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The primitive balanced-degree spectrum in the two-pair Image problem

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Abstract

Over every characteristic-zero field, the balanced degrees admitting radially primitive, absolutely nonpower two-pair Image counterexamples are precisely $4, 6, 7, 8, \ldots$. All pure balanced phases through degree three are safe, as are all pure radially primitive balanced quintics. One explicit integer family realizes the failure degrees with a common marker of balanced degree one that detects every positive power. We also prove that an abelian variety with Rosati-compatible rank-two multiplication by a CM field has ordinary primes of density one after finite extension if at most two conjugate embedding pairs have Hodge multiplicities $(1, 1)$. The full geometric endomorphism algebra may be larger than the acting field. In the quintic proof, this theorem supplies ordinary reduction for factors of cyclic covers whose Frobenius coefficients detect nonzero polynomial moments.

1 Introduction

Let K be a field of characteristic zero. For $\xi = (\xi_1, \xi_2)$ and $z = (z_1, z_2)$, define the contraction on monomials by $\mathcal{E}(\xi^\alpha z^\beta) = \partial_z^\alpha z^\beta$ and extend linearly. The Image space is the sum of the images of $\xi_i - \partial_{z_i}$, $i = 1, 2$, and equals $\ker \mathcal{E}$ [1, Theorem 3.3]. A phase F is pure if $\mathcal{E}(F^m) = 0$ for all $m \geq 1$. It is safe if for every fixed polynomial Q, $\mathcal{E}(QF^m) = 0$ for all sufficiently large m. Balanced degree d means bidegree (d, d) in (ξ, z). A nonzero balanced phase is radially primitive if $R = \xi_1 z_1 + \xi_2 z_2$ does not divide it. It is absolutely nonpower if it is no nonzero scalar multiple of a proper power over an algebraic closure.

Theorem 1. *Let K have characteristic zero.*

1. *Every pure two-pair balanced phase of degree at most three is safe.*
2. *Every pure, radially primitive two-pair balanced phase of degree five is safe.*
3. *In degree four and every degree at least six there are explicit integer, radially primitive, absolutely nonpower two-pair balanced phases F with $\mathcal{E}(F^m) = 0$ and $\mathcal{E}(ZF^m) \neq 0$ for every $m \geq 1$, with the common linear marker $Z = \xi_1 z_2$ in the balanced coordinates.*

Consequently the primitive, absolutely nonpower balanced counterexample degrees are $4, 6, 7, 8, \ldots$.

The Special Image Conjecture asserts safety for every pure phase in the original polynomial ring [1, Conjecture 3.1]. A single quartic witness already refutes that assertion in two pairs. The question here is which balanced degrees can carry a failure with the radial factor removed. Allowing radial factors, the balanced failure spectrum starts at four and contains every higher degree. Indeed, multiplying the quartic by R^{d-4} leaves its restriction to $R = 1$ unchanged, and (1) changes pure and marked contractions only by nonzero factorial ratios. Among phases divisible by R, the failure

degrees start at five, since a divisible quartic restricts to sphere degree at most three. Radial primitivity is the condition that degree in the original balanced ring agrees with least degree on the sphere.

Long's Hopf identity gives the quartic seed [3]. We give a joint endpoint identity that lets the seed be multiplied by selected quadratic and cubic factors while retaining the same marker. This produces degrees $4 + 2a + 3b$ for $a, b \geq 0$. Proving that five is genuinely missing requires controlling arbitrary primitive quintic coefficients.

An abelian variety of dimension g has ordinary reduction at a good prime when its Newton slopes consist of g zeros and g ones.

Theorem 2 (Ordinary reduction with rank-two CM multiplication)**.** *Let J be an abelian variety of dimension $2e > 0$ over a number field L. Suppose a CM field E of degree $2e$ embeds in $\mathrm{End}^0(J_{\overline{L}})$ and a polarization induces complex conjugation on E by its Rosati involution. For each conjugate pair of embeddings of E, consider the multiplicities of the two embeddings on holomorphic one-forms. Suppose at most two pairs have multiplicities $(1, 1)$. Then there is a finite extension of the number field over which the ordinary primes of J have natural density one.*

Theorem 2 concerns all abelian varieties with the stated action and signature. Its relative-rank-two range includes signatures with two balanced pairs. At a prime split in E, a nonordinary balanced pair forces divisibility of the Frobenius trace at *both* places of that pair. With at most two balanced pairs this supplies the remaining divisibility beyond the contribution from the definite pairs. The Weil bound puts the normalized trace norm in a finite set of integers. After adjoining enough torsion, the norm is congruent to its largest possible value, forcing equality. Equality contradicts the two local divisibilities at the nonordinary balanced pair.

The link to polynomial moments comes from the interval integration after angular averaging. Integration divides a term of height j by $j + 1$. At special moment orders and primes, the largest height is $p - 1$, so one prime denominator isolates one coefficient of a power of the leading face. The same coefficient occurs as the Frobenius scalar on a one-dimensional eigenspace of the coherent cohomology of a cyclic cover. An ordinary Jacobian makes that scalar nonzero. This rules out purity even though individual terms of the original moment can cancel in characteristic zero.

The proof of safety starts with the leading binary form on the conic at infinity. If its intersection orders are all small, the volume extends across a proper resolution and a nonzero constant exponential period is impossible. The borderline intersection order gives a single nonzero coefficient under the prime test. Higher order leaves finitely many supporting faces, where the cyclic-cover argument applies except in explicit cases that reduce to safe phases by a change of coordinates. The quadratic-height analysis below uses the general root and face tests from Section 5. Those tests are independent of the collision analysis, which uses quadratic height to finish the remaining cases.

