')$. So one of them is at most $(\ell(t) + \ell(t'))/2 \leq \max\{\ell(t), \ell(t')\}$. Let it be $c + c'$; the other case is the same with U' and U'' exchanged.

   By (iv) the edges of T^e in $Q_{U'}$ form a spanning tree of $Q_{U'}$. It contains a path R from x_1 to x_2. The edges t, t' are not edges of $Q_{U'}$: each has an end outside $V_{U'}$. So R contains

neither t nor t', and every blue edge of R is an edge e_l with $l \neq i$. Contract the expansion edges and replace the original edges of R by their threads. The result L is a connected subgraph of H (one vertex, if R has no original edge). It contains the ends of P_t and of $P_{t'}$ on the side of U'. It has no edge on P_t or $P_{t'}$. For a pair N_l with $l \neq i$ it has edges on the thread of e_l only.

Let J be a path from p to p' in the union of L, the part of P_t from p to its end on the side of U', and the part of $P_{t'}$ from p' to its end on the side of U'. This union is connected, so J exists. On the threads of N_i the path J has at most $c + c' \leq \max\{\ell(e_i), \ell(f_i)\}$ edges. On the threads of a pair N_l, $l \neq i$, it has at most $\ell(e_l) \leq \max\{\ell(e_l), \ell(f_l)\}$ edges.

The three cases cover all pairs $p \neq p'$. The bound "at most 3" holds because $\ell(t) \leq 3$ (i). □

The lift of one move of the contracted graph. Now H is a cycle or a subdivision. If H is a cycle of a multigraph G, a nowhere-zero Γ-flow f of G is *H-good* if $\overline{f}(e) \neq 0$ for every edge e of H (Definition 2.14).

Lemma 3.10 (The lift of one move of the contracted graph)**.** *Assume* $|A| \geq 8$, (R1) *and* (R2)*. Let G be a bridgeless multigraph and H a subgraph of G that is a cycle with at most* 5 *edges (case* CYC*), or a subdivision with a counted frame (case* SUB*). Let $G^* = G/H$ be the multigraph obtained from G by the identification of $V(H)$ to one vertex and the deletion of $E(H)$. Let f be an H-good Γ-flow of G with restriction f^* to G^*, and let $g^* = f^* + \lambda\chi_{C^*}$ be the result of a Σ-move of G^*. Then f is Σ-joined to an H-good Γ-flow g of G whose restriction to G^* is g^*.*

(*The frame form: the same holds with* (R1$^-$) *in the place of* (R1) *and, in the place of* (C1)*, the conclusion of Lemma* 3.8 *for this H and all f, Y.*)

Proof. We prove the frame form. It contains the first form: (R1) implies (R1$^-$), and for a subdivision with a counted frame (L1) holds by Lemma 3.8. So we assume: $|A| \geq 8$; (R1$^-$); (R2); and in the case SUB the frame has (C2), (C3), (C4), and (L1) holds. Let u be the A-coordinate of f.

(1) *The contracted graph.* $E(G^*) = E(G) \setminus E(H)$. Let v^* be the vertex of G^* made from $V(H)$. By (E5) the restriction of a flow of G to $E(G^*)$ is a flow of G^*. C^* is a cycle of G^* with a direction, λ is in Σ, and f^* and g^* are nowhere zero.

(2) *The cycle C.* If C^* does not contain v^*, its edges form a cycle of G. If C^* contains v^*, its edges form in G a path between two vertices p, p' of H whose other vertices are outside $V(H)$, or, if $p = p'$, a cycle of G. (C^* is a loop at v^*, or it has exactly two edges at v^*.)

If the edges of C^* form a cycle of G, direct it as C^* and put $g = f + \lambda\chi_{C^*}$. This flow is equal to g^* on $E(G^*)$ and to f on $E(H)$. So it is nowhere zero, the move is a Σ-move, g is H-good, and its restriction is g^*.

From now on the edges of C^* form a path between $p \neq p'$. For every path J of H from p to p' the union $C = C^* \cup J$ is a cycle of G, because all vertices of J are in $V(H)$. Direct C so that $\chi_C = \chi_{C^*}$ on $E(G^*)$. The path J is chosen below. In each case we reach an H-good flow f' that is equal to f outside $E(H)$, and then make the move $g = f' + \lambda\chi_C$. On $E(G^*)$ the flow g is equal to $f^* + \lambda\chi_{C^*} = g^*$; so it is not 0 there, and the restriction of g is g^*. It remains to look at the edges of H.

(3) *Case* CYC. Direct the cycle H. Let J be an arc of H between p and p' with at most 2 edges; it exists because H has at most 5 edges.

$\overline{\lambda} = 0$. Put $f' = f$. By (E2) the move $\lambda\chi_C$ does not change the A-coordinate, which is non-zero on $E(H)$. So g is nowhere zero (E1) and H-good.

$\overline{\lambda} \neq 0$. Put

$$X = \{\, c \in A : u + c\chi_H \text{ has a zero on } E(H) \,\},$$
$$X' = \{\, c \in A : u + c\chi_H + \overline{\lambda}\chi_C \text{ has a zero on } E(J) \,\}.$$

Every edge of H excludes one value of c, because χ_H is 1 or -1 on it. So $|X| \leq 5$, $|X'| \leq 2$, and 0 is not in X, because f is H-good. By the hypothesis $|A| \geq 8$ (it is (C0) of Theorem 3.11), $|X \cup X'| \leq 5 + 2 < |A|$; choose c outside $X \cup X'$. By ($R1^-$) there is a walk $0 = c_0, c_1, \ldots, c_\nu = c$ in $A \setminus X$ with steps $d_r = c_r - c_{r-1}$ in Σ_A. Make the moves $\gamma(d_r)\chi_H$, $r = 1, \ldots, \nu$. After the r-th move the A-coordinate is $u + c_r\chi_H$; it is non-zero on $E(H)$, because c_r is not in X; outside $E(H)$ nothing changes. So these are Σ-moves (E1). The result f' is H-good and equal to f outside $E(H)$. Now $g = f' + \lambda\chi_C$ has, on $E(H)$, the A-coordinate $u + c\chi_H$ outside J and $u + c\chi_H + \overline{\lambda}\chi_C$ on J. Both are non-zero, because c is not in X and not in X'. So g is nowhere zero and H-good.

(4) *Case* SUB, $\overline{\lambda} \neq 0$. Let J be the path of Lemma 3.9, and let j_i be the number of edges of J on the threads of the edges of N_i. So $j_i \leq \max\{\ell(e_i), \ell(f_i)\}$.

(4a) *The protected threads.* For an original blue edge t put

$$\Lambda_1(t) = \Lambda_f(t) \cup \{\, -u(e) - \overline{\lambda}\chi_C(e) : e \in E(P_t) \cap E(J) \,\},$$

and let Λ_1 be empty on the expansion edges. Then, for every node i,

$$|\Lambda_1(e_i)| + |\Lambda_1(f_i)| \;\leq\; \ell(e_i) + \ell(f_i) + j_i \;\leq\; \ell(e_i) + \ell(f_i) + \max\{\ell(e_i), \ell(f_i)\} \;<\; |A|$$

by (v) and (C2). By Lemma 3.4 (a), with $Y_0 = 0$, there is Y in $P(\Lambda_1)$. It is in $P(\Lambda_f)$, because Λ_1 contains Λ_f. By (L1), f is Σ-joined to an H-good flow f_1 with A-coordinate $u_1 = u + \pi(Y)$ that is equal to f outside $E(H)$. For an edge e of J on a protected thread P_t: $u_1(e) + \overline{\lambda}\chi_C(e) = u(e) + Y(t) + \overline{\lambda}\chi_C(e) \neq 0$, because $Y(t)$ is not in $\Lambda_1(t)$.

(4b) *The free threads.* Let E° be the set of the free edges ε such that P_ε has an edge on J. Their number is not bounded. For ε in E° put

$$W_\varepsilon = \{\, -f_1(e) : e \in E(P_\varepsilon),\ u_1(e) = 0 \,\},$$
$$T_\varepsilon = W_\varepsilon \cup \{\, -f_1(e) - \lambda\chi_C(e) : e \in E(P_\varepsilon) \cap E(J),\ u_1(e) + \overline{\lambda}\chi_C(e) = 0 \,\}.$$

T_ε is a subset of B: the A-coordinate of $f_1(e) + \lambda\chi_C(e)$ is $u_1(e) + \overline{\lambda}\chi_C(e)$. Every edge e of P_ε gives at most one element of T_ε: for an edge of J, $\chi_C(e)$ is 1 or -1 and $\overline{\lambda} \neq 0$, so $u_1(e) = 0$ and $u_1(e) + \overline{\lambda}\chi_C(e) = 0$ exclude each other. Hence $|T_\varepsilon| \leq \ell(\varepsilon)$. By Lemma 3.6 (it uses (R2) and (C4)), f_1 is Σ-joined to an H-good flow f_2 with A-coordinate u_1, equal to f_1 outside $E(H)$, with $f_2 = f_1 + y_\varepsilon$ on $E(P_\varepsilon)$ and $y_\varepsilon \notin T_\varepsilon$, for every ε in E°.

(4c) *The move.* Put $f' = f_2$ and $g = f_2 + \lambda\chi_C$. On an edge e of J the new A-coordinate is $u_1(e) + \overline{\lambda}\chi_C(e)$. If e is on a protected thread, this is not 0 by (4a). If e is on a free thread P_ε and this is 0, then $-f_1(e) - \lambda\chi_C(e)$ is in T_ε, so it is not y_ε, and $g(e) = f_1(e) + y_\varepsilon + \lambda\chi_C(e) \neq 0$. The edges of H outside J do not change. So g is nowhere zero (E1), the move is a Σ-move, and g is H-good: on the protected threads its A-coordinate is u_1 outside J and $u_1 + \overline{\lambda}\chi_C$ on J.

(5) *Case* SUB, $\overline{\lambda} = 0$. So λ is in Σ_B.

(5a) *The path J.* A thread has at most 3 edges (i). So every interior vertex of a thread is next to an end of the thread, and two interior vertices of one thread are next to each other. If p and p' are interior vertices of the same free thread, J is the edge of this thread between them. Otherwise: if p is an interior vertex of a free thread, let q be an end of this thread that is next to p, and let J start with the edge from p to q; if not, put $q = p$. In the same way p' gives q', and J ends with the edge from q' to p' if $q' \neq p'$. By (O6), q and q' are in H_P, and H_P is connected; between q and q' the path J runs inside H_P. (An interior vertex of a free thread is not in H_P; so J is a path.) A *free piece* of J is the part of J on one free thread. Hence: J has at most two free pieces; they lie on different free threads; *each has exactly one edge*; and all other edges of J are on protected threads.

(5b) *The A-coordinate on J.* Let E' be the set of the free edges ε such that P_ε has a free piece of J; so $|E'| \leq 2$. For ε in E' let e_ε be the edge of this free piece, and $\Phi(\varepsilon) = \{-u(e_\varepsilon)\}$: one element, not all of A by the hypothesis $|A| \geq 8$ ((C0)). For every node i,

$$|\Lambda_f(e_i)| + |\Lambda_f(f_i)| + \sum_{\varepsilon \in E',\ i(\varepsilon)=i} |\Phi(\varepsilon)| \ \leq\ \ell(e_i) + \ell(f_i) + 2 \ <\ |A|$$

by (v) and (C3). By Lemma 3.4 (b) there is Y in $P(\Lambda_f)$ with $Y(\varepsilon) \neq -u(e_\varepsilon)$ for every ε in E'. By (L1), f is Σ-joined to an H-good flow f_1 with A-coordinate $u_1 = u + \pi(Y)$ that is equal to f outside $E(H)$. u_1 is non-zero on the protected threads, and on a free piece $u_1(e_\varepsilon) = u(e_\varepsilon) + Y(\varepsilon) \neq 0$. So u_1 is non-zero on every edge of J.

(5c) *The move.* Put $f' = f_1$ and $g = f_1 + \lambda\chi_C$. By (E2) the A-coordinate does not change; it is non-zero on J. So g is nowhere zero (E1) and H-good. The group B and the condition (R2) are not used in this case, except inside (L1). □

The master theorem.

Theorem 3.11 (The master theorem)**.** *Let Γ be a finite abelian group, B a subgroup, $A = \Gamma/B$, and Σ a step system. Assume:*

(H1) *the step conditions* (R1), (R2), (R3*), *and the averaging condition* (AV″)*;*

(H2) *the counts:*

(C0) $|A| \geq 8$*;*

and in (AV″) *the good subgraph can always be chosen to be a cycle with at most* 5 *edges, or a subdivision that has a counted frame, that is a frame with* (C1), (C2), (C3), (C4).

Then on every connected bridgeless multigraph G (loops allowed) every nowhere-zero Γ-flow is Σ-joined to every Γ-flow whose A-coordinate is nowhere zero. Hence any two nowhere-zero Γ-flows of a bridgeless multigraph are Σ-joined, and $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G.

Theorem 3.12 (The master theorem, frame form)**.** *The conclusion of Theorem* 3.11 *holds also if, in* (H1), (R1) *is replaced by* (R1⁻), *and, in* (H2), (C1) *is replaced by*

(L1) *the conclusion of Lemma* 3.8 *holds for the subdivisions* H *and the frames of* (H2)*, for every multigraph* G *that contains* H*.*

((R1) *for six deleted elements and* (C1) *are used only in the walk lemma, and the walk lemma is used only in Lemma* 3.8.)

Proof of Theorems 3.11 and 3.12. The hypotheses of Theorem 3.11 imply those of Theorem 3.12: (R1) implies (R1$^-$), and (L1) holds by Lemma 3.8, which uses (R1), (R2), (C1) and (C4). So let the hypotheses of the frame form hold: (R1$^-$), (R2), (R3*), (AV$''$), (C0), and the good subgraph in (AV$''$) can be chosen *admissible*: a cycle with at most 5 edges, or a subdivision with a frame that has (C2), (C3), (C4) and (L1). For an admissible H Lemma 3.10 holds in its frame form.

A *target* of a multigraph is a Γ-flow whose A-coordinate is nowhere zero. A target is nowhere zero (E1).

Claim. On every connected bridgeless multigraph G, every nowhere-zero Γ-flow is Σ-joined to every target.

(1) *The claim gives the theorem.* The first statement is the claim. Let G be a bridgeless multigraph. It has a nowhere-zero A-flow, because $|A| \geq 6$ by (C0) (Theorem 0.1, the theorems of Seymour and of Tutte: every bridgeless multigraph, loops allowed, has a nowhere-zero flow with values in any abelian group of order at least 6). By (E4) it is the A-coordinate of a Γ-flow h: a target. A cycle of G lies in one component of G, and a Σ-move of a component is a Σ-move of G. So, component after component, the claim joins every nowhere-zero Γ-flow of G to h. The reverse of a Σ-move is a Σ-move; so any two nowhere-zero Γ-flows are Σ-joined through h. Σ-moves are moves of $F(G, \Gamma)$; so $F(G, \Gamma)$ is connected.

(2) *The induction.* We prove the claim by induction on the number n of vertices of G; for each n first for the loopless graphs, then for the graphs with loops. A connected loopless multigraph with one vertex has no edge and only the empty flow. A connected bridgeless loopless multigraph with $n \geq 2$ has a cut with exactly 2 edges or is 3-edge-connected. So three cases remain: (a), (b), (c). In all of them f is a nowhere-zero Γ-flow and h a target of G.

(a) *G has a loop.* Let G^- be G without its loops. It is connected, bridgeless and loopless, and it has n vertices; so the claim holds for G^- (the loopless graphs with n vertices come first). The restrictions of f and h to G^- are a nowhere-zero flow and a target. A Σ-move of G^- is a Σ-move of G: the values on the loops do not change. So f is Σ-joined to the flow that is equal to h on $E(G^-)$ and to f on the loops. Now let e be a loop, with the values $x = f(e)$ and $x' = h(e)$. Both are non-zero; x' because $\overline{x'} \neq 0$. By (R3*) there is a walk from x to x' in $\Gamma \setminus \{0\}$ with steps in Σ. A loop is a cycle with one edge. The moves $\lambda\chi_e$, for the steps λ of the walk, are Σ-moves: the value on e is never 0, and no other edge changes. Do this for every loop. The result is h.

(b) *G is loopless and has a cut with exactly two edges e_1, e_2.* Let G/e_1 be obtained from G by the identification of the two ends of e_1 and the deletion of e_1. It is connected, it has $n-1$ vertices, and it is bridgeless (a cycle of G through an edge $b \neq e_1$ becomes a closed trail of G/e_1 through b, and a closed trail through b contains a cycle through b); it can have loops. By induction the claim holds for G/e_1. By (E5) the restriction of a flow of G to $E(G/e_1)$ is a flow of G/e_1. Every flow of G/e_1 is the restriction of exactly one flow of G: the value on e_1 is fixed by conservation at one end of e_1, and conservation at the

other end then follows from conservation at the vertex of G/e_1 made from the two ends. In a flow of G the value on e_1 is the value on e_2 or its negative (conservation over one side of the cut). So a flow of G is nowhere zero if and only if its restriction is, and it is a target if and only if its restriction is. Let $g' \to g' + \lambda\chi_{C'}$ be a Σ-move of G/e_1. The edges of C' form a cycle of G, or a path of G between the two ends of e_1; in the second case add e_1. This gives a cycle C of G with $\chi_C = \chi_{C'}$ on $E(G/e_1)$. If g is the flow of G with restriction g', then $g + \lambda\chi_C$ is the flow of G with restriction $g' + \lambda\chi_{C'}$. Both are nowhere zero. So every Σ-move of G/e_1 lifts to a Σ-move of G, and the claim for G/e_1 gives the claim for G.

(c) *G is loopless and 3-edge-connected, $n \geq 2$.* By (AV″) and (H2), f is Σ-joined to a nowhere-zero flow f' with A-coordinate u' such that $(V(G), \mathrm{supp}(u'))$ contains an admissible good subgraph H. If H is a subdivision, fix a frame of H as in (H2). u' is non-zero on $E(H)$, so f' is H-good. H has at least 2 vertices (Definition 2.1). So $G^* = G/H$ has fewer vertices than G. It is connected. It is bridgeless: the edges outside $E(H)$ of a cycle of G through an edge b of G^* form a closed trail of G^* through b, and it contains a cycle through b. It can have loops. By induction the claim holds for G^*. The restriction of f' to G^* is nowhere zero, and the restriction h^* of h is a target of G^*. So there are Σ-moves of G^* that lead from the restriction of f' through flows $g_0, g_1, \ldots, g_\nu = h^*$. Lemma 3.10, used ν times, gives H-good flows $f' = f_0, f_1, \ldots, f_\nu$ of G, each Σ-joined to the one before it, such that the restriction of f_r to G^* is g_r. So f_ν is H-good and equal to h outside $E(H)$. Let u_ν be its A-coordinate.

End, case CYC. Direct the cycle H. The A-flow $\overline{h} - u_\nu$ is 0 outside $E(H)$, so it is $c\chi_H$ with c in A. The set $X = \{\, c' \in A : u_\nu + c'\chi_H$ has a zero on $E(H)\,\}$ has at most 5 elements. 0 is not in X, because f_ν is H-good; c is not in X, because $\overline{h}$ is nowhere zero. By (R1$^-$) there is a walk from 0 to c in $A \setminus X$ with steps d_r in Σ_A. The moves $\gamma(d_r)\chi_H$ are Σ-moves: after each of them the A-coordinate is non-zero on $E(H)$, and nothing changes outside $E(H)$. The result f'' has the A-coordinate $\overline{h}$ and is equal to h outside $E(H)$. So $h - f'' = y\chi_H$ with $\overline{y} = 0$, that is with y in B. By (R2) with X empty, y is a sum of elements of Σ_B. The moves on H with these elements are Σ-moves: the A-coordinate stays $\overline{h}$ (E2), which is nowhere zero (E1). The result is h.

End, case SUB. The A-flow $\overline{h} - u_\nu$ is 0 outside $E(H)$, so it is an A-flow of H. By Lemma 2.7 (c) there is an A-flow Y of Q with $\pi(Y) = \overline{h} - u_\nu$. Y is in $P(\Lambda_{f_\nu})$, because $u_\nu + \pi(Y) = \overline{h}$ has no zero (v). By (L1), f_ν is Σ-joined to a flow f'' with A-coordinate $\overline{h}$ that is equal to f_ν, hence to h, outside $E(H)$. So $h - f''$ is a flow of H with values in B. By (E3) it is a sum of flows $y\chi_O$ with O a cycle of H and y in B, and by (R2) with X empty every y is a sum of elements of Σ_B. Make these moves one by one. They are Σ-moves: the A-coordinate stays $\overline{h}$ (E2), which is nowhere zero (E1). The result is h. □

Remark 3.13 (Where the deletions are used)**.** (R1): the walk lemma deletes at most $\ell(e_i) + \ell(f_i) \leq 6$ elements of A; the case CYC of Lemma 3.10 and the end of the proof of Theorem 3.11 in the case CYC delete at most 5. (R2): the preparation in B (Lemma 3.8, and Lemma 3.10 for a step with non-zero A-part) deletes at most $\ell(\varepsilon) - 1 \leq 2$ elements of B; the end of the proof uses only that Σ_B generates B. (R3*) is used for loops only. (C0) is the count $5 + 2 < |A|$ of the case CYC; the target flow needs $|A| \geq 6$.

Remark 3.14 (Where the counts are used)**.** The proofs of this section use the order of A, the order of B and the numbers $\ell(t)$ at the following places, and at no other place. j_i is the number of edges of the path J on the threads of the edges of N_i.

(C0) Lemma 3.10, case CYC: at most $5+2$ values of c are excluded. Case SUB, $\overline{\lambda}=0$: the set $\Phi(\varepsilon)$ of one element is not all of A. Theorem 3.11: the target needs $|A|\geq 6$.

(C1) The walk lemma: an offset c_r outside a set of at most $2\,|\Lambda_f(e_i)|+|\Lambda_f(f_i)|\leq 2\,\ell(e_i)+\ell(f_i)$ elements. It is used only through Lemma 3.8.

(C2) Lemma 3.10, case SUB, $\overline{\lambda}\neq 0$: the reverse choice for lists with $\ell(e_i)+\ell(f_i)+j_i$ elements at the node i, and $j_i\leq \max\{\ell(e_i),\ell(f_i)\}$ by Lemma 3.9.

(C3) Lemma 3.10, case SUB, $\overline{\lambda}=0$: the reverse choice for lists with $\ell(e_i)+\ell(f_i)$ elements and at most two free edges with one excluded value each.

(C4) Lemma 3.6: an element of B outside a set of at most $\ell(\varepsilon)$ elements. It is used in Lemma 3.7, hence in Lemma 3.8, and in Lemma 3.10, case SUB, $\overline{\lambda}\neq 0$.

The bound $\ell(t)\leq 3$ is used for: at most 6 deleted elements in the walk lemma; at most 2 deleted elements in Lemma 3.6; the free pieces with one edge in Lemma 3.10, case SUB, $\overline{\lambda}=0$; Case 1 of Lemma 3.9. The frame form uses (C0), (C2), (C3), (C4) and, in the place of (C1), the hypothesis (L1).

Three corollaries. Each corollary is Theorem 3.11 together with a check of (H2). By Remark 3.14 this check is a check of the five inequalities (C0)–(C4).

Corollary 3.15 (No condition on the colouring)**.** *Let $|A|\geq 10$ and $|B|\geq 4$, and let Σ be a step system with* (R1)*,* (R2)*,* (R3*) *and* (AV$''$)*. Then the conclusion of Theorem* 3.11 *holds.*

Proof. (H1) is assumed. (C0): $|A|\geq 10\geq 8$. Let H be a subdivision that (AV$''$) gives. It has a frame (Definition 2.9: every truncated cubic 3-edge-connected graph has a well colouring, Lemma A.5). In every frame $\ell(t)\leq 3$ for all t. So the left sides of (C1), (C2), (C3) are at most $2\cdot 3+3=9$, $3+3+3=9$ and $3+3+2=8$; all are below $10\leq|A|$. The left side of (C4) is at most $3<4\leq|B|$. So every frame is counted, and (H2) holds. □

Corollary 3.16 (Quotient of order 9, condition (S))**.** *Let $|A|\geq 9$ and $|B|\geq 4$, and let Σ be a step system with* (R1)*,* (R2)*,* (R3*) *and with* (AV$''$) *in this form: the good subgraph can always be chosen to be a cycle with at most* 5 *edges, or a subdivision that has a frame with* (S)*. Then the conclusion of Theorem* 3.11 *holds.*

Proof. (H1) is assumed. (C0): $|A|\geq 9\geq 8$. Let H be a subdivision with a frame that has (S). Exchange the names at the nodes so that $\ell(e_i)\leq\ell(f_i)$ for every node i; the result is a frame (iii), and (S) does not depend on the names. If $\ell(e_i)=3$, then $\ell(f_i)=3$, and N_i consists of two original edges whose threads have 3 edges; (S) excludes this. So $\ell(e_i)\leq 2$ and $\ell(f_i)\leq 3$ for every i. The left sides: (C1) at most $2\cdot 2+3=7$; (C2) at most $2+3+3=8$; (C3) at most $2+3+2=7$; all are below $9\leq|A|$. (C4): at most $3<4\leq|B|$. So this frame is counted, and (H2) holds. □

Corollary 3.17 (Subgroup of order 3, condition (NR))**.** *Let $|A|\geq 10$ and $|B|=3$, and let Σ be a step system with* (R1)*,* (R2)*,* (R3*) *and with* (AV$''$) *in this form: the good subgraph can always be chosen to be a cycle with at most* 5 *edges, or a subdivision that has a frame with* (NR)*. Then the conclusion of Theorem* 3.11 *holds.*

Proof. (H1) is assumed. (C0): $|A|\geq 10\geq 8$. Let H be a subdivision with a frame that has (NR). The left sides of (C1), (C2), (C3) are at most 9, 9, 8, as in the proof of Corollary 3.15;

all are below $10 \le |A|$. By (NR) every free thread has at most 2 edges; so the left side of (C4) is at most $2 < 3 = |B|$. So this frame is counted, and (H2) holds. □

Remark 3.18. Corollaries 3.16 and 3.17 are conditional: no statement of this paper says that a frame with (S), or a frame with (NR), exists for every subdivision that the averaging step gives, so as they stand they give no new modulus and no new group.

The integer form.

Theorem 3.19 (The integer form)**.** *Let $k = ab$ with integers $a, b \ge 2$, $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (the multiples of a; b elements) and $A = \mathbb{Z}_a$ (reduction modulo a). Let Σ be a step system all of whose elements are liftable, and let Γ, B, Σ satisfy the hypotheses of Theorem* 3.11 *or of Theorem* 3.12*. Then*

(a) *$M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed);*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

Proof. The quotient $\mathbb{Z}_k/a\mathbb{Z}_k$ is $\mathbb{Z}_a$, and the quotient map is the reduction modulo a. (C0) gives $a \ge 8$, so $k \ge 16$.

(a) $M_k(G) = F(G, \mathbb{Z}_k)$. It is connected by Theorem 3.11 or Theorem 3.12.

(b) Let G be loopless, and let F, F' be two vertices of $I_k(G)$. Their reductions f, f' modulo k are nowhere-zero $\mathbb{Z}_k$-flows of G, because $1 \le |F(e)| \le k-1$. By Theorem 3.11 or Theorem 3.12 there are Σ-moves of G that lead from f through flows $f = f_0, f_1, \dots, f_\nu = f'$. For $r \ge 1$ the step from f_{r-1} to f_r is a move $f_{r-1} \to f_{r-1} + \lambda\chi_C$ of $M_k(G)$ with a liftable λ. By Definition 1.7, all integer lifts of f_{r-1} and all integer lifts of f_r lie in one component of $I_k(G)$. Every f_r has an integer lift (Lemma 1.12 (a)). So this is the same component for all r. It contains F and F'. For $\nu = 0$ use Lemma 1.12 (a): all integer lifts of one flow lie in one component.

Liftable steps are used only for the Σ-moves of the given loopless graph G. Graphs with loops occur only inside the induction in the proof of Theorem 3.11, where all flows have values in Γ. □

4. Averaging in a product

This section proves Lemma 4.1 and, as a first consequence, a theorem on groups (Theorem 4.5). Lemma 4.1 gives the averaging conditions (AV) and (AV″) of Definition 3.2 for every step system that contains generators of the two factors of a product, if $(|B|-1)/(|\Gamma|-1) < 1/9$. The subgroup B need not be a direct factor: the proof uses it only through this ratio.

The notation is that of Definition 3.1: Γ is a finite abelian group, B a subgroup, $A = \Gamma/B$, $x \mapsto \overline{x}$ the quotient map, and the A-coordinate of a Γ-flow f is the A-flow $\overline{f}$. A *zero* of the A-coordinate of f is an edge e with $\overline{f(e)} = 0$.

Lemma 4.1 (Averaging in a product)**.** *Let $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial, and let B be a subgroup of Γ with*

$$\frac{|B|-1}{|\Gamma|-1} < \frac{1}{9} \qquad \text{(true for every subgroup of index at least 9).}$$

Let Σ be a step system that contains a set of generators of $\Gamma_1 \times 0$ and a set of generators of $0 \times \Gamma_2$. Then on every multigraph G with m edges, every nowhere-zero Γ-flow is Σ-joined to a nowhere-zero Γ-flow whose A-coordinate has at most $m/9$ zeros. In particular (AV) *and* (AV″) *hold. (B need not be a direct factor, and it need not be one of the two factors.)*

Proof of Lemma 4.1. Let m be the number of edges of G and f a nowhere-zero Γ-flow of G. If $m = 0$, then f itself has no zero. Let $m \geq 1$, and put $g = |\Gamma|$.

We write an element x of Γ as $x = (x_{(1)}, x_{(2)})$ with $x_{(1)}$ in Γ_1 and $x_{(2)}$ in Γ_2, and a Γ-flow φ as $\varphi = (\varphi_{(1)}, \varphi_{(2)})$; here $\varphi_{(i)}$ is a Γ_i-flow. Put $\Sigma_1 = \Sigma \cap (\Gamma_1 \times 0)$ and $\Sigma_2 = \Sigma \cap (0 \times \Gamma_2)$. The given set of generators of $\Gamma_1 \times 0$ is a subset of Σ_1, and $\Sigma_1 = -\Sigma_1$ because $\Sigma = -\Sigma$. So every element of $\Gamma_1 \times 0$ is a finite sum of elements of Σ_1 (the empty sum for 0). In the same way every element of $0 \times \Gamma_2$ is a finite sum of elements of Σ_2.

For a nowhere-zero Γ-flow φ of G, let $G\langle\varphi_{(2)}\rangle$ be the spanning subgraph of G with the edge set $\mathrm{supp}(\varphi_{(2)})$, and $G\langle\varphi_{(1)}\rangle$ the one with the edge set $\mathrm{supp}(\varphi_{(1)})$. A Γ_1-flow of $G\langle\varphi_{(2)}\rangle$, with the value 0 on all other edges, is a Γ_1-flow of G; and in the same way for Γ_2.

(1) *Rounds.* A *round of type* 1 replaces a nowhere-zero Γ-flow φ by $(\varphi_{(1)} + \xi,\ \varphi_{(2)})$, where ξ is a Γ_1-flow of $G\langle\varphi_{(2)}\rangle$. A *round of type* 2 replaces φ by $(\varphi_{(1)},\ \varphi_{(2)} + \xi)$, where ξ is a Γ_2-flow of $G\langle\varphi_{(1)}\rangle$. In a *random* round, ξ is chosen uniformly at random in the finite group of these flows.

(2) *The result of a round is nowhere zero and Σ-joined to φ.* Take type 1; type 2 is the same with the coordinates exchanged. Let e be an edge. If e is in $\mathrm{supp}(\varphi_{(2)})$, the second coordinate of the new value is $\varphi_{(2)}(e) \neq 0$. If not, then $\xi(e) = 0$ and the value is $\varphi(e) \neq 0$. So the result is nowhere zero. By the cycle decomposition (Lemma 1.1 (a), for the graph $G\langle\varphi_{(2)}\rangle$ and the group Γ_1),

$$\xi = z_1\chi_{C_1} + \cdots + z_\nu\chi_{C_\nu}$$

with cycles $C_1, \ldots, C_\nu$ of $G\langle\varphi_{(2)}\rangle$, each with a direction, and z_r in Γ_1. Every $(z_r, 0)$ is a sum of elements of Σ_1. Add these elements one at a time, each one times χ_{C_r}, for $r = 1, \ldots, \nu$. Every flow in this sequence is $(\varphi_{(1)} + \xi',\ \varphi_{(2)})$, where ξ' is a Γ_1-flow that is 0 outside $\mathrm{supp}(\varphi_{(2)})$; as above, it is nowhere zero. So every addition is a Σ-move, and the last flow is the result of the round.

(3) *The law of one edge after one random round.* Define two matrices Π_1, Π_2 with rows and columns indexed by $\Gamma \setminus \{0\}$:

$$\Pi_1(x,y) = 1/|\Gamma_1| \ \text{ if } x_{(2)} = y_{(2)} \neq 0, \quad \Pi_1(x,x) = 1 \ \text{ if } x_{(2)} = 0, \quad \Pi_1(x,y) = 0 \ \text{ otherwise;}$$

$$\Pi_2(x,y) = 1/|\Gamma_2| \ \text{ if } x_{(1)} = y_{(1)} \neq 0, \quad \Pi_2(x,x) = 1 \ \text{ if } x_{(1)} = 0, \quad \Pi_2(x,y) = 0 \ \text{ otherwise.}$$

Claim: if φ' is the result of a random round of type i applied to φ, then for every edge e and every y in $\Gamma \setminus \{0\}$

$$\Pr[\,\varphi'(e) = y\,] = \Pi_i(\varphi(e), y).$$

Take type 1. If e is not in $\mathrm{supp}(\varphi_{(2)})$, then $\xi(e) = 0$, $\varphi'(e) = \varphi(e)$, and $\varphi(e)_{(2)} = 0$: this is the second case of the definition of Π_1. Let e be in $\mathrm{supp}(\varphi_{(2)})$. First, e lies on a cycle of $G\langle\varphi_{(2)}\rangle$. Indeed, suppose it does not. Then e is not a loop, because a loop is a cycle. Let X be the vertex set of the component of $G\langle\varphi_{(2)}\rangle - e$ that contains one end of e. The other end of e is not in X: otherwise a path of $G\langle\varphi_{(2)}\rangle - e$ between the two ends, together with e, is a cycle through e. So e is the only edge of $G\langle\varphi_{(2)}\rangle$ with exactly one end in X. Add the conservation equations of $\varphi_{(2)}$ at the vertices of X: an edge with both ends in X enters X once and leaves it once, so it drops out. The sum is $\pm\varphi_{(2)}(e)$ plus the values of $\varphi_{(2)}$ on the other edges of G with exactly one end in X; these edges are not in $\mathrm{supp}(\varphi_{(2)})$. So $\varphi_{(2)}(e) = 0$, a contradiction. Hence e lies on a cycle C of $G\langle\varphi_{(2)}\rangle$. For every x in Γ_1 the flow $x\chi_C$ has the value x or $-x$ on e. So the map $\xi \mapsto \xi(e)$ is a homomorphism of the group of the Γ_1-flows of $G\langle\varphi_{(2)}\rangle$ onto Γ_1. All

its fibres are cosets of its kernel, so they have the same size, and $\xi(e)$ is uniform on Γ_1. So $\varphi'(e) = (\varphi(e)_{(1)} + \xi(e),\ \varphi(e)_{(2)})$ is uniform on the $|\Gamma_1|$ elements y with $y_{(2)} = \varphi(e)_{(2)}$; these elements are not 0. This is the first case of the definition of Π_1.

(4) *The chain.* Put $f^{(0)} = f$. For $r \geq 0$ let $f^{(r+1)}$ be the result of a random round applied to $f^{(r)}$, of type 1 if r is even and of type 2 if r is odd, with a new independent choice. This is a finite probability space, and every sequence of choices has positive probability. By (2), every $f^{(r)}$ is a nowhere-zero Γ-flow that is Σ-joined to f, for every sequence of choices. Fix an edge e, and let μ_r be the law of $f^{(r)}(e)$: a row vector indexed by $\Gamma \setminus \{0\}$, with non-negative entries and sum 1. If $\Pr[f^{(r)} = \varphi] > 0$, then under the condition $f^{(r)} = \varphi$ the law of $f^{(r+1)}(e)$ is the row $\Pi_i(\varphi(e), \cdot)$, by (3). Summing over φ gives $\mu_{r+1} = \mu_r \Pi_i$, with $i = 1$ for even r and $i = 2$ for odd r. So, with $\Pi = \Pi_1 \Pi_2$,

$$\mu_{2h} = \mu_0 \Pi^h \qquad \text{for every integer } h \geq 0,$$

where μ_0 has the entry 1 at $f(e)$. Only the law of one edge is used, not the joint law of two edges.

(5) *Mixing.* The matrices Π_1, Π_2 have non-negative entries and row sums 1. They are symmetric: for $x \neq y$ the condition $x_{(2)} = y_{(2)} \neq 0$ is symmetric in x and y, and the same holds for Π_2. So their column sums are 1 too, and the same holds for Π. Let η be the row vector with all entries $1/(g-1)$. Then $\eta\Pi = \eta$.

Claim: every entry of Π^3 is positive. Four facts, for x, y in $\Gamma \setminus \{0\}$:

(i) $\Pi(x,x) > 0$. If $x_{(2)} \neq 0$: $\Pi_1(x,x) = 1/|\Gamma_1|$, and $\Pi_2(x,x)$ is $1/|\Gamma_2|$ or 1. If $x_{(2)} = 0$, then $x_{(1)} \neq 0$: $\Pi_1(x,x) = 1$ and $\Pi_2(x,x) = 1/|\Gamma_2|$.

