SPHERICAL STRUCTURES ON FUSION CATEGORIES OF RANK SIX

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Abstract. Every complex fusion category of rank six admits a spherical structure. For a fusion category of arbitrary rank, central induction places the formal codegrees of its sphericalization in the field generated by its Frobenius–Perron dimension. It also compares the ordinary and spherical dimensions on a common family of objects. Equality on these objects gives a fusion subcategory on which the spherical structure descends. In rank six, bounds on the Fourier coefficients of the reciprocal codegrees and integral trace identities for the dimension ratio restrict the remaining dimension fields. The last quadratic case uses a Morita dual.

GPT-6.2 Astra assisted with the research. Complete human verification is pending, and the results should be treated with caution.

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1. Introduction

A spherical structure is a monoidal identification of each object with its double dual for which the left and right traces agree [6, §4.7]. We prove the following existence theorem. The rank counts isomorphism classes of simple objects.

Theorem 1.1. *Every complex fusion category of rank six admits a spherical structure.*

In particular, every such category is pivotal. Pseudo-unitarity alone cannot explain the theorem. Schopieray exhibited a rank-six category whose fusion rules admit no pseudo-unitary categorification [22, Example 1 and Theorem 1].

Proof of Theorem 1.1. Write $D = \mathrm{FPdim}(\mathcal{C})$ and $E = \dim(\mathcal{C})$. The intrinsic dimension E is the sum of the squared norms of the simples. If $E = D$, pseudo-unitarity supplies a canonical spherical structure [7, Proposition 8.23]. It suffices to obtain this equality after Galois conjugation. Indeed $\mathcal{C}$ has an algebraic model, the fusion rules determine D, and a spherical structure transports back [7, Theorem 2.30 and the following paragraph].

There is also a direct way to obtain a spherical structure. Each simple of $\mathcal{C}$ has two lifts to its sphericalization $\mathcal{S}$. If a choice of one lift per simple spans a tensor subcategory, the forgetful functor identifies it with $\mathcal{C}$. Central induction places all ordinary and signed codegrees in $K = \mathbb{Q}(D)$. The two dimension characters have codegrees D and E. Evaluating them on the same induction object gives, for every $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, the inequality of Lemma 2.1

$$\frac{E}{\sigma(E)} \leq \frac{D}{\sigma(D)}.$$

Equality means that the two dimensions agree on every constituent of the corresponding induction object. If E/D lies in a proper subfield, these constituents generate enough dimensions to recover the larger field (Theorem 2.12). The remaining cases use bounds on the Fourier coefficients of the conjugates of $1/D$ and $1/E$ (Lemma 2.15).

The Perron character maps $K_0(\mathcal{C})_{\mathbb{Q}}$ onto a field containing K, so $[K : \mathbb{Q}] \leq 6$. In degree one D is integral and pseudo-unitarity follows from [7, Proposition 8.24]. Theorems 3.1 and 4.1 give sphericality in degrees three, five and six. In degrees two and four, Theorems 6.12 and 12.1 leave four fields.

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- In $\mathbb{Q}(\sqrt{2})$, passage to bimodules over $\mathbf{1} \oplus g$ for an invertible g of order two produces two additional codegrees. A modular bound excludes an irrational pair. For a rational pair, the sign bimodule exchanges simples in pairs, forcing even codegrees. A congruence for odd-index Pell numbers excludes the last possibility (Theorem 10.6).
- In $\mathbb{Q}(\zeta_{16})^+$ and $\mathbb{Q}(\sqrt{2},\sqrt{3})$, two characters of codegree four account for half the unit coefficient. The resulting integral moments on the quartic factor have incompatible residues (Theorem 13.1).
- In $\mathbb{Q}(\sqrt{2},\sqrt{5})$, the second Frobenius–Schur indicator makes the ordinary and signed induction columns agree modulo four in a suitable basis. Applying this congruence to the columns and their squares eliminates the remaining pairs of codegrees (Theorem 14.1). □

Finite calculations. Each computation links to its code or certificate. For example, if $z_i \geq 0$, $\sum_i z_i = 1$ and $Az \leq b$, a row $w \geq 0$ with $\min_i (wA)_i > wb$ gives a contradiction by taking the weighted average with weights z_i. These certificates use rational arithmetic. Interval checks round outwards. The instructions give the software requirements; `python verify.py` replays the checks.

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2. Two characters of the sphericalization

Write ε for the central sign of the sphericalization. The two lifts of a simple X have opposite spherical dimensions. Choose the lift $\widetilde{X}$ with positive dimension x_X. Its square is the squared norm of X [7, Theorem 2.3 and Remark 3.1]. With $d_X = \mathrm{FPdim}(X)$ we have $x_X \leq d_X$, $E = \sum_X x_X^2$ and $D = \sum_X d_X^2$. In the quotients

$$R = K_0(\mathcal{S})/([\varepsilon] - 1) \simeq K_0(\mathcal{C}), \qquad T = K_0(\mathcal{S})/([\varepsilon] + 1),$$

the classes $b_X = [\widetilde{X}]$ give bases indexed by the same simples. The coefficient of b_X in a product is the sum of the two lift multiplicities in R and their difference in T. In particular $R/2R \simeq T/2T$. We call characters of the two quotients ordinary and signed, respectively [20, §5.1].

2.1. **Central induction.** Let $F : Z(\mathcal{S}) \to \mathcal{S}$ be the forgetful functor, with right adjoint I. In either quotient the standard trace τ extracts the coefficient of the unit. On the complexified algebra, $\tau(a) = \sum_V \operatorname{Tr}(a|V)/f_V$, summed over the simple modules with their formal codegrees [19, §2]. For a linear character χ we have $f_\chi = \sum_X |\chi(b_X)|^2$. In the commutative case $\sum_\chi 1/f_\chi = 1$. We call the sum over a character orbit its *reciprocal mass*. On each summand cut out by $(1 \pm [\varepsilon])/2$, the unit trace of $K_0(\mathcal{S})$ restricts to $\tau/2$. Thus V has codegree $2f_V$ in $\mathcal{S}$.

For a simple module V write χ_V for its central character (the scalar action on V), and A_V for its label in the center. We also write A_χ when V is the linear character χ. By Ostrik's induction theorem [20, Example 2.9 and Theorems 2.5 and 2.13], A_V occurs $\dim V$ times in $I(\mathbf{1})$ and has twist one. With $S_{1,1} = 1/(2E)$ we have

$$\dim A_V = E/f_V, \qquad S_{A_V,Y} = \chi_V(FY)/(2f_V).$$

Thus symmetry of S becomes

$$\frac{\chi_V(FA_W)}{f_V} = \frac{\chi_W(FA_V)}{f_W}. \tag{2.1}$$

The central lift J of ε has dimension and twist -1. Since $I(\varepsilon) = JI(\mathbf{1})$, the simple summands of $I(\varepsilon)$ have negative dimension and those of $I(\mathbf{1})$ have positive dimension. Adjunction now gives $[FA_V : \mathbf{1}] = \dim V$ and $[FA_V : \varepsilon] = 0$.