2 Reduction to spherical moments

Contraction kills the image of each $\xi_i - \partial_{z_i}$. Applying this operator to $\xi^{\alpha - e_i} z^{\beta}$ replaces one outer factor ξ_i by ∂_{z_i} modulo its image. Induction on the ξ-degree shows that every $H - \mathcal{E}(H)$ belongs to the sum of the two images, proving the kernel identity.

Introduce the four balanced bilinears

$$R = \xi_1 z_1 + \xi_2 z_2, \quad Z = \xi_1 z_2, \quad W = 2\xi_2 z_1, \quad T = \xi_1 z_1 - \xi_2 z_2, \qquad T^2 + 2ZW = R^2.$$

They generate the balanced subring, with the displayed relation. Generation follows by pairing one ξ_i with one z_j. The normal monomials modulo the relation in homogeneous degree d number

$(d+1)^2$, the dimension of the original bidegree-(d,d) space, so there is no further relation. Restrict at $R=1$, writing $(x,y,t)=(Z,W,T)$ there. The sphere algebra and its trace are

$$A_K = K[x,y,t]/(t^2+2xy-1), \qquad \Lambda(h) = \mathrm{CT}_x\,\frac{1}{2}\int_{-1}^{1} h\left(x, \frac{1-t^2}{2x}, t\right) dt.$$

The integral is the rational polynomial rule with value $1/(j+1)$ on t^j for even j, and zero for odd j, including the factor $1/2$. If H has bidegree (d,d) and sphere restriction h, then

$$\mathcal{E}(H) = (d+1)!\,\Lambda(h). \tag{1}$$

To check this, take an original monomial with ξ_1-exponent i and z_1-exponent j. Both sides vanish for $i \neq j$ by angular weight. For $i=j$ the sphere integral is the beta integral $i!(d-i)!/(d+1)!$, and its contraction is $i!(d-i)!$.

The least sphere degree of h is d exactly when R does not divide H. Indeed homogenizing the normal monomials $x^a t^b$, $y^c t^b$ with $c>0$ recovers H, and each has filtration degree $a+b$ or $c+b$, respectively. We use *pure* for the condition $\Lambda(h^m)=0$ for every $m \geq 1$ and *safe* for the assertion that $\Lambda(gh^m)=0$ eventually for each fixed $g \in A_K$.

Sphere safety gives the required vanishing against arbitrary original polynomial markers. For a bihomogeneous marker Q of bidegree (a,b), contraction against H^m is zero if $a>b$. For $b \geq a$, its output has degree $b-a$ in z, and for every $|\gamma| = b-a$,

$$\partial_z^\gamma \mathcal{E}(QH^m) = \mathcal{E}(\xi^\gamma Q H^m).$$

The marker on the right is balanced. There are finitely many such derivatives, so their eventual vanishing by (1) forces eventual vanishing of the output. A general marker is the sum of finitely many bihomogeneous ones. Conversely any fixed sphere marker with infinitely many nonzero mixed moments lifts to a balanced marker by homogenization.

3 Quadratic height and the degrees below four

We first treat phases whose Laurent expression after eliminating y has degree at most two in t. A coefficient of x^{-j}, $j>0$, is divisible by $(1-t^2)^j$, so only a simple pole at $x=0$ is possible. With no pole, the angular core must vanish for purity by the interval polynomial theorem [10, Theorem 2.1]. All nonzero angular exponents are then positive, giving safety. With a pole, scale the phase and write

$$h = x^{-1}(A - Bt^2 + Ct), \qquad A,B,C \in K[x], \quad A(0)=B(0)=1, \quad C(0)=0.$$

Put $D = A + C^2/(4B)$ and $M = \sqrt{D/B}$, with $M(0)=1$. In completing the square, set $t = C/(2B) + Mz$. The endpoints in z are congruent to ± 1 modulo x. The integrand $(1-z^2)^m$ has contact order m there, so returning the endpoints to ± 1 changes its integral by a multiple of x^{m+1}. Therefore

$$\Lambda(h^m) = \frac{1}{2}\int_{-1}^{1} (1-z^2)^m\, dz\; [x^m] D^m M.$$

The factor given by the integral is a nonzero rational number. The conditions $[x^m]D^m M = 0$ determine each successive coefficient of M since $D(0)=1$. Differentiating D^m shows that the unique solution is $M = 1 - xD'/D$. In particular

$$B = \frac{D^3}{(D - xD')^2}. \tag{2}$$

Suppose the phase has least sphere degree at most five, so $\deg B \le 4$. At any finite zero or pole α of D of order $r \neq 0$, we have $\alpha \neq 0$, $\operatorname{ord}_\alpha(D - xD') = r - 1$, and $\operatorname{ord}_\alpha B = r + 2$. There are no other finite zeros of B by (2). If n is the degree of D at infinity and s counts its distinct finite zeros and poles over an algebraic closure, summing valuations gives $\deg B = n + 2s$. When $n \neq 1$, the leading coefficient of $D - xD'$ is $1 - n$ times that of D, hence $\deg B = n$ and D is constant. When $n = 1$, the bound $\deg B \le 4$ forces $s = 1$, so D is linear and $B = D^3$. Then B divides C^2, and $C = xD^2P$ for a polynomial P. Since $A = D - x^2DP^2/4$ has degree at most six, P has degree at most one. The phase has exact degree four, because its $x^{-1}Bt^2$ term has degree four and its other terms have degree at most four.

In the constant case $D = B = 1$. With $c = C/x$, the phase is $2y + ct - xc^2/4$, the image of $2y$ under the polynomial shear $(x, t, y) \mapsto (x, t + xa, y - at - xa^2/2)$ with $a = -c/2$. The derivation $xa\partial_t$ in the literal Laurent chart kills the trace by integration at $t = \pm 1$. Each boundary value is a polynomial in x since $y = 0$ there, and the extra factor x kills its constant term. The shear is its finite exponential on each regular polynomial, so both directions preserve the trace and safety. Thus in quadratic height every pure phase of least degree at most five is safe or has degree four.

