THE DIMENSION OF A EUCLIDEAN SPACE IS DETERMINED BY THE ISOMORPHIC STRUCTURE OF ITS LIPSCHITZ-FREE SPACE

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Abstract. We prove that the real Lipschitz-free space $\mathcal{F}(\mathbb{R}^m)$ embeds isomorphically into $\mathcal{F}(\mathbb{R}^n)$ if and only if $m \leq n$. Consequently, the Lipschitz-free spaces over Euclidean spaces of different positive dimensions are not linearly isomorphic.

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1. Introduction

Let $(M, d_M, 0)$ be a pointed metric space. We denote by $\mathrm{Lip}_0(M)$ the real Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ satisfying $f(0) = 0$, endowed with the norm

$$\|f\|_{\mathrm{Lip}} := \mathrm{Lip}(f) = \sup_{\substack{x,y \in M \\ x \neq y}} \frac{|f(x) - f(y)|}{d_M(x, y)}.$$

For each $x \in M$, let $\delta_x \in \mathrm{Lip}_0(M)^*$ be the evaluation functional given by $\delta_x(f) = f(x)$. The Lipschitz-free space over M is defined as

$$\mathcal{F}(M) := \overline{\mathrm{span}\{\delta_x : x \in M\}}^{\|\cdot\|_{\mathrm{Lip}_0(M)^*}}.$$

The identity

$$\|\delta_x - \delta_y\| = d_M(x, y), \qquad x, y \in M,$$

shows that the canonical map $x \mapsto \delta_x$ is an isometric embedding of M into $\mathcal{F}(M)$. Moreover, the canonical pairing

$$\langle \delta_x, f \rangle = f(x), \qquad x \in M, \quad f \in \mathrm{Lip}_0(M),$$

identifies $\mathcal{F}(M)^*$ isometrically with $\mathrm{Lip}_0(M)$.

Throughout this paper, $\mathbb{R}^d$ is equipped with the Euclidean metric and distinguished point 0, and we write

$$X_d := \mathcal{F}(\mathbb{R}^d, 0).$$

The question of whether X_m and X_n are linearly isomorphic for distinct integers $m, n \geq 2$ is explicitly raised in [3, Section 8.1] and [1, Section 1]. For $n \leq m$, the canonical coordinate inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ and the corresponding coordinate projection $\mathbb{R}^m \to \mathbb{R}^n$ induce an isometric embedding of X_n into X_m whose range is complemented by a projection of norm one. Furthermore, by a theorem of Kaufmann [5, Theorem 3.1], one has

$$X_d \cong \ell_1(X_d).$$

Consequently, the Pełczyński decomposition method reduces the isomorphism problem to the existence of a complemented embedding in the reverse direction. The nonembedding result for a one-dimensional target was already known. The planar earthmover theorem of Naor and Schechtman [6] implies that X_2 does not embed isomorphically into X_1, which is linearly isometric to $L_1(\mathbb{R})$. Since X_m contains a linearly isometric copy of X_2 for every $m \geq 2$, the same conclusion holds for all $m > n = 1$.

For distinct dimensions, the dimension result of Pernecká and Spěvák [7, Observation 2.1(ii) and Corollary 4.4] already rules out isomorphisms that map the algebraic span of the evaluation functionals onto the corresponding algebraic span. Our main result removes this additional

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hypothesis and, more strongly, excludes any isomorphic embedding of X_m into X_n whenever $m > n$.

Theorem 1. *Let $m, n \in \mathbb{N}$ be positive. The following conditions are equivalent:*

(1) *$m \le n$;*
(2) *X_m admits a complemented isometric embedding into X_n;*
(3) *X_m admits an isomorphic linear embedding into X_n.*

In particular, $X_m \cong X_n$ if and only if $m = n$.