Lemma 2.1 (Simultaneous induction column)**.** *The spherical dimension field M of $\mathcal{S}$ is abelian and contains the codegrees of both quotients. For $\sigma \in \operatorname{Gal}(M/\mathbb{Q})$ let $W = FA_{\sigma \circ \dim_{\mathcal{S}}}$. Its central characters and its two dimensions are*

$$\chi_V(W) = \frac{f_V}{\sigma(f_V)}, \qquad \dim W = \frac{E}{\sigma(E)}, \quad \operatorname{FPdim} W = \frac{D}{\sigma(D)}.$$

All these values are totally positive algebraic units. The formal codegrees of both quotients lie in $\mathbb{Q}(D)$.

Proof. The dimensions are cyclotomic integers [7, Theorem 8.51], and $E/f_V = \dim FA_V$ puts f_V in M. Reciprocity (2.1) gives $\chi_V(W)/f_V = \sigma(E/f_V)/\sigma(E)$. The codegrees $2f_V$ are totally positive, and their principal ideals are Galois invariant by Ostrik's d-number theorem [19]. If σ fixes D, then W has Perron dimension one and contains $\mathbf{1}$, so $W = \mathbf{1}$. The first identity now shows that σ fixes every f_V. Conversely D itself is a codegree. □

In particular $\alpha = E/D = \dim A_{\operatorname{FPdim}}$ is a totally positive algebraic integer. Since $\dim W \le \operatorname{FPdim} W$,

$$\alpha \le \sigma(\alpha) \qquad (\sigma \in \operatorname{Gal}(M/\mathbb{Q})). \tag{2.2}$$

If $\alpha < 1$, some conjugate exceeds one by integrality of the norm. We often pass to a Galois conjugate maximizing E. Then $1/E$ and $1/D$ are both least at the distinguished embedding, the latter by Perron domination.

Lemma 2.2 (Comparison of the two quotients)**.** *For any object Y of $\mathcal{S}$ its ordinary and signed classes satisfy $\|[Y]_+\|^2 - \|[Y]_-\|^2 \in 4\mathbb{Z}_{\ge 0}$. Every linear character satisfies $|\chi(Y)| \le \operatorname{FPdim} Y$. In particular $f_V/\sigma(f_V) \le D/\sigma(D)$ for every V.*

Proof. If the two lifts of X occur m_X^+ and m_X^- times in Y, the norm difference is $4\sum_X m_X^+ m_X^-$. Every eigenvalue of $[Y]$ on a simple module occurs in the regular representation, whose spectral radius on $[Y]$ is $\operatorname{FPdim} Y$. This applies in particular to the scalars by which $[W]$ acts. □

2.2. **Galois translates and their moments.**

Corollary 2.3 (Integral Galois translates)**.** *In either quotient $A = R, T$, let V_σ and U_σ be the classes in A of the forgotten induction labels of $\sigma \circ \mathrm{FPdim}$ and $\sigma \circ \dim_{\mathcal{S}}$. Here σ acts on a common Galois splitting field of the characters. Write $V = V_1$. Then U_σ is a unit of A, and*

$$\chi(U_\sigma^{-1} V_\sigma) = \sigma(\chi(V)) \qquad (\chi \text{ a central character of } A).$$

Every $\chi(V)$ is a totally positive algebraic integer.

In the signed quotient let M be the field of the spherical dimension character, $x = E/D$, $s = \mathrm{Tr}_{M/\mathbb{Q}}(1/D)$ and $t = \mathrm{Tr}_{M/\mathbb{Q}}(1/E)$. If σ fixes every codegree outside the dimension orbit, then

$$\mathrm{Tr}_{M/\mathbb{Q}}(\sigma(x)/E) \in s + \mathbb{Z}_{\geq 0}.$$

If all the complementary codegrees are rational, then $s \leq t$. At equality the complementary characters take value one on V and on every $U_\sigma^{-1} V_\sigma$.

Proof. The determinant of multiplication by U_σ is the product of $f_V/\sigma(f_V)$ with multiplicities $(\dim V)^2$. Galois conjugation permutes modules of each dimension, so this product is one. Reciprocity (2.1) gives

$$\frac{\chi(V_\sigma)}{f_\chi} = \sigma\left(\frac{\mathrm{FPdim}\, F(A_\chi)}{D}\right) = \sigma\left(\frac{\chi(V)}{f_\chi}\right).$$

This proves the first formula. At $\sigma = 1$ it gives $\chi(V) > 0$. Galois conjugation permutes the characters, so all conjugates are positive, and integrality follows from $V \in A$.

In the signed quotient the dimension orbit contributes s to the unit coefficient of V. Every other contribution is positive, so $s \leq 1$. If σ fixes the complementary codegrees, its action permutes the complementary values with their trace weights (module dimension divided by codegree) unchanged. The translated column therefore has unit coefficient $1 - s + \mathrm{Tr}_{M/\mathbb{Q}}(\sigma(x)/E)$. This is an integer and the trace is positive, giving the inclusion. If the complementary codegrees are rational, each complementary orbit has constant weight. By the arithmetic–geometric mean inequality its average value on V is at least one, with equality only if all values are one. Summing gives $1 - s \geq 1 - t$. □

Lemma 2.4 (Moments at equal reciprocal mass)**.** *Suppose all complementary signed codegrees are rational and that the two masses in Corollary 2.3 satisfy $s = t = u/v$ in lowest terms. Put $x = E/D$, $G = \mathrm{Gal}(M/\mathbb{Q})$ and $L(y) = \mathrm{Tr}_{M/\mathbb{Q}}(y/E)/t$. Consider the monomials in $g(x)$ and $E/g(E)$, $g \in G$, allowing negative powers of the units $E/g(E)$. Each such y satisfies*

$$L(y) \in 1 + t^{-1}\mathbb{Z}_{\geq 0}, \qquad \prod_{g \in G} (X - g(y)) \equiv (X - 1)^{|G|} \pmod{v}.$$

In particular all these moments are positive odd integers at $t = 1/2$.