(ii) If $x_{(2)} \neq 0$ and $y_{(1)} \neq 0$, then $\Pi(x,y) > 0$: Π_1 leads from x to $(y_{(1)}, x_{(2)})$, and Π_2 leads from there to y, because $y_{(1)} \neq 0$.

(iii) If $x_{(2)} = 0$, then $\Pi(x,x') > 0$ for $x' = (x_{(1)}, z)$ and every z in Γ_2: Π_1 keeps x, and Π_2 leads to x', because $x_{(1)} \neq 0$.

(iv) If $y_{(1)} = 0$, then $\Pi(x'',y) > 0$ for $x'' = (z, y_{(2)})$ and every z in Γ_1: Π_1 leads from x'' to y, because $y_{(2)} \neq 0$, and Π_2 keeps y, because $y_{(1)} = 0$.

Let x, y be given. There is x' with $x'_{(2)} \neq 0$ and $\Pi(x,x') > 0$: take $x' = x$ by (i) if $x_{(2)} \neq 0$, and otherwise $x' = (x_{(1)}, z)$ with $z \neq 0$ by (iii); such a z exists because Γ_2 is not trivial. If $y_{(1)} \neq 0$, then $\Pi(x',y) > 0$ by (ii), and $\Pi^3(x,y) \geq \Pi(x,x')\,\Pi(x',y)\,\Pi(y,y) > 0$ by (i). If $y_{(1)} = 0$, choose $z \neq 0$ in Γ_1 (it exists because Γ_1 is not trivial) and put $x'' = (z, y_{(2)})$. Then $\Pi(x',x'') > 0$ by (ii), and $\Pi(x'',y) > 0$ by (iv); so $\Pi^3(x,y) \geq \Pi(x,x')\,\Pi(x',x'')\,\Pi(x'',y) > 0$.

A bound. Every positive entry of Π_i is $1/|\Gamma_i|$ or 1, so it is at least $1/|\Gamma_i|$. A positive entry of Π is a sum of products $\Pi_1(x,z)\Pi_2(z,y)$, one of which is positive; so it is at least $1/(|\Gamma_1|\,|\Gamma_2|) = 1/g$. In the same way every entry of Π^3, which is positive, is at least $1/g^3$. Put

$$\delta = (g-1)/g^3, \qquad \text{so } 0 < \delta < 1,$$

and let Π_{u} be the matrix with all entries $1/(g-1)$. The matrix $\Pi^3 - \delta\Pi_{\mathrm{u}}$ has non-negative entries (each is at least $1/g^3 - \delta/(g-1) = 0$) and row sums $1-\delta$. So $\Pi^3 = \delta\Pi_{\mathrm{u}} + (1-\delta)\Pi_{\mathrm{r}}$ with a matrix Π_{r} that has non-negative entries and row sums 1. From $\eta\Pi^3 = \eta$ and $\eta\Pi_{\mathrm{u}} = \eta$ we get $\eta\Pi_{\mathrm{r}} = \eta$. For a row vector v put $\|v\| = \sum_y |v(y)|$. If R has non-negative

entries and row sums 1, then $\|vR\| \le \|v\|$. Let μ be a row vector with non-negative entries and sum 1. Then $\mu\Pi_{\mathrm{u}} = \eta$, and

$$\mu\Pi^3 - \eta = \delta\eta + (1-\delta)\,\mu\Pi_{\mathrm{r}} - \eta = (1-\delta)\,(\mu-\eta)\,\Pi_{\mathrm{r}},$$

and so $\|\mu\Pi^3 - \eta\| \le (1-\delta)\,\|\mu - \eta\|$. The vector $\mu\Pi^3$ again has non-negative entries and sum 1. By induction, for every integer $j \ge 0$,

$$\|\mu\Pi^{3j} - \eta\| \le (1-\delta)^j\,\|\mu - \eta\| \le 2\,(1-\delta)^j.$$

(6) *Zeros.* Every $f^{(r)}$ is nowhere zero. So an edge e is a zero of the A-coordinate of $f^{(r)}$ if and only if $f^{(r)}(e)$ is in $B \setminus \{0\}$. Let $j \ge 0$. After $6j$ rounds, that is $3j$ pairs of rounds, the law of $f^{(6j)}(e)$ is $\mu_{6j} = \mu_0\Pi^{3j}$ by (4). Let $\mathcal{E}_e$ be the event that e is a zero of the A-coordinate of $f^{(6j)}$. By (5),

$$\begin{aligned}\Pr[\mathcal{E}_e] &= \sum_{y\in B\setminus\{0\}} \mu_{6j}(y) \le \sum_{y\in B\setminus\{0\}} \Big(\frac{1}{g-1} + \Big|\mu_{6j}(y) - \frac{1}{g-1}\Big|\Big)\\ &\le \frac{|B|-1}{|\Gamma|-1} + \|\mu_{6j} - \eta\| \le \frac{|B|-1}{|\Gamma|-1} + 2\,(1-\delta)^j.\end{aligned}$$

By hypothesis $(|B|-1)/(|\Gamma|-1) < 1/9$. Fix j so large that $2\,(1-\delta)^j < 1/9 - (|B|-1)/(|\Gamma|-1)$. Then every edge is a zero with probability less than $1/9$. By linearity of expectation, the expected number of zeros of the A-coordinate of $f^{(6j)}$ is less than $m/9$. So some sequence of choices gives a flow $f^{(6j)}$ with fewer than $m/9$ zeros. By (4) it is a nowhere-zero Γ-flow that is Σ-joined to f. This proves the first statement.

(AV) is the first statement for the loopless 3-edge-connected multigraphs with at least two vertices. (AV″) follows from (AV) by Lemma 3.3. The group B is used only in step (6), through its order; it need not be a direct factor. □

Remark 4.2 (The estimate behind Lemma 4.1)**.** Steps (1)–(5) and the first display of step (6) of the proof of Lemma 4.1 do not use the hypothesis $(|B|-1)/(|\Gamma|-1) < 1/9$. They show: let $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial, let B be any subgroup of Γ, and let Σ be as in Lemma 4.1. Then for every multigraph G, every nowhere-zero Γ-flow f of G and every integer $j \ge 0$, every outcome of the random flow $f^{(6j)}$ of the proof is a nowhere-zero Γ-flow that is Σ-joined to f, and every edge of G is a zero of the A-coordinate of $f^{(6j)}$ with probability at most

$$\frac{|B|-1}{|\Gamma|-1} + 2\,(1-\delta)^j, \qquad \delta = \frac{|\Gamma|-1}{|\Gamma|^3}.$$

Part II of [Go1] uses this estimate also at the ratio $1/9$ [Go1, Lemma 14.3].

Remark 4.3 (The ratio)**.** Put $a = |A|$ and $b = |B|$, so $|\Gamma| = ab$. The condition $(b-1)/(ab-1) < 1/9$ is the inequality $9b - 9 < ab - 1$, that is $(9-a)\,b < 8$. It holds for every $a \ge 9$. More precisely, $(b-1)/(ab-1) < 1/a$ for every $a \ge 2$, because this inequality is $a(b-1) < ab - 1$, that is $a > 1$.

Remark 4.4 (Internal direct sums)**.** Lemma 4.1 speaks only of the group Γ, its subgroup B, the set Σ and flows; an isomorphism of groups carries all of them along. So the lemma applies to a group Γ with two non-trivial subgroups Γ_1, Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \{0\}$ and $\Gamma_1 + \Gamma_2 = \Gamma$: the map $(x_1, x_2) \mapsto x_1 + x_2$ is an isomorphism of $\Gamma_1 \times \Gamma_2$ onto Γ, and Σ must contain a set of generators of Γ_1 and a set of generators of Γ_2. For $\Gamma = \mathbb{Z}_k$ with $k = cc'$, where c, $c' \ge 2$ are coprime, take $\Gamma_1 = c'\mathbb{Z}_k$ and $\Gamma_2 = c\mathbb{Z}_k$. Then Γ_1 has order c and is

generated by c', and Γ_2 has order c' and is generated by c. Their intersection is $\{0\}$, because its order divides c and c'; so $|\Gamma_1 + \Gamma_2| = cc' = k$.

A consequence for groups.

Theorem 4.5 (The group theorem for a subgroup of index at least 10)**.** *Let Γ be a finite abelian group that is not cyclic of prime-power order, and let $|\Gamma|$ have a divisor d with $4 \le d \le |\Gamma|/10$. Then $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G (loops allowed). The smallest orders are* 40*,* 44*,* 48*,* 50*. (This is Corollary 3.15 with $\Sigma = \Gamma \setminus \{0\}$ and a subgroup B of order d;* (AV″) *by Lemma 4.1.)*

Proof. $|\Gamma| \ge 10\,d \ge 40$. The group Γ is a direct product of cyclic groups of prime-power order. It is not cyclic of prime-power order and not trivial, so there are at least two factors, and $\Gamma = \Gamma_1 \times \Gamma_2$ with both factors non-trivial. Let B be a subgroup of Γ of order d; it exists, because d divides $|\Gamma|$. Put $A = \Gamma/B$. Then $|B| = d \ge 4$ and $|A| = |\Gamma|/d \ge 10$.

Put $\Sigma = \Gamma \setminus \{0\}$. This is a step system. Every move of $F(G,\Gamma)$ is a Σ-move. Here $\Sigma_B = B \setminus \{0\}$, and $\Sigma_A = A \setminus \{0\}$, because every non-zero element of A is the image of a non-zero element of Γ. So (R1), (R2) and (R3*) hold with walks of one step: two different elements of $A \setminus X$, of $B \setminus X$, or of $\Gamma \setminus \{0\}$, differ by an element of Σ_A, of Σ_B, or of Σ. The set Σ contains all non-zero elements of $\Gamma_1 \times 0$ and of $0 \times \Gamma_2$, and the index of B is $|A| \ge 10$; so (AV″) holds by Lemma 4.1. Corollary 3.15 gives the connectivity of $F(G,\Gamma)$.

The orders: an order n of such a group has a divisor d with $4 \le d \le n/10$, so $n \ge 40$. For $40 \le n \le 50$ such a divisor exists exactly for $n = 40$, 44, 48 ($d = 4$) and $n = 50$ ($d = 5$); for these n the cyclic group $\mathbb{Z}_n$ is not of prime-power order. □

For the cyclic group $\mathbb{Z}_k$ the theorem is a statement on $M_k(G) = F(G,\mathbb{Z}_k)$ only. It says nothing about the integer graph $I_k(G)$: Theorem 3.19 needs a step system of liftable elements, and the lemmas of Section 1 do not show that every element of $\mathbb{Z}_k \setminus \{0\}$ is liftable (Remark 8.11).

5. The moduli 40, 48 and 66; the divisor criterion

This section proves that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G when $k = 40$, $k = 48$ or $k = 66$ (Theorems 5.3 and 5.4, Corollary 5.5), and Criterion 5.11 for $k = ab$ with $a \ge 10$ and a composite $b \ge 4$. Every result comes from Theorem 3.11, in the form of Corollary 3.15, and from Theorem 3.19. Lemma 5.1 says which hypotheses must be checked; for each modulus they are checked one by one. Criterion 5.11 uses the theorems of Cauchy and Davenport and of Mader and Watkins, but only in its last sentence; the proofs for 40, 48 and 66 use neither of them.

Setting. Throughout this section $k = ab$ with integers $a, b \ge 2$, $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (the multiples of a; b elements) and $A = \mathbb{Z}_a$; the quotient map is the reduction modulo a. This is the setting of Theorem 3.19. For an integer z we write z also for its residue in $\mathbb{Z}_k$. Because a divides k, the image in A of the residue of z is the residue of z modulo a.

Let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$. The *graph on $\mathbb{Z}_a$ with the step set D* has the vertex set $\mathbb{Z}_a$; two vertices x, y are adjacent if $y - x$ is in D. It *stays connected after the deletion of at most* 6 *elements* if for every set X of at most 6 elements of $\mathbb{Z}_a$, any two elements of $\mathbb{Z}_a \setminus X$ are joined by a walk in $\mathbb{Z}_a \setminus X$ all of whose steps are in D. For $D = \Sigma_A$ this is condition (R1) of Definition 3.2.

The master theorem for $\mathbb{Z}_k$ with a quotient of order at least 10.

Lemma 5.1 (The master theorem for $\mathbb{Z}_k$ with a quotient of order at least 10)**.** *Let $k = ab$ with integers $a \geq 10$ and $b \geq 4$, $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ and $A = \mathbb{Z}_a$. Let Σ be a step system with the following three properties.*

(a) *Every element of Σ is liftable.*

(b) *Σ has the properties* (R1)*,* (R2) *and* (R3*)*.*

(c) *$\mathbb{Z}_k$ has two non-trivial subgroups Γ_1 and Γ_2 whose orders are coprime and have the product k, and Σ contains a generator of Γ_1 and a generator of Γ_2.*

Then Γ, B and Σ satisfy the hypotheses of Theorem 3.11. Hence $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G, and $M_k(G)$ is connected also for every finite bridgeless multigraph G with loops.

Proof. The averaging condition. The subgroups of $\mathbb{Z}_k$ are cyclic; so by (c) the set Σ contains a set of generators of Γ_1 and a set of generators of Γ_2. The order of $\Gamma_1 \cap \Gamma_2$ divides the two coprime orders, so $\Gamma_1 \cap \Gamma_2 = \{0\}$, and $|\Gamma_1 + \Gamma_2| = |\Gamma_1|\,|\Gamma_2| = k$, so $\Gamma_1 + \Gamma_2 = \mathbb{Z}_k$. By Remark 4.4, Lemma 4.1 applies to these two factors. Its ratio condition holds:

$$\frac{|B|-1}{|\Gamma|-1} = \frac{b-1}{ab-1} < \frac{1}{a} \leq \frac{1}{10} < \frac{1}{9};$$

the first inequality is $a(b-1) < ab-1$, that is $a > 1$. So (AV$''$) holds by Lemma 4.1.

(H1). (R1), (R2) and (R3*) hold by (b), and (AV$''$) holds.

(H2). (C0): $|A| = a \geq 10 \geq 8$. Let H be a good subgraph that (AV$''$) gives. If H is a cycle with at most 5 edges, nothing is to be checked. If H is a subdivision, it has a frame (Lemma A.5; see Definition 2.9). Every frame of H is counted, because $|A| = a \geq 10$ and $|B| = b \geq 4$: the left sides of (C1), (C2), (C3), (C4) are at most 9, 9, 8, 3 (Remark 2.16). So (H2) holds. (This is the proof of Corollary 3.15.)

Conclusion. By (a), every element of Σ is liftable. So Theorem 3.19 applies: part (a) gives $M_k(G)$ for every finite bridgeless multigraph G, loops allowed, and part (b) gives $I_k(G)$ for every finite bridgeless loopless multigraph G. □

The moduli 40, 48 and 66. For these three moduli, every non-zero element of A except two is the image of a step. The following lemma gives (R1) in this case.

Lemma 5.2 (At most two non-steps)**.** *Let $a \geq 10$, and let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$ and $|D| \geq a-3$. Then the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.*

Proof. Let D^{c} be the set of the non-zero elements of $\mathbb{Z}_a$ that are not in D. Then $|D^{\mathrm{c}}| \leq 2$ and $D^{\mathrm{c}} = -D^{\mathrm{c}}$. Let X be a set of at most 6 elements of $\mathbb{Z}_a$, and suppose that two elements of $\mathbb{Z}_a \setminus X$ are not joined by a walk in $\mathbb{Z}_a \setminus X$ with steps in D. Let U be the set of the elements of $\mathbb{Z}_a \setminus X$ that are joined to the first of them, and $V = \mathbb{Z}_a \setminus (X \cup U)$. Both are not empty, and no step of D leads from U to V. So $v - u$ is in D^{c} for all u in U and v in V. For a fixed u this gives $|V| \leq |D^{\mathrm{c}}| \leq 2$, and for a fixed v it gives $|U| \leq 2$. But $|U| + |V| = |\mathbb{Z}_a \setminus X| \geq a-6 \geq 4$. Hence $a = 10$ and $|U| = |V| = |D^{\mathrm{c}}| = 2$. The group $\mathbb{Z}_a$ has at most one non-zero element x with $x = -x$ (namely $a/2$, if a is even). So $D^{\mathrm{c}} = \{d_0, -d_0\}$ with $d_0 \neq -d_0$. Write $U = \{u_1, u_2\}$. Then $V = u_1 + D^{\mathrm{c}} = u_2 + D^{\mathrm{c}}$. The element $u_2 + d_0$ is in $u_1 + D^{\mathrm{c}}$ and differs from $u_1 + d_0$; so $u_2 + d_0 = u_1 - d_0$. In the same way $u_2 - d_0 = u_1 + d_0$.

Subtracting the two equations gives $4d_0 = 0$ in $\mathbb{Z}_{10}$. So 5 divides $2d_0$, hence 5 divides d_0, and $2d_0 = 0$ in $\mathbb{Z}_{10}$, that is $d_0 = -d_0$. This is a contradiction. □

Theorem 5.3 (The modulus 40)**.** *$M_{40}(G)$ and $I_{40}(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Here $k = 40$, $a = 10$, $b = 4$, $\Gamma = \mathbb{Z}_{40}$, $B = 10\mathbb{Z}_{40} = \{0, 10, 20, 30\}$ and $A = \mathbb{Z}_{10}$. We check the hypotheses of Lemma 5.1.

(Set) $\Sigma = \{\pm1, \pm2, \pm4, \pm5, \pm8, \pm10, 20\}$, a set of 13 elements of $\mathbb{Z}_{40}$. They are not 0, and $\Sigma = -\Sigma$ because $-20 = 20$. So Σ is a step system.

(Lifts) ±1 are liftable by Lemma 1.12 (b). The numbers 2, 4, 5, 8, 10, 20 are divisors of 40 smaller than 40; so ±2, ±4, ±5, ±8, ±10 and 20 are liftable by Lemma 1.12 (d). This is hypothesis (a) of Lemma 5.1.

(Σ_A) The residues modulo 10 of $1, -1, 2, -2, 4, -4, 5, -5, 8, -8$ are $1, 9, 2, 8, 4, 6, 5, 5, 8, 2$; those of ±10 and 20 are 0. So $\Sigma_A = \{1, 2, 4, 5, 6, 8, 9\}$: all non-zero elements of $\mathbb{Z}_{10}$ except 3 and 7.

(Σ_B) $\Sigma_B = \Sigma \cap B = \{10, 20, 30\} = B \setminus \{0\}$.

(R1) $\Sigma_A = -\Sigma_A$ and $|\Sigma_A| = 7 = a - 3$. Lemma 5.2. With (R2) and (R3*) below, this is hypothesis (b).

(R2) Two different elements of B differ by an element of $B \setminus \{0\} = \Sigma_B$. So for every set X, any two elements of $B \setminus X$ are joined by a walk of one step.

(R3*) 1 is in Σ. For $1 \le x < x' \le 39$ the walk $x, x+1, \ldots, x'$ joins x to x' in $\mathbb{Z}_{40} \setminus \{0\}$.

(Product) $\Gamma_1 = 5\mathbb{Z}_{40}$ has order 8 and is generated by 5; $\Gamma_2 = 8\mathbb{Z}_{40}$ has order 5 and is generated by 8. The orders 8 and 5 are coprime, $8 \cdot 5 = 40$, and 5, 8 are in Σ. The ratio of Lemma 4.1 is $(|B| - 1)/(|\Gamma| - 1) = 3/39 = 1/13 < 1/9$. This is hypothesis (c) of Lemma 5.1.

(Counts) $a = 10 \ge 10$ and $b = 4 \ge 4$.

Lemma 5.1 gives the theorem. □

Theorem 5.4 (The modulus 48)**.** *$M_{48}(G)$ and $I_{48}(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Here $k = 48$, $a = 12$, $b = 4$, $\Gamma = \mathbb{Z}_{48}$, $B = 12\mathbb{Z}_{48} = \{0, 12, 24, 36\}$ and $A = \mathbb{Z}_{12}$.

(Set) $\Sigma = \{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, 24\}$, a set of 17 elements of $\mathbb{Z}_{48}$. They are not 0, and $\Sigma = -\Sigma$ because $-24 = 24$. So Σ is a step system.

(Lifts) ±1: Lemma 1.12 (b). The numbers 2, 3, 4, 6, 8, 12, 16, 24 are divisors of 48 smaller than 48; so the other elements are liftable by Lemma 1.12 (d). This is hypothesis (a).

(Σ_A) The residues modulo 12 of ±1, ±2, ±3, ±4 are $1, 11, 2, 10, 3, 9, 4, 8$; those of $6, -6, 8, -8, 16, -16$ are $6, 6, 8, 4, 4, 8$; those of ±12 and 24 are 0. So $\Sigma_A = \{1, 2, 3, 4, 6, 8, 9, 10, 11\}$: all non-zero elements of $\mathbb{Z}_{12}$ except 5 and 7.

(Σ_B) $\Sigma_B = \Sigma \cap B = \{12, 24, 36\} = B \setminus \{0\}$.

(R1) $\Sigma_A = -\Sigma_A$ and $|\Sigma_A| = 9 = a - 3$. Lemma 5.2.

(R2) As for $k = 40$: $\Sigma_B = B \setminus \{0\}$.

(R3*) 1 is in Σ; the walk with steps $+1$ inside $\{1, \dots, 47\}$.

(Product) $\Gamma_1 = 3\mathbb{Z}_{48}$ has order 16 and is generated by 3; $\Gamma_2 = 16\mathbb{Z}_{48}$ has order 3 and is generated by 16. The orders 16 and 3 are coprime, $16 \cdot 3 = 48$, and 3, 16 are in Σ. The ratio of Lemma 4.1 is $3/47 < 1/9$, because $27 < 47$. The subgroup B is not one of the two factors; Lemma 4.1 does not ask this. This is hypothesis (c).

(Counts) $a = 12 \geq 10$ and $b = 4 \geq 4$.

Lemma 5.1 gives the theorem. □

Corollary 5.5 (The modulus 66)**.** *$M_{66}(G)$ and $I_{66}(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Here $k = 66$, $a = 11$, $b = 6$, $\Gamma = \mathbb{Z}_{66}$, $B = 11\mathbb{Z}_{66} = \{0, 11, 22, 33, 44, 55\}$ and $A = \mathbb{Z}_{11}$.

(Set) $\Sigma = \{\pm 1, \pm 2, \pm 3, \pm 6, \pm 11, \pm 22, 33\}$, a set of 13 elements of $\mathbb{Z}_{66}$. They are not 0, and $\Sigma = -\Sigma$ because $-33 = 33$. So Σ is a step system.

(Lifts) ± 1: Lemma 1.12 (b). The numbers 2, 3, 6, 11, 22, 33 are divisors of 66 smaller than 66; Lemma 1.12 (d). This is hypothesis (a).

(Σ_A) The residues modulo 11 of $1, -1, 2, -2, 3, -3, 6, -6$ are $1, 10, 2, 9, 3, 8, 6, 5$; those of ± 11, ± 22 and 33 are 0. So $\Sigma_A = \{1, 2, 3, 5, 6, 8, 9, 10\}$: all non-zero elements of $\mathbb{Z}_{11}$ except 4 and 7.

(Σ_B) $\Sigma_B = \Sigma \cap B = \{11, 22, 33, 44, 55\} = B \setminus \{0\}$.

(R1) $\Sigma_A = -\Sigma_A$ and $|\Sigma_A| = 8 = a - 3$. Lemma 5.2.

(R2) As for $k = 40$: $\Sigma_B = B \setminus \{0\}$.

(R3*) 1 is in Σ; the walk with steps $+1$ inside $\{1, \dots, 65\}$.

(Product) $\Gamma_1 = 6\mathbb{Z}_{66}$ has order 11 and is generated by 6; $\Gamma_2 = 11\mathbb{Z}_{66}$ has order 6 and is generated by 11. The orders 11 and 6 are coprime, $11 \cdot 6 = 66$, and 6, 11 are in Σ. The ratio of Lemma 4.1 is $5/65 = 1/13 < 1/9$. This is hypothesis (c).

(Counts) $a = 11 \geq 10$ and $b = 6 \geq 4$.

Lemma 5.1 gives the corollary. □

The three proofs in one table (D^c is the set of the non-zero elements of A that are not in Σ_A; the last column is the ratio $(|B| - 1)/(|\Gamma| - 1)$ of Lemma 4.1):

k	a	b	Σ	D^c	Σ_B	generators	ratio
40	10	4	$\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, 20$	± 3	$B \setminus \{0\}$	5, 8	3/39
48	12	4	$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, 24$	± 5	$B \setminus \{0\}$	3, 16	3/47
66	11	6	$\pm 1, \pm 2, \pm 3, \pm 6, \pm 11, \pm 22, 33$	± 4	$B \setminus \{0\}$	6, 11	5/65

Remark 5.6 (What this section does not give)**.** (a) *The modulus* 44. The only factorization $44 = ab$ with $a \geq 10$ and $b \geq 4$ is $a = 11$, $b = 4$. The divisors of 44 smaller than 44 are 1, 2, 4, 11, 22; so the step set of the kind used here is $\{\pm 1, \pm 2, \pm 4, \pm 11, 22\}$, and its set Σ_A in $\mathbb{Z}_{11}$ is $\{\pm 1, \pm 2, \pm 4\}$, six elements. The deletion of the six neighbours 1, 2, 4, 7, 9, 10 of 0 leaves 0 without a neighbour. So (R1) fails for this step set, and Criterion 5.11 does not apply. No statement of this section concerns $I_{44}(G)$. (The group $\mathbb{Z}_{44}$ is covered by Theorem 4.5, with a subgroup of order 4; that is a statement on $M_{44}(G) = F(G, \mathbb{Z}_{44})$ only.)

(b) *Primes and prime powers.* Lemma 5.1 needs $k = ab$ with $a \geq 10$ and $b \geq 4$, so k is not a prime. It also needs two non-trivial subgroups of $\mathbb{Z}_k$ of coprime orders with the product k; a prime power k has no such subgroups. So no prime and no prime power is reached in this section.

The divisor criterion. The criterion uses one more elementary lemma, for (R2), and two quoted theorems, for its last sentence.

Lemma 5.7 ($\mathbb{Z}_b$ minus two elements)**.** *Let $b \geq 4$, let d be a divisor of b with $1 < d < b$, and let X be a set of at most 2 elements of $\mathbb{Z}_b$. Then any two elements of $\mathbb{Z}_b \setminus X$ are joined by a walk in $\mathbb{Z}_b \setminus X$ all of whose steps are in $\{1, -1, d, -d\}$.*

Proof. The steps ± 1 make $\mathbb{Z}_b$ a cycle of length b. If $|X| \leq 1$, then $\mathbb{Z}_b \setminus X$ is this cycle or a path, and the steps ± 1 are enough. Let $X = \{x, x'\}$ with $x \neq x'$. The cycle without x and x' consists of two arcs, and the steps ± 1 join the elements of each arc. If one arc is empty, we are done. Otherwise, exchange the names x, x' if necessary, and translate (a translation keeps the steps), so that $x = 0$, $x' = n + 1$, and the two arcs are $I = \{1, \ldots, n\}$ and $I' = \{n + 2, \ldots, b - 1\}$ with $1 \leq |I| \leq |I'|$. Then $n \leq b - n - 2$, that is $n \leq (b - 2)/2$. Because d is a divisor of b with $d < b$, $d \leq b/2$. So

$$n + 2 \;\leq\; n + d \;\leq\; \frac{b-2}{2} + \frac{b}{2} \;=\; b - 1,$$

and $n + d$ is in I'. The step d joins n in I to $n + d$ in I'. □

The two quoted theorems are those of Cauchy and Davenport (Theorem 0.2) and of Mader and Watkins (Theorem 0.4); both are stated in the list of quoted theorems at the end of Section 0.

Lemma 5.8 (Seven steps, prime quotient)**.** *Let a be a prime, and let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$ and $|D| \geq 7$. Then the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.*

Proof. Let X be a set of at most 6 elements of $\mathbb{Z}_a$, and suppose that two elements of $\mathbb{Z}_a \setminus X$ are not joined by a walk in $\mathbb{Z}_a \setminus X$ with steps in D. Let U be the set of the elements of $\mathbb{Z}_a \setminus X$ that are joined to the first of them, and $V = \mathbb{Z}_a \setminus (X \cup U)$; both are not empty. Put $D_0 = D \cup \{0\}$, so $|D_0| = |D| + 1 \geq 8$. Let u be in U and z in D_0. If $z = 0$, then $u + z = u$ is in U. If z is in D and $u + z$ is not in X, then $u + z$ is joined to u by one step, so it is in U. Hence $U + D_0$ is a subset of $U \cup X$, and

$$|U + D_0| \;\leq\; |U| + 6.$$

The set $U + D_0$ misses V, so $|U + D_0| < a$. By Theorem 0.2 (with $p = a$), $|U + D_0| \geq \min\{a,\ |U| + |D_0| - 1\}$; the minimum is not a, because $|U + D_0| < a$. So $|U + D_0| \geq |U| + |D_0| - 1 = |U| + |D| \geq |U| + 7$. This contradicts $|U + D_0| \leq |U| + 6$. □

Lemma 5.9 (Nine steps)**.** *Let $a \geq 10$, and let D be a set of non-zero elements of $\mathbb{Z}_a$ with $D = -D$, $1 \in D$ and $|D| \geq 9$. Then the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.*

Proof. Let Δ be the graph on $\mathbb{Z}_a$ with the step set D. It has no loop, because 0 is not in D. Two vertices x, y are joined by at most one edge, and x is adjacent to y if and only if y is adjacent to x, because $D = -D$. The neighbours of x are the $|D|$ different elements $x + z$ with z in D; so every vertex has $h = |D| \geq 9$ neighbours. Δ is connected: 1 is in D, and the

walk $x, x+1, x+2, \dots$ reaches every vertex. Δ is vertex-transitive: for every c in $\mathbb{Z}_a$ the translation $y \mapsto y + c$ maps edges to edges, and it maps 0 to c. By Theorem 0.4, the deletion of fewer than $2(h+1)/3$ vertices leaves Δ connected; and $2(h+1)/3 \geq 20/3 > 6$. So the deletion of a set X of at most 6 elements leaves a connected graph on $\mathbb{Z}_a \setminus X$; a path in it is a walk in $\mathbb{Z}_a \setminus X$ with steps in D.

The proof uses a only through the fact that $\mathbb{Z}_a$ has a set D as in the lemma; such a set exists only if $a - 1 \geq |D| \geq 9$, that is $a \geq 10$. No larger lower bound on a is used. □

Remark 5.10 (Nine steps for $a \leq 12$ without a quoted theorem)**.** For $a = 10$, 11, 12 Theorem 0.4 is not needed in Lemma 5.9: there $|D| \geq 9 \geq a - 3$, and Lemma 5.2 gives the same conclusion.

Criterion 5.11 (The divisor criterion)**.** *Let k be not a prime power, and $k = ab$ with $a \geq 10$ and a composite number $b \geq 4$. Let D be the set of the non-zero residues modulo a of ± 1, ± 2 and $\pm q$, for all divisors q of k with $1 \leq q < k$. If the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements, then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. This holds if D has at least 9 elements; and if a is a prime, it holds if D has at least 7 elements.*

Proof. Let $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$, $A = \mathbb{Z}_a$, and let Σ be the set of the residues in $\mathbb{Z}_k$ of the integers ± 1, ± 2 and $\pm q$, for all divisors q of k with $1 \leq q < k$. We check the hypotheses of Lemma 5.1.

(Set) $k = ab \geq 40$. So the residues of ± 1 and ± 2 are not 0, and the residue of $\pm q$ is not 0 because $1 \leq q < k$. $\Sigma = -\Sigma$ by its definition. So Σ is a step system.

(Lifts) ± 1: Lemma 1.12 (b). ± 2: Lemma 1.12 (c). $\pm q$: Lemma 1.12 (d). This is hypothesis (a).

(Σ_A) The image in A of the residue of an integer z is the residue of z modulo a. So Σ_A, the set of the non-zero images, is D.

(Σ_B) Because b is composite, it has a divisor d with $1 < d < b$. The numbers a and ad are divisors of k, and $1 \leq a < ad < ab = k$. So $\pm a$ and $\pm ad$ are in Σ. They are multiples of a, so they are in Σ_B.

(R1) It is the hypothesis, because $\Sigma_A = D$.

(R2) The map $j \mapsto aj$ is an isomorphism of $\mathbb{Z}_b$ onto B: it is well defined and additive, because $k = ab$; it is injective, because $aj = 0$ in $\mathbb{Z}_k$ means that b divides j; and its image is B. It maps 1, -1, d, $-d$ to a, $-a$, ad, $-ad$. Let X be a set of at most 2 elements of B, and X' its preimage in $\mathbb{Z}_b$. By Lemma 5.7 ($b \geq 4$), any two elements of $\mathbb{Z}_b \setminus X'$ are joined by a walk in $\mathbb{Z}_b \setminus X'$ with steps ± 1, $\pm d$. Its image is a walk in $B \setminus X$ with steps $\pm a$, $\pm ad$, which are in Σ_B.

(R3*) 1 is in Σ. For $1 \leq x < x' \leq k-1$ the walk $x, x+1, \dots, x'$ joins x to x' in $\mathbb{Z}_k \setminus \{0\}$.

(Product) Let p be a prime divisor of k, c the largest power of p that divides k, and $c' = k/c$. Then $c \geq 2$; $c' \geq 2$, because k is not a power of p; and c, c' are coprime. Both are divisors of k smaller than k, so c and c' are in Σ. The subgroup $c'\mathbb{Z}_k$ has order c and is generated by c'; the subgroup $c\mathbb{Z}_k$ has order c' and is generated by c (Remark 4.4). The ratio of Lemma 4.1 is $(b-1)/(ab-1) < 1/a \leq 1/10$. This is hypothesis (c).

(Counts) $a \geq 10$ and $b \geq 4$.

Lemma 5.1 gives the first statement. For the last sentence: 1 is in D, because $a \geq 10$; and $D = -D$. If D has at least 9 elements, Lemma 5.9 gives the deletion condition. If a is a prime and D has at least 7 elements, Lemma 5.8 gives it. □

Remark 5.12 (The three moduli as cases of the criterion)**.** The moduli 40, 48 and 66 are cases of Criterion 5.11. In each of the three proofs above, Σ is the set of the residues of ± 1, ± 2 and $\pm q$ for the divisors q of k with $1 \leq q < k$; the number b is composite (4, 4, 6); and $D = \Sigma_A$. For $k = 48$ the set D has 9 elements. For $k = 66$ the number $a = 11$ is a prime and D has 8 elements. For $k = 40$ the set D has 7 elements and $a = 10$ is not a prime, so the last sentence of the criterion does not apply; the deletion condition is Lemma 5.2. The proofs above use neither of the two quoted theorems.

6. The modulus $4p$ for a prime $p \geq 13$: the double step

This section proves Theorem 6.9: for every prime $p \geq 13$ and $k = 4p$, the graphs $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G ($M_k(G)$ also with loops). The theorem is Theorem 3.19 together with the frame form of the master theorem, Theorem 3.12.

The step system Σ of this section has only six A-parts: ± 1, ± 2, ± 4 modulo p. If six elements of A are deleted, one element can be isolated (Remark 6.2). So (R1) fails, and the walk lemma and Lemma 3.8 of Section 3 cannot be used. The frame form asks instead for (R1$^-$), which holds (Lemma 6.1), and for the conclusion (L1) of Lemma 3.8, which Lemma 6.7 proves. Its proof walks in the flows of the auxiliary graph that avoid the lists, with ordinary steps and with *double steps* (Lemma 6.5). Lemma 6.6 makes a double step in the graph. A double step leaves an isolated value through a middle flow that has exactly one zero on the protected threads. It has two routes, and its step ± 1 has two lifts, ± 1 and $\pm(1 + 2p)$. A parity argument in B shows that one of the four choices works.

Two conventions of Section 3 hold in this section. If H is a subgraph of a multigraph G, the reference orientation of G on $E(H)$ is the direction of the threads. And $u + \pi(Y)$ means: add $\pi(Y)$ on $E(H)$ and nothing outside $E(H)$.

The setting. Throughout this section p is a prime with $p \geq 13$, and

$$k = 4p, \qquad \Gamma = \mathbb{Z}_k, \qquad B = p\mathbb{Z}_k = \{0, p, 2p, 3p\}, \qquad A = \Gamma/B = \mathbb{Z}_p,$$

where the quotient map $x \mapsto \overline{x}$ is the reduction modulo p. So $k = ab$ with $a = p$ and $b = 4$, as in Theorem 3.19. As in Section 2, the A-part of x in Γ is $\overline{x}$. A *protected edge* is an edge of a protected thread. The step system of this section is

$$\Sigma = \{\pm 1,\ \pm(1 + 2p),\ \pm 2,\ \pm 4,\ \pm p,\ 2p\} \subseteq \Gamma.$$

(S1) *The group B.* B is cyclic of order 4, generated by p. Its only element of order 2 is $2p$. So $y + 2p \neq y$ for y in B, $2p + 2p = 0$, and $2p\,\xi = 2p$ for every odd integer ξ. For x in Γ: if $\overline{x} \neq 0$, then $x \neq 0$; if $\overline{x} = 0$, then x is in B. Adding an element of B does not change the A-part.

(S2) *Small integers.* An integer ξ with $1 \leq |\xi| \leq 12$ is not 0 modulo p, because $p \geq 13$.

(S3) *Σ is a step system of liftable elements.* 0 is not in Σ: $1 + 2p$ is odd, and 1, 2, 4, p, $2p$ and their negatives are not multiples of $4p$. $\Sigma = -\Sigma$, because $-2p = 2p$. Every element of Σ is liftable (Definition 1.7), by Corollary 1.13; in detail, by Lemma 1.12: ± 1 by (b); $\pm(1 + 2p)$ by (e); ± 2 by (c), and also by (d); ± 4, $\pm p$, $2p$ by (d), because 4, p, $2p$ are divisors of k smaller than k, $-p = 3p$ and $-2p = 2p$.