We establish a more precise operator statement. Fix a bounded open ball $U \subset \mathbb{R}^m$ and a nonnegative function $\rho \in C_c^\infty(U)$ with $\int_U \rho(x)\,\mathrm{d}x = 1$. For $\xi \in S^{m-1}$ and $k \in \mathbb{N}$, define

$$F_{k,\xi}(x) = \rho(x)\xi\cos(k\xi\cdot x), \qquad \mu_{k,\xi} = \int_{\mathbb{R}^m} (-\operatorname{div} F_{k,\xi})(x)\delta_x\,\mathrm{d}x \in X_m. \tag{1}$$

The integral is a Bochner integral. Its existence follows from the continuity of $x \mapsto \delta_x$ and the compact support of the smooth scalar density. In the following theorem, almost every direction refers to the usual surface measure on S^{m-1}.

Theorem 2. *Suppose $1 \le n < m$ and $J : X_m \to X_n$ is bounded and linear. For every U and ρ as above,*

$$\lim_{k\to\infty} \|J\mu_{k,\xi}\| = 0 \quad \textit{for almost every } \xi \in S^{m-1}. \tag{2}$$

On the other hand, for every $\xi \in S^{m-1}$,

$$\|\mu_{k,\xi}\| \le 1, \qquad \liminf_{k\to\infty} \|\mu_{k,\xi}\| \ge \tfrac{1}{2}. \tag{3}$$

2. Preliminaries and a vector-measure representation

For a scalar function g defined at 0, write $g^\circ = g - g(0)$. Gradients are column vectors, and the notation $\|\nabla g\|_\infty$ uses the Euclidean vector norm. For a Lipschitz function on $\mathbb{R}^d$, Rademacher's theorem gives its a.e. gradient, with essential supremum bounded by its Lipschitz constant. For smooth g, the Lipschitz constant is exactly $\|\nabla g\|_\infty$.

Lemma 3 (A norming family of smooth tests)**.** *For every $d \ge 1$ and $u \in X_d$,*

$$\|u\| = \sup\big\{|\langle u, g^\circ\rangle| : g \in C_c^\infty(\mathbb{R}^d),\ \|\nabla g\|_\infty \le 1\big\}. \tag{4}$$

Proof. The norm-controlled approximation in the proof of [2, Lemma 2.2], followed by standard mollification (cf. [4, Proposition 3.1(iii)]), shows that the indicated smooth tests are weak*-dense in the unit ball of $\mathrm{Lip}_0(\mathbb{R}^d)$. The assertion follows by duality. □

Lemma 4 (Representation on smooth gradients)**.** *Let $U \subset \mathbb{R}^m$ be a bounded open set. For every bounded linear map $T : \mathrm{Lip}_0(\mathbb{R}^n) \to \mathrm{Lip}_0(\mathbb{R}^m)$, there exist a finite positive Radon measure λ on $U \times \mathbb{R}^n$ and a bounded measurable matrix field $B : U \times \mathbb{R}^n \to \mathbb{R}^{m\times n}$ such that the first marginal of λ is absolutely continuous with respect to Lebesgue measure. For all $\alpha \in C_c^\infty(U)$ and $g \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_U \alpha(x)\nabla(Tg^\circ)(x)\,\mathrm{d}x = \int_{U\times\mathbb{R}^n} \alpha(x)B(x,z)\nabla g(z)\,\mathrm{d}\lambda(x,z). \tag{5}$$

Proof. Let $V = C_0(\mathbb{R}^n;\mathbb{R}^n)$ be the space of continuous vector fields vanishing at infinity, equipped with the supremum norm, and set

$$E = \overline{\{\nabla g : g \in C_c^\infty(\mathbb{R}^n)\}}^{\|\cdot\|_\infty} \subseteq V.$$

For each $i = 1, \dots, m$, the linear map

$$S_i(\nabla g) = \partial_i(Tg^\circ)|_U$$

is well defined on the space of smooth compactly supported gradients. Since

$$\|S_i(\nabla g)\|_\infty \le \|T\|\,\mathrm{Lip}(g) = \|T\|\|\nabla g\|_\infty,$$

it extends to a bounded linear map $S_i : E \to L^\infty(U)$.