Proof. Each generator has an integral lift with value one at every complementary character. For $g(x)$ take the translate of V in Corollary 2.3; for $E/g(E)$ take U_g, an integral unit. Consequently for any integral polynomial P in these generators and the indicated inverses, the unit coefficient of its lift gives

$$\mathrm{Tr}_{M/\mathbb{Q}}(P/E) - tP(1, \ldots, 1) \in \mathbb{Z}.$$

For a monomial, $P(1, \ldots, 1) = 1$ and the trace is positive. For the second assertion note that $g(E)/gh(E)$ is a quotient of two of the allowed units. All conjugates of y are thus monomials as well. Their kth elementary symmetric function a_k is a rational integer and its lift has complementary value $\binom{|G|}{k}$. Applying the trace identity to this lift gives $t(a_k - \binom{|G|}{k}) \in \mathbb{Z}$. □

2.3. **Discriminants and trace orders.** The relative discriminant gives a lower bound for the trace of a totally positive generator.

Lemma 2.5. *Let K/F be a totally real quadratic extension with $[F:\mathbb{Q}] = m$, and relative discriminant $\mathfrak{d}_{K/F}$. Put $\Delta = N_{F/\mathbb{Q}}(\mathfrak{d}_{K/F})$. For a totally positive $x \in \mathcal{O}_K$ with $K = F(x)$,*

$$\operatorname{Tr}_{K/\mathbb{Q}}(x) \;\geq\; m\sqrt{\Delta^{1/m} + 4N_{K/\mathbb{Q}}(x)^{1/m}}.$$

Proof. Pair the conjugates a_i, b_i of x above each embedding of F. The discriminant of $\mathcal{O}_F[x]$ is divisible by $\mathfrak{d}_{K/F}$, so $\prod_i (a_i - b_i)^2 \geq \Delta$. The geometric mean is superadditive on positive vectors by Hölder's inequality. Apply this to $(a_i + b_i)^2 = (a_i - b_i)^2 + 4a_i b_i$, then use the arithmetic–geometric mean inequality on the $a_i + b_i$. □

We shall also need the ordinary trace dual of an order. If a commutative quotient $A = R$ or T has rational algebra $\prod_i K_i$, choose one character in each field factor and write f_i for its codegree. Then $\tau = \sum_i \operatorname{Tr}_{K_i/\mathbb{Q}}(-/f_i)$. The pairing $\tau(ab)$ is unimodular: the dual classes of the simple basis elements form its dual basis.

Lemma 2.6 (The conductor of a trace order)**.** *Let $\mathcal{O} = \prod_i \mathcal{O}_{K_i}$ and $\mathfrak{D}_i$ be the different of K_i. Then*

$$(A : \mathcal{O}) = \prod_i f_i \mathfrak{D}_i^{-1}, \qquad [\mathcal{O} : A]^2 \,|\operatorname{disc}\mathcal{O}| = \prod_i N_{K_i/\mathbb{Q}}(f_i).$$

The conductor identity holds after completion at any prime. A local field factor on which its codegree is a unit splits off as an unramified maximal order. For a set I of factors, $A \cap \prod_{i\in I} K_i$ is the τ-dual of the projection of A to those factors, where the intersection uses zero in the other coordinates. In particular an isolated pair of rational p-adic axes with codegrees g, h has order $\{(x,y) \in \mathbb{Z}_p^2 : x \equiv y \pmod{p^v}\}$ with $v = v_p(g) = v_p(h)$. Its rows satisfy $p^{2v} \mid g + h$ when g, h are rational integers.

Proof. Write $B^{\#}$ for the dual of a lattice with respect to $\tau(ab)$. Since $A = A^{\#}$ and $A\mathcal{O} = \mathcal{O}$, one has $a\mathcal{O} \subset A$ exactly when $\tau(a\mathcal{O}) \subset \mathbb{Z}$. This gives $(A : \mathcal{O}) = \mathcal{O}^{\#}$, whose ith component is $f_i \mathfrak{D}_i^{-1}$. When f_i is a local unit, containment of the conductor in $\mathcal{O}$ forces $\mathfrak{D}_i$ to be a unit there as well. Taking determinants of the trace form gives the index identity. Finally an element supported on I pairs integrally with A exactly when it pairs integrally with its projection. On two rational axes the diagonal copy of $\mathbb{Z}_p$ lies in the order and the supported coordinate ideals are $g\mathbb{Z}_p$ and $h\mathbb{Z}_p$. Subtracting a diagonal element identifies these ideals and gives the congruence. Pairing the row difference with its dual gives $g + h$ by orthogonality. □

Lemma 2.7. *Let B be a commutative ordinary or signed based order, and let a rational character have formal codegree f with $v_p(f) = 1$. Its completed local factor is $\mathbb{Z}_p \times_{\mathbb{F}_p} \mathcal{O}_L$, with $L/\mathbb{Q}_p$ totally ramified. In particular an unramified quadratic branch cannot be its partner.*

Proof. Let A be the completed local factor, e the rational idempotent, and B' the projection of A to the other branches. Lemma 2.6 gives $A \cap \mathbb{Q}_p e = p\mathbb{Z}_p e$. Thus the rational coordinate modulo p defines a surjection $B' \to \mathbb{F}_p$ and $A = \mathbb{Z}_p \times_{\mathbb{F}_p} B'$.

As A is finite free over $\mathbb{Z}_p$, its canonical module is $\operatorname{Hom}_{\mathbb{Z}_p}(A, \mathbb{Z}_p)$. The perfect pairing identifies this module with A, so A is Gorenstein. The Gorenstein fiber-product theorem [16, Fact 2.6 and Corollary 2.7] now makes B' a discrete valuation ring. Its residue field is $\mathbb{F}_p$, hence its fraction field is totally ramified over $\mathbb{Q}_p$. □

2.4. **Dimension fields and spherical sections.**

Lemma 2.8. *With trivial universal grading the ordinary and signed dimension character fields both equal $K = \mathbb{Q}(D)$.*

Proof. For the ordinary field this is the dimensional grading of Gannon and Schopieray [8, §1.2, (13)]. For the signed field, which contains K by Lemma 2.1, let σ fix D. The label B of $\sigma \circ \dim_{\mathcal{S}}$ forgets to $\mathbf{1}$, so its half-braiding is a monoidal automorphism γ of the identity. On the central sign J its normalized S-row is

$$\gamma_\varepsilon \dim \varepsilon = S_{B,J}/S_{B,\mathbf{1}} = \sigma(\dim \varepsilon).$$

The first equality takes the trace of the double braiding, and the second is the induction character formula. Hence $\gamma_\varepsilon = 1$, so the scalars on the two lifts of each simple agree. They define a grading character of $\mathcal{C}$. Triviality of the universal grading gives $\gamma = 1$, hence $B = \mathbf{1}$. Injectivity of the labeling in [20, Theorem 2.13] gives $\sigma \circ \dim_{\mathcal{S}} = \dim_{\mathcal{S}}$. □