(S4) Σ_B, Σ_A *and the lifts.* The elements ±1, $\pm(1+2p)$, ±2, ±4 are ±1, ±2, ±4 modulo p, so they are not in B (S2). Hence $\Sigma_B = \Sigma \cap B = \{p, 2p, 3p\} = B \setminus \{0\}$. The A-parts are

$$\Sigma_A = \{1, -1, 2, -2, 4, -4\} \subseteq \mathbb{Z}_p$$

(the A-part of $1+2p$ is 1). These six elements are different and non-zero: they and their differences are integers ξ with $1 \le |\xi| \le 8$ (S2). The fixed lifts (Definition 3.1) are $\gamma(\sigma) = \sigma$, the integer σ in $\{\pm1, \pm2, \pm4\}$ read modulo k. The elements $\pm(1+2p)$ are used only in the double step (Lemma 6.6).

(S5) *The two lifts of* ±1. Let μ_1 be 1 or -1. The elements μ_1 and $\mu_1(1+2p)$ of Σ both have the A-part μ_1, and $\mu_1(1+2p) - \mu_1 = 2p\,\mu_1 = 2p$ by (S1).

(S6) *Condition* (R2). Let X be a set of at most 2 elements of B. Two elements of $B \setminus X$ differ by an element of $B \setminus \{0\} = \Sigma_B$: they are joined by a walk with one step. So (R2) holds. Also $|B| = 4$.

Six steps modulo p. The proof of the next lemma uses two quoted theorems on sums of sets in $\mathbb{Z}_p$: the theorem of Cauchy and Davenport (Theorem 0.2) and the theorem of Vosper (Theorem 0.3). For subsets $\mathcal{U}$, $\mathcal{V}$ of $\mathbb{Z}_p$ we write $\mathcal{U} + \mathcal{V} = \{\, x + x' : x \in \mathcal{U},\ x' \in \mathcal{V} \,\}$.

Lemma 6.1 (Six steps modulo p)**.** *Let X be a set of at most 6 elements of $A = \mathbb{Z}_p$, and consider the graph on $A \setminus X$ in which x and x' are adjacent if $x' - x$ is in $\Sigma_A = \{\pm1, \pm2, \pm4\}$.*

(a) *If $|X| \le 5$, this graph is connected. So* (R1$^-$) *holds for Σ.*

(b) *If $|X| = 6$ and $X \ne c + \Sigma_A$ for every c in A, this graph is connected.*

(c) *If $X = c + \Sigma_A$, then c is not in X, no other element c' has $X = c' + \Sigma_A$, and the graph has exactly two components: $\{c\}$ and $A \setminus (X \cup \{c\})$.*

Proof. Put $\Sigma_A^0 = \Sigma_A \cup \{0\}$; it has 7 elements (S4). Let Ω be the vertex set of a component of the graph, and assume $\Omega \ne A \setminus X$.

(1) $\Omega + \Sigma_A^0 \subseteq \Omega \cup X$. Reason: for x in Ω and σ in Σ_A^0, the element $x + \sigma$ is in X, or it is a vertex that is equal or adjacent to x, hence in Ω. So $|\Omega + \Sigma_A^0| \le |\Omega| + |X|$. Some vertex is not in Ω, so $\Omega \cup X \ne A$ and $|\Omega + \Sigma_A^0| \le p - 1$.

(2) By the theorem of Cauchy and Davenport, $|\Omega + \Sigma_A^0| \ge \min\{p,\ |\Omega| + 6\}$. By (1) this minimum is not p; so $|\Omega + \Sigma_A^0| \ge |\Omega| + 6$. With (1): $|X| \ge 6$. This proves (a): if $|X| \le 5$, no such Ω exists. If $|X| = 6$, then $|\Omega + \Sigma_A^0| = |\Omega| + 6$, and $\Omega + \Sigma_A^0 = \Omega \cup X$, because the two sets have the same number of elements (Ω and X are disjoint). From now on $|X| = 6$.

(3) Σ_A^0 is not an arithmetic progression. Suppose that $\Sigma_A^0 = \{\, b' + rd : r = 0, 1, \dots, 6 \,\}$ with $d \ne 0$. The sum of the seven elements is 0, because $\Sigma_A = -\Sigma_A$; it is also $7b' + 21d = 7(b' + 3d)$. Since $p \ne 7$, $b' = -3d$, and $\Sigma_A^0 = \{\, rd : -3 \le r \le 3 \,\}$. Let x be the inverse of d in $\mathbb{Z}_p$. Then $x\,\Sigma_A^0 = \{-3, \dots, 3\}$, so x, $2x$ and $4x$ are in $\{\pm1, \pm2, \pm3\}$. If $x = \pm1$, then $4x = \pm4$; if $x = \pm2$, then $2x = \pm4$; if $x = \pm3$, then $2x = \pm6$. But ±4 and ±6 are not in $\{\pm1, \pm2, \pm3\}$: the differences are integers ξ with $1 \le |\xi| \le 9$ (S2). This is a contradiction.

(4) By (2), $|\Omega + \Sigma_A^0| = |\Omega| + |\Sigma_A^0| - 1$. If $|\Omega| \ge 2$ and $|\Omega| + 6 \le p - 2$, the theorem of Vosper says that Σ_A^0 is an arithmetic progression, against (3). So $|\Omega| = 1$, or $|\Omega| + 6 \ge p - 1$; in the second case $|\Omega| + 6 = p - 1$ by (1).

(5) If $|\Omega| = 1$, say $\Omega = \{c\}$, then (2) gives $\{c\} \cup (c + \Sigma_A) = \{c\} \cup X$. Here c is not in X (it is a vertex) and not in $c + \Sigma_A$ (0 is not in Σ_A). So $X = c + \Sigma_A$. If $|\Omega| = p - 7$, then $|\Omega \cup X| = p - 1$, so exactly one element c of A is outside $\Omega \cup X$. It is a vertex. Its component is not Ω, and it lies in $A \setminus (\Omega \cup X) = \{c\}$. So $\{c\}$ is a component, and the first case, for this component, gives $X = c + \Sigma_A$.

(b) By (5), if the graph is not connected, then $X = c + \Sigma_A$ for some c.

(c) Let $X = c + \Sigma_A$. Then c is not in X, because 0 is not in Σ_A. If also $X = c' + \Sigma_A$, then the sum of the elements of X is $6c = 6c'$, so $c = c'$ by (S2). All neighbours $c + \sigma$ of c are in X, so $\{c\}$ is a component. The set $A \setminus (X \cup \{c\})$ has $p - 7 \geq 6$ elements; let Ω be a component that meets it. $\Omega \neq A \setminus X$, because $\{c\}$ is another component. By (4) and (5), $|\Omega| = 1$ or $|\Omega| = p - 7$. If $|\Omega| = 1$, then (5) gives $\Omega = \{c'\}$ with $X = c' + \Sigma_A$, so $c' = c$; but Ω meets $A \setminus (X \cup \{c\})$. So $|\Omega| = p - 7$, and $\Omega \subseteq A \setminus (X \cup \{c\})$ is all of this set. Hence the components are $\{c\}$ and $A \setminus (X \cup \{c\})$. □

Remark 6.2 (Condition (R1) fails)**.** Let $X = \Sigma_A$. Then 0 is not in X, every element $0 + \sigma$ with σ in Σ_A is in X, and $p - 7 \geq 6$ elements other than 0 are outside X. So 0 is isolated, and (R1) is false for Σ. Theorem 3.11 and Corollary 3.15 do not apply. By Lemma 6.1, this is the only way in which six deleted elements disconnect A: $X = c + \Sigma_A$, with the one isolated value c.

Double steps on the auxiliary graph. From here to the end of the proof of Lemma 6.7, H is a subdivision as in Definition 2.1 (SUB), with the graphs H_0, Q of Definition 2.6, and a frame of H is given (Definition 2.9). $C_1, \ldots, C_s$ are the official cycles (Definition 2.10). For a node j, $C_j^e = C_j$ and C_j^f are the two cycles of the node j (Definition 2.12), and $\Delta_j = \chi_{C_j^f} - \chi_{C_j^e}$. We use three facts of Lemma 2.13:

(TC1) $\chi_{C_j^e}$ and $\chi_{C_j^f}$ are equal on N_j (part (a)).

(TC2) If $h > j$, no edge of N_h lies on C_j^e or on C_j^f. If $i < j$, then C_j^e does not contain f_i, and C_j^f does not contain e_i. (Part (b): if i is below j, this is its first sentence; if not, no edge of N_i lies on the two cycles. A node $h > j$ is not below j.)

(TC3) Δ_j is 0 on $N_j \cup N_{j+1} \cup \cdots \cup N_s$. For every x in A, the A-flow $x\Delta_j$ is a sum of flows $x'\chi_C$ with x' in A and C a cycle of Q that has no edge in $N_j \cup \cdots \cup N_s$. (Part (c): the cycles C lie in $Q_{U_j'} \cup Q_{U_j''}$, whose edges are red edges and edges of pairs N_h with h below j, so $h < j$.)

The sets X_i. Let Λ be a list assignment (Definition 2.14). For a node i, an edge t of N_i and an A-flow Z of Q put

$$X_i^t(Z) = \{\, c \in A : Z(t) + c\,\chi_{C_i}(t) \in \Lambda(t) \,\}, \qquad X_i(Z) = X_i^{e_i}(Z) \cup X_i^{f_i}(Z).$$

These are the sets of the proof of the walk lemma of Section 3. By (O2) of Lemma 2.11, $\chi_{C_i}(t)$ is 1 or -1. Hence:

(Y1) $|X_i^t(Z)| = |\Lambda(t)|$, and $X_i^t(Z)$ depends only on the value $Z(t)$.

(Y2) For c in A, the value of $Z + c\chi_{C_i}$ on t is in $\Lambda(t)$ if and only if c is in $X_i^t(Z)$. In particular $Z(e_i) \notin \Lambda(e_i)$ and $Z(f_i) \notin \Lambda(f_i)$ if and only if 0 is not in $X_i(Z)$.

(Y3) $X_i(Z + c\chi_{C_i}) = X_i(Z) - c$ for c in A.

(Y4) If an A-flow Z' is equal to Z on N_i, then $X_i^t(Z') = X_i^t(Z)$ for both edges t of N_i.

Definition 6.3 (Charged node; double step)**.** Let Λ be a list assignment with $|\Lambda(t)| \le 3$ for every blue edge t, let Z be an A-flow of Q, and let j be a node.

(a) The node j is *charged* at Z if $X_j(Z) = \Sigma_A$. Then exactly one edge t of N_j has 1 in $X_j^t(Z)$ (Lemma 6.4 (a)). It is the *charged edge*, written t^*; it is an original blue edge. If $\Lambda = \Lambda_f$ for an H-good flow f, the protected thread P_{t^*} is the *charged thread*; it has 3 edges, because $3 = |\Lambda_f(t^*)| \le \ell(t^*) \le 3$.

(b) Let j be charged at Z. An *admissible pair* is a pair (μ_1, μ_2) of integers with $\mu_1 \in \{1, -1\}$ and $\mu_2 \in \{2, -2, 4, -4\}$ such that μ_1 and μ_2, read modulo p, are in $X_j^{t^*}(Z)$, and $\mu_1 + \mu_2$, read modulo p, is not in Σ_A.

(c) A *double step* of the node j at Z is given by Z, a node j that is charged at Z, and an admissible pair (μ_1, μ_2). Its four *flows* are

$$Z \text{ (start)}, \qquad Z^{(1)} = Z + \mu_1 \chi_{C_j^e} \quad \text{and} \quad Z^{(2)} = Z + \mu_2 \chi_{C_j^f} \text{ (middle)},$$
$$Z^{(3)} = Z + \mu_1 \chi_{C_j^e} + \mu_2 \chi_{C_j^f} \text{ (end)}.$$

The number $z = \mu_1 + \mu_2$ is its *end value*.

(d) The double step is *good* for Λ if Z and $Z^{(3)}$ are in $P(\Lambda)$, and $Z^{(1)}(t) \notin \Lambda(t)$ and $Z^{(2)}(t) \notin \Lambda(t)$ for every blue edge t other than t^*.

Lemma 6.4 (The four flows of a double step)**.** *Let Λ be a list assignment with $|\Lambda(t)| \le 3$ for every blue edge t, let Z be an A-flow of Q, and let the node j be charged at Z.*

(a) *Both edges of N_j are original, $|\Lambda(e_j)| = |\Lambda(f_j)| = 3$, and the sets $X_j^{e_j}(Z)$, $X_j^{f_j}(Z)$ are disjoint. So the charged edge t^* is well defined. An admissible pair exists. Every admissible pair has the end value $z \in \{3, -3, 5, -5\}$, and z is not in $\Sigma_A \cup \{0\}$.*

(b) *Let (μ_1, μ_2) be an admissible pair. On the edges of N_j the four flows Z, $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$ are equal to $Z + x\chi_{C_j}$ with $x = 0$, μ_1, μ_2, z. Hence Z and $Z^{(3)}$ have no value in Λ on N_j; $Z^{(1)}$ and $Z^{(2)}$ have a value in $\Lambda(t^*)$ on t^* and no value in Λ on the other edge of N_j. On N_h with $h > j$, the four flows are equal to Z.*

(c) *Let $i < j$ be a node. Then $Z^{(2)}(e_i) = Z(e_i)$, $Z^{(3)}(e_i) = Z^{(1)}(e_i)$, $Z^{(1)}(f_i) = Z(f_i)$ and $Z^{(3)}(f_i) = Z^{(2)}(f_i)$. So each edge of N_i has at most two values in the four flows.*

(d) $Z^{(3)} = Z + z\chi_{C_j} + \mu_2 \Delta_j$.

Proof. (a) By (Y1), $|X_j(Z)| \le |\Lambda(e_j)| + |\Lambda(f_j)| \le 6 = |\Sigma_A|$ (S4). Since $X_j(Z) = \Sigma_A$, equality holds everywhere: both lists have 3 elements, and the two sets are disjoint. A list assignment is empty on the expansion edges (Definition 2.14); so e_j and f_j are original. The element 1 of Σ_A lies in exactly one of the two disjoint sets; so t^* is well defined. Now $X_j^{t^*}(Z) = \{1, x, x'\}$ with two different elements x, x' of $\Sigma_A \setminus \{1\} = \{-1, 2, -2, 4, -4\}$. If one of x, x' is in $\{2, 4, -4\}$, the pair $(1, \text{this element})$ is admissible: its end value is 3, 5 or -3, not in Σ_A. If not, $\{x, x'\} \subseteq \{-1, -2\}$, so x and x' are -1 and -2, and $(-1, -2)$ is admissible: its end value is -3. For every pair (μ_1, μ_2) with $\mu_1 = \pm 1$ and $\mu_2 \in \{\pm 2, \pm 4\}$, the sum $\mu_1 + \mu_2$ is one of ± 1, ± 3, ± 5; for an admissible pair it is not ± 1, because ± 1 is in Σ_A. So z is one of ± 3, ± 5. Each of ± 3, ± 5 differs from each of 0, ± 1, ± 2, ± 4 by an integer ξ with $1 \le |\xi| \le 9$; so z is not in $\Sigma_A \cup \{0\}$ (S2).

(b) $C_j^e = C_j$, and by (TC1) $\chi_{C_j^f} = \chi_{C_j}$ on N_j. This gives the first sentence. By (Y2), $Z + x\chi_{C_j}$ has a value in $\Lambda(t)$ on the edge t of N_j if and only if x is in $X_j^t(Z)$. The number 0

is not in $\Sigma_A = X_j(Z)$, and z is not in Σ_A by (a). μ_1 and μ_2 are in $X_j^{t^*}(Z)$, and not in the set of the other edge, because the two sets are disjoint. The last sentence is (TC2).

(c) By (TC2), C_j^f does not contain e_i and C_j^e does not contain f_i. So the value on e_i changes only by the term $\mu_1\chi_{C_j^e}$, which $Z^{(1)}$ and $Z^{(3)}$ have and Z, $Z^{(2)}$ do not have; the value on f_i changes only by the term $\mu_2\chi_{C_j^f}$, which $Z^{(2)}$ and $Z^{(3)}$ have.

(d) $\mu_1\chi_{C_j^e} + \mu_2\chi_{C_j^f} = (\mu_1+\mu_2)\chi_{C_j} + \mu_2(\chi_{C_j^f} - \chi_{C_j^e})$, because $C_j^e = C_j$. □

The walk lemma with double steps.

Lemma 6.5 (Walk lemma with double steps)**.** *Let Λ be a list assignment with $|\Lambda(t)| \le 3$ for every blue edge t. Then any two Y, Y' in $P(\Lambda)$ are joined by a sequence $Y = Y_0, Y_1, \dots, Y_\nu = Y'$ in $P(\Lambda)$ in which every pair Y_{r-1}, Y_r is*

- *an* ordinary step*: $Y_r - Y_{r-1} = \sigma\chi_C$ with σ in Σ_A and a cycle C of Q with a direction; or*
- *a* double step*: Y_{r-1} and Y_r are the start and the end, in one of the two orders, of a double step that is good for Λ.*

Proof. For $i = 1, \dots, s+1$ let Λ_i be the list assignment that is equal to Λ on the edges of $N_i \cup \dots \cup N_s$ and empty on all other blue edges. So $\Lambda_1 = \Lambda$, because every blue edge lies in a pair, and $P(\Lambda_{s+1})$ is the set of all A-flows of Q. For $j \ge i$ the assignments Λ_i and Λ agree on N_j; so the sets X_j, the charged nodes j, their charged edges and the double steps of the node j are the same for Λ_i and for Λ. Claim(i): any two members of $P(\Lambda_i)$ are joined by a sequence in $P(\Lambda_i)$ whose steps are ordinary steps and double steps of nodes $j \ge i$ that are good for Λ_i. Claim(1) is the lemma.

(1) *Claim*$(s+1)$. Let Y, Y' be A-flows of Q. By (O5) of Lemma 2.11, $Y' - Y = x_1\chi_{C_1} + \dots + x_s\chi_{C_s}$ with x_l in A. Every x_l is a sum of copies of 1, and 1 is in Σ_A. Add these terms one by one: this is a sequence of ordinary steps from Y to Y'.

(2) *Notation.* Let $1 \le i \le s$ and write $\chi = \chi_{C_i}$. By (O3) of Lemma 2.11, C_i has no edge in $N_{i+1} \cup \dots \cup N_s$. With (Y2): for Z in $P(\Lambda_{i+1})$ and c in A,

$$(*) \qquad Z + c\chi \in P(\Lambda_{i+1}); \qquad \text{and} \qquad Z + c\chi \in P(\Lambda_i) \text{ if and only if } c \notin X_i(Z).$$

Also $|X_i(Z)| \le 6$ by (Y1).

(3) *From Claim*$(i+1)$ *to Claim*(i). Let Y, Y' be in $P(\Lambda_i)$. They are in $P(\Lambda_{i+1})$. By Claim$(i+1)$ there is a sequence $Y = Y_0, \dots, Y_\nu = Y'$ in $P(\Lambda_{i+1})$ whose steps are ordinary steps and double steps of nodes $j \ge i+1$ that are good for Λ_{i+1}.

(3a) *The corrections.* Choose $d_1, \dots, d_\nu$ in A as follows. If the step r is ordinary: d_r outside $X_i(Y_{r-1}) \cup X_i(Y_r)$, a set of at most 12 elements. If the step r is a double step of a node $j \ge i+1$, with the four flows Z, $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$: d_r outside $X_i(Z) \cup X_i(Z^{(1)}) \cup X_i(Z^{(2)}) \cup X_i(Z^{(3)})$. This set has at most 12 elements. Reason: by (Y1) the part $X_i^{e_i}$ of each of the four sets depends only on the value on e_i; by Lemma 6.4 (c), with $i < j$, the four flows have at most two values on e_i; so the four parts $X_i^{e_i}$ have at most $2\,|\Lambda(e_i)| \le 6$ elements together. The same holds for f_i. In both cases d_r exists, because $p \ge 13 > 12$. (This is the count $12 < p$. The proof uses $p \ge 13$ also through Lemmas 6.1 and 6.4.)

(3b) *The shifted steps.* Let $1 \le r \le \nu$. The flows $Y_{r-1} + d_r\chi$ and $Y_r + d_r\chi$ are in $P(\Lambda_i)$, by $(*)$: d_r is not in $X_i(Y_{r-1}) \cup X_i(Y_r)$ (for a double step, $\{Y_{r-1}, Y_r\} = \{Z, Z^{(3)}\}$).

If the step r is ordinary, their difference is the same step $\sigma\chi_C$. Let the step r be a double step of the node $j \geq i+1$ with start Z and pair (μ_1, μ_2). Then $Z + d_r\chi$ is the start of a double step of the node j with the same pair, and this double step is good for Λ_i. Reasons: C_i has no edge in N_j (O3), so $Z + d_r\chi = Z$ on N_j; by (Y4), $X_j^t(Z + d_r\chi) = X_j^t(Z)$ for both edges t of N_j; so j is charged at $Z + d_r\chi$, with the same charged edge t^*, and (μ_1, μ_2) is admissible. The four flows of the new double step are $Z^{(x)} + d_r\chi$, $x = 0, 1, 2, 3$ (with $Z^{(0)} = Z$). Start and end are in $P(\Lambda_i)$, as shown. The middle flow $Z^{(1)} + d_r\chi$ has no value in Λ on N_i, because $d_r \notin X_i(Z^{(1)})$ (Y2); on $N_{i+1} \cup \cdots \cup N_s$ it is equal to $Z^{(1)}$ (O3), which has no value in $\Lambda_{i+1} = \Lambda$ there except on t^*, because the old double step is good for Λ_{i+1}; and Λ_i is empty on the other blue edges. The same holds for $Z^{(2)} + d_r\chi$. So the shifted pair $Y_{r-1} + d_r\chi$, $Y_r + d_r\chi$ is a double step of a node $j \geq i$ that is good for Λ_i.

(3c) *The connections.* Put $d_0 = d_{\nu+1} = 0$. For $r = 0, \ldots, \nu$ a *connection at* Y_r is a sequence in $P(\Lambda_i)$ from $Y_r + d_r\chi$ to $Y_r + d_{r+1}\chi$ whose steps are ordinary steps and double steps of the node i that are good for Λ_i. We construct one. Write $d = d_r$, $d' = d_{r+1}$ and $X = X_i(Y_r)$. Then d and d' are not in X: for $r \geq 1$ and for $r + 1 \leq \nu$ by (3a); and $d_0 = 0$ is not in $X_i(Y_0)$, $d_{\nu+1} = 0$ is not in $X_i(Y_\nu)$, by $(*)$, because Y and Y' are in $P(\Lambda_i)$. For every c outside X the flow $Y_r + c\chi$ is in $P(\Lambda_i)$ $(*)$, and $Y_r + c'\chi - (Y_r + c\chi) = (c' - c)\chi$ is an ordinary step if $c' - c$ is in Σ_A.

- If $d = d'$, the connection has no step.
- If $A \setminus X$ has a walk from d to d' with steps in Σ_A, take the flows $Y_r + c\chi$ for the terms c of the walk. By Lemma 6.1 ($|X| \leq 6$), such a walk exists unless $X = c + \Sigma_A$ for some c and exactly one of d, d' is c.
- Let $X = c + \Sigma_A$ and $d = c \neq d'$. Put $Z = Y_r + c\chi$; it is in $P(\Lambda_i)$. By (Y3), $X_i(Z) = X - c = \Sigma_A$: the node i is charged at Z. Take an admissible pair (μ_1, μ_2) (Lemma 6.4 (a)) and the double step of the node i at Z with this pair. It is good for Λ_i: Z is in $P(\Lambda_i)$; by Lemma 6.4 (b), $Z^{(3)}$ has no value in Λ on N_i, $Z^{(1)}$ and $Z^{(2)}$ have none on N_i except on t^*, and on $N_{i+1} \cup \cdots \cup N_s$ all four flows are equal to Z; and Λ_i is empty on the other blue edges. By Lemma 6.4 (d), $Z^{(3)} = Y_r + (c + z)\chi + \mu_2\Delta_i$. The connection is: the double step from Z to $Z^{(3)}$, then Part 1, then Part 2.

  Part 1. $c + z$ is not in $c + (\Sigma_A \cup \{0\}) = X \cup \{c\}$ (Lemma 6.4 (a)), and d' is not in $X \cup \{c\}$. By Lemma 6.1 (c), $A \setminus (X \cup \{c\})$ is a component of the graph of that lemma; so it has a walk from $c + z$ to d' with steps in Σ_A. Take the flows $Y_r + w\chi + \mu_2\Delta_i$ for the terms w of this walk. By (TC3), Δ_i is 0 on $N_i \cup \cdots \cup N_s$; so on these edges such a flow is equal to $Y_r + w\chi$, which has no value in Λ there, because $w \notin X$ $(*)$. Λ_i is empty on the other blue edges. So these flows are in $P(\Lambda_i)$, and two consecutive ones differ by an ordinary step on C_i.

  Part 2. By (TC3), $\mu_2\Delta_i$ is a sum of flows $x'\chi_{C'}$ with x' in A and cycles C' of Q that have no edge in $N_i \cup \cdots \cup N_s$. Write every x' as a sum of copies of 1 and subtract the flows $\chi_{C'}$ one by one: these are ordinary steps with $\sigma = -1$. Every flow on the way is equal to $Y_r + d'\chi$ on $N_i \cup \cdots \cup N_s$, so it is in $P(\Lambda_i)$. The last flow is $Y_r + d'\chi$.
- Let $X = c + \Sigma_A$ and $d' = c \neq d$. Do the same in the reverse order: from $Y_r + d\chi$ add the flows $\chi_{C'}$ of Part 2 one by one (ordinary steps with $\sigma = 1$), then walk

from d to $c+z$ as in Part 1 (d is not in $X \cup \{c\}$), and end with the double step from $Z^{(3)}$ to $Z = Y_r + c\chi$.

(3d) *The new sequence.* Put together: the connection at Y_0, the shifted step 1, the connection at $Y_1, \dots$, the shifted step ν, the connection at Y_ν. Consecutive pieces fit: the shifted step r goes from $Y_{r-1} + d_r\chi$ to $Y_r + d_r\chi$. The sequence goes from $Y_0 + d_0\chi = Y$ to $Y_\nu + d_{\nu+1}\chi = Y'$, all its terms are in $P(\Lambda_i)$, and every step is an ordinary step or a double step of a node $j \geq i$ that is good for Λ_i. This is Claim(i). □

The double step in the graph. The next lemma makes a double step in a multigraph G that contains H. Its setting:

– G is a multigraph that contains H, and u_0 is an A-flow of G; in part (a), f_0 is an H-good Γ-flow of G with A-coordinate u_0;

– $\mathcal{O}^e$ and $\mathcal{O}^f$ are two sets of directed cycles of H; the cycles in $\mathcal{O}^e$ are pairwise edge-disjoint, and so are the cycles in $\mathcal{O}^f$ (a cycle of $\mathcal{O}^e$ and a cycle of $\mathcal{O}^f$ can have common edges); $\chi^e = \sum_{O\in\mathcal{O}^e} \chi_O$ and $\chi^f = \sum_{O\in\mathcal{O}^f} \chi_O$, integer flows of H with values in $\{0, 1, -1\}$;

– $\mu_1 \in \{1, -1\}$ and $\mu_2 \in \{2, -2, 4, -4\}$ are integers, read modulo p in A and modulo k in Γ; and, on $E(G)$ (all three are equal to u_0 outside $E(H)$),

$$u_1 = u_0 + \mu_1\chi^e, \qquad u_2 = u_0 + \mu_2\chi^f, \qquad u_3 = u_0 + \mu_1\chi^e + \mu_2\chi^f.$$

Hypothesis (J):

(J1) u_3 is non-zero on every edge of every protected thread;

(J2) there are edges e_1^* and e_2^* of one protected thread P_{t^*} such that $u_1(e_1^*) = 0$ and $u_2(e_2^*) = 0$; u_1 is non-zero on every other edge of every protected thread; and u_2 is non-zero on every other edge of every protected thread.

Lemma 6.6 (The double step in G)**.** *In this setting:*

(a) *Assume* (J). *Then f_0 is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate u_3 that is equal to f_0 outside $E(H)$.*

(b) *Let g be an H-good Γ-flow of G with A-coordinate u_3. Assume* (J2)*, and assume that u_0 is non-zero on every edge of every protected thread. Then g is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate u_0 that is equal to g outside $E(H)$.*

Proof. Part (b) *from part* (a). Use (a) for the flow g, with $-\mu_1$, $-\mu_2$ and the same $\mathcal{O}^e$, $\mathcal{O}^f$. The three flows of (a) are now $u_3 - \mu_1\chi^e = u_2$, $u_3 - \mu_2\chi^f = u_1$ and $u_3 - \mu_1\chi^e - \mu_2\chi^f = u_0$. (J1) for this use says that u_0 is non-zero on the protected threads: this is assumed. (J2) for this use holds with e_2^* in the place of e_1^* and e_1^* in the place of e_2^*. And $-\mu_1 \in \{1, -1\}$, $-\mu_2 \in \{2, -2, 4, -4\}$. So (a) gives (b).

Part (a). Assume (J). The proof has fourteen steps.

(1) *Constant values on threads.* Every cycle O of H is a union of threads, and χ_O is constant, 1 or -1, on each thread of O (Lemma 2.7 (a)). The cycles of $\mathcal{O}^e$ are pairwise edge-disjoint; so a thread lies on at most one of them. Hence χ^e is constant on every thread, with a value in $\{0, 1, -1\}$; the same holds for χ^f. For a free edge ε let c_ε^e and c_ε^f be the values of χ^e and χ^f on P_ε. Moreover $\chi^e(e_1^*) = \pm 1$: $u_0(e_1^*) \neq 0$ because f_0 is H-good, and $u_1(e_1^*) = 0$; so $\mu_1\chi^e(e_1^*) = -u_0(e_1^*) \neq 0$.

(2) *Moves, routes, choices.* Let γ be one of the two lifts μ_1, $\mu_1(1+2p)$ of μ_1 (not necessarily the fixed lift of μ_1, which is μ_1); both are in Σ and have the A-part μ_1, and they differ by $2p$ (S5). The integer μ_2 is in Σ, with the A-part μ_2. The moves are, in this order:

(i) the *preparation*: for every free edge ε the move $y_\varepsilon\chi_{D_\varepsilon}$, with y_ε in B (no move if $y_\varepsilon = 0$); in Case 1 below, also one move $y\chi_D$ with y in B and a cycle D of H_P;

(ii) *route 1*: the moves $\gamma\chi_O$ for O in $\mathcal{O}^e$, one after the other, and then the moves $\mu_2\chi_O$ for O in $\mathcal{O}^f$; *route 2*: first the moves $\mu_2\chi_O$ for O in $\mathcal{O}^f$, and then the moves $\gamma\chi_O$ for O in $\mathcal{O}^e$.

A non-zero element of B is in Σ_B (S4). So every move is $\lambda\chi_O$ with λ in Σ and a cycle O of H. A *choice* is a route together with a lift γ: there are four choices. If all flows reached are nowhere zero, all moves are Σ-moves, and the last flow has the A-coordinate $u_0 + \mu_1\chi^e + \mu_2\chi^f = u_3$ (the moves (i) do not change the A-coordinate, (S1)); it is equal to f_0 outside $E(H)$; and it is H-good by (J1). We show: for some choice there are y_ε (and y) such that all flows reached are nowhere zero.

(3) *The preparation.* Let f° be the flow after (i). Its A-coordinate is u_0 (S1). The cycle D_ε contains P_ε and no other free thread, and it is directed along P_ε (Definition 2.10); D lies in H_P. So the move of ε is the only move of (i) that changes an edge of P_ε, and $f^\circ = f_0 + y_\varepsilon$ on $E(P_\varepsilon)$. All other edges that (i) changes are on protected threads, where the A-coordinate u_0 is non-zero. So all flows of (i) are nowhere zero if

(W0) $f_0(e) + y_\varepsilon \neq 0$ for every edge e of every free thread P_ε with $u_0(e) = 0$.

(An edge with $u_0(e) \neq 0$ is never 0 in (i), by (S1).)

(4) *The routes.* Assume (W0). In route 1, after some of the moves $\gamma\chi_O$, an edge e has the value $f^\circ(e) + \gamma\chi^e(e)$, with the A-coordinate $u_1(e)$, if it lies on a cycle O already used, and the value $f^\circ(e)$ otherwise; this holds because the cycles of $\mathcal{O}^e$ are pairwise edge-disjoint. After all moves $\gamma\chi_O$ and some of the moves $\mu_2\chi_O$, an edge e has the value $f^\circ(e) + \gamma\chi^e(e) + \mu_2\chi^f(e)$, with the A-coordinate $u_3(e)$, if it lies on a cycle of $\mathcal{O}^f$ already used, and $f^\circ(e) + \gamma\chi^e(e)$ otherwise. Edges outside $E(H)$ do not change. A value with a non-zero A-coordinate is not 0 (S1). So all flows of route 1 are nowhere zero if

(W1) $f^\circ(e) + \gamma\chi^e(e) \neq 0$ for every edge e of H with $u_1(e) = 0$, and

(W3) $f^\circ(e) + \gamma\chi^e(e) + \mu_2\chi^f(e) \neq 0$ for every edge e of H with $u_3(e) = 0$.

In the same way all flows of route 2 are nowhere zero if (W3) holds and

(W2) $f^\circ(e) + \mu_2\chi^f(e) \neq 0$ for every edge e of H with $u_2(e) = 0$.

We say that a choice *works* for the given y_ε (and y) if (W0), (W3) and (W1) (route 1) or (W2) (route 2) hold. By (2)–(4) it suffices to find a choice and values that work.

(5) *The protected edges.* A protected edge e has $u_0(e) \neq 0$ (f_0 is H-good) and $u_3(e) \neq 0$ (J1). By (J2), $u_1(e) = 0$ only for $e = e_1^*$, and $u_2(e) = 0$ only for $e = e_2^*$. So on the protected threads the only conditions are (W1) at e_1^* in route 1 and (W2) at e_2^* in route 2. All other conditions are at edges of free threads.

(6) *One excluded value for each edge of a free thread.* Let e be an edge of P_ε, and write $c^e = c^e_\varepsilon$, $c^f = c^f_\varepsilon$. Then $f^\circ(e) = f_0(e) + y_\varepsilon$ by (3), and

$$u_1(e) = u_0(e) + \mu_1 c^e, \qquad u_2(e) = u_0(e) + \mu_2 c^f, \qquad u_3(e) = u_0(e) + \mu_1 c^e + \mu_2 c^f.$$

In route 1 the conditions at e say that y_ε is not

$$-f_0(e) \text{ if } u_0(e) = 0 \text{ (W0)}; \qquad -f_0(e) - \gamma c^e \text{ if } u_1(e) = 0 \text{ (W1)};$$
$$-f_0(e) - \gamma c^e - \mu_2 c^f \text{ if } u_3(e) = 0 \text{ (W3)}.$$

Each of these is an element of B, because its A-part is $-u_0(e)$, $-u_1(e)$ or $-u_3(e)$, which is 0 (S1). If two of the numbers $u_0(e)$, $u_1(e)$, $u_3(e)$ are 0, the two excluded elements are equal:

- $u_0(e) = u_1(e) = 0$: then $\mu_1 c^e = 0$, so $c^e = 0$;
- $u_1(e) = u_3(e) = 0$: then $\mu_2 c^f = 0$, so $c^f = 0$ ($\mu_2 \neq 0$ modulo p, (S2));
- $u_0(e) = u_3(e) = 0$: then $\mu_1 c^e + \mu_2 c^f = 0$. If $c^e \neq 0$ and $c^f \neq 0$, this number is one of $\pm 1 \pm 2$, $\pm 1 \pm 4$; if exactly one of them is 0, it is one of ± 1, ± 2, ± 4. None of these is 0 modulo p (S2). So $c^e = c^f = 0$.

So in route 1 every edge of a free thread excludes at most one value of y_ε. In route 2 the conditions exclude $-f_0(e)$ if $u_0(e) = 0$, $-f_0(e) - \mu_2 c^f$ if $u_2(e) = 0$, and $-f_0(e) - \gamma c^e - \mu_2 c^f$ if $u_3(e) = 0$; in the same way ($u_0(e) = u_2(e) = 0$ gives $c^f = 0$, $u_2(e) = u_3(e) = 0$ gives $c^e = 0$, $u_0(e) = u_3(e) = 0$ gives $c^e = c^f = 0$) every edge excludes at most one value. For a choice let AL_ε be the set of the elements of B that no edge of P_ε excludes. A thread has at most 3 edges and $|B| = 4$, so AL_ε is not empty. The free thread P_ε is *rigid* for the choice if AL_ε has exactly one element; then P_ε has 3 edges, and they exclude three different elements. The conditions at the edges of P_ε concern y_ε only.

(7) *The condition at e_1^* or e_2^*.* By (3), $f^\circ(e_1^*) = f_0(e_1^*) + \sum y_\varepsilon \chi_{D_\varepsilon}(e_1^*)$ ($+ y\chi_D(e_1^*)$ in Case 1), the sum over the free edges ε with e_1^* on D_ε; the same for e_2^*. In route 1 the condition (W1) at e_1^* is $\Psi_1 \neq 0$, and in route 2 the condition (W2) at e_2^* is $\Psi_2 \neq 0$, where

$$\Psi_1 = f^\circ(e_1^*) + \gamma\chi^e(e_1^*), \qquad \Psi_2 = f^\circ(e_2^*) + \mu_2\chi^f(e_2^*).$$

Both are in B, because their A-parts are $u_1(e_1^*) = 0$ and $u_2(e_2^*) = 0$. If D_ε contains P_{t^*}, then χ_{D_ε} is 1 or -1, the same on all edges of P_{t^*} (1). By (4)–(6): a choice works if one can take y_ε in AL_ε for every free edge ε (and y) such that $\Psi_1 \neq 0$ (route 1) or $\Psi_2 \neq 0$ (route 2).