Given an E-valued simple function

$$H(x) = \sum_{j=1}^{r} \mathbf{1}_{A_j}(x) v_j,$$

where the $A_j \subseteq U$ are pairwise disjoint measurable sets and $v_j \in E$, define

$$L_i(H) = \sum_{j=1}^{r} \int_{A_j} S_i(v_j)(x)\,\mathrm{d}x.$$

By refining partitions, one sees that L_i is well defined and linear. Moreover,

$$|L_i(H)| \le \|T\| \sum_{j=1}^{r} |A_j| \|v_j\|_\infty = \|T\| \int_U \|H(x)\|_\infty \,\mathrm{d}x, \tag{6}$$

where $|\cdot|$ denotes Lebesgue measure. Thus L_i extends by density to $L^1(U;E)$ and, by the Hahn–Banach theorem, to $L^1(U;V)$ with the same bound. We retain the notation L_i for this extension.

Every $H \in C_0(U \times \mathbb{R}^n; \mathbb{R}^n)$ can be identified with the Bochner integrable function $x \mapsto H(x,\cdot) \in V$. By (6),

$$|L_i(H)| \le \|T\| \int_U \|H(x,\cdot)\|_\infty \,\mathrm{d}x \le |U| \|T\| \|H\|_\infty.$$

The Riesz representation theorem, applied coordinatewise, therefore provides a finite $\mathbb{R}^n$-valued Radon measure K_i on $U \times \mathbb{R}^n$ such that

$$L_i(H) = \int_{U \times \mathbb{R}^n} H(x,z) \cdot \,\mathrm{d}K_i(x,z)$$

for every $H \in C_0(U \times \mathbb{R}^n; \mathbb{R}^n)$.

Let $|K_i|$ denote the total variation of K_i with respect to the Euclidean norm. For every open set $W \subseteq U$ and every $H \in C_c(W \times \mathbb{R}^n; \mathbb{R}^n)$ with $\|H\|_\infty \le 1$, extended by zero outside $W \times \mathbb{R}^n$, inequality (6) gives $|L_i(H)| \le \|T\| |W|$. By the characterization of total variation on open sets,

$$|K_i|(W \times \mathbb{R}^n) \le \|T\| |W|. \tag{7}$$

By outer regularity, this inequality extends to all Borel sets $W \subseteq U$.

Set

$$\lambda = \sum_{i=1}^{m} |K_i|, \qquad b_i = \frac{\mathrm{d}K_i}{\mathrm{d}\lambda},$$

and define $B(x,z) \in \mathbb{R}^{m \times n}$ to be the matrix whose ith row is $b_i(x,z)^{\mathsf{T}}$. Since $|K_i| \le \lambda$, we have $|b_i| \le 1$ λ-a.e. Thus, after modification on a λ-null set, B is bounded and measurable. Furthermore, by (7),

$$\lambda(W \times \mathbb{R}^n) = \sum_{i=1}^{m} |K_i|(W \times \mathbb{R}^n) \le m \|T\| |W|$$

for every Borel set $W \subseteq U$. Hence the first marginal of λ is absolutely continuous with respect to Lebesgue measure.

Finally, fix $\alpha \in C_c^\infty(U)$ and $g \in C_c^\infty(\mathbb{R}^n)$. The field

$$H(x,z) = \alpha(x) \nabla g(z)$$

belongs to $C_c(U \times \mathbb{R}^n; \mathbb{R}^n)$ and, when viewed as the function $x \mapsto \alpha(x)\nabla g$, also to $L^1(U; E)$. By the original definition of L_i and approximation of α by simple functions, we obtain

$$\begin{aligned}\int_U \alpha(x)\partial_i(Tg^\circ)(x)\,\mathrm{d}x &= \int_U \alpha(x)S_i(\nabla g)(x)\,\mathrm{d}x \\ &= L_i(H) \\ &= \int_{U\times\mathbb{R}^n} \alpha(x)\nabla g(z)\cdot \mathrm{d}K_i(x,z) \\ &= \int_{U\times\mathbb{R}^n} \alpha(x)b_i(x,z)^{\mathsf{T}}\nabla g(z)\,\mathrm{d}\lambda(x,z).\end{aligned}$$