Lemma 2.9 (A spherical section from a character)**.** *Let $\chi : T \to \mathbb{R}$ be a character. The simples X of $\mathcal{C}$ for which $|\chi(\widetilde{X})| = \mathrm{FPdim}(X)$ form a fusion subcategory admitting a spherical structure. In particular, if $f_\chi = D$, the whole category has a spherical structure.*

Proof. Write $d_X = \mathrm{FPdim}(X)$. The spectral radius of the fusion matrix of either lift gives $|\chi(\widetilde{X})| \le d_X$. Duality conjugates the character value, so equality is preserved by duals. For each simple attaining equality take the unique lift with value d_X. A product of two chosen lifts has character value equal to its Perron dimension. Equality in the triangle inequality forces every summand to be a chosen lift as well. These lifts therefore give a spherical section over the indicated subcategory. If $f_\chi = D$, the sum of the nonnegative differences $d_X^2 - |\chi(\widetilde{X})|^2$ vanishes. □

For a fusion category $\mathcal{A}$, let $K_{\mathcal{A}}$ be the field generated by the Frobenius–Perron dimensions of its simple objects.

Lemma 2.10. *Let $\mathcal{A} \subseteq \mathcal{C}$ be a fusion subcategory with Perron field $F = K_{\mathcal{A}}$. On each indecomposable component of the left action of $\mathcal{A}$, the rational span of the simple Perron dimensions is one-dimensional over F. If $x_A = d_A$ for every simple $A \in \mathcal{A}$, the ordinary and signed actions of these positive lifts agree and x_X/d_X is constant on each component.*

In particular a proper $\mathcal{A}$ in rank n satisfies

$$[F : \mathbb{Q}] \le \min\{\mathrm{rank}(\mathcal{A}), n - \mathrm{rank}(\mathcal{A})\}.$$

In a faithful grading each homogeneous component has rank at least $[\mathbb{Q}(D) : \mathbb{Q}]$.

Proof. On a component let P be the action matrix of the sum of the simples of $\mathcal{A}$, with the dimension column as right eigenvector. Its simple Perron root is $r = \sum_A d_A \in F$. Solving $(P - rI)y = 0$ over F shows that the ratios of the coordinates are in F. Their rational span is stable under multiplication by each d_A, so is an F-line.

Under the equality hypothesis the signed action matrix P' satisfies $P'x = rx$ and $P' \le P$ entrywise. Pair $(P - P')x$ with a positive left Perron vector. The result is zero; positivity of x gives $P' = P$, also term by term for each simple of $\mathcal{A}$. Simplicity of the eigenspace gives the constant ratio. The bounds follow by counting basis elements. For a G-grading take the neutral component as $\mathcal{A}$ and use $D = |G| \,\mathrm{FPdim}(\mathcal{A})$ [7, Proposition 8.20]. □

Lemma 2.11 (The regular idempotent of the integral subcategory)**.** *Let $\mathcal{A}$ be the full subcategory on the simples of integral Frobenius–Perron dimension and put $n = \mathrm{FPdim}(\mathcal{A})$. In both $R \otimes \mathbb{Q}$ and $T \otimes \mathbb{Q}$ it has a regular idempotent of unit coefficient $1/n$. A linear character takes value one on it precisely when its restriction to $\mathcal{A}$ is the Perron character, and zero otherwise.*

Proof. Closure of $\mathcal{A}$ is the dimension-subfield criterion of [8, §1] applied to $\mathbb{Q}$. Its integral global Perron dimension gives pseudo-unitarity [7, Propositions 8.23–8.24], so its positive lifts give a section by Lemma 2.9.

In either quotient take

$$e_{\mathcal{A}} = \frac{1}{n} \sum_{X \in \mathrm{Irr}(\mathcal{A})} d_X [\widetilde{X}].$$

The regular-object identity $[\widetilde{Y}]e_{\mathcal{A}} = d_Y e_{\mathcal{A}}$ for $Y \in \mathcal{A}$ gives the stated evaluations. The coefficient at the unit is $1/n$. □

2.5. **When the dimension ratio lies in a proper subfield.**

Theorem 2.12. *Let $K = \mathbb{Q}(D)$, $L = \mathbb{Q}(E/D)$ and $H = \mathrm{Gal}(K/L)$. If $E < D$ and $H \neq 1$, let $\mathcal{A}$ be the proper fusion subcategory of simples on which $x_X = d_X$. It is pseudo-unitary. The algebraic integer*

$$\delta = \sum_{\sigma \in H} \frac{D}{\sigma(D)}$$

belongs to $K_{\mathcal{A}}$, and

$$K = L(\delta), \qquad D = \delta w, \quad E = \delta z \quad (w, z \in L), \qquad \mathrm{Tr}_{K/L}(1/\delta) = 1.$$

In particular H acts faithfully on $\mathbb{Q}(\delta)$ and $|H| \leq [\mathbb{Q}(\delta) : \mathbb{Q}] \leq \lfloor \mathrm{rank}(\mathcal{C})/2 \rfloor$.

Proof. Lemma 2.9 identifies the positive lifts of $\mathcal{A}$ with a spherical section of Perron dimensions. For $\sigma \in H$, extend σ to the signed dimension field. The two dimensions of W_σ in Lemma 2.1 agree. Thus W_σ is supported on $\mathcal{A}$, so its Perron dimension $D/\sigma(D)$ belongs to $K_{\mathcal{A}}$. Their sum is $\delta = D\,\mathrm{Tr}_{K/L}(1/D)$, so $w = D/\delta$ and $z = (E/D)w$ lie in L and $\mathrm{Tr}_{K/L}(1/\delta) = 1$. Since $K = L(D)$, this also gives $K = L(\delta)$. The fields are abelian, so H acts faithfully on $\mathbb{Q}(\delta)$, and Lemma 2.10 gives the bound. □

Corollary 2.13. *Let n be the rank of $\mathcal{C}$. Suppose $K = \mathbb{Q}(D)$ is cyclic of prime-power degree $d > n/2$. If E/D belongs to a proper subfield of K, then $E = D$ and $\mathcal{C}$ is pseudo-unitary. In particular this holds for a cyclic quartic dimension field in rank six, with arbitrary complementary formal codegrees.*

Proof. If $E < D$, Theorem 2.12 writes K as the compositum of two proper subfields. In a cyclic prime-power extension every proper subfield lies in the same maximal proper subfield. □