(8) *Case 1: P_{t^*} lies on a cycle D of H_P.* Take route 1 and $\gamma = \mu_1$. Take any y_ε in AL_ε, for every free edge ε. Then Ψ_1 is a fixed element plus $y\chi_D(e_1^*)$, and $\chi_D(e_1^*) = \pm 1$. As y runs through B, Ψ_1 runs through B; choose y with $\Psi_1 \neq 0$. The move $y\chi_D$ changes only protected edges, so the conditions at the edges of the free threads still hold. This choice works.

(9) *Case 2: P_{t^*} lies on no cycle of H_P. Two sides.* Let x_1, x_2 be the ends of P_{t^*}; they are different vertices of H_0, because H_0 is loopless. H_P is connected ((O6) of Lemma 2.11). Let H_P^- be H_P without the edges and the interior vertices of P_{t^*}. An interior vertex of P_{t^*} has degree 2 in H, with both edges on P_{t^*}. So a path of H_P from a vertex of H_P^- to x_1 stays in H_P^- until it first meets x_1 or x_2; hence every vertex of H_P^- is joined in H_P^- to x_1 or to x_2. And x_1, x_2 are not joined in H_P^-: such a path and P_{t^*} would

form a cycle of H_P through P_{t^*}. So H_P^- has exactly two components; let $V_1 \ni x_1$ and $V_2 \ni x_2$ be their vertex sets. A path of H_P with both ends in $V_1 \cup V_2$ contains P_{t^*} if and only if it has one end in V_1 and the other in V_2. Reason: a path in H_P^- stays in one component, so a path from V_1 to V_2 has an edge of P_{t^*}; its ends are not interior vertices of P_{t^*}, so it contains all of P_{t^*}. Conversely, if the path contains P_{t^*}, the rest of it consists of a path of H_P^- that ends at x_1 and a path of H_P^- that ends at x_2; so one end is in V_1 and the other in V_2.

The ends of a free thread P_ε are vertices of H_0; so they are in H_P (O6) and are not interior vertices of P_{t^*}: they are in $V_1 \cup V_2$. Call ε *crossing* if P_ε has one end in V_1 and the other in V_2. The cycle D_ε without the edges and the interior vertices of P_ε is a path of H_P between the ends of P_ε, because its other threads are protected (Definition 2.10). So D_ε contains P_{t^*} if and only if ε is crossing; and e_1^*, e_2^*, which are edges of P_{t^*}, lie on D_ε if and only if ε is crossing. In Case 2 no move $y\chi_D$ is made. So, by (7),

$$\Psi_1 = f_0(e_1^*)+\gamma\chi^e(e_1^*)+\sum_{\varepsilon \text{ crossing}} y_\varepsilon \chi_{D_\varepsilon}(e_1^*), \qquad \Psi_2 = f_0(e_2^*)+\mu_2\chi^f(e_2^*)+\sum_{\varepsilon \text{ crossing}} y_\varepsilon \chi_{D_\varepsilon}(e_2^*).$$

(10) *Claim: if, for some choice, some crossing ε_0 is not rigid, then this choice works.* Take y_ε in AL_ε for every free edge $\varepsilon \neq \varepsilon_0$. Then Ψ_r (r the route) is a fixed element plus $y_{\varepsilon_0}\chi_{D_{\varepsilon_0}}(e_r^*)$, with $\chi_{D_{\varepsilon_0}}(e_r^*) = \pm 1$ by (9) and (7). Two different elements y_{ε_0} of $\mathrm{AL}_{\varepsilon_0}$ give two different values of Ψ_r; at most one of them is 0. Take the other one; by (7) the choice works.

(11) *From now on assume that no choice works* (in Case 2). By (10), every crossing thread is rigid for each of the four choices. Let ε be crossing and e an edge of P_ε. By (6), e excludes a value in each choice. We show: $u_0(e) = 0$ or $u_3(e) = 0$. In route 1, e excludes a value, so one of $u_0(e)$, $u_1(e)$, $u_3(e)$ is 0; in route 2, one of $u_0(e)$, $u_2(e)$, $u_3(e)$ is 0.

- $c^e \neq 0$ and $c^f \neq 0$. The differences of the four numbers $u_0(e)$, $u_1(e)$, $u_2(e)$, $u_3(e)$ are $\pm\mu_1 c^e$, $\pm\mu_2 c^f$, $\pm(\mu_1 c^e + \mu_2 c^f)$ and $\pm(\mu_2 c^f - \mu_1 c^e)$: integers ξ with $1 \le |\xi| \le 5$, not 0 modulo p (S2). So at most one of the four numbers is 0. If it is $u_1(e)$, route 2 excludes nothing at e; if it is $u_2(e)$, route 1 excludes nothing at e. So it is $u_0(e)$ or $u_3(e)$.
- $c^e \neq 0$ and $c^f = 0$. Then $u_2(e) = u_0(e)$ and $u_3(e) = u_1(e)$; route 2 gives $u_0(e) = 0$ or $u_3(e) = 0$.
- $c^e = 0$ and $c^f \neq 0$. Then $u_1(e) = u_0(e)$ and $u_3(e) = u_2(e)$; route 1 gives $u_0(e) = 0$ or $u_3(e) = 0$.
- $c^e = c^f = 0$. Then the four numbers are equal, and route 1 gives $u_0(e) = 0$.

(12) *The same allowed set in both routes.* Let ε be crossing and e an edge of P_ε. By (6) and (11), the value that e excludes is

- $-f_0(e)$ if $u_0(e) = 0$, in both routes (if also $u_1(e)$, $u_2(e)$ or $u_3(e)$ is 0, (6) gives the same value);
- $-f_0(e) - \gamma c^e - \mu_2 c^f$ if $u_0(e) \neq 0$ and $u_3(e) = 0$, in both routes (if also $u_1(e) = 0$ in route 1, or $u_2(e) = 0$ in route 2, (6) gives the same value).

So, for each lift γ, the set AL_ε is the same in the two routes. Call e *shifting* if $u_0(e) \neq 0$, $u_3(e) = 0$ and $c^e \neq 0$, and let κ_ε be the number of the shifting edges of P_ε. It depends

neither on the route nor on the lift. (All edges of P_ε have the same c^e; if $c^e = 0$, then $\kappa_\varepsilon = 0$.)

(13) *The other lift.* Replace γ by the other lift $\gamma + 2p$ (S5). The value that a shifting edge excludes changes by $-2p\,c^e = 2p$ (S1). The value that any other edge of P_ε excludes does not change: it is $-f_0(e)$, or $-f_0(e) - \mu_2 c^f$ because $c^e = 0$. The thread is rigid for both lifts: for each lift its three excluded values are different, and the allowed value is the fourth element of B.

- $\kappa_\varepsilon = 0$: the set of the excluded values does not change; so the allowed value does not change.
- $\kappa_\varepsilon = 3$: the three excluded values change by $2p$; so the allowed value changes by $2p$.
- $\kappa_\varepsilon = 1$: the shifting edge excludes w, then $w + 2p$; the other two edges exclude w' and w'' for both lifts. By rigidity, $w' \neq w''$, and w', w'' are not in $\{w, w + 2p\}$. So $\{w', w''\} = B \setminus \{w, w + 2p\}$, and the allowed value is $w + 2p$, then w: it changes by $2p$.
- $\kappa_\varepsilon = 2$: the shifting edges exclude w, w', then $w + 2p$, $w' + 2p$; the third edge excludes w'' for both lifts. Suppose $w' \neq w + 2p$. Then w, w', $w + 2p$, $w' + 2p$ are four different elements: $w \neq w'$ by rigidity, and $w' \neq w + 2p$, which is the same as $w \neq w' + 2p$ (S1). So they are all of B, and w'' is one of them; this contradicts rigidity for one of the two lifts. So $w' = w + 2p$, the set of the excluded values $\{w, w + 2p, w''\}$ does not change, and the allowed value does not change.

So the allowed value of P_ε changes by $2p$ if κ_ε is odd, and it does not change if κ_ε is even; by (12) this holds in both routes.

(14) *Parity.* Let ω be the number of the crossing threads with κ_ε odd. Every crossing thread is rigid, so in each choice its y_ε must be its allowed value. For the other free edges take any y_ε in AL_ε; they do not enter Ψ_1, Ψ_2 (9). Since no choice works, (7) gives $\Psi_1 = 0$ for both lifts in route 1 and $\Psi_2 = 0$ for both lifts in route 2, with these values of y_ε. When γ is replaced by $\gamma + 2p$, the sum over the crossing threads in (9) changes by $2p$ for every thread with κ_ε odd ($2p \cdot (\pm 1) = 2p$, (S1)) and by 0 for the others: by $2p\,\omega$ in total, in both routes.

Route 1: the term $\gamma\chi^e(e_1^*)$ changes by $2p\,\chi^e(e_1^*) = 2p$, by (1) and (S1). So Ψ_1 changes by $2p\,(1 + \omega)$. It is 0 for both lifts; so $2p\,(1 + \omega) = 0$, and ω is odd (S1).

Route 2: the term $\mu_2\chi^f(e_2^*)$ does not depend on γ. So Ψ_2 changes by $2p\,\omega$. It is 0 for both lifts; so $2p\,\omega = 0$, and ω is even.

This is a contradiction. So in Case 2 some choice works; in Case 1 a choice works by (8). By (2) this proves (a). □

In the proof, (W1) or (W2) at e_1^* or e_2^* is the only condition that involves more than one free thread. The preparation is the preparation in B of Lemma 3.6 with one move for each free thread; it is written out here because the values y_ε must be chosen, not only avoided.

Lemmas 3.8 and 3.10 for $k = 4p$.

Lemma 6.7 (Lemma 3.8 for $k = 4p$)**.** *Let H be a subdivision as in Definition 2.1* (SUB) *with a frame. Let G be a multigraph that contains H, f an H-good Γ-flow of G with A-coordinate u, and Y a flow in $P(\Lambda_f)$. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good*

flow with A-coordinate $u+\pi(Y)$ that is equal to f outside $E(H)$. So the conclusion (L1) *of Lemma* 3.8 *holds for every subdivision H and every frame of H.*

Proof. (1) *The sequence.* $|\Lambda_f(t)| \le \ell(t) \le 3$ for every original blue edge t, and Λ_f is empty on the expansion edges (Definition 2.14). 0 and Y are in $P(\Lambda_f)$. Lemma 6.5, for $\Lambda = \Lambda_f$, gives a sequence $0 = Y_0, Y_1, \dots, Y_\nu = Y$ in $P(\Lambda_f)$ of ordinary steps and double steps that are good for Λ_f. We build H-good flows $f = g_0, g_1, \dots, g_\nu$ of G, each Σ-joined to the one before it by Σ-moves on cycles of H and equal to f outside $E(H)$, such that g_r has the A-coordinate $u+\pi(Y_r)$. Let g_{r-1} be built.

(2) *An ordinary step,* $Y_r - Y_{r-1} = \sigma\chi_C$. Then $u+\pi(Y_r) = u+\pi(Y_{r-1}) + \pi(\sigma\chi_C)$, and it is non-zero on every edge of every protected thread, because Y_r is in $P(\Lambda_f)$ (Definition 2.14). Lemma 3.7, for the flow g_{r-1}, the cycle C and σ, gives g_r. Its hypotheses hold: (R2) by (S6), and (C4) because $\ell(\varepsilon) \le 3 < 4 = |B|$.

(3) *A double step.* Y_{r-1} and Y_r are Z and $Z^{(3)}$, in one of the two orders, for a double step of a node j at Z with a pair (μ_1, μ_2) that is good for Λ_f. Let t^* be its charged edge. By Lemma 2.7 (d) there are sets $\mathcal{O}^e$, $\mathcal{O}^f$ of pairwise edge-disjoint directed cycles of H with $\pi(x\chi_{C_j^e}) = x\chi^e$ and $\pi(x\chi_{C_j^f}) = x\chi^f$ for every x in A, where $\chi^e = \sum_{O\in\mathcal{O}^e}\chi_O$ and $\chi^f = \sum_{O\in\mathcal{O}^f}\chi_O$. Put $u_0 = u+\pi(Z)$. Then

$$u+\pi(Z^{(1)}) = u_0+\mu_1\chi^e = u_1, \qquad u+\pi(Z^{(2)}) = u_0+\mu_2\chi^f = u_2, \qquad u+\pi(Z^{(3)}) = u_3,$$

in the notation of Lemma 6.6. We check its hypotheses.

(3a) u_0 and u_3 are non-zero on every edge of every protected thread, because Z and $Z^{(3)}$ are in $P(\Lambda_f)$ (Definition 2.14). This is (J1), and the extra hypothesis of Lemma 6.6 (b).

(3b) For an original blue edge t and an edge e of P_t: $u_1(e) = u(e)+Z^{(1)}(t)$, so $u_1(e) = 0$ if and only if $Z^{(1)}(t) = -u(e)$, an element of $\Lambda_f(t)$. For $t \ne t^*$, $Z^{(1)}(t)$ is not in $\Lambda_f(t)$, because the double step is good; so u_1 is non-zero on P_t. On t^*, $Z^{(1)}(t^*)$ is in $\Lambda_f(t^*)$ (Lemma 6.4 (b)). The set $\Lambda_f(t^*) = \{-u(e) : e \in E(P_{t^*})\}$ has 3 elements (Lemma 6.4 (a)), and P_{t^*} has at most 3 edges; so the edges of P_{t^*} give three different elements, and exactly one edge e_1^* of P_{t^*} has $u_1(e_1^*) = 0$.

(3c) In the same way, u_2 is 0 on exactly one edge e_2^* of the protected threads, and e_2^* lies on P_{t^*}. So (J2) holds, with the charged thread P_{t^*}.

If $Y_{r-1} = Z$: g_{r-1} is H-good with A-coordinate u_0, and Lemma 6.6 (a) gives an H-good flow g_r with A-coordinate $u_3 = u+\pi(Y_r)$, equal to g_{r-1} outside $E(H)$. If $Y_{r-1} = Z^{(3)}$: g_{r-1} has A-coordinate u_3, and Lemma 6.6 (b) gives g_r with A-coordinate $u_0 = u+\pi(Y_r)$, equal to g_{r-1} outside $E(H)$.

(4) *The end.* g_ν is H-good, equal to f outside $E(H)$, and has the A-coordinate $u+\pi(Y)$. □

Corollary 6.8 (Lemma 3.10 for $k = 4p$)**.** *Let G be a bridgeless multigraph and H a subgraph of G that is a cycle with at most* 5 *edges* (*case* CYC)*, or a subdivision as in Definition 2.1* (SUB) *with any frame* (*case* SUB)*. Let $G^* = G/H$ be the multigraph obtained from G by the identification of $V(H)$ to one vertex and the deletion of $E(H)$. Let f be an H-good Γ-flow of G with restriction f^* to G^*, and let $g^* = f^* + \lambda\chi_{C^*}$ be the result of a Σ-move of G^*. Then f is Σ-joined to an H-good Γ-flow g of G whose restriction to G^* is g^*. This holds for every λ in Σ, also for $\lambda = \pm(1+2p)$.*

Proof. This is the frame form of Lemma 3.10. Its hypotheses: $|A| = p \geq 13 \geq 8$; (R1$^-$) by Lemma 6.1 (a); (R2) by (S6). In the case SUB the frame must have (C2), (C3), (C4) and (L1). Every thread has at most 3 edges, so the left sides of (C2), (C3), (C4) are at most $3+3+3 = 9 < 13 \leq |A|$, $3+3+2 = 8 < 13 \leq |A|$ and $3 < 4 = |B|$; and (L1) holds by Lemma 6.7. Lemma 3.10 is stated for every λ in Σ. □

The theorem.

Theorem 6.9 (The modulus $4p$)**.** *Let $p \geq 13$ be a prime and $k = 4p$. Then*

(a) *$M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed);*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

This is Theorem 3.19 *with the frame form of the master theorem (Theorem* 3.12*), for $a = p$, $b = 4$ and $\Sigma = \{\pm 1, \pm(1+2p), \pm 2, \pm 4, \pm p, 2p\}$.*

Proof. We check the hypotheses of Theorem 3.19 with Theorem 3.12, one by one.

(1) *The groups.* $k = ab$ with $a = p \geq 2$ and $b = 4 \geq 2$; $\Gamma = \mathbb{Z}_k$, $B = p\mathbb{Z}_k$ (the multiples of p; 4 elements), $A = \mathbb{Z}_p$ (reduction modulo p). This is the setting of this section.

(2) *The step system.* Σ is a step system, and all its elements are liftable: (S3). Which statement lifts each element: ± 1 by Lemma 1.12 (b), $\pm(1+2p)$ by (e), ± 2 by (c), ± 4, $\pm p$, $2p$ by (d).

(3) *Σ_A and Σ_B.* $\Sigma_A = \{\pm 1, \pm 2, \pm 4\}$ modulo p, six different elements, with the lifts $\gamma(\sigma) = \sigma$; $\Sigma_B = \{p, 2p, 3p\} = B \setminus \{0\}$: (S4).

(4) (R1$^-$) by Lemma 6.1 (a). ((R1) is false, Remark 6.2; the frame form does not need it.)

(5) (R2) by (S6).

(6) (R3*). 1 is in Σ. In $\Gamma \setminus \{0\} = \{1, 2, \ldots, k-1\}$ any two elements are joined by a walk with steps 1 or -1 that stays in $\{1, \ldots, k-1\}$.

(7) (AV$''$). By the Chinese remainder theorem, $x \mapsto (x \bmod p,\ x \bmod 4)$ is an isomorphism from Γ onto $\mathbb{Z}_p \times \mathbb{Z}_4$. It maps B onto $0 \times \mathbb{Z}_4$, the element 4 of Σ to $(4, 0)$, which generates $\mathbb{Z}_p \times 0$ (4 is invertible modulo p), and the element p of Σ to $(0, p \bmod 4)$, which generates $0 \times \mathbb{Z}_4$ (p is odd). And

$$\frac{|B|-1}{|\Gamma|-1} = \frac{3}{4p-1} \leq \frac{3}{51} < \frac{1}{9}.$$

So Lemma 4.1, with $\Gamma_1 = \mathbb{Z}_p$ and $\Gamma_2 = \mathbb{Z}_4$, gives (AV) and (AV$''$). (A group isomorphism maps flows to flows and moves to moves, so the lemma applies to Γ.)

(8) (C0): $|A| = p \geq 13 \geq 8$.

(9) *The good subgraphs and their frames* ((H2) in the frame form). A good subgraph is a cycle with at most 5 edges or a subdivision as in Definition 2.1 (SUB). Every subdivision has a frame (Lemma A.5; see Definition 2.9). We show that every frame has (C2), (C3), (C4) and (L1). Every thread has at most 3 edges. So: (C2): $\ell(e_i)+\ell(f_i)+\max\{\ell(e_i), \ell(f_i)\} \leq 9 < 13 \leq |A|$; (C3): $\ell(e_i) + \ell(f_i) + 2 \leq 8 < 13 \leq |A|$; (C4): $\ell(\varepsilon) \leq 3 < 4 = |B|$; (L1) by Lemma 6.7. ((C1) is not needed in the frame form; it would hold, $2 \cdot 3 + 3 = 9 < 13$, but (R1) fails.)

(10) *Conclusion.* By (1)–(9) the hypotheses of Theorem 3.12 hold for Γ, B and Σ, and all elements of Σ are liftable. Theorem 3.19 gives (a) and (b). □

Remark 6.10 (Where $p \geq 13$ is used)**.** The size of p is used only at these places: (S4) (six different elements of Σ_A, $|\xi| \leq 8$); Lemma 6.1 (p prime for the two quoted theorems, $p \neq 7$, (S2) for $|\xi| \leq 9$, and $p - 7 \geq 6$ in part (c)); Remark 6.2 ($p - 7 \geq 6$); Lemma 6.4 (a) ($z \notin \Sigma_A \cup \{0\}$, $|\xi| \leq 9$); Lemma 6.5, step (3a) (the count $12 < p$, the largest count of this section); Lemma 6.6 ((S2) for $|\xi| \leq 5$); Corollary 6.8 and the proof of Theorem 6.9: (C0), (C2), (C3) and $3/(4p-1) < 1/9$. For $p \leq 11$ the count $12 < p$ fails, and this section says nothing about $k = 4p$ with $p \leq 11$. The smallest modulus of the theorem is $k = 52$, with $\Sigma = \{\pm 1, \pm 27, \pm 2, \pm 4, \pm 13, 26\}$ in $\mathbb{Z}_{52}$. Theorem 6.9 is the case (B3) of Theorem 9.2.

7. Prime powers

This section proves that $M_k(G)$ and $I_k(G)$ are connected for every prime power $k = p^N$ with $N \geq 6$ (Theorem 7.4). The proof is Theorem 3.19 with Theorem 3.11, for $\Gamma = \mathbb{Z}_k$ and the subgroup $B = p^i\mathbb{Z}_k$ of the multiples of p^i, for a suitable i. The step conditions are checked by hand. The averaging condition needs a new argument. Lemma 4.1 asks for a product of two non-trivial groups, and a cyclic group of prime-power order is not such a product. Here (AV″) is proved by a deterministic argument: moves by the steps $1, p, \ldots, p^{i-1}$ lower the p-adic valuations of the values of the flow (the *flattening*), and a count over four levels of the valuation gives a good subgraph (Lemma 7.2).

Throughout this section p is a prime, $N \geq 2$ is an integer, and $k = p^N$.

Valuations and the flattening process. For an element $x \neq 0$ of $\mathbb{Z}_k$, its *valuation* $\mathrm{val}(x)$ is the largest j in $\{0, \ldots, N-1\}$ such that x is in $p^j\mathbb{Z}_k$. So $\mathrm{val}(-x) = \mathrm{val}(x)$, and for $x \neq 0$ and $0 \leq j \leq N-1$: x is in $p^j\mathbb{Z}_k$ if and only if $\mathrm{val}(x) \geq j$.

Fact (V). *Let x, y be non-zero elements of $\mathbb{Z}_k$ with $\mathrm{val}(x) \neq \mathrm{val}(y)$. Then $x + y \neq 0$ and $\mathrm{val}(x+y) = \min\{\mathrm{val}(x), \mathrm{val}(y)\}$.*

Proof. First, let z be an integer that p does not divide, and let $0 \leq j \leq N-1$. Then $p^j z$ is not 0 in $\mathbb{Z}_k$ and has the valuation j. Indeed, it is in $p^j\mathbb{Z}_k$. If $p^j z = p^{j+1}w$ in $\mathbb{Z}_k$ for an integer w (this includes the case $p^j z = 0$), then p^N divides $p^j(z - pw)$, so p divides $z - pw$ and z, which is false.

Next, an element $x \neq 0$ with $\mathrm{val}(x) = j$ is $x = p^j z$ for an integer z that p does not divide: x is $p^j z$ for some integer z; if p divided z, then x would be in $p^{j+1}\mathbb{Z}_k$, which contradicts the choice of j if $j \leq N-2$, and gives $x = 0$ if $j = N-1$.

Now let $\mathrm{val}(x) = j < \mathrm{val}(y) = l$, and write $x = p^j z$, $y = p^l w$ with integers z, w that p does not divide. Then $x + y = p^j(z + p^{l-j}w)$, and p does not divide $z + p^{l-j}w$. By the first paragraph, $x + y \neq 0$ and $\mathrm{val}(x+y) = j$. □

For a nowhere-zero $\mathbb{Z}_k$-flow g of a multigraph G and $0 \leq j \leq N-1$ put

$$E_j(g) = \{\, e \in E(G) : \mathrm{val}(g(e)) = j \,\},$$

and define $E_{<j}(g)$, $E_{>j}(g)$, $E_{\geq j}(g)$ in the same way. The valuation of $g(e)$ does not depend on the reference orientation of e, because $\mathrm{val}(-x) = \mathrm{val}(x)$. Every edge lies in exactly one set $E_j(g)$, and $E_{\geq j}(g) = E_{>j-1}(g)$ for $j \geq 1$.

Lemma 7.1 (The flattening process)**.** *Let $1 \leq i \leq N-1$, and let Σ be a step system of $\mathbb{Z}_k$ that contains 1 and p^j for $1 \leq j \leq i-1$. Let G be a finite loopless multigraph and f a nowhere-zero $\mathbb{Z}_k$-flow of G. Then f is Σ-joined to a nowhere-zero $\mathbb{Z}_k$-flow f' such that for every j in $\{0, \ldots, i-1\}$:*

(a) *every cycle of G with no edge in $E_j(f')$ has all its edges in $E_{<j}(f')$;*

(b) $|E_{>j}(f')| \le n_j - 1$, *where* n_j *is the number of components of the multigraph* $(V(G), E_{<j}(f'))$.

Proof. The move. Let g be a nowhere-zero $\mathbb{Z}_k$-flow of G, let $0 \le j \le i-1$, and let C be a cycle of G with a direction and with no edge in $E_j(g)$. Put $g' = g + p^j\chi_C$. The element p^j is in Σ (for $j = 0$ it is the step 1), and $p^j \ne 0$ with $\mathrm{val}(p^j) = j$, because $j \le i-1 \le N-2$ (Fact (V), first paragraph, with $z = 1$). For an edge e of C we have $g'(e) = g(e) \pm p^j$ and $\mathrm{val}(g(e)) \ne j = \mathrm{val}(\pm p^j)$. By Fact (V), $g'(e) \ne 0$ and $\mathrm{val}(g'(e)) = \min\{\mathrm{val}(g(e)), j\}$. The edges outside C keep their values. So g' is nowhere zero, and $g \to g'$ is a Σ-move. On C, an edge with valuation less than j keeps its valuation, and an edge with valuation greater than j gets the valuation j.

The process. Start with $g = f$. As long as there are j in $\{0, \dots, i-1\}$ and a cycle C of G with no edge in $E_j(g)$ and with at least one edge in $E_{>j}(g)$, make the move above with these j and C (in either direction of C). The sum of the valuations of all values of g is a non-negative integer, and every move lowers it by at least 1. So the process stops. Let f' be the last flow. It is Σ-joined to f. From now on E_j, $E_{<j}$, $E_{>j}$ are the sets of f'.

(a). Let C be a cycle with no edge in E_j. The process has stopped, so C has no edge in $E_{>j}$. Every edge has a valuation, so all edges of C are in $E_{<j}$.

(b). Let G_j be the multigraph whose vertices are the n_j components of $(V(G), E_{<j})$ and whose edges are the edges of $E_{>j}$; an edge of $E_{>j}$ joins the components of its two ends (it is a loop of G_j if they are the same component). We show that G_j has no cycle; then G_j is a forest with n_j vertices, and $|E_{>j}| = |E(G_j)| \le n_j - 1$.

Suppose that G_j has a cycle O, and let e_1 be an edge of O. Go once round O. At every vertex of O (a component of $(V(G), E_{<j})$) the edge of O that arrives and the edge of O that leaves have ends in this component; join these two ends by a path inside the component. (If O is a loop, join the two ends of e_1 by a path inside its component.) This gives a closed walk W of G that uses every edge of O exactly once and otherwise only edges of $E_{<j}$. So W uses e_1 exactly once, and it uses no edge of E_j. The two ends x, y of e_1 are different, because G is loopless. The walk W without its passage through e_1 is a walk from y to x that does not use e_1. It contains a path P from y to x, and P does not contain e_1. Then P and e_1 form a cycle of G that has no edge in E_j and that has the edge e_1 of $E_{>j}$. This contradicts (a). □

Four levels give a good subgraph.

Lemma 7.2 (Four levels)**.** *Let* $4 \le i \le N-1$ *(so* $N \ge 5$*), and let* Σ *be a step system of* $\mathbb{Z}_k$ *that contains* 1 *and* p^j *for* $1 \le j \le i-1$*. Let* G *be a loopless* 3*-edge-connected multigraph with* $n \ge 2$ *vertices, and* f *a nowhere-zero* $\mathbb{Z}_k$*-flow of* G*. Then* f *is* Σ*-joined to a nowhere-zero* $\mathbb{Z}_k$*-flow* f' *such that the multigraph*

$$\big(V(G),\ \{\, e : f'(e) \notin p^i\mathbb{Z}_k \,\}\big)$$

contains a good subgraph (Definition 2.1). So, for $\Gamma = \mathbb{Z}_k$*,* $B = p^i\mathbb{Z}_k$*,* $A = \Gamma/B$ *and this* Σ*, the condition* (AV″) *of Definition 3.2 holds.*

Proof. Let f' be the flow of Lemma 7.1 for this i, and let E_j, $E_{<j}$, $E_{>j}$ be the sets of f'. For $0 \le j \le i$ let n_j be the number of components of $(V(G), E_{<j})$; for $j \le i-1$ this is the number n_j of Lemma 7.1 (b). Then $n_0 = n$, because $E_{<0}$ is empty. Let m be the number of edges of G. Every vertex of G has degree at least 3, because G is loopless and 3-edge-connected with $n \ge 2$; so $m \ge 3n/2$.

An edge e has $f'(e) \notin p^i\mathbb{Z}_k$ if and only if $\mathrm{val}(f'(e)) < i$. So the multigraph of the lemma is $(V(G), E_{<i})$. Suppose that it contains no good subgraph. Let $1 \le j \le i$. The multigraph

$(V(G), E_{<j})$ is a subgraph of $(V(G), E_{<i})$, so it contains no good subgraph either; it is loopless and has n_j components.

Step 1: many edges. We have $E_{\geq j} = E_{>j-1}$, and $j-1$ is in $\{0, \dots, i-1\}$. By Lemma 7.1 (b),

$$|E_{<j}| \;=\; m - |E_{>j-1}| \;\geq\; \tfrac{3}{2}\, n - n_{j-1} + 1.$$

Step 2: few edges. Let a component of $(V(G), E_{<j})$ have n' vertices and m' edges. If $n' \geq 3$, then $m' \leq \frac{4}{3}n' - 2$: otherwise Lemma 2.2 (a), applied to this component (a loopless multigraph with $n' \geq 3$ vertices and more than $\frac{4}{3}n' - 2$ edges), gives a good subgraph of the component, and so of $(V(G), E_{<i})$. If $n' = 2$, then $m' \leq 1$: two parallel edges form a cycle with 2 edges and 2 vertices, which is a good subgraph of the kind (CYC). If $n' = 1$, then $m' = 0$, because there is no loop. In all three cases $m' \leq \frac{4}{3}\,(n' - 1)$. The sum over the n_j components gives

$$|E_{<j}| \;\leq\; \tfrac{4}{3}\,(n - n_j).$$

Step 3: the recursion. Steps 1 and 2 give $\frac{3}{2}n - n_{j-1} + 1 \leq \frac{4}{3}n - \frac{4}{3}n_j$, that is

$$n_j \;\leq\; \tfrac{3}{4}\big(n_{j-1} - \tfrac{1}{6}\, n - 1\big) \qquad (1 \leq j \leq i).$$

The right side grows with n_{j-1}. Define numbers α_j, β_j by $\alpha_0 = 1$, $\beta_0 = 0$ and $\alpha_j = \frac{3}{4}(\alpha_{j-1} - \frac{1}{6})$, $\beta_j = \frac{3}{4}(\beta_{j-1} + 1)$. Then $n_0 = \alpha_0 n - \beta_0$, and by induction on j, $n_j \leq \alpha_j n - \beta_j$ for $0 \leq j \leq 4$ (here $4 \leq i$ is used). The values are:

j	0	1	2	3	4
α_j	1	$\frac{5}{8}$	$\frac{11}{32}$	$\frac{17}{128}$	$-\frac{13}{512}$
β_j	0	$\frac{3}{4}$	$\frac{21}{16}$	$\frac{111}{64}$	$\frac{525}{256}$

(For example $\alpha_4 = \frac{3}{4}\big(\frac{17}{128} - \frac{1}{6}\big) = \frac{3}{4} \cdot \big(-\frac{13}{384}\big)$ and $\beta_4 = \frac{3}{4} \cdot \frac{175}{64}$.) So $n_4 \leq -\frac{13}{512}\, n - \frac{525}{256} < 0$. But $n_4 \geq 1$. This contradiction shows that $(V(G), E_{<i})$ contains a good subgraph.

The last sentence of the lemma: the A-coordinate of f' is 0 exactly on the edges e with $f'(e) \in p^i\mathbb{Z}_k = B$, so $\operatorname{supp}(\overline{f'}) = \{\, e : f'(e) \notin p^i\mathbb{Z}_k \,\}$; and f' is Σ-joined to f. □

The count uses part (b) of Lemma 7.1 only for $j \leq 3$. Three levels would not be enough for this count, because $\alpha_3 > 0$.

The steps $\pm 2^j$ in $\mathbb{Z}_{2^r}$.

Lemma 7.3 (The powers of 2)**.** *Let $r \geq 2$, and let Δ_r be the graph on $\mathbb{Z}_{2^r}$ in which x and y are adjacent if $y - x$ is 2^j or -2^j for some j with $0 \leq j \leq r-1$. Then for every set X of at most $2r-2$ elements of $\mathbb{Z}_{2^r}$, the graph $\Delta_r - X$ is connected.*

Proof. The degree. Let $r' \geq 2$. In $\mathbb{Z}_{2^{r'}}$ the elements $\pm 2^j$, $0 \leq j \leq r'-1$, are $2r'-1$ different non-zero elements: they are not 0; $2^j \neq 2^l$ for $j < l$, since $1 \leq 2^j < 2^l \leq 2^{r'-1}$; and $2^j = -2^l$ with $j \leq l$ means $2^j + 2^l = 2^{r'}$ (the sum is between 2 and $2^{r'}$), so $2^j(1 + 2^{l-j}) = 2^{r'}$; for $j < l$ the factor $1 + 2^{l-j}$ is odd and at least 3, which is impossible; so $j = l = r' - 1$. Hence every vertex of $\Delta_{r'}$ has degree $2r' - 1$.

Induction on r. Δ_2 is the complete graph on 4 vertices (the steps are ± 1 and 2); after the deletion of at most 2 vertices it is connected. Let $r \geq 3$, and let X be a set of at most $2r-2$ elements. Each of the two classes of parity of $\mathbb{Z}_{2^r}$ has $2^{r-1} \geq 4$ elements. One of the two classes of parity contains at most $r-1$ elements of X. The translation $x \mapsto x + 1$ is an automorphism of Δ_r that exchanges the two classes. So we may assume that the even class contains $w \leq r-1$ elements of X.

The two classes. The map $2x \mapsto x$ is a bijection from the even elements of $\mathbb{Z}_{2^r}$ onto $\mathbb{Z}_{2^{r-1}}$. Two even elements $2x$, $2y$ are adjacent in Δ_r if and only if $2(y-x) = \pm 2^j$ in $\mathbb{Z}_{2^r}$ for some $j \geq 1$ (± 1 is odd), that is $y - x = \pm 2^{j-1}$ in $\mathbb{Z}_{2^{r-1}}$ with $0 \leq j-1 \leq r-2$. So the even class induces a copy of Δ_{r-1}. In the same way the odd class induces a copy of Δ_{r-1} (by $2x+1 \mapsto x$).

The even survivors. We have $w \leq r-1 \leq 2(r-1)-2$, because $r \geq 3$. By induction the even elements outside X induce a connected graph. There are $2^{r-1} - w > 0$ of them.

Attached odd survivors. An odd element x has exactly two even neighbours, $x-1$ and $x+1$ (the only odd step is ± 1, and $x-1 \neq x+1$). Call an odd element x outside X *attached* if $x-1$ or $x+1$ is outside X. An attached element is joined to the connected set of the even survivors. If x is not attached, then $x-1$ and $x+1$ are both in X. The even elements form a cycle of length $2^{r-1} \geq 4$ in which y and $y+2$ are consecutive; so $\{x-1, x+1\}$ is an edge of this cycle with both ends in X, and different x give different edges. The $w < 2^{r-1}$ even elements of X induce a union of paths in this cycle, with at most $\max\{0, w-1\}$ edges. So at most $\max\{0, w-1\}$ odd elements outside X are not attached.

The end. If $w \leq 1$, every odd element outside X is attached, and $\Delta_r - X$ is connected. Let $w \geq 2$. There are at least $2^{r-1} - (|X| - w) \geq 2^{r-1} - 2r + 2 + w$ odd elements outside X, and at most $w-1$ of them are not attached; so at least $2^{r-1} - 2r + 3$ of them are attached. In the odd class, which induces a copy of Δ_{r-1}, every element has degree $2(r-1)-1 = 2r-3$, so it has exactly $(2^{r-1}-1) - (2r-3) = 2^{r-1} - 2r + 2$ non-neighbours in the odd class other than itself. Let x be an odd element outside X that is not attached. The attached elements are at least $2^{r-1} - 2r + 3$ odd elements other than x, so one of them is a neighbour of x. So every element outside X is joined to the even survivors, and $\Delta_r - X$ is connected. □

The theorem.

Theorem 7.4 (Prime powers)**.** *Let p be a prime, $N \geq 6$ and $k = p^N$. Then*

(a) *$M_k(G) = F(G, \mathbb{Z}_k)$ is connected for every finite bridgeless multigraph G (loops allowed);*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

For example $k = 64$, 128, 256, 729, 15625.

Proof. We apply Theorem 3.19 with $\Gamma = \mathbb{Z}_k$, $k = ab$, $B = a\mathbb{Z}_k$, $A = \mathbb{Z}_a$, and with the hypotheses of Theorem 3.11. The data of the two cases:

	$p = 2$	p odd
$a = \lvert A \rvert$	2^{N-2} (≥ 16)	p^4 (≥ 81)
$b = \lvert B \rvert$	4	p^{N-4} $(\geq p^2 \geq 9)$
i with $a = p^i$	$N-2$	4
Σ	$\{\pm 2^j : 0 \leq j \leq N-1\}$	$\{\pm 1, \pm 2\} \cup \{\pm p^j : 1 \leq j \leq N-1\}$

In both cases $B = a\mathbb{Z}_k = p^i\mathbb{Z}_k$, and $4 \leq i \leq N-1$ (for $p = 2$ because $N \geq 6$).