Here is the exclusion in degrees $d = 1, 2, 3$, using the leading-form test proved in Section 5.1. In a pure phase the binary leading form has a root of multiplicity greater than d. Move the root to the point $v = 0$ in the parametrization $(x, y, t) = (u^2, -v^2/2, uv)$. Every degree-$d$ normal monomial then has negative angle i. If the phase has no positive angle it is safe by the interval theorem and one-sided support. Otherwise, use the supporting-line and two-term tests of Section 5.2 with d in place of five. The allowed nodes have $-d \le i \le d - 1$ and $\max(0, -2i) \le j$, with $i + j \le d$ for $i < 0$ and $i + j \le d - 1$ otherwise. A crossing line $j + \rho i = H$ has $\rho > 1$ and $H < d$.

Joining opposite-sign nodes in that range gives no line with three or more nodes for $d = 1, 2$. For $d = 3$ the only such lines are $(\rho, H) = (2, 2)$, with angles $-1, 0, 1$, and $(\rho, H) = (3/2, 3/2)$, with angles $-3, -1, 1$. The former has quadratic height after interchanging x, y, so is safe by the preceding argument. The latter has face $Bx + Ayt + Cy^3$ with all three coefficients nonzero. Its order-three cover has branch exponents $2, 2, 2$ when the face quadratic has distinct roots, so its elliptic Jacobian supplies the ordinary-prime obstruction of Section 5.3. If the quadratic repeats, then $A^2 + 2BC = 0$. The face is B times the leading part of $x \mapsto x - cyt - c^2y^3/2$ with $c = -A/B$, in the shear fixing y and sending t to $t + cy^2$. Its inverse preserves the weights $(3, 1, 2)$ on (x, y, t) and removes the cubic part, leaving degree at most two. This last phase is safe by the preceding low-degree cases. Degree zero is immediate since a nonzero constant has nonzero first moment. The sharper mixed-moment bound for cubics is proved in [2].

4 Explicit primitive witnesses

The following construction uses the quartic endpoint mechanism of Long [3, Theorem 3.1]. A joint endpoint identity allows an additional cubic factor while keeping all positive power moments zero. Set

$$U = 1 + x, \qquad w = tU, \qquad e = \frac{U}{x}(1 - w^2) = U\big(2y - (2 + x)t^2\big), \qquad g = e + w^2 = U(2y - t^2).$$

For integers $r \ge 1$ and $s \ge 0$ put

$$J_{r,s}(V) = \frac{1}{2}\int_{-V}^{V} z^s(1 - z^2)^r \, dz.$$

Here the endpoints are evaluated in a polynomial primitive. Both endpoint integrands vanish to order at least r at $V = 1$, so $J_{r,s}(V) - J_{r,s}(1)$ is divisible by $(V - 1)^{r+1}$. The substitution $z = tU$

gives

$$\Lambda(e^r w^s) = [x^r]U^{r-1}J_{r,s}(U) = 0, \qquad \Lambda(xe^r w^s) = [x^{r-1}]U^{r-1}J_{r,s}(U) = J_{r,s}(1).$$

In each coefficient the endpoint error is of order at least $r+1$, and the remaining factor U^{r-1} is monic of degree $r-1$. Thus in all polynomials divisible by e in $K[e,w]$, the trace vanishes, and the x-marked trace agrees with the interval substitution $e = 1-z^2$, $w = z$. In particular, for $a, b \geq 0$, the phase $e(1+w)^a g^b$ is pure. Its x-marked mth power has trace

$$\frac{1}{2}\int_{-1}^{1}(1-z^2)^m(1+z)^{am}\,dz = 2^{(a+2)m}\frac{((a+1)m)!\,m!}{((a+2)m+1)!}. \tag{3}$$

The factor g^b disappears from this integral since $g = e + w^2 = 1$ under the marked substitution. Equation (3) follows by putting $z = 2u-1$ and integrating the beta monomial $u^{(a+1)m}(1-u)^m$.

The corresponding integer forms can be written using

$$L = R+Z, \quad D = 2WR^2 - (2R+Z)T^2, \quad E = 2WR - T^2, \quad A_2 = R^2 + TL, \quad G_3 = LE.$$

For $a, b \geq 0$ set $F_{a,b} = LD\,A_2^a G_3^b$ and $d = 4+2a+3b$. Each factor is balanced homogeneous, and the sphere restriction is $e(1+w)^a g^b$. Equations (1) and (3) give

$$\mathcal{E}(F_{a,b}^m) = 0, \qquad \mathcal{E}(ZF_{a,b}^m) = (dm+2)!\,2^{(a+2)m}\frac{((a+1)m)!\,m!}{((a+2)m+1)!} \neq 0 \qquad (m \geq 1).$$

At $(\xi_1, \xi_2, z_1, z_2) = (1, 1, 1, -1)$ we have $R = 0$ and $F_{a,b} = -4(-2)^a 4^b \neq 0$, which proves radial primitivity. Specializing $\xi_1 = z_1 = 1$, $z_2 = 0$, $\xi_2 = u$ yields $2^a(4u-2)(4u-1)^b$, with a simple zero at $u = 1/2$. Any scalar multiple of a proper power would specialize to zero or to a scalar multiple of a proper power. Hence these forms remain nonpowers over every field extension. Taking $a = b = 0$ realizes degree four, taking $b = 0$ realizes every even degree at least six, and taking $b = 1$ realizes every odd degree at least seven.