2.6. **Fourier bounds from the modular representation.** The following projection estimate is the only input from congruence representations. We use $U(r) = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$ and $s = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma 2.14 (Modular projection)**.** *Let z be a unit vector in a unitary representation π of $\mathrm{SL}_2(\mathbb{Z}/N)$. Suppose $U(r)$ fixes z for $r = 1$ or 2 and the diagonal subgroup acts on z by a character of conductor c. For the projection P_r onto the $U(r)$-fixed space one has $\|P_r \pi(s) z\|^2 \leq (c, r^2)/c$.*

Proof. Put $A = \mathbb{Z}/N$ and $L(b) = U(b)^{\mathsf{t}}$. The squared norm is the average of $\langle z, \pi(L(b))z \rangle$ over $b \in rA$. Group the terms by the ideal bA. If $u \in A^\times$ and $u - 1 \in rbA$, choose $x \in rA$ with $u = 1 + bx$. Then

$$U(x)L(b) = L(b/u)\,\mathrm{diag}(u, u^{-1})U(x/u).$$

On the entire stratum $b \mapsto b/u$ is a permutation. Its sum therefore vanishes unless the diagonal character is trivial on all $u \equiv 1 \pmod{rbA}$, or equivalently $c \mid rb$. These strata occupy a proportion $(c, r^2)/c$ of rA, and each term has modulus at most one [11, Lemma 4.1 and Proposition 4.3]. □

Suppose the ordinary and signed dimension characters have the same field K, and write $G = \mathrm{Gal}(K/\mathbb{Q})$ and $n = |G|$. For a nontrivial character η of G of conductor c, set

$$\widehat{a} = \sum_{g \in G} \eta(g) g(a), \qquad P = \widehat{1/D}, \qquad Q = \widehat{1/E}.$$

Every linear codegree lies in K by Lemma 2.1. To retain contributions from the complementary labels, put

$$O_\eta = \frac{1}{n} \sum_{\chi \text{ ordinary}} |\widehat{1/f_\chi}|^2, \qquad T_\eta = \frac{1}{n} \sum_{\chi \text{ signed}} |\widehat{1/f_\chi}|^2,$$

where the sums run over linear characters. The dimension orbits contribute $|P|^2$ and $|Q|^2$, respectively.

Lemma 2.15 (Simultaneous Fourier bounds)**.** *For each η as above,*

$$cO_\eta \le (c,2), \qquad cT_\eta \le (c,4), \qquad c(2O_\eta + T_\eta) \le 4. \tag{2.3}$$

The two dimension characters also satisfy

$$c|Q|^2 \le (2 - cO_\eta)(2 - c|P|^2), \qquad \sqrt{c}\,|Q| + c|P|^2 \le 2.$$

Proof. Write B_g, A_g for the labels of the conjugates of the signed and ordinary dimension characters. Their translates by J give two more disjoint orbits. Only the JB_g have twist -1; the other three orbits have twist one. In the unitary center representation [7, Proposition 2.12], define $b = n^{-1/2}\sum_g \eta(g) e_{B_g}$ and $a = n^{-1/2}\sum_g \eta(g) e_{A_g}$. Each is a diagonal eigenvector of conductor c, as are its J-translates: the Galois symmetry of the modular data permutes the dimension rows by conjugation with positive sign, and $S_{J,Y}/S_{1,Y}$ is rational [3, Theorem II and Remark 4.5].

Reciprocity gives $S_{B_g,A_\chi} = 1/(2g(f_\chi))$. Multiplication of a source label by J preserves its S-entry at an ordinary target and reverses it at a signed target. Multiplication of the target by J preserves the modulus. Thus in the twist-one projection of Sb, the ordinary labels and their translates contribute $O_\eta/2$ and the signed labels contribute $T_\eta/4$. For $(b + Jb)/\sqrt{2}$ and $(b - Jb)/\sqrt{2}$ the twist-square projection instead retains contributions O_η and T_η, respectively. Lemma 2.14 now gives (2.3), using $cO_\eta \le 2$ from the mixed bound when $4 \mid c$.

For the last bounds use the twist-one source vectors b and $(a - Ja)/\sqrt{2}$. Project their images onto the span of the conjugate-weighted B-vector and all the ordinary linear labels and their J-translates. The second image vanishes at the ordinary targets. After unitary changes of target coordinates, the resulting matrix has the form

$$\begin{pmatrix} Q/2 & P/\sqrt{2} \\ \sqrt{O_\eta/2} & 0 \end{pmatrix}$$

up to conjugation of the first row. Indeed the B_g, B_h and B_g, A_h entries are $1/(2gh(E))$ and $1/(2gh(D))$. Lemma 2.14 bounds the norm of this matrix M by $1/\sqrt{c}$. The determinant of $I - cM^*M$ gives the first inequality. Since $|P|^2 \le O_\eta \le 2/c$, taking square roots gives the second. □

When K is real quadratic, $O_\eta = \Delta_R^2$ and $T_\eta = \Delta_T^2$, where each Δ^2 sums $(1/f - 1/f')^2$ over the conjugate pairs, with multiplicity. We shall use (2.3) in the form

$$\Delta_R^2 \le (c,2)/c, \qquad \Delta_T^2 \le (c,4)/c, \qquad 2\Delta_R^2 + \Delta_T^2 \le 4/c. \tag{2.4}$$

For a quartic dimension orbit with a quadratic complementary pair f, f' in the ordinary sector, take η cutting out $\mathbb{Q}(f)$. Each embedding of that subfield occurs twice in G. Hence $O_\eta = |P|^2 + 2(1/f - 1/f')^2$; the signed formula is identical with T_η, Q. For example, the roots of $X^2 - 15X + 25$ have squared reciprocal gap $1/5$. In conductor five they contribute $2/5$ to O_η, already exceeding the bound $1/5$.

3. Odd-degree dimension fields

Theorem 3.1. *Let $\mathcal{C}$ have rank six and put $D = \mathrm{FPdim}(\mathcal{C})$. If $[\mathbb{Q}(D) : \mathbb{Q}]$ is odd, then $\mathcal{C}$ has a spherical structure. In degrees one and five a Galois conjugate is pseudo-unitary. In degree three the same conclusion holds unless the signed algebra is $\mathbb{Q}(\zeta_9)^+ \times \mathbb{Q}(\zeta_9)^+$ and its dimension codegree is nine.*

Write $K = \mathbb{Q}(D)$, $p = 1/D$, $q = 1/E$, $\alpha = E/D$, $s = \mathrm{Tr}_K(p)$ and $t = \mathrm{Tr}_K(q)$. Suppose first that K has degree three or five and the universal grading is trivial. By Lemma 2.8, K is a factor of both character algebras. Their complements have dimension at most three, so both algebras are commutative. All codegrees lie in the cyclic field K by Lemma 2.1.