(Σ) *A step system.* No element of Σ is 0 in $\mathbb{Z}_k$: the integers 1, 2 and p^j with $j \leq N-1$ lie between 1 and $k-1$. And $\Sigma = -\Sigma$. So Σ is a step system (Definition 3.1).

(Lift) *Liftable.* The elements ± 1 are liftable by Lemma 1.12 (b), ± 2 by Lemma 1.12 (c), and $\pm p^j$ for $0 \leq j \leq N-1$ by Lemma 1.12 (d), because p^j is a divisor of k with $1 \leq p^j < k$. So every element of Σ is liftable.

(Σ_A) *The A-parts.* The quotient map is the reduction modulo $a = p^i$, and p^j is 0 modulo p^i for $j \geq i$. For $p = 2$: Σ_A is the set of the residues of $\pm 2^j$, $0 \leq j \leq N-3$, modulo

2^{N-2}. For p odd: Σ_A is the set of the residues of ±1, ±2, $\pm p$, $\pm p^2$, $\pm p^3$ modulo p^4.

(Σ_B) *The steps in B.* For $p = 2$: $B = \{0, a, 2a, 3a\}$, and $a = 2^{N-2}$, $3a = -2^{N-2}$, $2a = 2^{N-1}$ are in Σ; so $\Sigma_B = B \setminus \{0\}$. For p odd: $\Sigma_B = \{\, \pm p^j : 4 \le j \le N-1 \,\}$, because ±1, ±2, $\pm p$, $\pm p^2$, $\pm p^3$ are not 0 modulo p^4; it contains $\pm a = \pm p^4$ and $\pm ap = \pm p^5$ ($5 \le N-1$).

(R3*) Let $x \neq x'$ be non-zero elements of $\mathbb{Z}_k$; take their representatives in $\{1, \dots, k-1\}$, with $x < x'$. The walk $x, x+1, \dots, x'$ stays in $\{1, \dots, k-1\}$, so it avoids 0, and its steps are $1 \in \Sigma$.

(R2) For $p = 2$: every two different elements of B differ by a non-zero element of B, which is in Σ_B; so any two elements of $B \setminus X$ are joined by one step, for every X. For p odd: the map $x \mapsto ax$ is an isomorphism of $\mathbb{Z}_b$ onto B, and it maps ±1, $\pm p$ into Σ_B. The prime p divides $b = p^{N-4}$, and $1 < p < b$ because $N - 4 \ge 2$. By Lemma 5.7 (for $b \ge 4$ and a divisor d of b with $1 < d < b$, the set $\mathbb{Z}_b$ minus at most 2 elements stays connected under the steps ±1, $\pm d$), used with $d = p$, the set B minus at most 2 elements stays connected under the steps of Σ_B.

(R1) For $p = 2$: the graph on $A = \mathbb{Z}_{2^{N-2}}$ with the steps Σ_A is the graph Δ_{N-2} of Lemma 7.3. Put $r = N - 2 \ge 4$; then $2r - 2 = 2N - 6 \ge 6$. By Lemma 7.3 it stays connected after the deletion of at most 6 elements. This is (R1).

For p odd: the ten residues in Σ_A are different and not 0. Indeed: $0 < p^j < p^4$ for $0 \le j \le 3$, so $p^j \neq 0$, and $p^j \neq p^l$ for $j \neq l$. If $p^j = -p^l$ modulo p^4 with $j \le l \le 3$, then $p^j + p^l = p^4$, because the sum is between 2 and $2p^3 < 2p^4$; so $p^j(1 + p^{l-j}) = p^4$; for $j < l$ the factor $1 + p^{l-j}$ is not divisible by p and is greater than 1, which is impossible; and $j = l$ gives $2p^j = p^4$, which is impossible for odd p. Next, $2 = \pm p^j$ modulo p^4 would mean that p^4 divides $p^j \mp 2$; this number is not 0, because p is odd, and its absolute value is at most $p^3 + 2 < p^4$. The same holds for -2. Finally $2 \neq 0$ and $2 \neq -2$ modulo p^4, because p^4 divides neither 2 nor 4. So $\Sigma_A = -\Sigma_A$, 1 is in Σ_A, and $|\Sigma_A| = 10 \ge 9$. By Lemma 5.9 (for $a \ge 10$, a set of at least 9 non-zero elements of $\mathbb{Z}_a$ that is closed under $x \mapsto -x$ and contains 1, used as a step set: $\mathbb{Z}_a$ minus at most 6 elements stays connected), used with $a = p^4 \ge 81$ and the step set Σ_A, (R1) holds.

(AV″) Σ contains 1 and p^j for $1 \le j \le i-1$: for $p = 2$ the elements 2^j with $j \le N-3$; for p odd the elements p, p^2, p^3. So Lemma 7.2, with this i, gives (AV″) for $\Gamma = \mathbb{Z}_k$, $B = p^i\mathbb{Z}_k = a\mathbb{Z}_k$.

(Counts) *The counts.* $|A| = a \ge 16 \ge 10$ and $|B| = b \ge 4$. So (C0) holds, and by Remark 2.16 every frame is counted. Every subdivision has a frame (Lemma A.5; see Definition 2.9). So in (AV″) the good subgraph is a cycle with at most 5 edges or a subdivision with a counted frame: (H2) holds. This is the check in the proof of Corollary 3.15.

So Γ, B, Σ satisfy the hypotheses (H1) and (H2) of Theorem 3.11, and every element of Σ is liftable. Theorem 3.19 gives (a) and (b). □

Remark 7.5. (1) Where $N \ge 6$ is used. For $p = 2$: Lemma 7.2 needs $i = N - 2 \ge 4$, Remark 2.16 needs $|A| = 2^{N-2} \ge 10$, and (R1) needs $2(N-2) - 2 \ge 6$ in Lemma 7.3. For p

odd: Lemma 5.7 needs a divisor of $b = p^{N-4}$ strictly between 1 and b, so $N - 4 \geq 2$. The case $N \leq 5$ is not treated here.

(2) What is quoted. For $p = 2$ the proof uses no quoted theorem beyond those of Sections 1 to 3. For p odd, Lemma 5.9 rests on the theorem of Mader and Watkins on vertex-transitive graphs. For $p = 2$ the steps ± 2 are of the form $\pm p^j$; for p odd they are needed in this argument, because without them $|\Sigma_A| = 8$.

8. Two coprime factors

This section proves connectivity for a modulus $k = ab$ whose two factors a and b are coprime and each large enough (Theorem 8.8 and Corollary 8.9). It gives every multiple of $1120 = 32 \cdot 35$, every product pq of two different primes $p, q \geq 43$, and every p^2q with different primes $p \geq 7$ and $q \geq 43$. For a product of two different primes it is the only result of this paper that gives the integer graphs $I_k(G)$ for all graphs G.

The proof is not a case of Theorem 3.11. Every move changes only one of the two residues of the flow, modulo a or modulo b, on a cycle on which the other residue has no zero. We call such a move a *guarded move*. The divisor extension (Theorem 1.5) lifts every guarded move to the integer flows (Lemma 8.2). Remark 8.11 says what Theorem 3.11 gives for these moduli, and why Theorem 3.19 does not give the integer flows here.

Guarded moves. In this section $a, b \geq 2$ are coprime integers and $k = ab$, except in the subsection 'List avoidance on a 3-edge-connected graph'. For x in $\mathbb{Z}_k$, $\rho_a(x)$ and $\rho_b(x)$ are its residues modulo a and modulo b. By the Chinese remainder theorem the map $x \mapsto (\rho_a(x), \rho_b(x))$ is a bijection from $\mathbb{Z}_k$ onto $\mathbb{Z}_a \times \mathbb{Z}_b$, and it is additive. The map ρ_a is a bijection from the subgroup $b\mathbb{Z}_k$ (it has a elements) onto $\mathbb{Z}_a$: if $\rho_a(bx) = 0$, then a divides x, because a and b are coprime, and so $bx = 0$ in $\mathbb{Z}_k$. Let $\iota_a \colon \mathbb{Z}_a \to b\mathbb{Z}_k$ be its inverse. So $\rho_a(\iota_a(z)) = z$ and $\rho_b(\iota_a(z)) = 0$ for every z in $\mathbb{Z}_a$. In the same way $\iota_b \colon \mathbb{Z}_b \to a\mathbb{Z}_k$ is the inverse of the restriction of ρ_b to $a\mathbb{Z}_k$. Every x in $\mathbb{Z}_k$ is $\iota_a(\rho_a(x)) + \iota_b(\rho_b(x))$.

For a map f from $E(G)$ to $\mathbb{Z}_k$, the maps $\rho_a(f)$ and $\rho_b(f)$ are taken edge by edge. The map f is a flow if and only if $\rho_a(f)$ and $\rho_b(f)$ are flows, and $f(e) = 0$ if and only if $\rho_a(f)(e) = 0$ and $\rho_b(f)(e) = 0$. For a $\mathbb{Z}_a$-flow w of G, the map $\iota_a(w)$ is a $\mathbb{Z}_k$-flow with values in $b\mathbb{Z}_k$, because ι_a is additive; in the same way for ι_b.

Definition 8.1 (Guarded moves)**.** Let f be a nowhere-zero $\mathbb{Z}_k$-flow of a multigraph G. A move $f \to f + \lambda\chi_C$ of $M_k(G)$ is a *guarded a-move* if λ is in $b\mathbb{Z}_k$ and $\rho_b(f)(e) \neq 0$ for every edge e of C. It is a *guarded b-move* if λ is in $a\mathbb{Z}_k$ and $\rho_a(f)(e) \neq 0$ for every edge e of C. A *guarded move* is a guarded a-move or a guarded b-move.

A guarded a-move does not change the residues modulo b, so its reverse is a guarded a-move. If $\lambda \neq 0$ is in $b\mathbb{Z}_k$ and $\rho_b(f)$ has no zero on C, then $f + \lambda\chi_C$ is nowhere zero: on C its residue modulo b is that of f, and outside C nothing changes. So $f \to f + \lambda\chi_C$ is then a guarded a-move. The same holds with a and b exchanged.

Lemma 8.2 (Guarded moves)**.** (a) *Let f be a nowhere-zero $\mathbb{Z}_k$-flow of a multigraph G, let E' be a set of edges with $\rho_b(f)(e) \neq 0$ for every e in E', and let w be a $\mathbb{Z}_a$-flow of G that is 0 outside E'. Then f is joined to $f + \iota_a(w)$ by a sequence of guarded a-moves. All flows of the sequence are equal to f outside E', and their residues modulo b are those of f. The same holds with a and b exchanged.*

(b) *Let $f \to g$ be a guarded move of $M_k(G)$. Then all integer lifts of f and all integer lifts of g lie in one component of $I_k(G)$.*

Proof. (a) The restriction of w to the graph $(V(G), E')$ is a $\mathbb{Z}_a$-flow of this graph. By Lemma 1.1 (a), $w = x_1\chi_{C_1} + \cdots + x_\nu\chi_{C_\nu}$ with x_r in $\mathbb{Z}_a$ and cycles C_r all of whose edges are in E'. Put $f_r = f + \sum_{r' \le r} \iota_a(x_{r'})\chi_{C_{r'}}$. Then $f_0 = f$ and $f_\nu = f + \iota_a(w)$, because ι_a is additive. Every f_r has the residues of f modulo b, because the values of ι_a are in $b\mathbb{Z}_k$; so $f_r(e) \neq 0$ for e in E'. Outside E' we have $f_r = f$, which is not 0. If $x_r \neq 0$, the step from f_{r-1} to f_r is the move with $\lambda = \iota_a(x_r) \neq 0$ on C_r, and $\rho_b(f_{r-1}) = \rho_b(f)$ has no zero on C_r: a guarded a-move. If $x_r = 0$, then $f_r = f_{r-1}$, and we skip the step.

(b) Let $g = f + \lambda\chi_C$ be a guarded a-move. Then f and g have the same reduction $w = \rho_b(f)$ modulo b. If $w(e) = 0$, then e is not on C, so $f(e) = g(e)$. Theorem 1.5 with $m = a$ and $q = b$ gives the claim. For a guarded b-move use Theorem 1.5 with $m = b$ and $q = a$. □

Lemma 8.3 (Averaging in one coordinate)**.** *Let G be a multigraph with m edges and f a nowhere-zero $\mathbb{Z}_k$-flow of G. Then f is joined by guarded a-moves to a flow f' with $\rho_b(f') = \rho_b(f)$ such that $\rho_a(f')$ has at most m/a zeros.*

Proof. Let E' be the set of the edges e with $\rho_b(f)(e) \neq 0$, and $G' = (V(G), E')$. Every edge e of E' lies on a cycle of G'. If not, e is not a loop, and its two ends are in different components of $G' - e$. Let X be the vertex set of the component of $G' - e$ that contains the head of e. Add the conservation equations of the $\mathbb{Z}_b$-flow $\rho_b(f)$ over the vertices of X. An edge with both ends in X drops out, the edges outside E' carry 0, and e is the only edge of E' with exactly one end in X. So $\rho_b(f)(e) = 0$, a contradiction.

Let $\mathcal{F}$ be the finite group of the $\mathbb{Z}_a$-flows of G'. For an edge e of E' the map $y \mapsto y(e)$ is a homomorphism from $\mathcal{F}$ onto $\mathbb{Z}_a$: e lies on a cycle C of G', and $x\chi_C$ is in $\mathcal{F}$ with the value x or $-x$ on e. So for every e in E' exactly $|\mathcal{F}|/a$ flows y in $\mathcal{F}$ have $\rho_a(f)(e) + y(e) = 0$. Summed over e in E': the number of pairs (y, e) with y in $\mathcal{F}$, e in E' and $\rho_a(f)(e) + y(e) = 0$ is $|E'|\,|\mathcal{F}|/a$. So some y in $\mathcal{F}$ has $\rho_a(f)(e) + y(e) = 0$ for at most $|E'|/a \le m/a$ edges e of E'. Extend y by 0 to a $\mathbb{Z}_a$-flow of G. For an edge e outside E' we have $\rho_b(f)(e) = 0$, so $\rho_a(f)(e) \neq 0$, because $f(e) \neq 0$; and $y(e) = 0$. So $\rho_a(f) + y$ has at most m/a zeros. By Lemma 8.2 (a) with this E', f is joined by guarded a-moves to $f' = f + \iota_a(y)$, with $\rho_a(f') = \rho_a(f) + y$ and $\rho_b(f') = \rho_b(f)$. □

List avoidance on a 3-edge-connected graph. In this subsection Γ is a finite abelian group and B is a subgroup of Γ; a, b and k are not used.

Definition 8.4 (One-value avoidance)**.** A finite abelian group Γ has *one-value avoidance* if for every good subgraph H of a multigraph (Definition 2.1) and every map $z\colon E(H) \to \Gamma$ there is a Γ-flow w of H with $w(e) \neq z(e)$ for every edge e of H.

If Γ has one-value avoidance, then $|\Gamma| \ge 6$: on a cycle C with 5 edges the flows are the maps $x\chi_C$, and every edge excludes one value of x.

The next two lemmas use the objects of Section 2 for a loopless 3-edge-connected multigraph G on at least two vertices, seen as the subdivision $H = G$ of $H_0 = G$ in which every thread has one edge; this is a good subgraph of the kind (SUB) of Definition 2.1. So there are the auxiliary graph Q (Definition 2.6), a frame (Definition 2.9) and its official cycles $C_1, \ldots, C_s$ (Definition 2.10). The original edges of Q are the edges of G; the free edges are the original red edges. The reference orientation of G is the direction of the threads; a change of the reference orientation of an edge e replaces the values of all flows on e, and a set of values excluded on e, by their negatives, and changes nothing else. By Lemma 2.7 (c), the restriction of a Γ-flow of Q to the original edges is a Γ-flow of G. The reverse choice, Lemma 3.4,

applies with the trivial subgroup $\{0\}$ in the place of B, so that Γ is in the place of A; we use its part (a) in this form. For a free edge ε, let F_ε be the cycle formed by ε and the path between the ends of ε in the blue spanning tree of the component of Q that contains ε (Definition 2.9 (F1)), directed along ε. Its only red edge is ε.

Lemma 8.5 (Subgroup list avoidance)**.** *Let $j \geq 1$ be an integer, and let $|B| \geq j+1$ and $|\Gamma| > 2j\,|B|$. Let G be a loopless 3-edge-connected multigraph on at least two vertices, and for every edge e of G let $\Phi(e)$ be a set of at most j elements of Γ. Then G has a Γ-flow w with $w(e) \notin \Phi(e)$ for every edge e.*

Proof. Use the frame and the notation above. Define a list assignment: $\Lambda(t) = \Phi(t) + B = \{\, z + y : z \in \Phi(t),\ y \in B \,\}$ for every original blue edge t, and $\Lambda(t)$ empty for every blue expansion edge. Then $|\Lambda(t)| \leq j\,|B|$, so $|\Lambda(e_i)| + |\Lambda(f_i)| \leq 2j\,|B| < |\Gamma|$ for every node i. By Lemma 3.4 (a) with $Y_0 = 0$ there is a Γ-flow Y of Q with $Y(t) \notin \Lambda(t)$ for every blue edge t. So $Y(t) + y \notin \Phi(t)$ for every original blue edge t and every y in B.

Let ε be a free edge. The $|B| \geq j+1$ elements $Y(\varepsilon)+y$, y in B, are different, and at most j of them are in $\Phi(\varepsilon)$. Choose y_ε in B with $Y(\varepsilon)+y_\varepsilon \notin \Phi(\varepsilon)$. Put $Y' = Y + \sum_\varepsilon y_\varepsilon \chi_{F_\varepsilon}$, the sum over all free edges. A free edge ε lies on F_ε and on no other cycle $F_{\varepsilon'}$, so $Y'(\varepsilon) = Y(\varepsilon) + y_\varepsilon \notin \Phi(\varepsilon)$. On a blue edge t the difference $Y'(t) - Y(t)$ is a sum of elements $\pm y_\varepsilon$, so it is in B; hence $Y'(t) \notin \Phi(t)$ if t is original. Every original edge is blue or free. By Lemma 2.7 (c) the restriction w of Y' to the original edges is a Γ-flow of G, and $w(e) \notin \Phi(e)$ for every edge e of G. □

Lemma 8.6 (Cyclic list avoidance)**.** *Let $j \geq 1$ and $N > 2j(2j+1)$ be integers. Let G be a loopless 3-edge-connected multigraph on at least two vertices, and for every edge e of G let $\Phi(e)$ be a set of at most j elements of $\mathbb{Z}_N$. Then G has a $\mathbb{Z}_N$-flow w with $w(e) \notin \Phi(e)$ for every edge e.*

Proof. Use the frame and the notation above, with $\Gamma = \mathbb{Z}_N$, and put $d = j + 1$. For every original blue edge t let $\Lambda(t)$ be the set of the elements $z - y$ of $\mathbb{Z}_N$ with z in $\Phi(t)$ and an integer y with $|y| \leq j$; for a blue expansion edge $\Lambda(t)$ is empty. Then $|\Lambda(t)| \leq j(2j+1)$, so $|\Lambda(e_i)| + |\Lambda(f_i)| \leq 2j(2j+1) < N$ for every node i. By Lemma 3.4 (a) with $Y_0 = 0$ there is a $\mathbb{Z}_N$-flow Y of Q with $Y(t) \notin \Lambda(t)$ for every blue edge t. So $Y(t) + y \notin \Phi(t)$ for every original blue edge t and every integer y with $|y| \leq j$.

The $2j+1$ integers y with $|y| \leq j$ have different residues modulo N, because $2j < N$. Let ε be a free edge, and let $\Psi(\varepsilon)$ be the set of the residues modulo d of the integers y with $|y| \leq j$ and $Y(\varepsilon) + y$ in $\Phi(\varepsilon)$. For every z in $\Phi(\varepsilon)$ at most one such y has $Y(\varepsilon) + y = z$; so $|\Psi(\varepsilon)| \leq j < d$. Choose x_ε in $\mathbb{Z}_d$ outside $\Psi(\varepsilon)$. Let η be the $\mathbb{Z}_d$-flow $\sum_\varepsilon x_\varepsilon \chi_{F_\varepsilon}$ of Q, the sum over all free edges; so $\eta(\varepsilon) = x_\varepsilon$ for every free edge ε. By Corollary 1.3 (for the modulus $d \geq 2$, zeros allowed) there is an integer flow L of Q with $L = \eta$ modulo d and $|L(e)| \leq d - 1 = j$ for every edge e. Put $Y' = Y + L$, where L is read modulo N; this is a $\mathbb{Z}_N$-flow of Q.

On an original blue edge t: $|L(t)| \leq j$, so $Y'(t) = Y(t) + L(t) \notin \Phi(t)$. On a free edge ε: if $Y(\varepsilon) + L(\varepsilon)$ were in $\Phi(\varepsilon)$, the residue of $L(\varepsilon)$ modulo d would be in $\Psi(\varepsilon)$, because $|L(\varepsilon)| \leq j$; but this residue is x_ε. So $Y'(\varepsilon) \notin \Phi(\varepsilon)$. By Lemma 2.7 (c) the restriction w of Y' to the original edges is a $\mathbb{Z}_N$-flow of G with $w(e) \notin \Phi(e)$ for every edge e of G. □

The integer flow L is what makes the corrections on the free edges work in the two moduli d and N at once: it is one integer flow, and it is small on every edge.

Corollary 8.7 (One-value avoidance)**.** *Let $N = de$ with integers $d \geq 4$ and $e \geq 7$, or let $N \geq 43$. Then $\mathbb{Z}_N$ has one-value avoidance.*

Proof. Let H be a good subgraph of a multigraph, and $z\colon E(H) \to \mathbb{Z}_N$. In both cases $N \geq 28$.

H is a cycle with at most 5 edges. Direct it. Its $\mathbb{Z}_N$-flows are the maps $x\chi_H$, x in $\mathbb{Z}_N$ (conservation at its vertices of degree 2). An edge e excludes the one value $x = \chi_H(e)\, z(e)$. At most $5 < N$ values are excluded; take another one.

H is a subdivision as in Definition 2.1 (SUB). Give the edges of every thread the direction of the thread; this changes z by signs only. By Lemma 2.7 (a) the $\mathbb{Z}_N$-flows of H are the maps with the value $w_0(t)$ on every edge of P_t, for a $\mathbb{Z}_N$-flow w_0 of H_0. Put $\Phi(t) = \{\, z(e) : e \in E(P_t) \,\}$; it has at most 3 elements. H_0 is loopless and 3-edge-connected, with at least two vertices. If $N = de$ with $d \geq 4$ and $e \geq 7$, let B be the subgroup of $\mathbb{Z}_N$ of order d (the multiples of e): $|B| = d \geq 4 = 3+1$ and $N = de \geq 7d > 6d = 2 \cdot 3 \cdot |B|$. Lemma 8.5 with $j = 3$ gives w_0 with $w_0(t) \notin \Phi(t)$ for every edge t of H_0. If $N \geq 43$, then $N > 42 = 2 \cdot 3 \cdot (2 \cdot 3 + 1)$, and Lemma 8.6 with $j = 3$ gives such a w_0. The flow of H that has the value $w_0(t)$ on P_t avoids z. □

The theorem of two coprime factors. From here on a, b and $k = ab$ are as in the subsection 'Guarded moves'.

Theorem 8.8 (Two coprime factors)**.** *Let $a \geq 9$ and $b \geq 6$ be coprime integers such that $\mathbb{Z}_a$ and $\mathbb{Z}_b$ have one-value avoidance, and let $k = ab$.*

(a) *On every bridgeless multigraph G (loops allowed) any two nowhere-zero $\mathbb{Z}_k$-flows are joined by a sequence of guarded moves. In particular $M_k(G)$ is connected.*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

Proof. A *target* of a multigraph is a $\mathbb{Z}_k$-flow h such that $\rho_a(h)$ and $\rho_b(h)$ are both nowhere zero. A target is nowhere zero.

Claim. On every connected bridgeless multigraph G (loops allowed), every nowhere-zero $\mathbb{Z}_k$-flow is joined to every target by a sequence of guarded moves.

(1) *The claim gives the theorem.* Let G be bridgeless. By Theorem 0.1 (Seymour, Tutte), every bridgeless multigraph, loops allowed, has a nowhere-zero flow with values in any abelian group of order at least 6. So G has a nowhere-zero $\mathbb{Z}_a$-flow h_a and a nowhere-zero $\mathbb{Z}_b$-flow h_b, because $a \geq 9$ and $b \geq 6$. Then $h = \iota_a(h_a) + \iota_b(h_b)$ is a target. A cycle of G lies in one component of G, and a guarded move of a component is a guarded move of G. So, component after component, the claim joins every nowhere-zero $\mathbb{Z}_k$-flow of G to h by guarded moves. The reverse of a guarded move is a guarded move. This gives (a), because guarded moves are moves of $M_k(G)$.

For (b) let G be loopless, and let F, F' be vertices of $I_k(G)$. Their reductions f, f' modulo k are nowhere-zero $\mathbb{Z}_k$-flows, because $1 \leq |F(e)| \leq k-1$. By (a) a sequence of guarded moves leads from f to f'. By Lemma 8.2 (b), all integer lifts of two consecutive flows of the sequence lie in one component of $I_k(G)$. Every flow of the sequence has an integer lift, and all integer lifts of one flow lie in one component (Lemma 1.12 (a); $k \geq 54$). So all of them lie in one component, and it contains F and F'.

(2) *The induction.* We prove the claim by induction on the number n of vertices of G; for each n first for the loopless graphs, then for the graphs with loops. A connected loopless multigraph with one vertex has no edge and only the empty flow. A connected

bridgeless loopless multigraph with $n \geq 2$ vertices has a cut with exactly 2 edges or is 3-edge-connected: a cut with one edge consists of a bridge. So three cases remain: (I), (II), (III). In all of them f is a nowhere-zero $\mathbb{Z}_k$-flow and h is a target of G.

(I) *G has a loop.* Let G^- be G without its loops. It is connected, bridgeless and loopless, and it has n vertices; so the claim holds for G^-. The restrictions of f and h to G^- are a nowhere-zero flow and a target. A guarded move of G^- is a guarded move of G: its cycle has no loop, and the values on the loops do not change. So f is joined by guarded moves to the flow that is equal to h on $E(G^-)$ and to f on the loops. Now let e be a loop, directed, with the values $x = f(e) \neq 0$ and $x' = h(e)$; a loop is a cycle with one edge. If $\rho_b(x) \neq 0$, first add $\iota_a(\rho_a(x') - \rho_a(x))\,\chi_e$: a guarded a-move (if it is not 0), because $\rho_b(x) \neq 0$; the new value x_1 has $\rho_a(x_1) = \rho_a(x') \neq 0$. Then add $\iota_b(\rho_b(x') - \rho_b(x_1))\,\chi_e$: a guarded b-move, because $\rho_a(x_1) \neq 0$. The new value is x'. If $\rho_b(x) = 0$, then $\rho_a(x) \neq 0$, and the two moves are made in the other order: first the b-move, guarded by $\rho_a(x) \neq 0$, then the a-move, guarded by $\rho_b(x') \neq 0$. No other edge changes. Do this for every loop. The result is h.

(II) *G is loopless and has a cut with exactly two edges e_1, e_2.* Let G/e_1 be obtained from G by the identification of the two ends of e_1 and the deletion of e_1. It is connected and has $n-1$ vertices. It is bridgeless: if an edge $e \neq e_1$ lies on a cycle O of G, then the edges of O other than e_1 form a closed trail of G/e_1 through e, and every edge of a closed trail lies on a cycle of the trail. It can have loops. By induction the claim holds for G/e_1. The restriction of a flow of G to $E(G/e_1)$ is a flow of G/e_1 (add the two conservation equations at the ends of e_1), and every flow of G/e_1 is the restriction of exactly one flow of G: conservation at one end of e_1 fixes the value on e_1. In a flow of G, with values in any abelian group, the value on e_1 is the value on e_2 or its negative (conservation over one side of the cut). So a flow of G is nowhere zero, or a target, if and only if its restriction is. Let $g' \to g' + \lambda\chi_{C'}$ be a guarded move of G/e_1. The edges of C' form a cycle of G, or a path of G between the two ends of e_1; in the second case add e_1. This gives a cycle C of G with $\chi_C = \chi_{C'}$ on $E(G/e_1)$. In the second case the path crosses the cut and does not use e_1, so C' contains e_2. Let g be the flow of G with restriction g'. Then $g + \lambda\chi_C$ is the flow of G with restriction $g' + \lambda\chi_{C'}$, and $g \to g + \lambda\chi_C$ is a guarded move of the same kind: the residue that guards is not 0 on the edges of C', and, if e_1 is on C, also on e_1, because its value there is $\pm$ its value on e_2. So the claim for G/e_1 gives the claim for G.

(III) *G is loopless and 3-edge-connected, $n \geq 2$.* Let m be the number of edges. Every vertex has degree at least 3, so $m \geq 3n/2$. By Lemma 8.3, f is joined by guarded a-moves to a flow f' such that $\rho_a(f')$ has at most $m/a \leq m/9$ zeros. So $\operatorname{supp}(\rho_a(f'))$ has at least $(8/9)\,m \geq (4/3)\,n > (4/3)\,n - 2$ edges. If $n \geq 3$, Lemma 2.2 (a) gives a good subgraph H of the loopless multigraph $(V(G), \operatorname{supp}(\rho_a(f')))$. If $n = 2$, then $m \geq 3$, and the support has at least $8m/9 > 2$ edges; two of them are parallel and form a good subgraph H of the kind (CYC). In both cases H is connected, it has at least two vertices, and $\rho_a(f')$ has no zero on $E(H)$.

Let $G^* = G/H$ be obtained from G by the identification of $V(H)$ to one vertex v_H and the deletion of $E(H)$. It has fewer vertices than G, and it is connected. It is bridgeless: if an edge e of G^* lies on a cycle O of G, the edges of O outside $E(H)$ form a closed trail of G^* through e, and every edge of a closed trail lies on a cycle of the trail. It can have loops. By induction the claim holds for G^*. The restriction of a flow of G to $E(G^*)$ is a

flow of G^* (add the conservation equations at the vertices of H; the edges of H drop out). So the restriction of f' is a nowhere-zero flow of G^*, and the restriction h^* of h is a target of G^*. So there are guarded moves of G^* that lead from the restriction of f' through flows $g_0, g_1, \dots, g_\nu = h^*$.

The cycles. For a cycle C^* of G^* we fix a cycle C of G with $\chi_C = \chi_{C^*}$ on $E(G^*)$ and $E(C) \subseteq E(C^*) \cup E(H)$. If C^* does not pass v_H, its edges form a cycle of G; take it. Otherwise C^* passes v_H once, and its edges form, in G, a cycle or a path P between two different vertices v_1, v_2 of H whose inner vertices are not in $V(H)$; a loop of G^* at v_H is such a path with one edge, because G is loopless. Add a path of H between v_2 and v_1; H is connected. Direct C along C^*.

The invariant. We build flows $x_0 = f', x_1, \dots, x_\nu$ of G, each joined to the one before it by guarded moves, such that the restriction of x_r to $E(G^*)$ is g_r and $\rho_a(x_r)$ has no zero on $E(H)$. Such a flow is nowhere zero. $x_0 = f'$ has these properties. Let $x = x_{r-1}$ be built, and let $g_r = g_{r-1} + \lambda\chi_{C^*}$, with the cycle C of C^*.

A guarded b-move of G^* (λ in $a\mathbb{Z}_k$, $\rho_a(g_{r-1})$ has no zero on C^*). Then $\rho_a(x)$ has no zero on $E(C)$: not on $E(H)$ by the invariant, and not on $E(C^*)$, where x is equal to g_{r-1}. So $x \to x + \lambda\chi_C$ is a guarded b-move. Put $x_r = x + \lambda\chi_C$. It has the residues of x modulo a, and its restriction to $E(G^*)$ is g_r.

A guarded a-move of G^* (λ in $b\mathbb{Z}_k$, $\rho_b(g_{r-1})$ has no zero on C^*).

(i) $\mathbb{Z}_b$ has one-value avoidance. So there is a $\mathbb{Z}_b$-flow w_b of H with $w_b(e) \neq -\rho_b(x)(e)$ for every edge e of H. Extend it by 0 to a flow of G. By Lemma 8.2 (a), with a and b exchanged and $E' = E(H)$ ($\rho_a(x)$ has no zero there), x is joined by guarded b-moves to $x' = x + \iota_b(w_b)$. Now $\rho_b(x')$ has no zero on $E(H)$; $\rho_a(x') = \rho_a(x)$; and $x' = x$ outside $E(H)$.

(ii) $\mathbb{Z}_a$ has one-value avoidance. So there is a $\mathbb{Z}_a$-flow w_a of H with $w_a(e) \neq -\rho_a(x')(e) - \rho_a(\lambda)\chi_C(e)$ for every edge e of H. Extend it by 0. By Lemma 8.2 (a) with $E' = E(H)$ ($\rho_b(x')$ has no zero there), x' is joined by guarded a-moves to $x'' = x' + \iota_a(w_a)$. Now $\rho_b(x'') = \rho_b(x')$, and $x'' = x$ outside $E(H)$. ($\rho_a(x'')$ can have zeros on $E(H)$.)

(iii) $\rho_b(x'')$ has no zero on $E(C)$: not on $E(H)$ by (i), and not on $E(C^*)$, where x'' is equal to g_{r-1}. So $x'' \to x'' + \lambda\chi_C$ is a guarded a-move. Put $x_r = x'' + \lambda\chi_C$. Its restriction to $E(G^*)$ is $g_{r-1} + \lambda\chi_{C^*} = g_r$. On $E(H)$, $\rho_a(x_r) = \rho_a(x') + w_a + \rho_a(\lambda)\chi_C$, which has no zero by the choice of w_a.

The end. The flow x_ν is equal to h outside $E(H)$, and $\rho_a(x_\nu)$ has no zero on $E(H)$. So $\rho_a(x_\nu)$ has no zero at all, because $\rho_a(h)$ has none. By Lemma 8.2 (a), with a and b exchanged and $E' = E(G)$, x_ν is joined by guarded b-moves to the flow x' with $\rho_b(x') = \rho_b(h)$ and $\rho_a(x') = \rho_a(x_\nu)$ (add ι_b of the $\mathbb{Z}_b$-flow $\rho_b(h) - \rho_b(x_\nu)$). Now $\rho_b(x') = \rho_b(h)$ has no zero. By Lemma 8.2 (a) with $E' = E(G)$, x' is joined by guarded a-moves to the flow with the residues $\rho_a(h)$ and $\rho_b(h)$, that is to h. Together: f is joined to f', to x_ν, and to h by guarded moves. □

Corollary 8.9 (Two admitted factors)**.** *Call an integer $N \geq 2$ an* admitted factor *if $N = de$ with integers $d \geq 4$ and $e \geq 7$, or if $N \geq 43$.*

(a) *Let $k = ab$ with coprime admitted factors a and b. Then $M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed), and $I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

(b) *Every multiple of a product of two coprime admitted factors is again such a product.*

(c) *In particular* (a) *holds for every multiple of* $1120 = 32 \cdot 35$*, for* $k = pq$ *with different primes* $p, q \geq 43$*, and for* $k = p^2q$ *with different primes* $p \geq 7$ *and* $q \geq 43$*.*

Proof. (a) An admitted factor is at least $4 \cdot 7 = 28$. So $a \geq 9$ and $b \geq 6$. By Corollary 8.7, $\mathbb{Z}_a$ and $\mathbb{Z}_b$ have one-value avoidance. Theorem 8.8 gives the claim.

(b) Let $k = ab$ with coprime admitted factors a, b, and let k' be a multiple of k. For every prime p that divides k'/k, let p^v be the largest power of p that divides k'/k. Put it to b if p divides b, and to a otherwise. This gives $k' = a'b'$, where a' is a multiple of a and b' is a multiple of b. A prime that divides a' divides a or was put to a; in both cases it does not divide b, and it was not put to b. So a' and b' are coprime. A multiple of an admitted factor is an admitted factor: a multiple of de is de' with a multiple e' of e, so $e' \geq 7$; and a multiple of a number $N \geq 43$ is at least 43.

(c) $32 = 4 \cdot 8$ and $35 = 5 \cdot 7$ are coprime admitted factors; use (b). A prime $p \geq 43$ is an admitted factor. For a prime $p \geq 7$, $p^2 = p \cdot p$ with $p \geq 4$ and $p \geq 7$ is an admitted factor. □

Remark 8.10 (The admitted factors)**.** The admitted factors below 43 are 28, 32, 35, 36, 40, 42. So the admitted factors are the integers $N \geq 28$ other than 29, 30, 31, 33, 34, 37, 38, 39, 41. The smallest product of two coprime admitted factors is 1120. Indeed, let $ab < 1120$ with admitted factors $a \leq b$. Then $a^2 < 1120$ gives $a \leq 33$, so a is 28 or 32, and $b < 1120/28 = 40$ gives b in $\{28, 32, 35, 36\}$; of these pairs only 32, 35 is coprime, and $32 \cdot 35$ is not smaller than 1120. Every multiple of 1120 is also a multiple of 4 from 52 on, so Theorem 9.2 gives it as well. The two lemmas give one-value avoidance only for the admitted factors; other orders are not treated.