5 Safety in least sphere degree five

Write a sphere phase in the normal monomials $x^i t^j$ with $i \geq 0$ and $y^i t^j$ with $i > 0$. Its leading symbol of degree d, restricted to the cone $t^2 + 2xy = 0$, gives a binary form $h_d(u,v)$ of degree $2d$ by $(x, y, t) = (u^2, -v^2/2, uv)$. This identification sends the degree-d normal monomials to distinct binary monomials. The SL_2 action on (u, v) induces linear isometries of the sphere, by congruence on $\left(\begin{smallmatrix} x & t \\ t & -2y \end{smallmatrix}\right)$. They preserve Λ, since the sphere trace in each even homogeneous degree is proportional to the Gaussian trace with covariance inverse to $t^2 + 2xy$, and the odd-degree traces are zero. This proportionality follows by integrating each normal monomial. In particular we may move a geometric root of h_d to either coordinate point after scalar extension.

5.1 The leading root threshold

Suppose every root of h_d has multiplicity less than d. Embed the finitely generated coefficient field in $\mathbb{C}$. Compactify the sphere by the smooth projective quadric with boundary conic D. Its affine residue volume ω has divisor $-2D$ by adjunction. In local coordinates (u, z) with $D = \{u = 0\}$, write the phase as F/u^d. At a boundary root of multiplicity $r > 0$, Weierstrass preparation writes F as a unit times a monic degree-r polynomial in z. For fixed u, the set $|F| \leq \delta$ lies in the union of

r planar disks of radius $O(\delta^{1/r})$. Thus on $|f| \le M$ the squared volume density has local integral bounded by a constant times

$$\int_{|u|<\epsilon} |u|^{-4+2d/r}\, d^2u,$$

which is finite since $r < d$. Away from the roots, the region $|f| \le M$ stays away from D.

Resolve the pencil and remove its fibre over infinity. On the resulting smooth surface Y, $f : Y \to \mathbb{A}^1$ is proper. The pulled-back volume form is algebraic and holomorphic on Y, since a divisorial pole would give an infinite squared integral in a bounded-f chart. Take the compact cycle $\Gamma = \{y = \overline{x},\ t \in \mathbb{R},\ 2|x|^2 + t^2 = 1\}$, which lies in the original affine chart. For $-1 < t < 1$, write $x = \sqrt{(1-t^2)/2}\, e^{i\theta}$. In these coordinates the residue form is a nonzero constant multiple of $d\theta \wedge dt$, using its expression $\pm dx \wedge dt/(2x)$ in the chart $x \neq 0$. Consequently $\int_\Gamma \omega \neq 0$, and integrating $f^m\omega$ against Γ gives $(\int_\Gamma \omega)\Lambda(f^m)$. Since the exponential series converges uniformly on the compact cycle on bounded τ-sets, purity would give the nonzero constant period $\int_\Gamma e^{-\tau f}\omega = \int_\Gamma \omega$. Lemma 3, proved below, excludes this. Thus a pure phase has a leading root of multiplicity at least d.

Here is the prime-isolation rule used in the remaining cases. Work over a number field at a good odd prime p where all coefficients are integral. If the angular constant term of h^m is a polynomial $C_m(t)$ of degree at most $p-1$, then

$$p\Lambda(h^m) \equiv [t^{p-1}]C_m(t) \pmod{\mathfrak{p}}. \tag{4}$$

Indeed the integral kills odd heights and divides even-height coefficients by $j+1$. The only denominator divisible by p in the given range is the denominator at the even height $p-1$.

An exact multiplicity d is also impossible under purity. Move that root to $u = 0$. In the literal Laurent substitution $y = (1-t^2)/(2x)$, every monomial $x^i t^j$ has $i + j \le d$, and the part on equality is $t^d L(x/t)$ for $L(z) = z^{-d}h_d(z, 1)$. With the chosen root, L is a polynomial with nonzero constant. At a good prime $p = 1 \bmod d$ put $m = (p-1)/d$. The coefficient at height $p-1$ is $\mathrm{CT}\, L^m = L(0)^m$, a unit, so (4) applies. This argument may be made after one number-field specialization preserving the leading identity and its specified nonzero constants. Indeed a finitely generated coefficient algebra over $\mathbb{Q}$, localized to keep these constants nonzero, has a maximal-ideal quotient that is a number field, and all rational moment identities specialize. We use the same specialization argument in the prime tests below, keeping the required distinct roots and support coefficients nonzero. Primes in the progression 1 modulo a positive integer are infinite, for example by the elementary cyclotomic argument.

5.2 Supporting faces for multiplicity at least six

Let $d = 5$ and move a root of multiplicity at least six to $v = 0$. All degree-five normal terms now have negative angular exponent, since their v-exponents are $5-i$ for angular exponent i. If the phase has no positive angular terms, purity forces its zero-angle core to vanish by the interval moment theorem [10, Theorem 2.1]. The negative support then gives eventual mixed vanishing for every fixed marker. Otherwise take the upper supporting line $j + \rho i = H$ of the literal Laurent support over angular exponent zero. A convex combination of nodes reaching angle zero has height less than five, since all equality nodes of $i + j \le 5$ have negative angle. This gives $H < 5$. For any degree-five node $j + \rho i = 5 + (\rho - 1)i \le H$ with $i < 0$, so $\rho > 1$. In negative angle $-b$, only the highest Laurent term of each $y^b t^r$ can lie on the upper boundary. Thus the possible face nodes satisfy

$$-5 \le i \le 4, \qquad \max(0, -2i) \le j \le \begin{cases} 5-i, & i < 0, \\ 4-i, & i \ge 0. \end{cases}$$

A face with no opposite signs must contain a core node at angle zero. Its height H is a positive integer, since the support contains nodes of both signs and all negative-angle nodes have positive height. At $m = (p-1)/H$ for good primes $p \equiv 1 \bmod H$, the core coefficient to the mth power occupies the top angular-constant slot. Equation (4) excludes it. For a two-node crossing face, write its angles $-b, a$ and heights j_-, j_+. Set $M = aj_- + bj_+$ and choose a good prime $p \equiv 1 \bmod M$, writing $k = (p-1)/M$. At exponent $m = (a+b)k$, angular cancellation uses ak negative and bk positive terms, giving height $p-1$. The coefficient is the product of $\binom{(a+b)k}{ak}$ and a unit monomial in the two coefficients. The factorials have arguments below p because $M \geq 2ab \geq a+b$. Equation (4) excludes this case.