Lemma 3.2 (A cyclic rearrangement)**.** *Let u, v be real vectors indexed cyclically by two or three indices. If both have a strict minimum at zero, then some cyclic shift j satisfies $\sum_i u_i v_{i+j} < \sum_i u_i v_i$.*

Proof. Subtracting the respective minima changes every sum by the same constant. For two indices the shifted sum is now zero. For three write the vectors as $(0, a, b)$ and $(0, c, d)$. The three sums are $ac + bd$, ad and bc, and $\min(ad, bc) \le \sqrt{acbd} < ac + bd$. □

Lemma 3.3. *Suppose the ordinary and signed algebras are commutative and their dimension characters both generate a cubic field $K = \mathbb{Q}(D)$. If every complementary signed codegree is rational, a Galois conjugate is pseudo-unitary. The same conclusion holds for $T_{\mathbb{Q}} = K \times K$ if $(1, -1) \in T$.*

Proof. Put $h_\sigma = \mathrm{Tr}(q\sigma(\alpha))$, so $h_1 = s$. For rational complementary codegrees Corollary 2.3 gives $s \le t$ and $h_\sigma \ge s$.

For the second assertion write $g = (1, -1)$. The positive involution of K^2 is trivial. Thus g has squared norm one and is a signed basis element distinct from ± 1. Its unit coefficient $2t - 1$ vanishes. Moreover multiplication by $1 + g$ doubles the first coordinate and kills the second. Applied to the integral translates of the Perron column in Corollary 2.3, this shows $2h_\sigma \in \mathbb{Z}_{>0}$. Since the ordinary dimension orbit is proper, $0 < s < 1$, whence $s = 1/2 = t$ and again $h_\sigma \ge s$.

If E is rational, $\mathrm{Tr}(\alpha) = Es \le Et = 3$. Total positivity and the integral norm force $\alpha = 1$. Otherwise maximize E by transport. A rational α is a positive integer at most one. If α is irrational, q and α both have a strict minimum at the distinguished embedding, by (2.2). Lemma 3.2 gives $h_\sigma < h_1 = s$, a contradiction. Thus $E = D$ after transport. □

All nontrivial characters of the cubic and quintic fields have odd conductor c. By (2.3), in either sector the sum of squared Fourier coefficients of the reciprocal codegrees, one for each full orbit, is at most $1/c$.

We shall use the same integral coordinates at conductors $7, 11, 13$. For an odd prime $\ell = 2m+1$, put $b_j = \zeta_\ell^j + \zeta_\ell^{-j}$ for $1 \le j \le m$. Their trace Gram matrix is $\ell I - 2J$, where J is the all-ones matrix, by the identity $b_i b_j = b_{i+j} + b_{i-j}$. Its determinant ℓ^{m-1} is the discriminant of $\mathbb{Q}(\zeta_\ell)^+$, so the b_j form an integral basis. Writing a totally positive integer $z = -\sum c_j b_j$ of trace A gives

$$\sum c_j = A, \qquad \ell c_j = \mathrm{Tr}((2 - b_j)z) \in (0, 4A). \tag{3.1}$$

These coordinates at eleven are also used in [11, §8.2].

3.1. **Two cubic orbits.** Put $P(x) = \sum_i x_i^2 - \sum_{i<j} x_i x_j$ for triples, and write $\nu(f) = P(1/f)/\mathrm{Tr}(1/f)^2$ for a totally positive cubic element, using its three embeddings.

Proposition 3.4. *Suppose K is cubic, $T_{\mathbb{Q}} = K^2$, and the complementary signed codegree H is irrational. Then the category has a pseudo-unitary Galois conjugate at conductor seven. The only other possibility has conductor nine and $E = 9$, and the category is spherical there.*

Proof. Bounding the conductor. Write $w = 1 - t$. The cubic field has discriminant c^2. The character Gram determinant of the dimension orbit is $N(E)$. Factoring the character matrix through an integral basis of K gives $c^2 \mid N(E)$. Thus rational E is divisible by c (a cubic conductor is squarefree away from a possible factor 9). The preceding Fourier bound becomes

$$t^2\nu(E) + w^2\nu(H) \le 1/c, \qquad P(p) \le 1/c.$$

Suppose $c \ge 13$. Some irrational $f \in \{E, H\}$ has $\nu(f) \le 2/c$: if E is irrational use $t^2+w^2 \ge 1/2$, and otherwise use $w \ge 1 - 3/c > 1/\sqrt{2}$. Normalize the reciprocal triple of f to $z_0 + z_1 + z_2 = 1$ and put $v = \sqrt{P(z)} \le \sqrt{2/13} < 2/5$. The unit z_1/z_0 and its inverse have integral traces A, B, with $A + B + 3 = \sum_i 1/z_i$. At fixed v the pair sum of z is fixed, and the product is least at $(1 - 2v, 1 + v, 1 + v)/3$, up to permutation (the extremum has two equal entries). Thus

$$A + B + 3 \le \frac{9(1 - v)}{(1 - 2v)(1 + v)} < \frac{135}{7}.$$

Up to inversion $4 \le A \le B$ and $A + B \le 16$. The polynomial $X^3 - AX^2 + BX - 1$ is irreducible with positive square discriminant. In this range the only triples $(A, B, \sqrt{\Delta})$ are $(5, 6, 7)$ and $(6, 9, 9)$, by substitution in $\Delta = A^2B^2 - 4(A^3 + B^3) + 18AB - 27$. But $c \mid \sqrt{\Delta}$. Thus only the conductors seven and nine remain.

The gap at seven and nine. There are sharper gaps in $K_p = \mathbb{Q}(\zeta_p)^+$ for $p = 7, 9$. Scale $1/f$ by a positive rational to clear the common ideal valuations off the ramified prime and reduce its ramified exponent modulo three. The resulting integer has norm $C = 1, \ell, \ell^2$, with $\ell = 7, 3$, respectively. Write its polynomial as $X^3 - AX^2 + BX - C$. Then $\nu(f) = 1 - 3B/A^2$, $C \mid A^3$ and $C^2 \mid B^3$. If $\nu(f) < 1/4$, then $m = 4B - A^2$ is positive and is 0 or 3 modulo four. Schur's inequality for the three distinct positive roots gives $Am < 9C$. For $C = 1, 9, 49$ the divisibilities give $m \ge 3, 27, 147$, respectively, impossible since $A > 3C^{1/3}$. At $C = 3$ we have $3 \mid A, B$ and $m \ge 3$, so $A = 6$; now $12 \mid m$, again impossible. At $C = 7$ one has $7 \mid A, B, m$, so $A = 7$, $B = 14$.