Remark 8.11 (The master theorem for these moduli)**.** Let $a \geq 10$ and $b \geq 4$ be coprime and $k = ab$. The connectivity of $M_k(G)$ in Theorem 8.8 (a) then also follows from Theorem 3.11. Put $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (order b), $A = \mathbb{Z}_a$, and $\Sigma = \Gamma \setminus \{0\}$. Then $\Sigma_A = A \setminus \{0\}$ and $\Sigma_B = B \setminus \{0\}$, so (R1), (R2) and (R3*) hold with walks of one step. By the Chinese remainder theorem $\mathbb{Z}_k = \mathbb{Z}_a \times \mathbb{Z}_b$; the element b of Σ generates the factor $\mathbb{Z}_a \times 0$ and the element a generates $0 \times \mathbb{Z}_b$, and $(|B| - 1)/(|\Gamma| - 1) = (b - 1)/(ab - 1) < 1/9$ because $a \geq 10$. So Lemma 4.1 gives (AV) and (AV″). And $|A| = a \geq 10$, $|B| = b \geq 4$: Corollary 3.15 applies. This is a special case of Theorem 4.5, for $\Gamma = \mathbb{Z}_k$.

Theorem 3.19 does not give the integer graphs $I_k(G)$ in this way, because it asks that every element of $\Sigma = \Gamma \setminus \{0\}$ is liftable (Definition 1.7), and this is not shown. For $k = pq$ a smaller Σ does not help either. Let $k = pq$ with different primes $p, q \geq 5$. The elements that Lemma 1.12 shows to be liftable are ± 1, ± 2, $\pm p$, $\pm q$: in part (e), $c = 1$ modulo p and modulo q means $c = 1$ modulo k. With $B = p\mathbb{Z}_k$, the only ones of them in B are $\pm p$. The map $jp \mapsto j$ identifies B with $\mathbb{Z}_q$ and $\pm p$ with the steps ± 1, and $\mathbb{Z}_q$ without the two elements 1, 3 is not connected by these steps. So (R2) fails; the same holds for $B = q\mathbb{Z}_k$. The guarded moves do not need liftable steps: Lemma 8.2 (b) lifts every guarded move, whatever its step.

9. Every multiple of four from 52 on; the moduli by their number of prime factors

This section collects the integer moduli. Theorem 9.2 gives $k = 40$, $k = 48$ and every multiple of 4 from 52 on. Theorem 9.5 gives every k with at least four prime factors that is not a prime power, except 24, 36 and 54. Proposition 9.7 and Corollary 9.8 say exactly

which k these results reach (Definition 9.6), by the number of prime factors. Corollary 9.9: the moduli k for which $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G have natural density one.

$\Omega(k)$ is the number of prime factors of k counted with multiplicity, $\omega(k)$ the number of different prime factors, and $\tau(a)$ the number of positive divisors of a. For a set D of non-zero elements of $\mathbb{Z}_a$ with $D = -D$, the *graph on $\mathbb{Z}_a$ with the step set D* has the vertex set $\mathbb{Z}_a$; two vertices x, y are adjacent if $y - x$ is in D. The *sign class* of an element x is $\{x, -x\}$. For a split $k = ab$, D always denotes the step set of Criterion 5.11: the non-zero residues modulo a of ± 1, ± 2 and $\pm q$, for all divisors q of k with $1 \le q < k$.

How Criterion 5.11 is used. Most cases below are uses of Criterion 5.11, which is Theorem 3.19 together with Corollary 3.15. For every use we name the split $k = ab$ (with k not a prime power, $a \ge 10$ and a composite $b \ge 4$) and the reason why the deletion condition holds. The other data of the proof do not depend on the case. We list them once, so that each hypothesis of Theorem 3.11 can be checked.

(D1) $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k$ (order b), $A = \mathbb{Z}_a$. Σ is the set of the elements ± 1, ± 2, $\pm q$ of $\mathbb{Z}_k$, for the divisors q of k with $1 \le q < k$. They are liftable by Lemma 1.12 (b), (c), (d).

(D2) $\Sigma_A = D$, and (R1) is the deletion condition: the graph on $\mathbb{Z}_a$ with the step set D stays connected after the deletion of at most 6 elements.

(D3) Let d be the least prime factor of b; so $1 < d < b$. The divisors a and ad of k give $\pm a$, $\pm ad$ in Σ_B; under $j \mapsto aj$ these are the steps ± 1, $\pm d$ of $\mathbb{Z}_b$, and (R2) is Lemma 5.7.

(D4) (R3*): two non-zero elements of $\mathbb{Z}_k$ are joined by steps 1 or -1 without passing 0.

(D5) (AV): let p be a prime factor of k, k_1 the largest power of p that divides k, and $k_2 = k/k_1$. Both are divisors of k with $1 < k_1, k_2 < k$, because k is not a prime power. By the Chinese remainder theorem $\mathbb{Z}_k = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$; the element k_2 of Σ generates the factor $\mathbb{Z}_{k_1} \times 0$, and k_1 generates $0 \times \mathbb{Z}_{k_2}$. And $(|B|-1)/(|\Gamma|-1) = (b-1)/(ab-1) < 1/9$, because $a \ge 10$. So Lemma 4.1 gives (AV) and (AV″).

(D6) The counts: $|A| = a \ge 10$ and $|B| = b \ge 4$. So (C0) holds, and every frame is counted: the left sides of (C1), (C2), (C3), (C4) are at most 9, 9, 8, 3 (Remark 2.16). This is Corollary 3.15, and Theorem 3.19 gives the connectivity of $M_k(G)$ and $I_k(G)$.

Criterion 5.11 says that the deletion condition holds if D has at least 9 elements, and, if a is a prime, if D has at least 7 elements. If D has at most 6 elements, the deletion condition fails: delete the elements of D, the neighbours of 0; then 0 has no neighbour, and other vertices remain, because $a \ge 10$.

Lemma 9.1 (Many divisors)**.** *Let k be not a prime power, and $k = ab$ with $a \ge 10$ and a composite number $b \ge 4$. If $\tau(a) \ge 6$, or if a is odd and $\tau(a) \ge 5$, then D has at least 9 elements. So, by Criterion 5.11, $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Every divisor q of a with $1 \le q < a$ is a divisor of k with $q < k$, so the residues $\pm q$ modulo a are in D. The residue of q is q itself, not 0, and different q give different residues. Let $q \equiv -q'$ modulo a for two such divisors. Then $q + q' = a$, because $2 \le q + q' \le 2a - 2$. So q divides $a - q = q'$ and q' divides q, hence $q = q' = a/2$. Therefore the residues $\pm q$ form a set R_0 with $2\,(\tau(a) - 1)$ elements if a is odd, and with $2\,(\tau(a) - 1) - 1$ elements if a is even (only $a/2$ is its own negative). If $\tau(a) \ge 6$, then $|R_0| \ge 9$.

Let a be odd and $\tau(a) \geq 5$; so $|R_0| \geq 8$. The residues 2 and -2 are in D. They are different and not 0, because a is odd and $a \geq 10$. They are not in R_0: 2 is not a divisor of the odd number a; and $2 \equiv -q$ with a divisor $q < a$ gives $q = a - 2$, so q divides 2, and $a = q + 2$ is 3 or 4, which is not the case. As $R_0 = -R_0$, also -2 is not in R_0. So $|D| \geq 10$. □

Every multiple of four from 52 on.

Theorem 9.2 (The bases)**.** *Let $k = 40$, or $k = 48$, or let k be a multiple of 4 with $k \geq 52$. Then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. The cases and the theorems that give them:*

(B1) *$k = 40$: Theorem 5.3, that is Theorem 3.19 with Corollary 3.15, $a = 10$, $b = 4$, $\Sigma = \{\pm1, \pm2, \pm4, \pm5, \pm8, \pm10, 20\}$.*

(B2) *$k = 48$: Theorem 5.4, that is Theorem 3.19 with Corollary 3.15, $a = 12$, $b = 4$, $\Sigma = \{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, 24\}$.*

(B3) *$k = 4p$ with a prime $p \geq 13$: the theorem of the double step (Theorem 6.9), that is Theorem 3.19 with Theorem 3.12, $a = p$, $b = 4$, $\Sigma = \{\pm1, \pm(1 + 2p), \pm2, \pm4, \pm p, 2p\}$.*

(B4) *$k = 2^N$ with $N \geq 6$: the theorem of the prime powers (Theorem 7.4), that is Theorem 3.19 with Corollary 3.15, $a = 2^{N-2}$, $b = 4$, $\Sigma = \{\pm 2^j : 0 \leq j \leq N - 1\}$, and (AV″) by the flattening of valuations.*

(B5) *k a multiple of 8, $k \geq 56$, k not a power of 2: Criterion 5.11 with $a = k/4$, $b = 4$ (the nine residues ±1, ±2, ±4, ±8, $a/2$).*

(B6) *$k = 4m$ with m odd and composite (so $m \geq 15$): Criterion 5.11 with $a = m$, $b = 4$ (the ten residues ±1, ±2, ±4, $\pm d_0$, $\pm 2d_0$, d_0 the least prime factor of m).*

Every multiple of 4 from 52 on is in one of the cases (B3)–(B6)*.*

Proof. (B1) to (B4) are the theorems named there, with their proofs in their sections.

(B5) Let k be a multiple of 8 with $k \geq 56$ that is not a power of 2. It is not a prime power: it is even and has an odd prime factor. Use Criterion 5.11 with $a = k/4$ and $b = 4$ (composite; in (D3) $d = 2$; in (D5) the prime 2). Here a is even, $a \geq 14$, and $a \neq 16$, because $k \neq 64$. The numbers 4, 8 and $a/2 = k/8$ are divisors of k smaller than k. So the nine residues ±1, ±2, ±4, ±8, $a/2$ modulo a are in D. They are different and not 0. Indeed, $0 < a/2 < a$ and $8 < a$. The difference of two of the eight elements ±1, ±2, ±4, ±8 is, up to its sign, one of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16; the number a divides none of them, because $a \geq 14$ and $a \neq 16$. And $a/2 \equiv x$ modulo a, with x one of the eight, gives $a - 2x = 2ja$ for an integer j, so a divides $2x$; with $2 \leq |2x| \leq 16$ and $a \geq 14$ this means $a = 16$, which is excluded. So $|D| \geq 9$, and Criterion 5.11 applies.

(B6) Let $k = 4m$ with m odd and composite, and $k \geq 52$. Then $m \geq 13$, and so $m \geq 15$, because 13 is a prime. k is not a prime power. Use Criterion 5.11 with $a = m$ and $b = 4$ (in (D3) $d = 2$; in (D5) the prime 2, $\mathbb{Z}_k = \mathbb{Z}_4 \times \mathbb{Z}_m$). Let d_0 be the least prime factor of m; it is odd, $d_0 \geq 3$, and $d_0^2 \leq m$. The numbers 4, d_0 and $2d_0$ are divisors of $k = 4m$ smaller than k. The five integers 1, 2, 4, d_0, $2d_0$ are different, because d_0 is odd and at least 3, and $2d_0 \geq 6$. They are smaller than $m/2$: $4 < 15/2 \leq m/2$; if $m = 15$, then $2d_0 = 6 < m/2$; if $m > 15$, then $m \geq 21$ and $2d_0 \leq 2\sqrt{m} < m/2$. For two integers x, y with $0 < x, y < m/2$, the residues of x and $-y$ modulo m are different, because $0 < x + y < m$; and those of x and y are different if $x \neq y$. So the ten residues ±1, ±2, ±4, $\pm d_0$, $\pm 2d_0$ are different and not 0: $|D| \geq 10$, and Criterion 5.11 applies.

The last sentence. Let $k = 4m \geq 52$, so $m \geq 13$. If m is odd and a prime, then $k = 4p$ with $p = m \geq 13$: case (B3). If m is odd and composite: case (B6). If m is even, then 8 divides k and $k \geq 56$; if k is a power of 2, then $k = 2^N$ with $2^N \geq 64$, so $N \geq 6$: case (B4); otherwise case (B5). □

Criterion 5.11 also gives, in the same way as (B5) and (B6), every $k = 2^e m$ with $e \geq 2$ and an odd composite $m \geq 15$ ($a = m$, $b = 2^e$); all these k are already in (B5) or (B6).

Twice a prime cube.

Lemma 9.3 (The steps ± 1, ± 2, $\pm p$, $\pm 2p$ in $\mathbb{Z}_{p^2}$)**.** *Let $p \geq 5$ be an integer. The graph on $\mathbb{Z}_{p^2}$ with the step set $\{\pm 1, \pm 2, \pm p, \pm 2p\}$ stays connected after the deletion of at most 6 elements.*

Proof. Let X be the set of the deleted elements, $|X| \leq 6$. The *rows* are the p residue classes modulo p; each row has p elements. For a row i let d_i be the number of its elements in X.

A row with at most 3 deleted elements stays connected. Order the elements of a row as $x, x + p, x + 2p, \ldots, x + (p-1)p$ in a cycle. The steps $\pm p$, $\pm 2p$ join two elements at distance 1 or 2 along this cycle. Between two consecutive remaining elements (along the cycle) lie the deleted elements of a gap. If at most one gap has 2 or more elements, the remaining elements are joined along the cycle, around the one long gap. Two gaps with 2 or more elements need at least 4 deleted elements. Every such row keeps at least $p - 3 \geq 2$ elements. (For $p = 5$ the row is a complete graph.)

Call a row *good* if $d_i \leq 3$. At most one row is not good, because two such rows need at least 8 deleted elements. The step $+1$ maps the row i onto the row $i + 1$ (modulo p), one to one; this is a set of p disjoint edges between the two rows, also from the row $p - 1$ to the row 0. A deleted element meets at most one of these edges.

Case 1: one row i_0 is not good. Then $d_{i_0} \geq 4$, and the other rows together contain at most 2 deleted elements. The good rows form a path in the cycle of the rows $0, 1, \ldots, p - 1$ (the cycle without the row i_0). Two consecutive good rows are joined by an edge between two remaining elements, because $2 < p$. Each good row is connected. So all remaining elements of the good rows lie in one component. A remaining element x of the row i_0 has the four different neighbours $x + 1$, $x - 1$, $x + 2$, $x - 2$; they lie in other rows, because $p \geq 5$ does not divide 1 or 2. At most 2 of them are deleted. So x lies in the same component.

Case 2: all rows are good. Then every row is connected and has remaining elements. Say that the link between the rows i and $i + 1$ *fails* if none of the p edges between them has two remaining ends. Then $d_i + d_{i+1} \geq p \geq 5$. Two failed links without a common row need at least 10 deleted elements. Two failed links with the common row i need $d_{i-1} + d_i + d_{i+1} \geq 2p - d_i \geq 10 - 3 = 7$ deleted elements. So at most one link fails. A cycle without one edge is connected. So all remaining elements lie in one component. □

Proposition 9.4 ($k = 2p^3$)**.** *Let $p \geq 5$ be a prime and $k = 2p^3$. Then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. Use Criterion 5.11 with $a = p^2 \geq 25$ and $b = 2p$ (composite, $b \geq 10$; in (D3) $d = 2$; in (D5) the prime 2, $\mathbb{Z}_k = \mathbb{Z}_2 \times \mathbb{Z}_{p^3}$, with the generators p^3 and 2). k is not a prime power. The divisors q of k with $q < k$ are 1, 2, p, $2p$, p^2, $2p^2$, p^3; the last three are 0 modulo p^2. So $D = \{\pm 1, \pm 2, \pm p, \pm 2p\}$ modulo p^2, and Lemma 9.3 is the deletion condition. □

At least four prime factors.

Theorem 9.5 (Four prime factors)**.** *Let k be not a prime power, $\Omega(k) \geq 4$, and $k \notin \{24, 36, 54\}$. Then $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G.*

Proof. In every case below we give a split $k = ab$ for Criterion 5.11 with $a \geq 10$ and a composite $b \geq 4$, and we show $|D| \geq 9$; or we name another theorem. The data (D1)–(D6) are those of the subsection 'How Criterion 5.11 is used'.

Step 1: $\Omega(k) \geq 5$. Write k as a product of $\Omega(k)$ primes, the *entries*; not all entries are equal. If $k = 48$, use Theorem 9.2 (B2). Otherwise we put two entries into b and the others into a; then b is a product of two primes, so $b \geq 4$ and b is composite. We use one fact: if a has two different prime factors and $\Omega(a) \geq 3$, then $\tau(a) \geq 6$. (If $a = y^e y'^f \cdots$ with different primes y, y', then $\tau(a) \geq (e+1)(f+1)$; with exactly two different primes $e + f \geq 3$ gives $(e+1)(f+1) \geq 6$; with three or more different primes $\tau(a) \geq 8$.)

(1a) $\Omega(k) \geq 6$. Choose two different primes y, y' among the entries. Put y, y' and $\Omega(k)-4 \geq 2$ further entries into a, and the two remaining entries into b. Then a has two different prime factors, $\Omega(a) \geq 4$, and $a \geq 2 \cdot 3 \cdot 2 \cdot 2 = 24$.

(1b) $\Omega(k) = 5$, and some entry y is at least 5. Choose an entry $y' \neq y$ and a third entry y''. Put $a = yy'y''$, and the two remaining entries into b. Then a has two different prime factors, $\Omega(a) = 3$, and $a \geq 5 \cdot 2 \cdot 2 = 20$.

(1c) $\Omega(k) = 5$, and all entries are 2 or 3. Then k is $2^4 \cdot 3 = 48$, $2^3 \cdot 3^2 = 72$, $2^2 \cdot 3^3 = 108$ or $2 \cdot 3^4 = 162$. For 72, 108, 162 put $a = 18$ and $b = 4$, 6, 9; $\tau(18) = 6$.

In all three cases $\tau(a) \geq 6$ and $a \geq 18$, and Lemma 9.1 applies.

Step 2: $\Omega(k) = 4$. There are four types; p, q, r, u, v, w, z are primes.

Type p^3q, $p \neq q$. If $p = 2$, then $k = 8q$: $q = 3$ gives 24, excluded; $q = 5$ gives 40, which is Theorem 9.2 (B1); $q \geq 7$ gives a multiple of 4 with $k \geq 56$, which is Theorem 9.2. If $q = 2$, then $k = 2p^3$: $p = 3$ gives 54, excluded; $p \geq 5$ is Proposition 9.4. Now let p and q be odd.

If $p \not\equiv \pm 1$ modulo q, put $a = pq \geq 15$ and $b = p^2 \geq 9$. The six residues ± 1, ± 2, $\pm q$ modulo pq are in D (q divides k). They are different: 1, 2, q are different and smaller than $pq/2$. None of them is a multiple of p. The four residues $\pm p$, $\pm p^2$ are in D; they are non-zero multiples of p, because q does not divide p or p^2. They are different: $p \equiv -p$ or $p^2 \equiv -p^2$ modulo pq would mean that q divides 2 or $2p$, and $p \equiv \pm p^2$ modulo pq means that q divides $1 \mp p$, that is $p \equiv \pm 1$ modulo q. So $|D| \geq 10$.

If $p \equiv \pm 1$ modulo q, then $p > q$: for $p < q$ it would mean $p = 1$ or $p = q - 1$, and $q - 1$ is even. Also $q \neq p - 1$, because $p - 1$ is even; so $q \leq p - 2$. Put $a = p^2 \geq 25$ and $b = pq \geq 15$. The six residues ± 1, ± 2, $\pm q$ modulo p^2 are different (1, 2, q are different and smaller than $p^2/2$) and not multiples of p. The four residues $\pm p$, $\pm pq$ are in D. They are the residues $p \cdot x$ with x in $\{1, -1, q, -q\}$, and these four values of x are different modulo p, because $3 \leq q \leq p - 2$ and p is odd. So the four residues are different non-zero multiples of p, and $|D| \geq 10$.

Type p^2q^2, $p < q$. If $p = 2$ and $q = 3$, then $k = 36$, excluded. If $p = 2$ and $q \geq 5$, then k is a multiple of 4 with $k \geq 100$: Theorem 9.2. Now let p and q be odd. Put $a = b = pq \geq 15$. As above, ± 1, ± 2, $\pm q$ are six different residues modulo pq, none a multiple of p. The four residues $\pm p$, $\pm p^2$ are different non-zero multiples of p, because $p \not\equiv \pm 1$ modulo q: $3 \leq p \leq q - 2$. So $|D| \geq 10$.

Type p^2qr, p, q, r different. If $p = 2$, then $k = 4qr \geq 60$: Theorem 9.2. If one of q, r is 2, say $r = 2$, then $k = 2p^2q$ with different odd primes p, q; put $a = pq \geq 15$ and $b = 2p \geq 6$.

The residues ± 1, ± 2 modulo pq are four different units. The residues $\pm p$, $\pm 2p$ are non-zero multiples of p, and they are four different residues unless $q = 3$, when they are two; in the same way $\pm q$, $\pm 2q$ give four residues, or two if $p = 3$. Multiples of p and multiples of q have only 0 in common, and units are neither. At most one of p, q is 3. So $|D| \geq 4 + 2 + 4 = 10$.

Now let p, q, r be odd. If $p \not\equiv \pm 1$ modulo qr, put $a = qr \geq 15$ and $b = p^2 \geq 9$. The eight residues ± 1, ± 2, $\pm q$, $\pm r$ are different: the first four are units, the others are non-zero multiples of q or of r, and qr is odd. The residue of p is a unit, so it is not $\pm q$ or $\pm r$. If $p \not\equiv \pm 2$ modulo qr, then $\pm p$ are two new residues. If $p \equiv \pm 2$, then $p^2 \equiv 4$, and ± 4 are two new residues: 4 is a unit, and $4 \not\equiv \pm 1, \pm 2, -4$ modulo $qr \geq 15$. So $|D| \geq 10$. If $p \equiv \pm 1$ modulo qr, then $p \geq qr - 1 > q, r$. Put $a = pq \geq 15$ and $b = pr \geq 15$. The eight residues ± 1, ± 2, $\pm p$, $\pm q$ modulo pq are different. The residues $\pm r$ are two more: r is a unit modulo pq, so not $\pm p$, $\pm q$; and $3 \leq r < p$ gives $r \not\equiv \pm 1, \pm 2$ ($pq - 2 > p$). So $|D| \geq 10$.

Type uvwz, $u < v < w < z$. Put $a = wz \geq 35$ and $b = uv \geq 6$. The eight residues ± 1, ± 2, $\pm w$, $\pm z$ modulo wz are different: wz is odd, the first four are units, and $\pm w$, $\pm z$ are non-zero multiples of w or of z. The divisor b of k is a unit modulo a, so it is not $\pm w$, $\pm z$. We have $w \geq u + 2$ and $z \geq v + 2$, so $a - b \geq (u + 2)(v + 2) - uv = 2u + 2v + 4 > 2$; with $6 \leq b < a$ this gives $b \not\equiv \pm 1, \pm 2$ modulo a. And $b \not\equiv -b$, because a is odd and $0 < b < a$. So $\pm b$ are two new residues, and $|D| \geq 10$.

In every case of Step 2 with a split, $a \geq 15$ and $b \geq 6$ is composite; $|D| \geq 10 \geq 9$, and Criterion 5.11 applies. The four types are all the non-prime-powers with $\Omega(k) = 4$. □

The moduli by their number of prime factors.

Definition 9.6 (Covered)**.** An integer $k \geq 2$ is *covered* if one of the following four results applies to it:

(1) Theorem 9.2: $k = 40$, $k = 48$, or k is a multiple of 4 with $k \geq 52$;

(2) Theorem 7.4: $k = p^N$ with a prime p and $N \geq 6$;

(3) Corollary 8.9 (a): k is the product of two coprime admitted factors;

(4) Criterion 5.11: k is not a prime power, and some split $k = ab$ with $a \geq 10$ and a composite $b \geq 4$ has the deletion condition.

For every covered k, $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. The proofs of Theorem 9.5, Lemma 9.1 and Proposition 9.4 use only (1) and (4); so every k that they give is covered.

Proposition 9.7 (Three prime factors)**.** *Let p, q, r be primes.*

(i) *Let $p \neq q$. Then $k = p^2 q$ is covered if and only if*

  (α) *$p = 2$ and $q \geq 13$; or*

  (β) *$p \geq 7$ and $q \geq 43$; or*

  (γ) *$q \geq 11$, and none of p, p^2 is congruent to 1, -1, 2 or -2 modulo q.*

(ii) *Let $p < q < r$ be odd. Then $k = pqr$ is covered, except $k = 105 = 3 \cdot 5 \cdot 7$, which is not covered.*

(iii) *Let $3 \leq p < q$ be odd. Then $k = 2pq$ is covered, except the ten values 30, 42, 70, 110, 182, 286, 506, 646, 1406, 1798, which are not covered.*

In particular: $4q$ is covered exactly for the primes $q \geq 13$; $9q$ exactly for the primes $q \geq 13$; $25q$ exactly for the primes $q \geq 11$ other than 13 and 23; no $p^2 q$ with q in $\{2, 3, 5, 7\}$ is covered;

and among the products of three different primes exactly 30*,* 42*,* 70*,* 105*,* 110*,* 182*,* 286*,* 506*,* 646*,* 1406*,* 1798 *are not covered.*

Proof. Let $\Omega(k) = 3$ and let k not be a prime power. We first see which of the four results can apply. (1) needs $4 \mid k$, so $k = 4q$ with an odd prime q; and $4q$ is not 40 or 48; so (1) applies exactly to $k = 4q$ with $q \geq 13$. (2) does not apply. For (3), the coprime splits of k come from its prime powers. A prime is an admitted factor if and only if it is at least 43, because a number de with $d, e \geq 2$ is not a prime. For (4), b is composite, so $\Omega(b) \geq 2$ and $\Omega(a) \leq 1$; with $a \geq 10$, a is a prime, $a \geq 11$, and $\Omega(b) = 2$. A divisor q' of k that is a multiple of a gives the residue 0. So D is the set of the residues of ± 1, ± 2 and $\pm q'$ for the divisors q' of k that are not multiples of a. As a is an odd prime, D is a union of sign classes with two elements each. If D has at least four sign classes, then $|D| \geq 8 \geq 7$ and the deletion condition holds by Criterion 5.11; if it has at most three, then $|D| \leq 6$, and the deletion condition fails (the paragraph before Lemma 9.1).

(i) $k = p^2 q$. (1) is exactly (α). (3): the only coprime split is p^2 and q. The prime q is admitted exactly if $q \geq 43$. The number p^2 is admitted exactly if $p \geq 7$: 4, 9, 25 are smaller than 28, and for $p \geq 7$, $p^2 = p \cdot p$ with $p \geq 4$ and $p \geq 7$. So (3) is exactly (β). (4): $a = p$ or $a = q$. For $a = p$ the divisors that are not multiples of p are 1 and q, so D has at most the three sign classes of 1, 2, q: no. For $a = q \geq 11$ they are 1, p, p^2, so $D = \{\pm 1, \pm 2, \pm p, \pm p^2\}$ modulo q; all are non-zero, because $q \geq 11$ and $q \neq p$. The sign classes of 1 and 2 are different, because $q \geq 5$. The class of p^2 is that of p exactly if $p \equiv \pm 1$, since p is invertible modulo q. So there are four classes exactly if neither p nor p^2 is ± 1 or ± 2 modulo q, that is (γ).

The readings. For $p = 2$: (β) fails, and (γ) fails because $p = 2$. For $p = 3$, (α) and (β) fail; for $q \geq 11$, 3 is not ± 1, ± 2, and 9 is ± 1 or ± 2 exactly if q divides 8, 10, 7 or 11, that is $q = 11$. For $p = 5$: 5 is not ± 1, ± 2 modulo $q \geq 11$, and 25 is exactly if q divides 24, 26, 23 or 27, that is $q = 13$ or $q = 23$. For $q \leq 7$ none of (α), (β), (γ) holds.

(ii) $k = pqr$ with odd $p < q < r$. (1) does not apply. If $r < 11$, then $k = 3 \cdot 5 \cdot 7 = 105$: (4) needs a prime $a \geq 11$, and the nontrivial coprime splits for (3) are $3 \cdot 35$, $5 \cdot 21$ and $7 \cdot 15$; each has a factor smaller than 28, so Corollary 8.9(a) does not apply. Thus 105 is not covered. Let $r \geq 11$, and take $a = r$, $b = pq$. The divisors that are not multiples of r are 1, p, q, pq. The classes of 1, 2, p, q modulo r are different unless $q = r - 2$: p and q are odd and smaller than r, so neither is $\equiv 1$ or $\equiv 2$, neither is $r - 1$, $p = r - 2$ is impossible because $p < q < r$ and q is odd, and $p \equiv -q$ would mean $p + q = r$, but $p + q$ is even. So if $q \neq r - 2$, there are four classes. Let $q = r - 2$. Then $pq \equiv -2p$, and its class is new unless $2p \equiv \pm 1$: $2p \equiv \pm 2$ would give $p \equiv \pm 1$, and $2p \equiv \pm p$ would give $r \mid p$ or $r \mid 3$. As $2p < 2r$, $2p \equiv \pm 1$ means $r = 2p - 1$ or $r = 2p + 1$. If $r = 2p + 1$, then $q = 2p - 1$. One of the three primes p, $2p - 1$, $2p + 1$ is divisible by 3, so it is 3; as $2p - 1$ and $2p + 1$ are larger than 3, $p = 3$, and $r = 7 < 11$, which is not the case. If $r = 2p - 1$, then $q = 2p - 3$, and $p \geq 7$ ($p = 3$ gives $q = p$, $p = 5$ gives $r = 9$), so $q \geq 11$. Take instead $a = q$ and $b = pr$. Modulo q: $r \equiv 2$ and $pr \equiv 2p \equiv 3$. The classes of 1, 2, 3, p are different, because $q \geq 11$ and $p \equiv \pm 1, \pm 2, \pm 3$ modulo $q = 2p - 3$ would give $p \leq 6$. So in all cases with $r \geq 11$ there are four classes, and (4) applies.

(iii) $k = 2pq$ with odd $3 \leq p < q$. (1) does not apply, because $4 \nmid k$. (3): the coprime splits are 2 and pq, p and $2q$, q and $2p$. The factor 2 is not admitted. A number $2s$ with an odd prime s is admitted exactly if $2s \geq 43$, because $2s = de$ with $d \geq 4$ and $e \geq 7$ is impossible. So (3) applies exactly if $p \geq 43$, or $q \geq 43$ and $p \geq 23$; that is, exactly if $p \geq 23$ and $q \geq 43$. (4): a is p or q ($a = 2$ is too small). For $a = q \geq 11$, $D = \{\pm 1, \pm 2, \pm p, \pm 2p\}$ modulo q. Its four classes are different unless $q = p + 2$, $q = 2p - 1$ or $q = 2p + 1$: $p \equiv \pm 1$

is impossible; $p \equiv \pm 2$ only as $p = q - 2$; $2p \equiv \pm 1$ only as $q = 2p \mp 1$; $2p \equiv \pm 2$ would give $p \equiv \pm 1$; $2p \equiv \pm p$ would give $q \mid p$ or $q \mid 3$. In these three exceptional forms, also $a = p$ fails if $p \geq 11$: $D = \{\pm 1, \pm 2, \pm q, \pm 2q\}$ modulo p, and $q \equiv 2$ or $q \equiv \pm 1$ modulo p, so there are at most three classes. If $q < 11$, then $p < 11$ too, and (4) needs $a \geq 11$. So (4) applies exactly if $q \geq 11$ and q is none of $p + 2$, $2p - 1$, $2p + 1$.

So k is not covered exactly if $[q < 11$, or $q \in \{p + 2, 2p - 1, 2p + 1\}]$ and $[p < 23$ or $q < 43]$. With $q < 11$: $(3, 5)$, $(3, 7)$, $(5, 7)$, that is 30, 42, 70. With $q = p + 2 \geq 11$ and $q < 43$ (if $p < 23$, then $q < 43$ as well): $(11, 13)$, $(17, 19)$, $(29, 31)$, that is 286, 646, 1798; the pair $(41, 43)$ has $q = 43$ and $p \geq 23$. With $q = 2p - 1 \geq 11$ and $p \leq 19$: $p = 7$ and $p = 19$ ($q = 13$, 37), that is 182, 1406; for $p = 11$, 13, 17 the number $2p - 1$ is not a prime. With $q = 2p + 1 \geq 11$ and $p \leq 19$: $p = 5$ and $p = 11$ ($q = 11$, 23), that is 110, 506; for $p = 7$, 13, 17, 19 the number $2p + 1$ is not a prime. For $p \geq 23$ the forms $q = 2p \pm 1$ have $q \geq 45$. These are the ten values. □

Corollary 9.8 (The covered moduli)**.** *Let $k \geq 2$.*

(a) *A prime power $k = p^e$ is covered if and only if $e \geq 6$. In particular no prime is covered.*

(b) *A product $k = pq$ of two different primes is covered if and only if $p \geq 43$ and $q \geq 43$.*

(c) *For $\Omega(k) = 3$, k not a prime power: Proposition 9.7.*

(d) *If $\Omega(k) \geq 4$ and k is not a prime power, then k is covered if and only if $k \notin \{24, 36, 54\}$.*

Proof. (a) (1) needs $4 \mid k$, so $k = 2^e$; 2^e is not 40 or 48, and $2^e \geq 52$ means $e \geq 6$. (2) is $e \geq 6$. (3) and (4) do not apply to a prime power.

(b) (1) does not apply: $4 \nmid pq$. (2) does not apply. (4) needs a composite b, so $\Omega(b) \geq 2$ and $a = 1$: no. (3): the only coprime split is p and q, and a prime is an admitted factor exactly if it is at least 43.

(d) Theorem 9.5 and the sentence after Definition 9.6 give every other k. The three values are not covered: (1) does not apply, because 24 and 36 are smaller than 40, and $4 \nmid 54$; (2) does not apply; (3) needs two admitted factors, so $k \geq 28 \cdot 28$; (4) needs $k = ab \geq 40$ with $4 \leq b \leq k/10$, which fails for 24 and 36; for 54 it gives $b = 4$ or $b = 5$, and neither divides 54. □

Density one.

Corollary 9.9 (Density one)**.** *Let S^* be the set of the integers $k \geq 2$ such that $M_k(G)$ and $I_k(G)$ are connected for every finite bridgeless loopless multigraph G. Then S^* has natural density 1: the number of elements of S^* that are at most x, divided by x, tends to 1 as x tends to infinity.*

Proof. If $\omega(k) \geq 5$, then $\Omega(k) \geq 5$, k is not a prime power, and $k \geq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$; so k is in S^* by Theorem 9.5. So it is enough to show that the set T of the positive integers n with $\omega(n) \leq 4$ has density 0.

Fix $y \geq 2$, and let P_y be the product of the primes $p \leq y$. Let T_y be the set of the positive integers that are divisible by at most 4 of the primes $p \leq y$. Then T is a subset of T_y. Whether n is in T_y depends only on the residue of n modulo P_y; so T_y has the density $\delta_y = |T_y \cap [1, P_y]|/P_y$. Let n be a uniformly random element of $\{1, \ldots, P_y\}$. By the Chinese remainder theorem the events "p divides n", for the primes $p \leq y$, are independent, and the event for p has the probability $1/p$. Let X be the number of these events that occur. Then its mean is $\mu_y = \sum_{p \leq y} 1/p$, and its variance is $\sum_{p \leq y} (1/p)(1 - 1/p) \leq \mu_y$. If $\mu_y > 4$,

Chebyshev's inequality (Theorem 0.7) gives

$$\delta_y = \Pr[X \le 4] \;\le\; \Pr\big[\,|X-\mu_y| \ge \mu_y - 4\,\big] \;\le\; \frac{\mu_y}{(\mu_y-4)^2}.$$

The sum of $1/p$ over all primes diverges (Euler; Theorem 0.6), so μ_y tends to infinity, and δ_y tends to 0. For every y the upper density of T is at most δ_y. So T has density 0, and S^* has density 1. □

Remark 9.10 (Moduli that are not covered)**.** The moduli that are not covered include the primes, the prime powers p^e with $e \le 5$, the values 24, 36, 54 and the eleven products of three different primes of Proposition 9.7. For $k \ge 6$ in these cases, the connectivity of $I_k(G)$ for all graphs is open (Section 12). For example $k = 44 = 4 \cdot 11$ is not covered (Proposition 9.7 (i): $4q$ is covered only for $q \ge 13$): it is smaller than 52 and is not 40 or 48; its only split for Criterion 5.11 is $a = 11$, $b = 4$, with $D = \{\pm1, \pm2, \pm4\}$ modulo 11; and 44 is not the product of two coprime admitted factors. (The modular graph $M_k(G)$ with $k = 44$ is connected by Theorem 4.5.)

10. One graph: larger moduli, few edges, few vertices, short cycles

Theorem 9.2 gives moduli k for which $I_k(G)$ and $M_k(G)$ are connected for *every* finite bridgeless loopless multigraph G: $k = 40$, $k = 48$ and every multiple of 4 from 52 on. We call them the *bases*. This section fixes one graph G and gives *every* k from a bound on, prime numbers k included. The bound depends on G:

(a) on the length of a longest cycle: every $k \ge \max(48, c(G) + 4)$, and every $k \ge 40$ if every cycle has at most 39 edges (Corollary 10.6);

(b) on the number of edges: every $k \ge 40$ if G has at most 189 edges (Corollary 10.12);

(c) on the number of vertices: every $k \ge 40$ if G has at most 149 vertices (Corollary 10.16);

(d) on short cycles: every $k \ge 40$ if every edge lies on a cycle with at most 8 edges, or if G is connected with diameter at most 3 (Corollaries 10.22, 10.23, 10.24).

All four go up one level at a time, from $k-1$ to k, starting at a base. This section does *not* prove that $I_k(G)$ is connected for every bridgeless G and every large k.

Setting. In this section G is a finite bridgeless loopless multigraph, and $k \ge 3$ is an integer, unless something else is said. The conventions are those of Section 1. In particular $I_k(G)$ is the graph on the integer flows F of G with $1 \le |F(e)| \le k-1$ for every edge e, and F, F' are adjacent if $F' - F = \lambda\chi_C$ for a cycle C with a direction and an integer $\lambda \ne 0$ (a move).