The remaining lines are listed below. To obtain the list, take the line through every pair of allowed nodes of opposite signs, keep those with $\rho > 1$, $H < 5$, at least one degree-five node on or below the line, and at least three equality nodes. An actual crossing face occupies both signs and at least three listed nodes.

ρ	H	i	ρ	H	i
5/3	5/3	$-5,-2,1$	2	2	$-3,-2,-1,0,1$
3/2	5/2	$-5,-3,-1,1$	3/2	3	$-4,-2,0,2$
2	3	$-2,-1,0,1$	3	3	$-1,0,1$
4/3	10/3	$-5,-2,1$	3/2	7/2	$-3,-1,1$
4/3	11/3	$-4,-1,2$	5/4	15/4	$-5,-1,3$
4/3	4	$-3,0,3$	3/2	4	$-2,0,2$
2	4	$-1,0,1,2$	3/2	9/2	$-1,1,3$

At $\rho = H = 2$, normal terms have weight at most two for weights $(x, y, t) = (2, 0, 1)$. After exchanging x, y, the literal Laurent height is at most two. Regularity then allows at most a simple negative angular pole. Section 3 proves safety for every such pure phase of least sphere degree five.

5.3 The cyclic-cover obstruction on a face

For a noninteger height $H = n/q$ in lowest terms, the angular Laurent polynomial of the face is $z^{-b}P(z^q)$, where $-b, a$ are its occupied extreme angles. The table gives the same q as the denominator of ρ. In an integer-height row, first divide the angles by their common step if the displayed angles are all multiples of two or three, and then put $q = 1$ and $n = H$. Angular compression leaves constant terms unchanged. In these coordinates $\deg P = (a+b)/q$, because both extreme terms are occupied. For a prime $p = nk + 1$ and moment order qk, the constant angular term has height at most $qkH = p-1$. A product contributing at that height uses only equality nodes of the supporting line. The only interval coefficient with denominator divisible by p is the one of height $p-1$. Thus multiplying the moment by p and reducing gives

$$[z^{bk}]P(z)^{qk} = [z^{p-1}]\big(z^{n-b}P(z)^q\big)^k. \tag{*}$$

The raw degree-$2n$ branch form in the second expression has endpoint exponents $n-b, n-a$, and each root of P has its multiplicity multiplied by q. When all nonzero exponents are less than n, the normalized cyclic cover $w^n = z^{n-b}P(z)^q$ detects (*) in its character-one $H^1(O)$ line. In fact the character-one sheaf is $O(-2)$ and its Čech generator w/z maps to $wz^{-p}(z^{n-b}P(z^q)^k$ under Frobenius. Projection to the line gives exactly the selected coefficient. More generally the character-i dimension is $-1 + \sum_j \{ia_j/n\}$ for branch exponents a_j, by dividing the local Kummer generator w^i by the branch coordinate to power $\lfloor ia_j/n \rfloor$. All the covers used here are connected, as the branch exponents have greatest common divisor one with n. The rational idempotents of the cyclic

deck group give Jacobian factors up to isogeny. The factor for a divisor $s > 1$ of n has action by $\mathbb{Q}(\zeta_s)$, and its dimension is the sum of the coherent character dimensions for characters of order s. Complex conjugation interchanges the coherent and holomorphic-one-form characters, preserving the unordered multiplicities of each conjugate pair. Ordinary reduction makes Frobenius invertible on coherent H^1. It suffices for the primitive factor to be ordinary because the character-one line belongs to that factor. The nonzero scalar then excludes the face by (4) and (*). Work away from the finitely many primes of bad reduction for the curve, cover, and deck action. Adjoin the required roots of unity and take ordinary primes of absolute degree one, so they lie in the required prime progression.

For heights three and four the compressed angular sets, up to sign, are subsets of $\{-2, -1, 0, 1\}$ or $\{-1, 0, 1\}$. The branch exponents are displayed below. The first two cases of the cubic partition are all distinct and one double root. The third is a triple root.

H	$\deg P$	distinct roots	coincident roots
3	2	$(2, 1, 1, 2)$	$(2, 2, 2)$
3	3	$(1, 1, 1, 1, 2)$	$(1, 2, 1, 2),\ (1, 3, 2)$
4	2	$(3, 1, 1, 3)$	$(3, 2, 3)$
4	3	$(2, 1, 1, 1, 3)$	$(2, 2, 1, 3),\ (2, 3, 3)$

At height three the distinct quadratic and cubic root configurations give genus two and three, respectively. In genus three the Jacobian has imaginary quadratic action by $\mathbb{Q}(\zeta_3)$. At height four the distinct-root quadratic configuration has genus three, with an elliptic order-two factor and a surface with order-four action. The distinct-root cubic configuration has genus four, with an elliptic order-two factor and an imaginary quadratic threefold. All repeated configurations have genus at most two, except the triple cubic root at height three whose raw exponent equals three. There (*) is a unit times $\binom{3k}{2k}$. The genus at most two cases have ordinary primes of density one after finite extension [7]. For the genus-four case, first pass to a common extension containing the roots of unity over which the elliptic factor has density-one ordinarity. Apply Fité's theorem over that same field to the threefold with imaginary quadratic multiplication. Its positive-density ordinary set intersects both the elliptic ordinary set and the absolute degree-one primes in a set of positive density. The same theorem gives the required primes for the genus-three case [8, Theorem 1].