Thus $\nu(f) \ge 1/4$ except for $f = n\sigma(F)$ at seven, where

$$F^3 - 14F^2 + 49F - 49 = 0, \qquad \nu(F) = 1/7, \qquad \mathrm{Tr}(1/F) = 1.$$

The rational scalar n is integral since $N(F) = 49$.

Conductor nine. Both codegrees cannot be irrational, by $(t^2 + w^2)/4 \ge 1/8$. Thus $9 \mid E$ and $(1 - 3/E)^2/4 \le 1/9$ give $E = 9$, $w = 2/3$, $\nu(H) = 1/4$. Writing $C = \mathrm{N}(H)$ gives $C \mid 9^3$ by integrality of $9/H$, and its polynomial is $X^3 - (C/9)X^2 + (2C/3)X - C$. The positive discriminant forces $C > 162$, hence $C = 3^5$ or 3^6. Since the order discriminant 9^3C differs from 81^2 by a square factor, $C = 3^6$. The ordinary estimate $s^2/4 \le 1/9$ gives $\mathrm{Tr}(\alpha) = 9s \le 6$. Its norm divides 9^3 and is at most eight. The trace ideal of K_9 is $3\mathbb{Z}$ by its integral power basis in $\zeta_9 + \zeta_9^{-1}$. Thus the trace of the irrational α is six, and its middle coefficient $B' = (36 - \mathrm{Tr}(\alpha^2))/2$ is divisible by three. The arithmetic–geometric mean inequalities give $3 < B' < 12$. At $B' = 6$ the discriminants for norms $1, 3$ are $189, -459$, respectively, neither a positive square. Thus α has polynomial $X^3 - 6X^2 + 9X - C'$ with $C' = 1, 3$.

Here $s = 2/3$, so a second irrational ordinary orbit would require $1/9 \ge ((2/3)^2 + (1/3)^2)/4$. The three rational ordinary complementary codegrees divide nine and their reciprocal sum is one third, so all equal nine. If $C' = 3$, the ordinary order discriminant $\mathrm{N}(D)9^3 = 3^{11}$ is nonsquare. Every simple is self-dual because $T_{\mathbb{Q}} = K^2$ is totally real. Hence the ordinary complement consists of totally real abelian field factors of total degree three. No real quadratic field ramifies only at three and each abelian cubic discriminant is a square, a contradiction. If $C' = 1$, $D = 9/\alpha$ satisfies the same cubic polynomial as H. There is therefore a real signed character of codegree D. Lemma 2.9 gives a spherical section.

Conductor seven. If E is rational, $7 \mid E$ gives $w \ge 4/7$. The exceptional orbit has mass $1/n$ and cannot give this $w \in [4/7, 1)$. Thus $w^2/4 \le 1/7$, forcing $E = 7$. Then $7/H$ is a totally positive

integer of trace four. In (3.1) its coefficients must be $2, 1, 1$ in some order. Hence it is a conjugate of $1 - b_1$, which is negative since $2\cos(2\pi/7) > 1$.

Suppose one orbit has the exceptional codegree $n\sigma(F)$. Its mass is $1/n$. If the other is exceptional too, their masses sum to one only at $n = 2$. Otherwise $1/(7n^2) + (1 - 1/n)^2/4 \le 1/7$, so $n = 2, 3$. For $\theta^3 - \theta^2 - 2\theta + 1 = 0$ take $F = 1 + \theta + 2\theta^2$. In the integral basis $(1, \theta, \theta^2 - 1)$ the trace form weighted by $1/F$ has Gram matrix I (trace calculation). The coordinate matrix Y of the six character values satisfies $YY^{\mathsf{t}} = nI$, with first column $(1, 0, 0)^{\mathsf{t}}$. For $n = 3$ choose column signs and order so the first row is $(1, 1, 1, 0, 0, 0)$. Each other row has the form $(0, a, -a, w)$ with $2a^2 + \|w\|^2 = 3$. If both a vanish their dot product is odd; if just one vanishes it is ± 1; otherwise it is ± 2 plus 0 or ± 1. Thus they cannot be orthogonal. For $n = 2$ the first row, twice the orbit idempotent, is $1 \pm b_j$. Hence $(1, -1) \in T$, and Lemma 3.3 applies.

It remains to handle $\nu(E), \nu(H) \ge 1/4$. Order $q_0 \le q_1 \le q_2$ by maximizing E under transport. Since $P(q) \ge (q_2 - q_1)^2$,

$$1 - 3q_1 \le (1 - t) + (q_2 - q_1) \le \sqrt{5((1 - t)^2/4 + P(q))} \le \sqrt{5/7}.$$

Thus $q_2 \ge q_1 > 1/20$. Using $\alpha_0 \le 1$ and $\sum q_i\alpha_i = s < 1$ gives $\mathrm{Tr}(\alpha) < 21$. In the power basis used by the ratio enumeration the corresponding trace bound gives coordinate bounds $20, 10, 10$ by the inverse trace Gram diagonal $(1, 2/7, 2/7)$. Positivity, $\alpha_0 \le 1$ and the d-number divisibilities leave fifteen irrational ratio polynomials.

For each polynomial order its roots $a_0 < a_1 < a_2$ and put $p_i = a_iq_i$, where q_0 is least. They must satisfy

$$P(q) + (1 - t)^2/4 \le 1/7, \quad P(p) \le 1/7, \quad P(q) \ge t^2/4, \quad \sum p_i < 1.$$

The quadratic-form certificates exclude thirteen polynomials and give the bounds in the second column below. They use nonnegative combinations of the four inequalities and $q_i \ge q_0$. For $X^3 - 7X^2 + 14X - 7$ first pair the larger of q_1, q_2 with a_1, which minimizes $P(p)$, and split at $q_2/q_0 = 4, 9$.

polynomial of α	q_0	candidates before $P(p) \le 1/7$
$X^3 - 5X^2 + 6X - 1$	$> 1/2000$	7
$X^3 - 6X^2 + 5X - 1$	$> 1/44$	0

In either case $\mathrm{Tr}(E) < 6000$. Put $\delta = 1 - \theta + \theta^2$, of norm seven. The unit signatures of $-1, \theta, \sigma(\theta)$ span all signatures, so Dirichlet's theorem makes every totally positive unit a square. Removing rational ideal valuations therefore writes $E = n\delta^j u^2$ with $n \ge 1$ integral, $0 \le j < 3$, and u a unit. The ellipsoid enumeration tests $n\,\mathrm{Tr}(\delta^j u^2) < 6000$, $49 \mid N(E)$, the bounds on q, and the minimum conditions on p_0, q_0. It gives the last column of the table. The smallest $P(p)$ among the seven remaining values is $9/49 > 1/7$. □

3.2. **The quintic field and its rational complement.**

Proposition 3.5. *If the ordinary and signed dimension character fields both have degree five and equal $K = \mathbb{Q}(D)$, then a Galois conjugate of the category is pseudo-unitary.*

Proof. The two rational algebras are $K \times \mathbb{Q}$. Write f, g for their ordinary and signed rational codegrees, respectively, so that $s = 1 - 1/f$ and $t = 1 - 1/g$. The rational signed character evaluated at the ordinary Frobenius–Perron induction column gives a positive integer r. Its unit coefficient yields $1 = s + r/g$, hence $g = rf$.