For a flow F of $I_k(G)$, G_F is the orientation of G in which F is positive: every edge is directed so that its value is in $[1, k-1]$. We change the reference orientation freely (Section 1). In a statement about one flow F the values are read in G_F; an edge of G with this direction is an *arc*, and its *value* $F(e)$ is its size $|F(e)|$. An arc is a *top* if its value is $k-1$, a 1-*arc* if its value is 1, and a *middle arc* if its value is in $[2, k-2]$. For a set X of vertices, $\mathrm{in}(X)$ and $\mathrm{out}(X)$ are the numbers of arcs of G_F that enter X and that leave X; $\mathrm{in}_v(X)$ and $\mathrm{out}_v(X)$ count only the arcs of value v. The sum of the conservation equations over the vertices of X gives

$$\sum_{e \text{ leaves } X} F(e) \;=\; \sum_{e \text{ enters } X} F(e) \qquad \text{(the flow out of } X \text{ is the flow into } X\text{)}.$$

By Lemma 1.1 (b), applied to F with the reference orientation G_F, every arc of G_F lies on a *directed cycle* of G_F, that is a cycle that passes all its arcs in their direction.

Joined means: in the same component of $I_k(G)$, for the k of the context. A flow of $I_k(G)$ *descends* if it is joined to a flow of $I_{k-1}(G)$. $c(G)$ is the number of edges of a longest cycle of G, and $c(G) = 0$ if G has no cycle. For an integer $k_0 \geq 2$ put

$$\sigma(k_0) = \sum_{\lambda=1}^{k_0-1} \left\lceil \frac{k_0}{\lambda} \right\rceil .$$

One quoted theorem is used: Hoffman's circulation theorem, Theorem 0.5.

Lemma 10.1 (One level)**.** *Let G be any finite multigraph and $k \geq 3$.*

(a) *The vertices of $I_{k-1}(G)$ are the flows of $I_k(G)$ with all sizes at most $k-2$, and two of them are adjacent in $I_{k-1}(G)$ if and only if they are adjacent in $I_k(G)$.*

(b) *If $I_{k-1}(G)$ is connected and every flow of $I_k(G)$ descends, then $I_k(G)$ is connected.*

Proof. (a) holds by the definitions: the adjacency is the same rule in both graphs. (b) If $I_{k-1}(G)$ has no vertex, then no flow of $I_k(G)$ descends, so $I_k(G)$ has no vertex and is connected. Otherwise, by (a), the connected graph $I_{k-1}(G)$ lies in one component of $I_k(G)$, and every flow of $I_k(G)$ is joined to it. □

Lemma 10.2 (From integers to residues)**.** *Let G be any finite multigraph and $k \geq 2$. If $I_k(G)$ is connected, then $M_k(G)$ is connected. So each statement of this section that $I_k(G)$ is connected gives that $M_k(G)$ is connected.*

Proof. Let f, f' be vertices of $M_k(G)$. By Corollary 1.3 they have integer lifts F, F'. Take a path $F = F_0, F_1, \ldots, F_\nu = F'$ in $I_k(G)$, and let f_r be the reduction of F_r modulo k. It is a nowhere-zero $\mathbb{Z}_k$-flow, because $1 \leq |F_r(e)| \leq k-1$. If $F_r - F_{r-1} = \lambda\chi_C$, then $f_r - f_{r-1}$ is $\lambda\chi_C$ modulo k: either $f_r = f_{r-1}$, or f_{r-1} and f_r are adjacent in $M_k(G)$. So $f = f_0$ and $f' = f_\nu$ are joined in $M_k(G)$. □

Lemma 10.3 (Same orientation)**.** *Let F and T be flows of $I_k(G)$ with $G_T = G_F$. Then F and T are joined, by moves of the form χ_O.*

Proof. Use the reference orientation G_F. By Lemma 1.1 (b), $T - F = \chi_{O_1} + \cdots + \chi_{O_j}$, where every directed cycle O_r passes its edges in the direction of the sign of $T - F$. So on every edge e, every partial sum $F + \chi_{O_1} + \cdots + \chi_{O_r}$ has a value between $F(e)$ and $T(e)$, that is in $[1, k-1]$. All partial sums are in $I_k(G)$, and two consecutive ones differ by a move. □

Larger moduli on one graph.

Lemma 10.4 (Top-value reduction)**.** *Let F be a flow of $I_k(G)$ with a top e, and let C be a directed cycle of G_F through e. Let λ be an integer with $1 \leq \lambda \leq k-2$ that is not the value of any arc of C. Then $F - \lambda\chi_C$ is a flow of $I_k(G)$ that is adjacent to F (χ_C is read in G_F, so it is 1 on the arcs of C), and its tops are the tops of F that are not on C.*

Proof. An arc of C has a value v with $1 \leq v \leq k-1$ and $v \neq \lambda$. Its new value $v - \lambda$ is not 0, and $1 - \lambda \leq v - \lambda \leq k - 1 - \lambda$; so $|v - \lambda| \leq \max\{\lambda - 1, k - 1 - \lambda\} \leq k - 2$. The arcs outside C keep their values. So $F - \lambda\chi_C$ is in $I_k(G)$, no arc of C is a top of it, and the tops outside C do not change. It is adjacent to F because $\lambda \neq 0$. □

Theorem 10.5 (Larger moduli on one graph)**.** *Let $k_0 \geq 2$ be an integer with $k_0 \geq c(G) + 1$ such that $I_{k_0}(G)$ is connected. Then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq k_0$.*

Proof. Induction on k; for $k = k_0$ this is the hypothesis and Lemma 10.2. Let $k > k_0$, so $k \geq 3$, and let $I_{k-1}(G)$ be connected. A flow of $I_k(G)$ without a top is in $I_{k-1}(G)$. Let F be a flow of $I_k(G)$ with a top e. Take a directed cycle C of G_F through e. It has at most $c(G) \leq k_0 - 1 \leq k - 2$ arcs, so at most $k - 3$ arcs other than e. The value of e is $k - 1$. So at most $k - 3$ of the $k - 2$ integers $1, \dots, k - 2$ are values of arcs of C, and some λ in $[1, k - 2]$ is not. By Lemma 10.4, F is adjacent to a flow with fewer tops. Repeat this with the new flow and its own orientation. The number of tops goes down at every step; so F is joined to a flow without tops, that is a flow of $I_{k-1}(G)$. So every flow of $I_k(G)$ descends, and $I_k(G)$ is connected by Lemma 10.1 (b). $M_k(G)$ is connected by Lemma 10.2. □

Corollary 10.6 (One graph: the bound by the longest cycle)**.** *Let G be a finite bridgeless loopless multigraph, and $c = c(G)$. Then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq \max(48, c + 4)$. If $c \leq 39$, they are connected for every $k \geq 40$. Since $c \leq |V(G)|$, this holds in particular for every $k \geq \max(48, |V(G)| + 4)$.*

Proof. Let k_0 be the least multiple of 4 with $k_0 \geq \max(48, c + 1)$. It is a base, so $I_{k_0}(G)$ is connected by Theorem 9.2, and $k_0 \geq c + 1$. If $c + 1 \leq 48$, then $k_0 = 48$; otherwise $k_0 \leq (c + 1) + 3 = c + 4$. So $k_0 \leq \max(48, c + 4)$, and Theorem 10.5 gives every $k \geq k_0$. If $c \leq 39$, take the base $k_0 = 40 \geq c + 1$ in Theorem 10.5. □

One missing value; few edges.

Theorem 10.7 (One missing value)**.** *Let λ be an integer with $1 \leq \lambda \leq k - 2$, and let F be a flow of $I_k(G)$ with no arc of size λ. Then F is joined, by moves of the form $-\lambda\chi_O$ with directed cycles O of G_F, to a flow with all sizes at most $k - 2$. In particular F descends.*

Proof. Claim $(*)$. Let F be any flow of $I_k(G)$ and $1 \leq \lambda \leq k - 2$. Let Y be an integer flow of G with values in $\{0, 1\}$ in the orientation G_F, with $Y(e) = 1$ on every top and $Y(e) = 0$ on every arc of value λ. Then F is joined, by moves $-\lambda\chi_O$ with directed cycles O of G_F, to a flow with all sizes at most $k - 2$.

Proof of the claim. Use the reference orientation G_F, and write $v(e) = F(e)$, so $1 \leq v(e) \leq k - 1$. By Lemma 1.1 (b), $Y = \chi_{O_1} + \cdots + \chi_{O_\nu}$ with pairwise edge-disjoint directed cycles $O_1, \dots, O_\nu$ of G_F (Y has values in $\{0, 1\}$), and every arc with $Y(e) = 1$ is on exactly one of them. Put $F_r = F - \lambda(\chi_{O_1} + \cdots + \chi_{O_r})$ for $r = 0, \dots, \nu$. An arc on one of $O_1, \dots, O_r$ has in F_r the value $v(e) - \lambda$. It is not 0, because $Y(e) = 1$ and so $v(e) \neq \lambda$. And $v(e) - \lambda \leq k - 1 - \lambda \leq k - 2$, $\lambda - v(e) \leq \lambda - 1 \leq k - 3$. Every other arc keeps its value. So every F_r is in $I_k(G)$, and $F_r - F_{r-1} = -\lambda\chi_{O_r}$ is a move. In F_ν an arc on some O_r has a size at most $k - 2$, and an arc on no O_r has $Y(e) = 0$; so it is not a top of F, and its size $v(e)$ is at most $k - 2$. This proves the claim.

Now let F have no arc of size λ. In the orientation G_F put $x(e) = F(e)/(k - 1)$. Then $0 < x(e) \leq 1$, $x(e) = 1$ exactly on the tops, and $\partial x = 0$. By Lemma 1.2 there is $Y\colon E(G) \to \{0, 1\}$ with $\partial Y = 0$ and $Y(e) = 1$ on every top. This Y is an integer flow, and the condition on the arcs of value λ is empty. The claim gives the theorem. □

Corollary 10.8 (Every size occurs)**.** *Let F be a flow of $I_k(G)$ that does not descend. Then F, and every flow that is joined to F, has arcs of every size $1, 2, \dots, k - 1$.*

Proof. A flow joined to F does not descend either. If it has no arc of a size $\lambda \leq k - 2$, it descends by Theorem 10.7. If it has no arc of size $k - 1$, it is in $I_{k-1}(G)$, so it descends. □

Corollary 10.9 (The cut for one size)**.** *Let $1 \le \lambda \le k-2$, and let F be a flow of $I_k(G)$ that is* not *joined by moves of the form $-\lambda\chi_O$, with directed cycles O of G_F, to a flow with all sizes at most $k-2$. Then there is a set X of vertices with*

$$\mathrm{out}_{k-1}(X) \;\ge\; \mathrm{in}(X) - \mathrm{in}_\lambda(X) + 1 \qquad \textit{and} \qquad \lambda \cdot \mathrm{in}_\lambda(X) \;\ge\; k-1.$$

So at least $\lceil (k-1)/\lambda \rceil$ arcs of value λ enter X.

Proof. On the arcs of G_F put $l(e) = u(e) = 1$ on the tops, $l(e) = u(e) = 0$ on the arcs of value λ, and $l(e) = 0$, $u(e) = 1$ on all other arcs (a top does not have the value $\lambda \le k-2$). An integer flow Y with $l \le Y \le u$ is a flow as in Claim $(*)$ of the proof of Theorem 10.7; the claim would join F by moves $-\lambda\chi_O$ to a flow with all sizes at most $k-2$. This is excluded. So by Theorem 0.5 some set X has

$$(\text{sum of } l \text{ over the arcs that leave } X) > (\text{sum of } u \text{ over the arcs that enter } X),$$

that is $\mathrm{out}_{k-1}(X) > \mathrm{in}(X) - \mathrm{in}_\lambda(X)$. All values are in $[1, k-1]$, and the flow out of X is the flow into X:

$$(k-1)\,\mathrm{out}_{k-1}(X) \;\le\; \lambda\,\mathrm{in}_\lambda(X) + (k-1)\big(\mathrm{in}(X) - \mathrm{in}_\lambda(X)\big) \;\le\; \lambda\,\mathrm{in}_\lambda(X) + (k-1)\big(\mathrm{out}_{k-1}(X) - 1\big).$$

So $\lambda\,\mathrm{in}_\lambda(X) \ge k-1$. □

Corollary 10.10 (The count of the edges)**.** *Let F be a flow of $I_k(G)$ that does not descend. Then for every λ with $1 \le \lambda \le k-2$ the flow F has at least $\lceil (k-1)/\lambda \rceil$ arcs of size λ, and it has a top. So $|E(G)| \ge 1 + \sigma(k-1)$.*

Proof. F is not joined to any flow with all sizes at most $k-2$, because such a flow is in $I_{k-1}(G)$. Corollary 10.9 gives the arcs of size λ, for every λ. Arcs of different sizes are different arcs. F has a top, because F is not in $I_{k-1}(G)$. So $|E(G)| \ge 1 + \sum_{\lambda=1}^{k-2} \lceil (k-1)/\lambda \rceil = 1 + \sigma(k-1)$. □

Corollary 10.11 (One level at a time)**.** *Let $k_0 \ge 3$, let $I_{k_0}(G)$ be connected, and let $|E(G)| \le \sigma(k_0)$. Then $I_k(G)$ and $M_k(G)$ are connected for every $k \ge k_0$.*

Proof. First, $\sigma(k+1) \ge \sigma(k)$ for $k \ge 2$: for $\lambda \le k-1$ the term $\lceil (k+1)/\lambda \rceil$ is at least $\lceil k/\lambda \rceil$, and there is one more term. Induction on k; $k = k_0$ is the hypothesis. Let $k > k_0$, and let $I_{k-1}(G)$ be connected. If a flow of $I_k(G)$ does not descend, Corollary 10.10 gives $|E(G)| \ge 1 + \sigma(k-1) \ge 1 + \sigma(k_0)$, a contradiction. So every flow descends, and Lemma 10.1 (b) gives the step. $M_k(G)$: Lemma 10.2. □

Corollary 10.12 (At most 189 edges)**.** *Let G be a finite bridgeless loopless multigraph.*

(a) *If G has at most* 189 *edges, then $I_k(G)$ and $M_k(G)$ are connected for every $k \ge 40$.*

(b) *If G has at most* 235 *edges, then they are connected for every $k \ge 48$.*

(c) *In general, they are connected for every $k \ge k_0$, for every base k_0 with $\sigma(k_0) \ge |E(G)|$.*

Proof. (c) is Corollary 10.11 with Theorem 9.2. (a) and (b) are (c) with $\sigma(40) = 189$ and $\sigma(48) = 235$. The terms $\lceil 40/\lambda \rceil$ and $\lceil 48/\lambda \rceil$ are:

λ	1	2	3	4	5	6	7	8, 9	10, …, 13	14, …, 19	20, …, 39
$\lceil 40/\lambda \rceil$	40	20	14	10	8	7	6	5	4	3	2

λ	1	2	3	4	5	6	7	8, 9	10, 11	12, …, 15	16, …, 23	24, …, 47
$\lceil 48/\lambda \rceil$	48	24	16	12	10	8	7	6	5	4	3	2

So $\sigma(40) = 40+20+14+10+8+7+6+2\cdot5+4\cdot4+6\cdot3+20\cdot2 = 115+16+18+40 = 189$, and $\sigma(48) = 48+24+16+12+10+8+7+2\cdot6+2\cdot5+4\cdot4+8\cdot3+24\cdot2 = 147+16+24+48 = 235$. □

Few vertices. In this subsection $k \geq 4$. For a flow F of $I_k(G)$ let $\Phi(F) = \sum_e |F(e)|$, the sum of the sizes. A *failure* (for G and k) is a component of $I_k(G)$ that contains no flow of $I_{k-1}(G)$. Put

$$S(k) = \sum_{\lambda=2}^{k-2} \left\lceil \frac{k-1}{\lambda} \right\rceil = \sigma(k-1) - (k-1).$$

Lemma 10.13 (A potential)**.** *Let $\mathcal{U}$ be a component of $I_k(G)$, and let F be a flow of $\mathcal{U}$ with the least Φ in $\mathcal{U}$. Then there is a map $\theta\colon V(G) \to \mathbb{Z}$ such that for every arc e of G_F from x to y:*

$$\begin{aligned} &\theta(y) - \theta(x) = 1 && \text{if } e \text{ is a middle arc;} \\ &\theta(y) - \theta(x) \leq 1 && \text{if } e \text{ is a 1-arc;} \\ &\theta(y) - \theta(x) \geq 1 && \text{if } e \text{ is a top.} \end{aligned}$$

Proof. Let R be the directed multigraph on $V(G)$ with the following arcs and costs. Every arc e of G_F from x to y gives an arc $x \to y$ of cost $+1$ if $F(e) \leq k-2$, and an arc $y \to x$ of cost -1 if $F(e) \geq 2$. A middle arc gives both.

Every directed cycle of R has a cost of at least 0. Let O be a directed cycle of R (distinct vertices; R has no loop, because G has none). If O uses both arcs that come from one arc e of G_F, then after the first of them O is at an end of e, and the only arc of O that leaves this vertex is the second one; so O consists of these two arcs, and its cost is 0. Otherwise O uses pairwise different edges of G and passes pairwise different vertices, so it is a cycle C of G with a direction (two parallel edges give a cycle with 2 edges). Put $F' = F + \chi_C$, with χ_C read in G_F. If O uses the arc $x \to y$ of e, then C passes e in its direction and $F'(e) = F(e) + 1 \leq k-1$. If O uses the arc $y \to x$ of e, then C passes e against its direction and $F'(e) = F(e) - 1 \geq 1$. All other values are equal. So F' is in $I_k(G)$, positive on G_F, and adjacent to F; so F' is in $\mathcal{U}$, and $\Phi(F') - \Phi(F)$ is the cost of O. Since F has the least Φ in $\mathcal{U}$, the cost is at least 0.

The potential. For a vertex y let $\theta(y)$ be the least cost of a directed path of R that ends at y (the path with the one vertex y and no arc has cost 0; there are finitely many paths). Let $x \to y$ be an arc of R with cost κ, and let P be a path that ends at x with cost $\theta(x)$. If y is not on P, then P followed by the arc is a path that ends at y, so $\theta(y) \leq \theta(x) + \kappa$. If y is on P, let P_1 be the part of P up to y and P_2 the part from y to x. Then P_2 followed by the arc $x \to y$ is a directed cycle of R, so $\text{cost}(P_2) + \kappa \geq 0$, and $\theta(y) \leq \text{cost}(P_1) = \theta(x) - \text{cost}(P_2) \leq \theta(x) + \kappa$. In both cases $\theta(y) \leq \theta(x) + \kappa$.

An arc e of G_F from x to y with $F(e) \leq k-2$ gives $\theta(y) - \theta(x) \leq 1$. If $F(e) \geq 2$, the arc $y \to x$ of cost -1 gives $\theta(x) \leq \theta(y) - 1$. A middle arc has both; a 1-arc has the first $(1 \leq k-2)$; a top has the second $(k-1 \geq 2)$. □

Lemma 10.14 (A forest of middle arcs)**.** *Every component $\mathcal{U}$ of $I_k(G)$ contains a flow F^* that has the least Φ in $\mathcal{U}$ and whose middle arcs contain no cycle of G.*

Proof. Among the flows of $\mathcal{U}$ with the least Φ, take a flow F with the least number of middle arcs. Suppose that C is a cycle of G all of whose edges are middle arcs of F. Give C a direction, and take θ from Lemma 10.13. Every arc of C has $\theta(y) - \theta(x) = 1$. The sum of $\theta(y) - \theta(x)$ over the arcs that C passes in their direction, minus the same sum over the arcs that C passes against their direction, is 0, because it is a sum of differences of θ once round C. So C passes as many arcs in their direction as against it. C has at least 2 edges, so it passes at least one arc in its direction.

Put $F_j = F + j\chi_C$ (χ_C read in G_F) for $j = 0, 1, 2, \dots$, and let $j_0 \geq 1$ be the least j such that F_j has the value 1 or $k-1$ on some arc of C. It exists, because the value of an arc that C passes in its direction grows by 1 at every step. For $j < j_0$ all values of F_j on C are in $[2, k-2]$, so all values of F_{j_0} on C are in $[1, k-1]$. The values outside C do not change. So $F_0, \dots, F_{j_0}$ are in $I_k(G)$, all positive on G_F, and two consecutive ones differ by a move. Each step adds 1 to the values of the arcs that C passes in their direction and subtracts 1 from the values of the other arcs of C; there are as many arcs of the first kind as of the second, so $\Phi(F_j) = \Phi(F)$ for $0 \leq j \leq j_0$. So F_{j_0} is in $\mathcal{U}$ and has the least Φ. Its middle arcs are among those of F, and an arc of C is not a middle arc of F_{j_0}. So F_{j_0} has fewer middle arcs than F, a contradiction. □

Theorem 10.15 (A failure needs many vertices)**.** *Let $k \geq 4$, and let $\mathcal{U}$ be a failure for G and k. Then*

$$|V(G)| \geq 1 + S(k) \qquad \text{and} \qquad |E(G)| \geq k + S(k).$$

Proof. Take F^* in $\mathcal{U}$ from Lemma 10.14. Let $1 \leq \lambda \leq k-2$. The flow F^* is not joined to any flow with all sizes at most $k-2$ (such a flow is in $I_{k-1}(G)$), in particular not by moves $-\lambda\chi_O$ with directed cycles O of G_{F^*}. By Corollary 10.9, F^* has at least $\lceil (k-1)/\lambda \rceil$ arcs of size λ. For $2 \leq \lambda \leq k-2$ these are middle arcs, and arcs of different sizes are different. So F^* has at least $S(k)$ middle arcs. They form a forest of G, which has at most $|V(G)| - 1$ edges. So $|V(G)| - 1 \geq S(k)$. For the edges: F^* has in addition at least $k-1$ arcs of size 1 ($\lambda = 1$) and at least one top (F^* is not in $I_{k-1}(G)$). □

Corollary 10.16 (At most 149 vertices)**.** *Let G be a finite bridgeless loopless multigraph with n vertices.*

(a) *If $n \leq 149$, then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq 40$ (with any number of edges). If $n \leq 187$, they are connected for every $k \geq 48$.*

(b) *They are connected for every $k \geq k_0$, for every base k_0 with $S(k_0 + 1) = \sigma(k_0) - k_0 \geq n$.*

(c) *$S(k_0 + 1) \geq k_0(\ln k_0 - 1)$ for every $k_0 \geq 4$. So* (b) *holds for every base k_0 with $k_0(\ln k_0 - 1) \geq n$.*

(d) *If $n \geq 150$, they are connected for every $k \geq 2n/\ln n + 4$.*

Proof. First, $S(k+1) \geq S(k)$ for $k \geq 4$: $S(k+1) = \sum_{\lambda=2}^{k-1} \lceil k/\lambda \rceil$, every term $\lceil k/\lambda \rceil$ with $\lambda \leq k-2$ is at least $\lceil (k-1)/\lambda \rceil$, and there is one more term.

(b) Induction on $k \geq k_0$; for $k = k_0$, $I_{k_0}(G)$ is connected by Theorem 9.2. Let $k > k_0$, and let $I_{k-1}(G)$ be connected. Then $S(k) \geq S(k_0 + 1) \geq n$, so by Theorem 10.15 there is no failure for G and k: every flow of $I_k(G)$ descends. Lemma 10.1 (b) gives the step, and Lemma 10.2 gives $M_k(G)$.

(a) $S(41) = \sigma(40) - 40 = 149$ and $S(49) = \sigma(48) - 48 = 187$ (Corollary 10.12). Use (b) with the bases 40 and 48.

(c) $S(k_0{+}1) = \sum_{\lambda=2}^{k_0-1} \lceil k_0/\lambda \rceil \geq k_0\big(\frac{1}{2} + \dots + \frac{1}{k_0-1}\big) \geq k_0(\ln k_0 - 1)$, because $1 + \frac{1}{2} + \dots + \frac{1}{k_0-1} \geq \int_1^{k_0} \frac{dx}{x} = \ln k_0$.

(d) Put $\eta = 2n/\ln n$, and let k_0 be the least multiple of 4 with $k_0 \geq \eta$; so $k_0 < \eta + 4$. The function $\xi/\ln\xi$ grows for $\xi \geq 3$, and $\ln 150 < 5.02$; so $\eta \geq 300/5.02 > 59$, and $k_0 \geq 60$ is a base. The function $\xi(\ln\xi - 1)$ grows for $\xi \geq 1$, so $k_0(\ln k_0 - 1) \geq \eta(\ln\eta - 1)$. And $\eta(\ln\eta - 1) \geq n$ means $2(\ln 2 + \ln n - \ln\ln n - 1) \geq \ln n$, that is

$$\ln n - 2\ln\ln n \;\geq\; 2 - 2\ln 2.$$

Here $\ln n \geq \ln 150 > 5$. The function $\xi - 2\ln\xi$ grows for $\xi > 2$, and at $\xi = 5$ it is $5 - 2\ln 5 > 1.7$, while $2 - 2\ln 2 < 0.62$. So $k_0(\ln k_0 - 1) \geq n$, and (c) gives every $k \geq k_0$, in particular every $k \geq \eta + 4$. □

Short cycles. In this subsection F is a flow of $I_k(G)$. A *lift* of F is a flow $\widetilde{F}$ of $I_k(G)$ with $\widetilde{F} = F$ modulo k (edge by edge). Let $\Psi(F)$ be the number of arcs of G_F that are 1-arcs or tops. An edge e is counted in $\Psi(F)$ if and only if $F(e)$ is 1 or -1 modulo k; so $\Psi(F)$ depends only on F modulo k.

Residues. Let C be a cycle of G with a direction. For an arc e of G_F on C put $r(e) = F(e)$ if C passes e in its direction, and $r(e) = k - F(e)$ if C passes e against its direction. So $1 \leq r(e) \leq k-1$, and $r(e)$ is the value of F on e measured in the direction of C, taken modulo k. It is the same for all lifts of F. If the direction of C is reversed, $r(e)$ becomes $k - r(e)$. Write $C_{<\lambda}$ and $C_{>\lambda}$ for the sets of the arcs e of C with $r(e) < \lambda$ and with $r(e) > \lambda$. For a set X of vertices and an arc e of the cut of X (the edges with one end in X) put $r_X(e) = F(e)$ if e leaves X and $r_X(e) = k - F(e)$ if e enters X. If C passes the arc e of the cut *into* X, then $r_X(e) = k - r(e)$; if C passes e *out of* X, then $r_X(e) = r(e)$. (Check the four cases: e in the direction of C or against it, and e entering or leaving X in G_F.)

Lemma 10.17 (Lifts)**.** (a) *The lifts of F are exactly the flows $F - kY$, where Y is an integer flow of G with values in $\{0,1\}$ in the orientation G_F. All lifts of F are joined to F.*

(b) *For every set X of vertices, $k\,\mathrm{in}(X)$ is the sum of $r_X(e)$ over the arcs e of the cut of X.*

Proof. (a) Let $\widetilde{F}$ be a lift, read in G_F. On an arc e the number $\widetilde{F}(e)$ is $F(e)$ modulo k and lies in $[-(k-1), k-1] \setminus \{0\}$; so it is $F(e)$ or $F(e) - k$. So $Y = (F - \widetilde{F})/k$ is an integer flow with values in $\{0,1\}$. Conversely, for such a Y the flow $F - kY$ has on every arc the value $F(e)$ in $[1, k-1]$ or $F(e) - k$ in $[-(k-1), -1]$; so it is a lift of F. The lifts of F are the integer lifts of the reduction of F modulo k, so they are joined by Lemma 1.4.

(b) The flow out of X is the flow into X. So the sum of r_X is $\sum_{e \text{ leaves } X} F(e) + \sum_{e \text{ enters } X} \big(k - F(e)\big) = k\,\mathrm{in}(X)$. □

Theorem 10.18 (A step after a change of lift)**.** *Let C be a cycle of G with a direction, and λ an integer with $1 \leq \lambda \leq k-1$ such that $r(e) \neq \lambda$ for every arc e of C. The following are equivalent:*

(i) *some lift $\widetilde{F}$ of F has $\widetilde{F} - \lambda\chi_C$ in $I_k(G)$;*

(ii) *for every set X of vertices, the number of arcs of $C_{<\lambda}$ that C passes into X is at most $\mathrm{in}(X)$.*

If (i) *holds, then F is joined to $\widetilde{F} - \lambda\chi_C$, and this flow is $F - \lambda\chi_C$ modulo k.*

Proof. Measure every value on an arc e of C in the direction of C. Measured so, F has a value that is $r(e)$ modulo k. A lift $\widetilde{F}$ has a value that is $r(e)$ modulo k and whose absolute value is in $[1, k-1]$; so it has the value $r(e)$ or $r(e) - k$. After the step it has $r(e) - \lambda$ or $r(e) - k - \lambda$. The first number is not 0 and lies in $[-(k-2), k-2]$. The second is negative, and it is at least $-(k-1)$ exactly if $r(e) \geq \lambda + 1$. The values outside C do not change. So $\widetilde{F} - \lambda\chi_C$ is in $I_k(G)$ if and only if $\widetilde{F}$ has the value $r(e)$, measured in the direction of C, on every arc e of $C_{<\lambda}$.

Write $\widetilde{F} = F - kY$ as in Lemma 10.17 (a). Let C^+ be the set of the arcs of $C_{<\lambda}$ that C passes in their direction, and C^- the set of those that C passes against their direction. On an arc of C^+, F has the value $r(e)$ along C, and $\widetilde{F}$ has it if and only if $Y(e) = 0$. On an arc

of C^-, F has the value $-F(e) = r(e) - k$ along C, and $\widetilde{F}$ has the value $r(e)$ if and only if $Y(e) = 1$. So (i) holds if and only if there is an integer flow Y in G_F with $l \le Y \le u$, where $l = u = 0$ on C^+, $l = u = 1$ on C^-, and $l = 0$, $u = 1$ on all other arcs. By Theorem 0.5 this holds if and only if for every X:

$$\text{(number of arcs of } C^- \text{ that leave } X) \;\le\; \text{in}(X) - \text{(number of arcs of } C^+ \text{ that enter } X).$$

An arc of C^- that leaves X is passed by C into X, and so is an arc of C^+ that enters X; and every arc of $C_{<\lambda}$ that C passes into X is of one of these two kinds. So this is (ii).

The last sentence: F is joined to $\widetilde{F}$ by Lemma 10.17 (a), and $\widetilde{F} \to \widetilde{F} - \lambda\chi_C$ is a move. □

Corollary 10.19 (A sufficient condition)**.** *Let C and λ be as in Theorem 10.18. If the sum of $r(e) - 1$ over the arcs e of $C_{<\lambda}$ is less than k, then* (ii) *of Theorem 10.18 holds. In particular* (ii) *holds for $\lambda = 1$ ($C_{<1}$ is empty) and for $\lambda = 2$ (every arc of $C_{<2}$ has $r(e) = 1$).*

Proof. Fix X. The cycle C passes arcs of the cut of X into X as often as out of X; say j times each. Let j' of the j arcs that C passes into X be in $C_{<\lambda}$. By Lemma 10.17 (b), and because $r_X \ge 1$ on every arc of the cut,

$$\begin{aligned} k\,\text{in}(X) \;&\ge \sum_{\substack{e\in C_{<\lambda}\\ \text{passed into } X}} \big(k - r(e)\big) + (j - j') + j \;=\; j'k - \sum_{\substack{e\in C_{<\lambda}\\ \text{passed into } X}} \big(r(e) - 1\big) + 2(j - j') \\ &\ge\; j'k - \sum_{e\in C_{<\lambda}} \big(r(e) - 1\big) \;>\; j'k - k. \end{aligned}$$

(The j' arcs of $C_{<\lambda}$ passed into X have $r_X = k - r(e)$; the other $j - j'$ arcs passed into X, the j arcs passed out of X, and the arcs of the cut that are not on C have $r_X \ge 1$, so they add at least $(j - j') + j$; and every term $r(e) - 1$ is at least 0.) So $\text{in}(X) > j' - 1$, that is $\text{in}(X) \ge j'$. □

Theorem 10.20 (The cleaning step)**.** *Let e_0 be a 1-arc or a top of F, and let C be a cycle through e_0, directed so that $r(e_0) = 1$ (in the direction of a 1-arc, against the direction of a top).*

(a) *Let λ be an integer with $3 \le \lambda \le k - 3$ such that no arc of C has $r(e)$ in $\{\lambda - 1, \lambda, \lambda + 1\}$, and such that the sum of $r(e) - 1$ over $C_{<\lambda}$ is less than k, or the sum of $k - 1 - r(e)$ over $C_{>\lambda}$ is less than k. Then F is joined to a flow F' with $F' = F - \lambda\chi_C$ modulo k and $\Psi(F') \le \Psi(F) - 1$.*

(b) *Let $h \ge 0$ be an integer with $3h^2 < k$ and $6h + 8 < k$, and put $I_{\text{low}} = [2, 3h + 4]$ and $I_{\text{high}} = [k - 3h - 4, k - 2]$ (sets of integers; they are disjoint, because $3h + 4 < k - 3h - 4$). If at most h arcs of C have $r(e)$ in I_{low}, or at most h arcs of C have $r(e)$ in I_{high}, then F is joined to a flow F' with $\Psi(F') \le \Psi(F) - 1$.*

Proof. (a) First case: the sum over $C_{<\lambda}$ is less than k. No arc of C has $r(e) = \lambda$. By Theorem 10.18 and Corollary 10.19, F is joined to $F' = \widetilde{F} - \lambda\chi_C$, which is $F - \lambda\chi_C$ modulo k. Second case: the sum over $C_{>\lambda}$ is less than k. Let C' be C with the other direction; its residues are $k - r(e)$, none of them is $k - \lambda$, and $3 \le k - \lambda \le k - 3$. The arcs with $k - r(e) < k - \lambda$ are the arcs of $C_{>\lambda}$, and the sum of $(k - r(e)) - 1$ over them is less than k. Theorem 10.18 and Corollary 10.19 for C' and $k - \lambda$ give F' joined to F with $F' = F - (k - \lambda)\chi_{C'} = F + (k - \lambda)\chi_C = F - \lambda\chi_C$ modulo k.

The count. An edge outside C keeps its value modulo k. An arc e of C has in F' the value $r(e) - \lambda$ modulo k, measured in the direction of C. It is counted in $\Psi(F')$ if and only

if $r(e) - \lambda$ is 1 or -1 modulo k. Since $r(e) - \lambda$ lies in $[4-k, k-4]$, this means $r(e) = \lambda + 1$ or $r(e) = \lambda - 1$, which is excluded. So no edge of C is counted in $\Psi(F')$, while e_0 is counted in $\Psi(F)$.

(b) First case: at most h arcs of C have $r(e)$ in I_{low}. Look at the $3h+1$ integers λ in $[3, 3h+3]$. An arc with $r(e)$ in I_{low} has $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$ for at most 3 of them. An arc with $r(e) = 1$ or $r(e) \geq 3h+5$ has this for none of them, because $\lambda - 1 \geq 2$ and $\lambda + 1 \leq 3h+4$. So at most $3h$ values of λ are excluded, and some λ in $[3, 3h+3]$ has no arc of C with $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$. For this λ, an arc of $C_{<\lambda}$ has $r(e) = 1$, or $2 \leq r(e) \leq \lambda - 2$; the arcs of the second kind have $r(e)$ in I_{low}, so there are at most h of them, each with $r(e) - 1 \leq \lambda - 3 \leq 3h$. So the sum of $r(e) - 1$ over $C_{<\lambda}$ is at most $3h^2 < k$. And $3 \leq \lambda \leq 3h + 3 \leq k - 3$, because $6h + 8 < k$. Part (a) applies.

Second case: at most h arcs of C have $r(e)$ in I_{high}. Look at the $3h+1$ integers λ in $[k-3h-3, k-3]$. An arc with $r(e) = k-1$ or $r(e) \leq k-3h-5$ has $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$ for none of them, because $\lambda + 1 \leq k-2$ and $\lambda - 1 \geq k - 3h - 4$; an arc with $r(e)$ in I_{high} for at most 3. So some λ has no arc of C with $r(e)$ in $\{\lambda-1, \lambda, \lambda+1\}$. An arc of $C_{>\lambda}$ has $r(e) = k-1$, or $\lambda + 2 \leq r(e) \leq k-2$; the arcs of the second kind have $r(e)$ in I_{high}, so there are at most h of them, each with $k - 1 - r(e) \leq k - 3 - \lambda \leq 3h$. So the sum of $k - 1 - r(e)$ over $C_{>\lambda}$ is at most $3h^2 < k$. And $k - 3 \geq \lambda \geq k - 3h - 3 \geq 3$. Part (a) applies. □

Theorem 10.21 (Short cycles)**.** *Let $h \geq 0$ be an integer with $3h^2 < k$ and $6h + 8 < k$. If every edge of G lies on a cycle of G with at most $2h+2$ edges, then every flow of $I_k(G)$ is joined to a flow with all sizes in $[2, k-2]$. In particular every flow of $I_k(G)$ descends.*

Proof. Induction on $\Psi(F)$. If $\Psi(F) = 0$, then F has no 1-arc and no top. Otherwise take a 1-arc or a top e_0, and a cycle C through e_0 with at most $2h+2$ edges, directed so that $r(e_0) = 1$. The residue 1 is in neither I_{low} nor I_{high}. The other arcs of C are at most $2h+1$, and the two sets are disjoint; so one of them contains the residues of at most h arcs of C. Theorem 10.20 (b) gives F' joined to F with $\Psi(F') < \Psi(F)$. A flow with all sizes in $[2, k-2]$ is in $I_{k-1}(G)$. □

Corollary 10.22 (Every edge on a short cycle)**.** (a) *If every edge of G lies on a cycle with at most 8 edges, then every flow of $I_k(G)$ descends for every $k \geq 28$, and $I_k(G)$ and $M_k(G)$ are connected for every $k \geq 40$.*

(b) *Let $h \geq 4$. If every edge of G lies on a cycle with at most $2h+2$ edges, then $I_k(G)$ and $M_k(G)$ are connected for every $k \geq k_h$, where k_h is the least multiple of 4 with $k_h \geq 3h^2$ (so $48 \leq k_h \leq 3h^2 + 3$).*

Proof. (a) Take $h = 3$: $3h^2 = 27 < k$ and $6h + 8 = 26 < k$ for $k \geq 28$, so Theorem 10.21 applies. $I_{40}(G)$ is connected by Theorem 9.2. For $k \geq 41$, every flow of $I_k(G)$ descends, and Lemma 10.1 (b) gives every $k \geq 40$ by induction. $M_k(G)$: Lemma 10.2.