The noninteger heights with $n \in \{7, 9, 10, 11, 15\}$ have quadratic P. For distinct roots the primitive signatures in increasing character order are as follows.

n	branch exponents	primitive signature
7	$(4, 2, 2, 6)$	$(1, 1, 2, 0, 1, 1)$
9	$(8, 2, 2, 6)$	$(1, 1, 2, 0, 1, 1)$
10	$(5, 3, 3, 9)$	$(1, 2, 0, 1)$
11	$(7, 3, 3, 9)$	$(1, 1, 2, 0, 0, 2, 2, 0, 1, 1)$
15	$(10, 4, 4, 12)$	$(1, 1, 0, 2, 0, 2, 1, 1)$

In each row the primitive factor has rank two over $\mathbb{Q}(\zeta_n)$ and at most two conjugate pairs of multiplicities $(1, 1)$, so Theorem 2 applies. The deck action preserves the Jacobian polarization, so its Rosati involution inverts the deck generator and induces complex conjugation on $\mathbb{Q}(\zeta_n)$. For $n = 9$ there is also an elliptic factor of order three. For $n = 10$ the additional factors have orders two and five, with dimensions one and two, respectively. For $n = 15$ they have orders three and five, again with dimensions one and two. There are no lower-order factors for $n = 7, 11$. If the roots coincide, merge the two equal interior exponents. The character formula gives dimension zero at every lower order in all five cases. The primitive factor has rank one over its cyclotomic field. After

adjoining the field, its crystalline components at good degree-one primes have height one and are ordinary. This proves exclusion in all five rows, including the repeated-root cases.

For example, the height $7/2$ face has angles $-3, -1, 1$. Writing it as $z^{-3}P(z^2)$, at $p = 7k + 1$ the moment of order $2k$ detects $[z^{3k}]P(z)^{2k}$. The associated cover is $w^7 = z^4P(z)^2$. When P has distinct roots, the Jacobian is a sixfold with rank-two $\mathbb{Q}(\zeta_7)$-action. Its character pairs $(1, 6)$ and $(2, 5)$ are balanced, while $(3, 4)$ has multiplicities $(2, 0)$. The two balanced pairs are precisely the situation in which the extra trace divisibility in Theorem 2 is used.

5.4 The order-five collisions

At height $5/3$ a distinct-root quadratic gives exponents $(3, 3, 4)$ and genus two. At height $5/2$ the positive extreme is 1 and the negative extreme is -3 or -5. In both cases distinct roots give exponents $(4, 2, 2, 2)$ and a rank-two order-five Jacobian, so Theorem 2 applies. Merging two of the exponents 2 gives genus two. There remain only the raw exponents exceeding five: the repeated quadratic root at height $5/3$ and the triple cubic root at height $5/2$.

For these cases use the sphere automorphism

$$\theta_{c,r} : (x, y, t) \longmapsto \big(x - cy^r t - c^2 y^{2r+1}/2,\ y,\ t + cy^{r+1}\big).$$

It preserves Λ: its locally nilpotent derivation in the Laurent y-chart is $cy^{r+1}\partial_t$, and integration gives zero by the endpoint formula, since $x = 0$ at $t = \pm 1$. Thus it also preserves purity and safety by transporting markers.

At height $5/3$, the face is $Bx + Ay^2t + Cy^5$, with $A^2 + 2BC = 0$ in the repeated-root case. It is the leading part of $B\theta_{-A/B,2}(x)$ for weights $(5, 1, 3)$. The inverse shear gives $Bx + P_4(y) + (a + by)t$, where $\deg P_4 \le 4$. The possible y^4 term is excluded by the two-node test. The next possible tie is between y^3 and yt, since both have slope $3/2$ from the node of Bx. If both occur, a distinct-root face gives the order-three cover with exponents $(2, 2, 2)$, and a repeated-root face is the leading part of $B\theta_{-b/B,1}(x)$ for weights $(3, 1, 2)$. Invert this shear as well. Any pure phase left has degree at most two.

At height $5/2$, the triple-root face is the leading part of $B\theta_{c,1}(xt)$ for weights $(3, 1, 2)$. Indeed its face cubic is $B(z + c/4)^3$, as follows by expanding the shear and replacing $xy = (1 - t^2)/2$. The inverse shear gives $x(Bt + a) + P_4(y) + tP_2(y) + dt^2$ with $\deg P_i \le i$. The slopes from Bxt to y^4, y^2t, y^3 are $7/5, 4/3, 5/4$, respectively, each strictly largest once the earlier terms have been removed. The two-node test removes all three. Again the remaining degree is at most two.

Section 3 gives safety for every pure phase of degree at most two. This completes the unstable-leading branch and proves safety for every pure phase of least sphere degree five.

6 Proof of the ordinary-reduction theorem

Fix a complex embedding of the base field. The rational Betti H^1 is two-dimensional over E, and Hodge conjugation exchanges the embeddings in each pair. The two holomorphic multiplicities of such a pair therefore sum to two. We call the pair *balanced* when both are one.

Proof of Theorem 2. Work over a finite extension containing a normal closure of E, with the endomorphisms and polarization defined. Choose a rational prime ℓ away from the discriminant of E and the index of the endomorphism order in $\mathcal{O}_E$. Take N with $\ell^N > 2^{2e+1}$ and further extend the base to make $J[\ell^N]$ rational and to contain the ℓ^N-th roots of unity. The absolute degree-one primes have density one, since primes of higher residue degree have norm at least the square of their underlying rational prime. At a good such prime of residue characteristic p, outside a finite set,

$p \neq \ell$ and $E \otimes \mathbb{Q}_p$ splits. The actual endomorphism order in E has completion $\mathcal{O}_E \otimes \mathbb{Z}_p$ away from finitely many primes. Its idempotents split the p-divisible group, and the crystalline summands have rank two.