Combining these identities for $i = 1, \dots, m$ yields (5). □

3. Localization of oscillatory gradients

Lemma 5. *Let T, U, λ, B be as in Lemma 4, and fix $\xi \in S^{m-1}$. Let $g_k \in C_c^\infty(\mathbb{R}^n)$ satisfy $\|\nabla g_k\|_\infty \le 1$, and put $f_k = g_k^\circ$ and*

$$h_k(x,z) = \cos(k\xi\cdot x)\nabla f_k(z).$$

If h is a weak-star limit of a subsequence of (h_k) in $L^\infty(\lambda; \mathbb{R}^n)$, then

$$B(x,z)h(x,z) \in \mathbb{R}\xi \quad \text{for } \lambda\text{-almost every } (x,z). \tag{8}$$

Proof. For any $v \in \mathbb{R}^m$ with $v\cdot\xi = 0$, we will prove that

$$\lim_{k\to\infty} \int \alpha(x)\beta(z)v^{\mathsf{T}}B(x,z)h_k(x,z)\,\mathrm{d}\lambda(x,z) = 0 \tag{9}$$

for every $\alpha \in C_c^\infty(U)$ and $\beta \in C_c^\infty(\mathbb{R}^n)$. Since $f_k(0) = 0$ and $\mathrm{Lip}(f_k) \le 1$, we have $|f_k(z)| \le |z|$. By the product rule,

$$\beta\nabla f_k = \nabla(\beta f_k) - f_k\nabla\beta. \tag{10}$$

Denote the integral in (9) by I_k. Substituting the definition of h_k and using (10), we obtain

$$I_k = A_k - C_k,$$

where

$$\begin{aligned}A_k &= \int_{U\times\mathbb{R}^n} \alpha(x)\cos(k\xi\cdot x)v^{\mathsf{T}}B(x,z)\nabla(\beta f_k)(z)\,\mathrm{d}\lambda(x,z), \\ C_k &= \int_{U\times\mathbb{R}^n} \alpha(x)\cos(k\xi\cdot x)v^{\mathsf{T}}B(x,z)f_k(z)\nabla\beta(z)\,\mathrm{d}\lambda(x,z).\end{aligned}$$

We show that both terms converge to zero.

Let $H_k = T((\beta f_k)^\circ)$. The Lipschitz constants of H_k are uniformly bounded, since

$$\mathrm{Lip}(\beta f_k) \le \|\beta\|_\infty + \sup_{z\in\mathbb{R}^n} |z|\,|\nabla\beta(z)|.$$

Moreover, $H_k(0) = 0$, so the functions H_k are uniformly bounded on compact sets.

Since $\beta f_k \in C_c^\infty(\mathbb{R}^n)$ and $x \mapsto \alpha(x)\cos(k\xi\cdot x)$ belongs to $C_c^\infty(U)$, we may apply (5) to obtain

$$\begin{aligned}A_k &= \int_U \alpha(x)\cos(k\xi\cdot x)v^{\mathsf{T}}\nabla H_k(x)\,\mathrm{d}x \\ &= \int_U \alpha(x)\cos(k\xi\cdot x)\partial_v H_k(x)\,\mathrm{d}x,\end{aligned}$$

where $\partial_v H_k = v^{\mathsf{T}}\nabla H_k$ denotes the directional derivative in the direction v, defined almost everywhere. Integration by parts in the v direction, using $v\cdot\xi = 0$, gives

$$A_k = -\int_U (\partial_v\alpha)(x)\cos(k\xi\cdot x)H_k(x)\,\mathrm{d}x.$$

As $\partial_\xi \sin(k\xi \cdot x) = k\cos(k\xi \cdot x)$, a further integration by parts yields

$$|A_k| \leq \frac{1}{k}\big(\|\partial_\xi \partial_v \alpha\|_{L^1(U)} \sup_{\operatorname{supp}\alpha} |H_k| + \|\partial_v \alpha\|_{L^1(U)} \operatorname{Lip}(H_k)\big) \longrightarrow 0.$$