(b) $k_h \geq 3 \cdot 16 = 48$ is a multiple of 4, so $I_{k_h}(G)$ is connected by Theorem 9.2. For $k \geq k_h + 1$: $3h^2 \leq k_h < k$, and $6h + 8 \leq 3h^2 < k$, because $3h^2 - 6h - 8 \geq 16$ for $h \geq 4$. So Theorem 10.21 applies, and the induction is as in (a). □

Examples for (a): complete graphs on at least 3 vertices (every edge is on a triangle), complete bipartite graphs with at least 2 vertices on each side and hypercubes of dimension at least 2 (every edge is on a cycle with 4 edges), and every bridgeless loopless multigraph whose cycles all have at most 8 edges.

Corollary 10.23 (Diameter)**.** *Let G be connected, with at least one edge, and let* $\operatorname{diam}(G)$ *be its diameter. Then every edge of G lies on a cycle with at most* $2\operatorname{diam}(G)+1$ *edges. So, if* $\operatorname{diam}(G)\le 3$*, then $I_k(G)$ and $M_k(G)$ are connected for every $k\ge 40$; if* $\operatorname{diam}(G)\ge 4$*, they are connected for every $k\ge k_{\operatorname{diam}(G)}$ (Corollary 10.22 (b)).*

Proof. Write dist for the distance in G, and $d_G=\operatorname{diam}(G)$. Let e be an edge with ends $x\ne y$.

Case 1: some vertex w has $\operatorname{dist}(w,x)=\operatorname{dist}(w,y)=j$. Then $j\le d_G$. Take shortest paths P from w to x and P' from w to y. Every vertex of a shortest path from w has, as its distance from w, its position on the path. So P does not contain y (y has the distance j, and the only vertex of P with the distance j is x), P' does not contain x, and neither path uses e. Let z be the last vertex of P that lies on P' (w lies on both); $z\ne x$, $z\ne y$. The part of P from z to x, the edge e, and the part of P' from y back to z form a cycle through e: the first part meets P' only in z. It has at most $2j+1\le 2d_G+1$ edges.

Case 2: there is no such w. Then $V(G)$ is the disjoint union of $V_x=\{w:\operatorname{dist}(w,x)<\operatorname{dist}(w,y)\}$ and $V_y=\{w:\operatorname{dist}(w,y)<\operatorname{dist}(w,x)\}$, and the two distances of a vertex differ by exactly 1, because x and y are adjacent. The edge e is not a bridge, so $G-e$ has a path from x in V_x to y in V_y, and some edge $e'\ne e$ of this path joins a vertex x_1 in V_x to a vertex y_1 in V_y. Now $\operatorname{dist}(y_1,y)+1=\operatorname{dist}(y_1,x)\le\operatorname{dist}(x_1,x)+1$ and $\operatorname{dist}(x_1,x)+1=\operatorname{dist}(x_1,y)\le\operatorname{dist}(y_1,y)+1$; so $\operatorname{dist}(x_1,x)=\operatorname{dist}(y_1,y)=i'$, and $i'+1=\operatorname{dist}(x_1,y)\le d_G$. A shortest path from x to x_1 stays in V_x: a vertex z on it has $\operatorname{dist}(z,y)\ge\operatorname{dist}(x_1,y)-\operatorname{dist}(z,x_1)=i'+1-(i'-\operatorname{dist}(z,x))=\operatorname{dist}(z,x)+1$. In the same way a shortest path from y_1 to y stays in V_y. These two paths, with e' and e, form a cycle through e with $2i'+2\le 2d_G$ edges.

The rest: if $d_G\le 3$, every edge lies on a cycle with at most $7\le 8$ edges, and Corollary 10.22 (a) applies. If $d_G\ge 4$, use Corollary 10.22 (b) with $h=d_G$, because $2d_G+1\le 2d_G+2$. □

Corollary 10.24 (All but a few edges on short cycles)**.** *Let $h\ge 0$ be an integer with $3h^2<k$ and $6h+8<k$. Call an edge* short *if it lies on a cycle with at most $2h+2$ edges. An* edge cut *is the set of the edges with one end in X and one end outside X, for a set X of vertices.*

(a) *Every flow of $I_k(G)$ is joined to a flow F' in which no short edge is a 1-arc or a top.*

(b) *If no edge cut of G contains k or more edges that are not short, then every flow of $I_k(G)$ descends. This holds in particular if G has at most $k-1$ edges that are not short.*

(c) *If all edges of G except at most 40 lie on cycles with at most 8 edges, then $I_k(G)$ and $M_k(G)$ are connected for every $k\ge 40$.*

Proof. (a) As long as a short edge e_0 is a 1-arc or a top, apply Theorem 10.20 (b) with a cycle through e_0 that has at most $2h+2$ edges, as in the proof of Theorem 10.21. Every time Ψ goes down.

(b) Take F' from (a). In this proof in, out are counted in $G_{F'}$. If $G_{F'}$ has an integer flow T with $1\le T(e)\le k-2$ on every arc, then T is in $I_{k-1}(G)$ and $G_T=G_{F'}$, so T is joined to F' by Lemma 10.3, and F' descends. Otherwise, by Theorem 0.5 with $l=1$ and $u=k-2$ on all arcs, some set X has $\operatorname{out}(X)>(k-2)\operatorname{in}(X)$. Then $\operatorname{out}(X)\ge 1$, the flow out of X is positive, so is the flow into X, and $\operatorname{in}(X)\ge 1$; hence $\operatorname{out}(X)\ge k-1$. Every arc that leaves X and is not a 1-arc has a value of at least 2:

$$(k-1)\operatorname{in}(X)\ \ge\ \text{flow into } X\ =\ \text{flow out of } X\ \ge\ \operatorname{out}_1(X)+2\big(\operatorname{out}(X)-\operatorname{out}_1(X)\big).$$

With $\mathrm{in}(X) \le (\mathrm{out}(X)-1)/(k-2)$ this gives

$$\begin{aligned}\mathrm{out}_1(X) &\ge 2\,\mathrm{out}(X) - \frac{(k-1)(\mathrm{out}(X)-1)}{k-2} = \frac{(k-3)\,\mathrm{out}(X)+(k-1)}{k-2}\\ &\ge \frac{(k-3)(k-1)+(k-1)}{k-2} = k-1.\end{aligned}$$

Also the flow into X is at least $\mathrm{out}(X) > (k-2)\,\mathrm{in}(X)$, so some arc that enters X has a value greater than $k-2$: it is a top. So the cut of X contains at least k arcs that are 1-arcs or tops of F'. By (a) none of them is short. This contradicts the hypothesis. So F', and hence every flow, descends.

(c) Take $h = 3$. For every $k \ge 41$: $27 < k$, $26 < k$, and G has at most $40 \le k-1$ edges that are not short; so by (b) every flow of $I_k(G)$ descends. $I_{40}(G)$ is connected by Theorem 9.2. Lemma 10.1 (b) and induction give every $k \ge 40$, and Lemma 10.2 gives $M_k(G)$. □

Remarks.

Remark 10.25 (How the bounds compare)**.** The four kinds of bound do not compare in general. For a graph with a Hamiltonian cycle on n vertices, Theorem 10.5, with the base k_0 of the proof of Corollary 10.6, gives every $k \ge \max(48, 4\lceil (n+1)/4\rceil)$, and Corollary 10.6 gives every $k \ge 40$ if $n \le 39$; Corollary 10.16 (d) gives every $k \ge 2n/\ln n + 4$ for $n \ge 150$. A graph with many vertices and only short cycles is better for Corollary 10.6 or 10.22. A graph with at most 149 vertices can have any number of edges, and a graph with at most 189 edges can have a cycle with more than 39 edges. Corollary 10.22 is a local condition: the graph can be as large as we like and can have long cycles.

Remark 10.26 (Computations; not part of a proof)**.** No proof of this section uses a computer. The following values were computed by a short program, as a check of the arithmetic only: $\sigma(52) = 262$, $\sigma(56) = 286$, $\sigma(60) = 308$, $\sigma(100) = 572$; and $\sigma(k_0) - k_0 = 149$, 187, 210, 230, 248, 472, 1085 for $k_0 = 40$, 48, 52, 56, 60, 100, 200. The least base $k_0 \ge 48$ with $\sigma(k_0) - k_0 \ge n$ (Corollary 10.16 (b)) is 188 for $n = 1000$, 1360 for $n = 10{,}000$, and 10,612 for $n = 100{,}000$.

11. The limits of the method

This section says where the method of this paper stops. It contains no new result. As in Section 0, "k is reached" means that one of the results of this paper applies to k. We do not know a finite bridgeless loopless multigraph G and an integer $k \ge 6$ such that $I_k(G)$ is not connected (the modulus 5 fails [EHLMNS]).

The bound 40**.** Corollary 3.15 needs $|A| \ge 10$ and $|B| \ge 4$. The reason is in the counts (Definition 2.15, Remark 2.16): a thread has up to 3 edges, so the left sides of (C1), (C2), (C3) can be 9, 9, 8, and the left side of (C4) can be 3. For $\Gamma = \mathbb{Z}_k$ with $B = a\mathbb{Z}_k$ this means $k = ab \ge 40$. Theorem 3.11 itself asks only $|A| \ge 8$ and counted frames; but no statement of this paper controls the frames of the good subgraphs that the averaging step gives. So all results for all graphs have $k \ge 40$; for $6 \le k < 40$ the question for all graphs is open (Problem (P1) of Section 12). For a group with $2A = 0$ and $2B = 0$, Part II of [Go1] avoids these counts ([Go1, Theorem 13.1]: $|A| \ge 8$ and any frame) and reaches the order 32 [Go1, Corollary 13.2]. This does not help $\mathbb{Z}_k$: its elements x with $2x = 0$ form a subgroup with at most 2 elements, so $\mathbb{Z}_k$ has no subgroup B with $2B = 0$ and $|B| \ge 4$.

What a condition on the colouring would give. Corollaries 3.16 and 3.17 are conditional (Remark 3.18). Suppose that every subdivision that the averaging step gives had a frame with (S) (no pair consists of two original edges whose threads both have 3 edges). Then Corollary 3.16 would apply with $|A| = 9$. For $k = 36 = 9 \cdot 4$, the residues of ± 1, ± 2 and $\pm q$ for the divisors $q < 36$ of 36 form the step system $\Sigma = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, 18\}$ of liftable elements (Lemma 1.12); modulo 9 it gives every non-zero residue, so (R1) holds with walks of one step; $\Sigma_B = \{9, 18, 27\} = B \setminus \{0\}$; and $\mathbb{Z}_{36} = \mathbb{Z}_4 \times \mathbb{Z}_9$ with the generators 9 and 4 and the ratio $3/35 < 1/9$ gives (AV$''$) by Lemma 4.1. So the frame condition (S) would give $k = 36$. In the same way, a frame with (NR) (no free thread with 3 edges) would, by Corollary 3.17, give $k = 30 = 10 \cdot 3$ (Σ_A is all of $\mathbb{Z}_{10} \setminus \{0\}$, $\Sigma_B = \{10, 20\}$, $\mathbb{Z}_{30} = \mathbb{Z}_2 \times \mathbb{Z}_{15}$, ratio $2/29$) and $k = 36 = 12 \cdot 3$ (Σ_A misses only ± 5, Lemma 5.2; $\Sigma_B = \{12, 24\}$; ratio $2/35$). In a graph without cycles with at most 5 edges, Lemma 2.2 (b) gives a good subgraph (a vertex-minimal dense subgraph) with the properties (X2)–(X4), which limit the threads with 3 edges; they do not give (S) or (NR). So $k = 30$ and $k = 36$ remain open.

The modulus 44 and $4p$ with $p \leq 11$. The only split $44 = ab$ with $a \geq 10$ and $b \geq 4$ is $a = 11$, $b = 4$. The liftable steps of Section 1 give $\Sigma_A = \{\pm 1, \pm 2, \pm 4\}$ in $\mathbb{Z}_{11}$, and the deletion of the six neighbours of 0 isolates 0; so (R1) fails (Remark 5.6). The double step of Section 6 replaces (R1) for $k = 4p$, but its walk lemma needs the count $12 < p$ (Remark 6.10); so it does not reach $4p$ with $p \leq 11$, that is 8, 12, 20, 28 and 44. And 44 is not a product of two coprime admitted factors. So $I_{44}(G)$ is not reached. The modular graph $M_{44}(G)$ is connected for every bridgeless G (Theorem 4.5); whether this implies the connectivity of $I_{44}(G)$ is a case of Problem 2.6 of [EHLMNS].

Primes and prime powers. Every result of Sections 5–9 for all graphs uses a factorization of k. Theorem 3.19 needs $k = ab$ with $a, b \geq 2$; Lemma 4.1 needs a product of two non-trivial groups; the guarded moves of Section 8 need two coprime factors. So no prime k is reached for all graphs (Corollary 9.8 (a)). For a prime k, Section 10 gives the connectivity only for single graphs, from a base on.

A cyclic group of prime-power order is not a product of two non-trivial groups, so Lemma 4.1 does not apply to $\mathbb{Z}_{p^e}$. Section 7 replaces it by the flattening of valuations; the proof of Theorem 7.4 needs $e \geq 6$ (Remark 7.5: for $p = 2$, Lemma 7.2 needs $e - 2 \geq 4$, the count needs $|A| = 2^{e-2} \geq 10$, and (R1) needs $2(e-2) - 2 \geq 6$; for an odd p, Lemma 5.7 needs a divisor of p^{e-4} strictly between 1 and p^{e-4}). So p^e with $e \leq 5$ is not reached; for example 81, 125 and 243 are not reached.

Two prime factors. For $k = pq$ with different primes, Criterion 5.11 does not apply, because it needs a composite factor b. Theorem 8.8 needs factors with one-value avoidance, and the lemmas of Section 8 give it for a prime only from 43 on. For the modular graph alone, more moduli are known: for coprime $a \geq 10$ and $b \geq 4$, Theorem 3.11 gives the connectivity of $M_k(G)$ (Remark 8.11); but for primes $p, q \geq 5$ the liftable steps that Section 1 provides in $B = p\mathbb{Z}_k$ are only $\pm p$, so (R2) fails, and Theorem 3.19 does not give $I_k(G)$.

One graph. The bounds of Section 10 depend on G: on its longest cycle, its number of edges or of vertices, or its short cycles. They do not give one bound for all graphs beyond the bases; in particular they do not give every $k \geq 41$ for all graphs.

12. Open problems

We end with the questions that this paper leaves open. We know no finite bridgeless loopless multigraph G and no $k \geq 6$ for which $I_k(G)$ or $M_k(G)$ is not connected.

(P1) *The least modulus.* Is $I_6(G)$ connected for every finite bridgeless loopless multigraph G? Is there any k with $6 \leq k < 40$ with this property? The modulus 5 fails [EHLMNS, Theorem 3.4].

(P2) *All moduli from* 40 *on.* Is $I_k(G)$ connected for every $k \geq 40$ and every such G? The moduli from 41 to 47 are not reached, and no prime is reached (Proposition 9.7, Corollary 9.8 and Section 11). For $k = 44$ the graph $M_{44}(G)$ is connected (Theorem 4.5), and the question for $I_{44}(G)$ is a case of Problem 2.6 of [EHLMNS].

(P3) *Small groups.* What is the least order $n \geq 6$ of a finite abelian group Γ such that $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G? By [Go1, Corollary 13.2] it is at most 32. Is $F(G, \mathbb{Z}_6)$ connected for every such G? We ask for $n \geq 6$ for these reasons. A group with at most two elements has at most one nowhere-zero flow on G. For order 3, orient $K_{3,3}$ from one side to the other: its only nowhere-zero $\mathbb{Z}_3$-flows are the two constant flows, which differ on all nine edges and hence are not adjacent. Neither group of order 4, nor the group of order 5, has the property [EHLMNS, Sections 2.4, 3 and 4]. By [CLSWX, Theorem 5.1], for each fixed abelian group Γ of order at least five it suffices to test this connectivity on simple, 2-connected, 3-edge-connected cubic graphs. For $\mathbb{F}_2^3$ and $\mathbb{F}_2^4$, Part II of [Go1] proves the connectivity for every 4-edge-connected G [Go1, Theorem 13.3], and for every G whose components have at most 44, respectively 19022, vertices of odd degree [Go1, Theorems 22.1 and 28.6].

Appendix A. The closure lemma of Jaeger, Linial, Payan and Tarsi

This appendix proves the two facts on truncated cubic 3-edge-connected graphs that Section 2 uses: every such graph has a well colouring (Lemma A.5), and a well colouring gives a closure sequence of cycles (Lemmas A.6 and A.7). Both are results of Jaeger, Linial, Payan and Tarsi [JLPT]: the generation and the colouring are [JLPT, Lemmas 3.2 and 3.3, pp. 175 and 177], and the closure sequence is the observation at the start of the proof of their Theorem 3.2 [JLPT, p. 179]. The proofs are written out in full. This appendix contains no new result.

Definitions and counts. TC3 is the class of the truncated cubic 3-edge-connected graphs of Definition 2.4: the graphs $Q = Q^+ - v$, where Q^+ is a cubic 3-edge-connected graph on at least four vertices and v is a vertex of Q^+. The words 2-join, 2-constructible, 2-construction, stage, node and operand are those of Definition 2.8, and 2C is the class of the 2-constructible graphs. Graphs can have parallel edges (a 2-constructible graph often has them).

Definition A.1 (Well colouring)**.** Let Q be a graph in which every edge is blue or red. A *red component* of Q is a component of the graph with the vertex set $V(Q)$ and the red edges. Q/red is the graph whose vertices are the red components of Q and whose edges are the blue edges of Q; the ends of a blue edge in Q/red are the red components of its two ends. The colouring is a *well colouring* if

(i) the blue edges form a spanning tree of Q, and

(ii) Q/red is 2-constructible.

This is the notion of Definition 2.9 (F1).

Lemma A.2 (Counts)**.** (a) *Let $Q = Q^+ - v$ be in* TC3*, with n vertices. Then Q^+ and Q are simple graphs, Q is connected, n is odd, Q has $3(n-1)/2$ edges, exactly three vertices of Q have degree 2 (the neighbours of v), and all other vertices of Q have degree 3. The smallest graph of* TC3 *is the triangle ($Q^+ = K_4$).*

(b) *Let U_0 be a graph on j vertices with a 2-construction. Then U_0 is connected, has no loop, and has $2(j-1)$ edges. Two stages are disjoint, or one is a subgraph of the other. Every edge of U_0 is an edge of exactly one node. Every stage U is an induced subgraph of U_0, and the edges of U are the edges of the nodes that are subgraphs of U.*

(c) *In a well coloured graph no blue edge has both ends in one red component. If a graph of* TC3 *with n vertices is well coloured and has j red components, then $2(j-1) = n-1$, and the red edges form a forest.*

Proof. (a) A loop at a vertex x of the cubic graph Q^+ makes the third edge at x a bridge. Two parallel edges with the ends x, y leave at most 2 edges between $\{x, y\}$ and the other vertices, and there are other vertices because Q^+ has at least four. Both are impossible in a 3-edge-connected graph. So Q^+ is simple, and Q is simple. A cubic graph has an even number of vertices, so $n = |V(Q^+)| - 1$ is odd, and Q has $3(n+1)/2 - 3 = 3(n-1)/2$ edges. The three neighbours of v are different; they lose one edge, and no other vertex does. If Q had more than one component, one of them would get at most one of the three edges at v; this edge alone, or no edge, would separate it from the rest of Q^+. So Q is connected.

(b) Induction on the 2-construction. For one vertex all is clear. Let U_0 be the 2-join of U_0' and U_0'', with j' and j'' vertices. The two graphs are connected and disjoint, and a new edge joins them; so U_0 is connected. A new edge has its ends in two different graphs, so it is no loop. The number of edges is $2(j'-1) + 2(j''-1) + 2 = 2(j-1)$. The stages other than U_0 are the stages of U_0' and the stages of U_0''; the first are subgraphs of U_0' and the second are subgraphs of U_0''. This gives the statement on two stages, and the statements on the edges. Let $U \neq U_0$ be a stage, say a stage of U_0', and let t be an edge of U_0 with both ends in $V(U)$. Then t is not a new edge and not an edge of U_0''. So t is an edge of U_0', and by induction it is an edge of U.

(c) A blue edge with both ends in one red component is a loop of Q/red; by (b) there is none. Let Q be in TC3. By (b), Q/red has $2(j-1)$ edges. They are the blue edges, and the blue spanning tree has $n-1$ edges. So $2(j-1) = n-1$. By (a) there are $3(n-1)/2 - (n-1) = (n-1)/2 = n-j$ red edges. A graph with n vertices, j components and $n-j$ edges is a forest. □

The graphs of the stages. Let Q be a graph with a well colouring and with a 2-construction of Q/red. For a stage U let V_U be the union of the red components that are the vertices of U, and let Q_U be the subgraph of Q induced on V_U. We call Q_U the *graph of* U. For $U = Q/\text{red}$ it is Q.

Lemma A.3 (The graphs of the stages)**.** *Let Q be a graph with a well colouring and with a 2-construction of Q/red, and let U be a stage.*

(a) *The edges of Q_U are the red edges with both ends in V_U and the edges of the nodes that are subgraphs of U.*

(b) *Q_U is connected.*

(c) *If U is a node with the operands U', U'' and the two edges t_1, t_2, then V_U is the disjoint union of $V_{U'}$ and $V_{U''}$, each of t_1, t_2 has one end in $V_{U'}$ and one end in $V_{U''}$, and $E(Q_U) = E(Q_{U'}) \cup E(Q_{U''}) \cup \{t_1, t_2\}$.*

Proof. (a) A red edge has both ends in one red component. A blue edge with both ends in V_U is an edge of Q/red whose two ends are vertices of U. By Lemma A.2 (b), U is an induced subgraph of Q/red, so this edge is an edge of U, and the edges of U are the edges of the nodes that are subgraphs of U.

(c) follows from (a) and Lemma A.2 (b).

(b) Induction on the 2-construction. If U has one vertex, Q_U is one red component with its red edges. If U is a node, then $Q_{U'}$ and $Q_{U''}$ are connected, and t_1 joins them. □

Two operations. Let Q' and Q'' be disjoint graphs, and let d be 2 or 3.

- A *d-sticking* of Q' and Q'': z is a vertex of Q' of degree d with the edges $t_1, \dots, t_d$, and $z_1, \dots, z_d$ are d different vertices of degree 2 of Q''. The result is the graph $(Q' - z) \cup Q''$ in which every t_i keeps its end in $Q' - z$ and gets z_i as its other end.
- A *growing* of Q' at an edge y_1y_2 and a vertex x' of degree 2: put a new vertex y on the edge y_1y_2 (the edge becomes the path y_1, y, y_2), and add a new vertex x with two edges, to x' and to y.

Generation.

Lemma A.4 (Generation)**.** *Every graph Q of* TC3 *other than the triangle is a 3-sticking or a 2-sticking of two graphs of* TC3 *with fewer vertices, or a growing of a graph of* TC3 *with fewer vertices.*

Proof. Let $Q = Q^+ - v$, and let Q be not the triangle. Then $|V(Q^+)| \geq 6$.

Case 1: $V(Q^+)$ is the disjoint union of two sets V_1, V_2 with $|V_1|, |V_2| \geq 2$ and with exactly three edges between V_1 and V_2.

(1a) The three edges are pairwise disjoint. Suppose that a vertex x of V_1 meets two or three of them (the same for V_2). Move x to the other side. Then 2 or 0 edges join the two sides, and both sides are non-empty because $|V_1| \geq 2$. This is impossible.

(1b) $|V_i|$ is odd: three times $|V_i|$ is twice the number of edges inside V_i, plus 3. So $|V_i| \geq 3$.

(1c) Let Q^+/V_2 be the graph obtained from Q^+ by the identification of all vertices of V_2 to one new vertex and the deletion of the edges inside V_2. The new vertex has degree 3 (the three edges between V_1 and V_2), and every other vertex keeps its degree 3. Put $Q_1^+ = Q^+/V_2$ and $Q_2^+ = Q^+/V_1$. Both graphs are cubic. They are 3-edge-connected: a partition of the vertex set of Q_1^+ into two non-empty sets gives such a partition of $V(Q^+)$ with the same edges between the two sets (put V_2 on the side of the new vertex); the same for Q_2^+. They have $|V_i| + 1 \geq 4$ vertices.

By symmetry let v be in V_1, and let z be the vertex of Q_1^+ that comes from V_2. Put $Q' = Q_1^+ - v$, and let Q'' be the subgraph of Q^+ induced on V_2; this is Q_2^+ minus the vertex that comes from V_1. Both are in TC3. The vertices of degree 2 of Q'' are the three ends in V_2 of the edges between V_1 and V_2. By (1a), at most one of these three edges is at v.

- No one is at v: then z has degree 3 in Q', and Q is the 3-sticking of Q' and Q'' at z.
- One of them is at v; let y' be its end in V_2: then z has degree 2 in Q', and Q is the 2-sticking of Q' and Q'' at z. The two edges of z go to two of the three vertices of degree 2 of Q''; the third one is y'.

Q' has $|V_1|$ vertices and Q'' has $|V_2|$ vertices. Q has $|V_1| + |V_2| - 1$ vertices; by (1b) this is more than each of the two numbers.

Case 2: there is no such pair V_1, V_2. Then Q^+ has no triangle: exactly three edges leave a triangle of a simple cubic graph, and $|V(Q^+)| \geq 6$, so a triangle would give a pair as in Case 1. Let x be a neighbour of v, and xy an edge of Q. In Q^+ the neighbours of x are v, y, x', and the neighbours of y are x, y_1, y_2. Because Q^+ is simple and has no triangle, v, x', y_1, y_2 are four different vertices, and they are different from x and y. Let $\widehat{Q}^+$ be the graph obtained from Q^+ by the deletion of the edge xy and the suppression of x and y: the path v, x, x' becomes a new edge vx', and the path y_1, y, y_2 becomes a new edge y_1y_2.

Claim: $\widehat{Q}^+$ is cubic and 3-edge-connected, and it has $|V(Q^+)| - 2 \geq 4$ vertices.

Proof of the claim. $\widehat{Q}^+$ is cubic by its construction. Suppose that $V(\widehat{Q}^+)$ is the disjoint union of two non-empty sets V', V'' such that the set E^* of the edges between them has at most 2 elements (E^* can be empty). Put x back: on the side of v and x' if they are on the same side; on any side if the edge vx' is in E^* (then exactly one of the edges vx, xx' joins the two sides, in the place of vx'). Do the same for y with y_1, y_2. This is a partition of $V(Q^+)$ into two non-empty sets, and the number of edges of Q^+ between them is $|E^*|$ if x and y are on the same side, and $|E^*| + 1$ if they are not.

If vx' or y_1y_2 is in E^*, we can choose the sides such that x and y are together. Then at most 2 edges of Q^+ join the two sides: impossible. In the other case x and y have no choice. If they are on the same side, again at most 2 edges join the sides: impossible. If they are on different sides and $|E^*| \leq 1$, at most 2 edges join the sides: impossible. If they are on different sides and $|E^*| = 2$, exactly three edges join the sides, and the sides contain $\{x, v, x'\}$ and $\{y, y_1, y_2\}$: a pair as in Case 1, which does not exist.

So $\widehat{Q}^+$ is connected (E^* empty) and 3-edge-connected. This proves the claim.

The vertex v is a vertex of $\widehat{Q}^+$, so $\widehat{Q} = \widehat{Q}^+ - v$ is in TC3. The vertex x' is a neighbour of v in $\widehat{Q}^+$, so it has degree 2 in $\widehat{Q}$. And y_1y_2 is an edge of $\widehat{Q}$, because y_1 and y_2 are not v. The graph Q is the growing of $\widehat{Q}$ at the edge y_1y_2 and the vertex x', with the new vertices y and x. $\widehat{Q}$ has two vertices fewer than Q. □

Well colouring.

Lemma A.5 (Well colouring)**.** *Every graph of* TC3 *has a well colouring.*

Proof. Induction on the number of vertices. By Lemma A.4 it is enough to show three things: the triangle has a well colouring; a graph of TC3 that is a sticking of two well coloured graphs of TC3 has a well colouring; a graph of TC3 that is a growing of a well coloured graph of TC3 has a well colouring.

Triangle. Colour two edges blue and one edge red. The blue edges form a spanning tree. There are two red components, and the two blue edges join them: Q/red is the 2-join of two single vertices.

Sticking. Let Q' and Q'' in TC3 be well coloured, with the blue trees T' and T''. Let Q be the d-sticking of Q' and Q'' at z, with the edges $t_1, \ldots, t_d$ of z and their new ends $z_1, \ldots, z_d$. Keep all colours.

(i) The blue edges of Q are the edges of T' and of T''; the edges of T' at z have a new end. Their number is $(|V(Q')| - 1) + (|V(Q'')| - 1) = |V(Q)| - 1$. The tree T'' connects $V(Q'')$. Every component of $T' - z$ contains a neighbour of z in T', and the edge from this neighbour to z ends now in $V(Q'')$. So the blue edges form a connected spanning subgraph of Q with $|V(Q)| - 1$ edges: a spanning tree.

(ii) Let Ω be the red component of z in Q'; it is a vertex of Q'/red. By Lemma A.2 (c) the red edges of Q' form a forest. So Ω without z is the disjoint union of trees Ω_i, one for every *red* edge t_i at z; the end of t_i other than z is in Ω_i. (If no edge at z is red, then $\Omega = \{z\}$ and there is no Ω_i.) In Q, a red edge t_i joins Ω_i to the red component of z_i in Q'', and a blue edge t_i is a blue edge at the red component of z_i. All other red components of Q' and of Q'' are not changed. So the red components of Q are: the red components of Q' other than Ω; and the red components of Q'', some of them enlarged by trees Ω_i. An edge of Q'/red at Ω is a blue edge of Q' with one end in Ω. This end is z or a vertex of some Ω_i. In both cases the edge ends now in a red component that comes from Q''. Its other end is a vertex of Q'/red other than Ω, because Q'/red has no loop. So Q/red is obtained in this way: take Q'/red without the vertex Ω, and Q''/red, and every edge of Q'/red at Ω, with its end Ω moved to some vertex of Q''/red.

It remains to show that such a graph is 2-constructible. Claim: let U_a and U_b be disjoint 2-constructible graphs and ω a vertex of U_a; let U be obtained from U_a without ω, from U_b, and from every edge of U_a at ω, with its end ω moved to some vertex of U_b. Then U is 2-constructible. Proof by induction on a 2-construction of U_a. If U_a has only the vertex ω, then $U = U_b$. If U_a is the 2-join of U'_a and U''_a and ω is in U'_a, make the same change in U'_a; by induction the result U' is 2-constructible. Each of the two new edges of the 2-join has one end in U''_a and one end in U'_a; if this end was ω, it is now a vertex of U_b, which is in U'. So U is the 2-join of U' and U''_a. This proves the claim. So Q is well coloured.

Growing. Let Q' in TC3 be well coloured with the blue tree T', and let Q be the growing of Q' at the edge y_1y_2 and the vertex x', with the new vertices y and x. Colour the edges xx' and xy blue. Keep the colours of the old edges other than y_1y_2.

The edge y_1y_2 is red. Colour y_1y and yy_2 red. The blue edges are those of T', and xx', and xy: a spanning tree of Q (x hangs on x', and y hangs on x). The vertex y is now in the red component of y_1 and y_2, and x is a new red component with one vertex. So Q/red is Q'/red and one new vertex x with two edges, to the red components of x' and of y: a 2-join of Q'/red and a single vertex.

The edge y_1y_2 is blue. T' without the edge y_1y_2 has two components, T'_1 with y_1 and T'_2 with y_2. If x' is in T'_1, colour yy_2 blue and y_1y red. If x' is in T'_2, colour y_1y blue and yy_2 red. Look at the first case; the second one is the same with y_1 and y_2 exchanged. The number of blue edges is $(|V(Q')| - 1) - 1 + 3 = |V(Q)| - 1$, and the blue edges connect $V(Q)$: T'_1, then the path x', x, y, y_2, then T'_2. So they form a spanning tree. The vertex y is in the red component of its red neighbour y_1. The vertices y_1 and y_2 are in different red components of Q', by Lemma A.2 (c). So the blue edge yy_2 joins the same two red components that y_1y_2 joined. And x is a new red component with one vertex and with two blue edges. Again Q/red is a 2-join of Q'/red and a single vertex.

In all cases Q is well coloured. □

From a 2-construction to a closure sequence.

Lemma A.6 (Closure sequence)**.** *Let Q be a graph with a well colouring and with a 2-construction of Q/red. Number the nodes $1, \dots, s$ such that every node comes after all nodes that are subgraphs of its two operands. Then there are cycles $C_1, \dots, C_s$ of Q such that, for every i, the cycle C_i contains the two edges of the node i, and all other edges of C_i are red edges or edges of nodes with a smaller number. So, with the set R of the red edges,*

$$E(Q) = R \cup E(C_1) \cup \cdots \cup E(C_s),$$

and each C_i has at most two edges outside $R \cup E(C_1) \cup \cdots \cup E(C_{i-1})$. (Here it is not used that the red edges form a forest.)

Proof. Let i be a node with the operands U', U'' and the edges t_1, t_2. By Lemma A.3 (b), (c) the graphs $Q_{U'}$ and $Q_{U''}$ are connected and disjoint, and each of t_1, t_2 has one end in each of them. Take a path of $Q_{U'}$ between the ends of t_1 and t_2 in $V_{U'}$, and a path of $Q_{U''}$ between their ends in $V_{U''}$; a path can have length 0. The two paths with t_1 and t_2 form a cycle C_i. (If both paths have length 0, it is the cycle of length 2 with the edges t_1, t_2.) By Lemma A.3 (a), an edge of $Q_{U'}$ or of $Q_{U''}$ is red, or it is an edge of a node that is a subgraph of U' or of U''; such a node has a number $h < i$, and its edges are on C_h. By Lemma A.2 (b) every blue edge is an edge of a node, so it is on one of the cycles. □

Lemma A.7 (The closure lemma)**.** *Let Q be a truncated cubic 3-edge-connected graph. Then Q has a spanning tree T_0 and cycles $C_1, \dots, C_s$ such that, with $R = E(Q) \setminus E(T_0)$,*

$$E(Q) = R \cup E(C_1) \cup \cdots \cup E(C_s),$$

and each C_i has at most two edges outside $R \cup E(C_1) \cup \cdots \cup E(C_{i-1})$.

Proof. Lemma A.5 gives a well colouring of Q. Let T_0 be the tree of the blue edges; then R is the set of the red edges. Take a 2-construction of Q/red and number its nodes as in Lemma A.6. This lemma gives the cycles. □

Remark A.8 (How the closure sequence is used)**.** In the words of [JLPT], Lemma A.7 says that Q is the 2-closure of the spanning subgraph with the edge set R. For $i = 1, \dots, s$ let N_i be the set of the edges of C_i outside $R \cup E(C_1) \cup \cdots \cup E(C_{i-1})$. The sets N_i are disjoint, each has at most two edges, and their union is $E(T_0)$. For a group A and elements $x_1, \dots, x_s$ of A, the value of the A-flow $x_1\chi_{C_1} + \cdots + x_s\chi_{C_s}$ on an edge of N_i does not depend on $x_1, \dots, x_{i-1}$. So the coefficients can be chosen in the order $x_s, \dots, x_1$: when x_i is chosen, the values on N_i are final, and every value that is forbidden on an edge of N_i forbids one value of x_i. This is the reverse choice.

For a frame (Definition 2.9), the official cycles of Lemma 2.11 are a sequence as in Lemma A.6, with $N_i = \{e_i, f_i\}$; they are fixed by the frame, and they have the additional property (O4).

Discussion

This version corrects one sentence in the proof of Proposition 9.7, for $k = 105$: the earlier version listed 35 among the factors smaller than 28, but $35 = 5 \cdot 7$ is an admitted factor. The conclusion stands, since each coprime split $3 \cdot 35$, $5 \cdot 21$, $7 \cdot 15$ of 105 has a factor smaller than 28. No other mathematical statement changed.

Three directions stay open (Section 12). *The least modulus.* $I_k(G)$ and $M_k(G)$ are connected for $k = 40$, $k = 48$ and every multiple of 4 from 52 on, and the modulus 5 fails [EHLMNS, Theorem 3.4]; whether $I_6(G)$ is connected for every G is unknown. *All moduli from 40 on.* For $I_k(G)$ the moduli from 41 to 47 and the primes are not reached; for $k = 44$ the graph $M_{44}(G)$ is connected (Theorem 4.5). *Small groups.* The least order $n \geq 6$ of a finite abelian group whose flow graphs are all connected is at most 32 [Go1, Corollary 13.2]; for $\mathbb{F}_2^3$ and $\mathbb{F}_2^4$ the connectivity is proved only for some graphs: every 4-edge-connected G [Go1, Theorem 13.3], and every G whose components have at most 44, respectively 19022, vertices of odd degree [Go1, Theorems 22.1 and 28.6].

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