Here are the trace comparisons used below. For an actual endomorphism h of the reduction, the characteristic polynomial on H^1 is the degree polynomial of h, both in ℓ-adic and in crystalline cohomology. Its value at a sufficiently large integer m equals $\deg([m] - h)$, by the exterior algebra description of cohomology and the action on top cohomology. Take a rational basis of E made of integral endomorphisms a and apply this to $a\pi_p$, where π_p is the p-power Frobenius of the reduction. The nondegenerate field-trace pairing identifies the relative traces in the two cohomologies as the scalar extensions of one element $\tau \in E$, determined by $\mathrm{Tr}_{E/\mathbb{Q}}(a\tau) = \mathrm{Tr}(a\pi_p \mid H^1)$. The same argument with a alone identifies the ranks of the crystalline summands with the rank-two Betti comparison ranks. Every embedding of τ gives a sum of two Frobenius eigenvalues, which are algebraic integers of absolute value $\sqrt{p}$ under every complex embedding. Hence $\tau \in \mathcal{O}_E$ and

$$|N_{E/\mathbb{Q}}(\tau)| \leq 2^{2e} p^e.$$

The split crystalline components have slopes in $[0, 1]$, with sum given by the dimension of the corresponding p-divisible factor. The dimensions agree with the characteristic-zero Hodge multiplicities in the split idempotent pieces, since the ranks in the Lie algebra are preserved by good reduction. In each conjugate pair of type $(0, 2)$, the component of dimension two has both slopes one. Its trace is divisible by p. In a nonordinary pair of type $(1, 1)$, the two slopes of the nonordinary component lie strictly between zero and one, since they sum to one. The polarization pairs that component with slopes obtained by subtracting its slopes from one. Both component traces are therefore divisible by p. The slopes show that each trace has positive p-adic valuation, and integrality of the relative trace at each split place makes that valuation at least one. If b is the number of pairs of type $(1, 1)$, nonordinarity implies

$$p^{e-b+2} \mid N_{E/\mathbb{Q}}(\tau).$$

Indeed all $e - b + 2$ places counted here are distinct and have degree one. For $b \leq 2$ the normalized trace norm belongs to a fixed finite set of integers at every nonordinary degree-one prime.

We now use a torsion congruence, as in the ordinary-reduction argument attributed to Katz by Ogus [6, Proposition 2.7.2]. At our good degree-one prime, Frobenius acts as the identity modulo ℓ^N on the Tate module, and $p \equiv 1 \pmod{\ell^N}$. For every element a of the integral endomorphism order in E we obtain

$$\mathrm{Tr}_{E/\mathbb{Q}}(a\tau) = \mathrm{Tr}(a\pi_p \mid H^1) \equiv \mathrm{Tr}(a \mid H^1) = \mathrm{Tr}_{E/\mathbb{Q}}(2a) \pmod{\ell^N}.$$

The order completes at ℓ to $\mathcal{O}_E \otimes \mathbb{Z}_\ell$, where the field-trace pairing is perfect by the choice of ℓ. Thus $\tau \equiv 2$ in this completion modulo ℓ^N. Taking the norm and using the congruence for p gives

$$p^{-e} N_{E/\mathbb{Q}}(\tau) \equiv 2^{2e} \pmod{\ell^N}.$$

If reduction were nonordinary, the integer on the left would lie in $[-2^{2e}, 2^{2e}]$. The modulus forces it to equal 2^{2e}. Equality in the product Weil bound forces $|\sigma(\tau)| = 2\sqrt{p}$ for every complex embedding σ of E, so $\tau\overline{\tau} = 4p$ in E. At either split place of a nonordinary balanced pair, the left side has valuation at least two because both traces are divisible by p. The right side has valuation one for $p \neq 2$. This contradiction excludes all nonordinary absolute degree-one primes outside the finite exclusions over the chosen extension, proving the density claim. □

6.1 Comparison with ordinary-reduction theorems

Sawin determines the density of ordinary primes for abelian surfaces and uses a Weil bound and algebraic Chebotarev to control an integral, normalized Frobenius trace [7, Theorem 1 and Corollary 2]. Fité's results apply to certain threefolds, fourfolds, and higher-dimensional varieties with extra endomorphisms. In particular, a Galois endomorphism subfield of degree $g = \dim J$ gives positive density of p-rank at least $\lceil g/2 \rceil$ [8, Theorems 1–4 and Corollary 5]. These results supply several small-dimensional cases of the cover argument.

Cantoral Farfán, Li, Mantovan, Pries, and Tang use a relative trace norm to prove density-one ordinarity after finite extension for simple signature with full geometric endomorphism algebra a CM field [9, Assumptions 2.1 and 2.5, Proposition 3.12, and Theorem 4.18]. In relative rank two, simple signature permits one balanced conjugate pair. Theorem 2 extends the potentially density-one part of that rank-two result to two balanced pairs and to an acting field E inside an arbitrary full geometric endomorphism algebra. The additional divisibility is the trace contribution of the second member of the nonordinary balanced pair. The proof works at split primes after base extension. The results of [9, Theorems 4.18–4.19] also address higher relative rank, ordinary density over the original field in certain cases, and μ-ordinary reduction.

The primitive factors of cyclic orders $7, 9, 11, 15$ in the quintic argument have dimensions $6, 6, 10, 8$, respectively, and the indicated cyclotomic action has exactly two balanced pairs in each case. The theorem applies to each smooth member with those branch exponents, including members with additional endomorphisms. By comparison, the order-five four-branch case has one balanced pair. Its exponents $(4, 2, 2, 2)$ become $(1, 3, 3, 3)$ on changing the deck generator, giving the Moonen family treated in [9, Corollary 5.1 and §5.2] under the endomorphism hypotheses there.