All integrations by parts are in the distributional sense; the compact support of α in U removes boundary terms. It remains to show that

$$C_k \longrightarrow 0. \tag{11}$$

If $K = \operatorname{supp}\nabla\beta$ is empty there is nothing to prove. Otherwise K is compact, and the restrictions $f_k|_K$ form a relatively compact subset of $C(K)$ by the Arzelà–Ascoli theorem. For each fixed $a \in C(K)$, the signed measure

$$\mathrm{d}\nu_a(x,z) = \alpha(x)a(z)v^{\mathsf{T}}B(x,z)\nabla\beta(z)\,\mathrm{d}\lambda(x,z)$$

is interpreted as zero outside $U \times K$. Its pushforward by $(x,z) \mapsto x$ is a finite signed measure absolutely continuous with respect to Lebesgue measure, by the first-marginal property of λ. Its density, extended by zero outside U, is in $L^1(\mathbb{R}^m)$. The Riemann–Lebesgue lemma therefore gives

$$\int \cos(k\xi \cdot x)\,\mathrm{d}\nu_a(x,z) \longrightarrow 0.$$

As functions of $a \in C(K)$, these integrals are linear functionals with norms bounded uniformly in k, by $\|\alpha\|_\infty |v| \|B\|_\infty \|\nabla\beta\|_\infty \lambda(U \times \mathbb{R}^n)$. Their pointwise convergence to zero is thus uniform on the compact closure of $\{f_k|_K : k \in \mathbb{N}\}$ (use a finite ε-net). This proves (11).

Combining the two contributions in (10) proves (9). Now pass to the specified weak-star convergent subsequence. Because $\alpha\beta B^{\mathsf{T}}v$ is in $L^1(\lambda;\mathbb{R}^n)$, it follows that

$$\int \alpha(x)\beta(z)v^{\mathsf{T}}B(x,z)h(x,z)\,\mathrm{d}\lambda(x,z) = 0.$$

Finite sums of products $\alpha(x)\beta(z)$, with the stated smooth compact supports, form an algebra separating points and vanishing nowhere on $U \times \mathbb{R}^n$. By the locally compact version of the Stone–Weierstrass theorem this algebra is uniformly dense in $C_0(U \times \mathbb{R}^n)$, and hence dense in $L^1(\lambda)$ by Radon regularity. It follows that $v^{\mathsf{T}}Bh = 0$ λ-a.e. Applying this to a finite basis of $\xi^\perp$ proves (8). □

4. The dimensional obstruction

Proof of Theorem 2. For $f \in \operatorname{Lip}_0(\mathbb{R}^m)$, integration by parts in (1) yields

$$\langle \mu_{k,\xi}, f\rangle = \int_U \rho(x)\cos(k\xi \cdot x)\xi \cdot \nabla f(x)\,\mathrm{d}x. \tag{12}$$

In particular, $\|\mu_{k,\xi}\| \leq 1$. The function

$$f_{k,\xi}(x) = k^{-1}\sin(k\xi \cdot x)$$

is 1-Lipschitz and vanishes at the base point. Testing with this function gives

$$\|\mu_{k,\xi}\| \geq \int_U \rho(x)\cos^2(k\xi \cdot x)\,\mathrm{d}x \longrightarrow \tfrac{1}{2}$$

by the Riemann–Lebesgue lemma. This establishes (3).