7 A proper-volume obstruction

The obstruction uses the fact that the original volume form extends regularly to a proper model over the line of phase values. The V-filtration records this regularity at the zero fiber. A nonzero polynomial period would force the same volume class to have a nonzero image in a module supported at zero, where its filtration degree is impossible.

Lemma 3. *Let Y be a smooth irreducible complex algebraic variety of dimension n, let $f : Y \to \mathbb{A}^1_s$ be a proper nonconstant morphism, and let ω be a regular algebraic n-form on Y. For any compact closed n-cycle γ on $Y(\mathbb{C})$, the entire function*

$$P_\gamma(\tau) = \int_\gamma e^{-\tau f}\omega$$

cannot be a nonzero polynomial.

Proof. Factor $f = \pi \circ i_f$, with $i_f : Y \hookrightarrow Y \times \mathbb{A}^1_s$ its graph and π the projection. Put $\mathcal{B}_f = i_{f+}\mathcal{O}_Y$ and $\mathcal{M} = \mathcal{H}^0(\pi_+\mathcal{B}_f)$, using left $\mathcal{D}$-modules. Use the decreasing Kashiwara–Malgrange filtration along $s = 0$, with $\partial_s s - \alpha$ nilpotent on gr_V^α. The graph generator $e_f = \delta(s - f)$ belongs to $V^{>0}\mathcal{B}_f$. This is the graph-generator consequence of strict negativity of the Bernstein–Sato roots [5, Corollary 5.2] [4, Corollary 5.15 and Example 5.19]. In particular at each point of the graph over zero the generator lies in V^α for some rational $\alpha > 0$ locally. Quasi-compactness of the fiber gives a common positive index by taking the minimum over finitely many neighborhoods. Off $s = 0$ every filtration piece is the whole graph module, and off the graph there is nothing to check. Thus the top relative form ωe_f represents a degree-zero class in the relative de Rham complex with coefficients in $V^\alpha\mathcal{B}_f$. The

direct-image theorem for a smooth projection [4, Theorem 8.1] requires properness on the support of the module. Here the support is the graph and the restriction of π to it identifies with the proper map f. The theorem therefore identifies the image of the filtered complex with the corresponding filtration piece of $\mathcal{M}$. Consequently the class $m = [\omega e_f]$ satisfies

$$m \in V^{\alpha}\mathcal{M} \quad \text{at } s = 0, \qquad \alpha > 0.$$

This step applies to the actual volume class even if the zero fiber has a positive-dimensional critical locus.

We next check that the period respects the relations on the Fourier transform of m. Write $M = \Gamma(\mathbb{A}^1_s, \mathcal{M})$. The graph action [4, Equation (27)] identifies the graph module along Y with $\mathcal{O}_Y[\partial_s]e_f$, with actions

$$\begin{aligned} v(a\partial_s^j e_f) &= v(a)\partial_s^j e_f - a\, v(f)\partial_s^{j+1} e_f, \\ s(a\partial_s^j e_f) &= fa\partial_s^j e_f - ja\partial_s^{j-1} e_f \end{aligned}$$

for any local vector field v on Y. The Fourier convention is $\tau = \partial_s$ and $\partial_\tau = -s$, which preserves the Weyl relation. Therefore $\widehat{M}$ is represented by degree-n hypercohomology on Y of

$$\big(\Omega_Y^{\bullet}[\tau],\, d_Y - \tau\, df\wedge\big), \qquad \partial_\tau \text{ acting by } \partial/\partial\tau - f.$$

Indeed the relative de Rham complex of the projection uses the shift by n, and global sections on the affine target are exact. One can compute this hypercohomology on a finite affine open cover of Y, whose finite intersections are affine since Y is separated. This description uses the full polynomial parameter ring $\mathbb{C}[\tau]$.

At any $\tau_0 \in \mathbb{C}$, substituting $\tau = \tau_0$ in the total Čech complex commutes with both differentials. After analytification, multiplying by $e^{-\tau_0 f}$ turns the specialized twisted differential into the ordinary holomorphic de Rham differential. The affine intersections analytify to Stein spaces, so the resulting Čech complex computes holomorphic de Rham hypercohomology. Its class pairs with γ by analytic de Rham comparison. A Čech coboundary thus has zero period even when there is no global algebraic primitive. For a Weyl operator A, the class $A\widehat{m}$ is represented by a global polynomial family of top forms. Differentiating its exponential integral on the fixed compact cycle realizes ∂_τ as $\partial/\partial\tau - f$ on such a family. It follows that evaluation on γ sends $A\widehat{m}$ to AP_γ and respects all relations on the cyclic submodule $\widehat{N} = \mathbb{C}[\tau]\langle\partial_\tau\rangle\,\widehat{m}$.

If P_γ were a nonzero polynomial, it would generate the polynomial module $\mathbb{C}[\tau]$ under Weyl operators, since differentiation first produces a nonzero constant. The period evaluation would give a surjection $\widehat{N} \to \mathbb{C}[\tau]$ taking $\widehat{m}$ to P_γ. Inverse Fourier transform sends the polynomial module $D_\tau/D_\tau\partial_\tau$ to the point module $\delta_0 = D_s/D_s s$, where $D_u = \mathbb{C}[u]\langle\partial_u\rangle$. It would give a quotient of a submodule of M sending m nontrivially to δ_0. The induced filtration on submodules and strictness for morphisms [4, Proposition 3.19(ii) and Corollary 3.20] put the image of m in $V^{\alpha}\delta_0$. For $\alpha > 0$ the point module has zero such piece [4, Example 3.18]. The contradiction proves the lemma. □

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