Now let $T = J^*$ under the canonical dual identifications, and fix a representation λ, B as in Lemma 4. Since $n < m$, for each (x,z) where B is defined, the range of $B(x,z)$ is a proper linear subspace of $\mathbb{R}^m$. Its intersection with S^{m-1} has surface measure zero. The incidence set

$$\{(x,z,\xi) : \xi \in \operatorname{ran} B(x,z)\}$$

is measurable: the distance to the range is the infimum of the countable family $|\xi - B(x,z)q|$, for $q \in \mathbb{Q}^n$. Here one may take a measurable representative of B defined everywhere. Fubini's theorem gives, for almost every $\xi \in S^{m-1}$,

$$\xi \notin \operatorname{ran} B(x,z) \quad \text{for } \lambda\text{-a.e. } (x,z). \tag{13}$$

Fix such a ξ for the remainder of the argument. Importantly, this choice is made independently of the sequence of test functions below.

For an arbitrary sequence $g_k \in C_c^\infty(\mathbb{R}^n)$ with $\|\nabla g_k\|_\infty \le 1$, set $f_k = g_k^\circ$ and let h_k be the sequence from Lemma 5. We have $\|h_k\|_{L^\infty(\lambda;\mathbb{R}^n)} \le 1$. Since λ is a finite Radon measure on a second countable locally compact metric space, $L^1(\lambda;\mathbb{R}^n)$ is separable. By the Banach–Alaoglu theorem and weak-star metrizability on bounded sets, every subsequence of (h_k) has a further weak-star convergent subsequence, say with limit h. By Lemma 5, $Bh \in \mathbb{R}\xi$ λ-a.e. On the other hand, $Bh \in \operatorname{ran} B$ pointwise wherever B and h are defined. Condition (13) implies

$$\mathbb{R}\xi \cap \operatorname{ran} B(x,z) = \{0\} \quad \text{for } \lambda\text{-a.e. } (x,z).$$

Thus every such limit satisfies $Bh = 0$ λ-a.e.

By (12), the adjoint identity, and (5), we have

$$\langle J\mu_{k,\xi}, f_k\rangle = \int_{U\times\mathbb{R}^n} \rho(x)\xi^{\mathsf{T}} B(x,z)h_k(x,z)\,\mathrm{d}\lambda(x,z). \tag{14}$$

The test $\rho B^{\mathsf{T}}\xi$ belongs to $L^1(\lambda;\mathbb{R}^n)$. Along every weak-star convergent subsequence the right side converges to zero, because $Bh = 0$. The subsequence principle consequently gives $\langle J\mu_{k,\xi}, g_k^\circ\rangle \to 0$ for every sequence of smooth tests with $\|\nabla g_k\|_\infty \le 1$.

Finally, $J\mu_{k,\xi} \in X_n$, so Lemma 3 applies to it. For each k one may choose a smooth compactly supported g_k with $\|\nabla g_k\|_\infty \le 1$ such that

$$\|J\mu_{k,\xi}\| \le |\langle J\mu_{k,\xi}, g_k^\circ\rangle| + k^{-1}.$$

The preceding conclusion for arbitrary sequences of tests now gives (2). This proves the theorem. □

Proof of Theorem 1. If $m \le n$, the maps $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x) = (x,0)$, and $p : \mathbb{R}^n \to \mathbb{R}^m$, $p(x,y) = x$, are pointed and 1-Lipschitz. They induce contractive linear maps $\widehat{i} : X_m \to X_n$ and $\widehat{p} : X_n \to X_m$ by $\widehat{i}\delta_x = \delta_{i(x)}$ and $\widehat{p}\delta_z = \delta_{p(z)}$. The relation $\widehat{p}\widehat{i} = I_{X_m}$ follows on point evaluations and hence by continuity. Therefore $\widehat{i}$ is an isometry onto the range of the norm-one projection $\widehat{i}\widehat{p}$. This proves (1) ⇒ (2), and (2) ⇒ (3) is immediate.

Suppose instead that $m > n$ and $J : X_m \to X_n$ were an isomorphic embedding. There would be $c > 0$ such that $\|Ju\| \ge c\|u\|$ for all $u \in X_m$. Choose any direction satisfying (2) in Theorem 2; then (3) contradicts this lower bound. Hence (3) ⇒ (1). Finally, a linear isomorphism gives embeddings in both directions, so the equivalence also proves the last assertion. □

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