Let A be the induction label of the rational signed character. On the signed dimension labels its row is constant $1/(2g)$. The unit coefficient of $F(A)$ determines its remaining signed entry $S_{A,A} = (g-5)/(2g)$. The six entries and their J-translates have squared norm $[5 + (g-5)^2]/(2g^2)$, greater than one at $g = 2$. Thus $g \ge 3$ by unitarity and $t > 0$. A nonidentity invertible would exchange $1, h$ and fix the other four simples, since the kernel of the dimension map

has rank one. Its positive lift has the same permutation action, producing a signed character $(1,-1,0,0,0,0)$ of codegree two. Hence all nonunit simples have dimension greater than one, and Perron orthogonality excludes $f=2$.

If $g=3$ then $f=g=3$ and $r=1$. Now $\mathrm{FPdim}(FA)=D/f$ by symmetry, so its ordinary row likewise has entries $1/(2f)$ five times and $(f-5)/(2f)$ once. Each sector with its translates accounts for half the squared norm. In $(STS)_{A,A}$ the signed translates cancel by their opposite twists and the ordinary ones contribute $1/2$. But A has twist one, so $(STS)_{A,A}=S_{A,A}=-1/3$. Thus $g\geq 4$.

The field discriminant is c^4 by the conductor–discriminant formula. Write B_T, B_R for the projections of the two orders onto K. Lemma 2.6 gives $T\cap\mathbb{Q}=g\mathbb{Z}$, hence $[B_T\times\mathbb{Z}:T]=g$; the ordinary index is similarly f. Its discriminant identity gives

$$N(E)=gc^4[\mathcal{O}_K:B_T]^2, \qquad N(D)=fc^4[\mathcal{O}_K:B_R]^2.$$

In particular $rN(\alpha)$ is a square.

Assume no conjugate is pseudo-unitary and maximize E by transport. Then q_0,α_0 are their respective minima and $0<\alpha_0<1$. If E were rational, $s\leq t$ would give $\mathrm{Tr}(\alpha)\leq 5$, hence $\alpha=1$. Also α cannot be rational, so it has degree five. Write its polynomial $M(X)=X^5-AX^4+BX^3-CX^2+HX-W$, where $A>5$. The integral columns of Corollary 2.3 are $(\sigma(\alpha),r)$. Taking elementary symmetric polynomials and subtracting the corresponding scalar integers gives

$$M(X)\equiv(X-r)^5 \pmod g.$$

Taking unit coefficients gives $\mathrm{Tr}(q\sigma(\alpha))=s+n_\sigma$ for nonnegative integers n_σ, with $n_1=0$ and $\sum n_\sigma=N=At-5s$. Fourier convolution, Parseval, and $|\widehat{q}(\eta)|^2\leq 1/c$ yield

$$N^2/4\leq 5\sum n_\sigma^2-N^2\leq(5\,\mathrm{Tr}(\alpha^2)-A^2)/c, \qquad cN^2\leq 16A^2-40B.$$

Let $i\neq 0$. Fourier inversion and $\sum_{k=1}^4|1+\zeta_5^k|=2\sqrt{5}$ give

$$q_0\geq(t-4/\sqrt{c})/5, \qquad q_0+q_i\geq(2/5)(t-\sqrt{5/c}).$$

If these two lower bounds are at least $m,a>0$ respectively and $m\leq a/2$, then $\prod_i q_i\geq\min\{m(a-m)^4,(a/2)^5\}$. Indeed the minimum of $x(a-x)^4$ on $[m,a/2]$ occurs at an endpoint. For $c\geq 31$, one may take $m=1/220$, $a=2/15$, and both products exceed $1/(4\cdot 31^4)$, contradicting the norm formula. The only smaller conductors are $11,25$. At $c=25$, $g\geq 6$ gives $m=1/150$, $a=23/150$, again contradicting $\prod q_i\leq 1/(6\cdot 25^4)$. Thus at 25 one has $f=g=4$ or 5. At either conductor $q_i\geq(t-\sqrt{5/c})/5$ for every $i\neq 0$, so

$$A<1+5s/(t-\sqrt{5/c}).$$

Using $\sqrt{5/11}<27/40$ and $\sqrt{1/5}<9/20$ bounds A by 50 at eleven and by $13,12$ at twenty-five for $g=4,5$.

At eleven (3.1) gives $1\leq c_j<4A/11$, $\sum c_j=A\leq 50$. Retain totally positive d-numbers with least conjugate below one. For each polynomial test $r\mid A$, rW square, $g\mid M(X)-(X-r)^5$ coefficientwise, and $g/r=f\in\mathbb{Z}_{\geq 3}$. The exact enumeration leaves two profiles, both violating the moment inequality.

A	B	W	f	g	N	$11N^2$	$16A^2-40B$
16	76	1	11	11	10	1100	1056
18	112	32	4	8	12	1584	704

At twenty-five the prime five is totally ramified, hence $5\mid A$. The trace bounds and $A\equiv 5 \pmod g$ give $f=g=5$ and $A=10$. The congruence $M(X)\equiv(X-1)^5 \pmod 5$ makes α congruent to one at the ramified prime. We show directly that α cannot be totally positive.

For each of the five cosets C_i of $\langle 7\rangle$ in $(\mathbb{Z}/25)^\times$ put $b_i = \sum_{j\in C_i} \zeta_{25}^j$. These periods sum to zero, and their trace Gram matrix is $25I - 5J$. Indeed each coset reduces bijectively to $(\mathbb{Z}/5)^\times$. Among the sixteen sums of exponents from two cosets, four vanish modulo five; these vanish modulo twenty-five exactly when the cosets coincide. The cyclotomic traces give the stated matrix. Consequently one and any four periods have Gram determinant $5^8 = \operatorname{disc}(K)$ and form an integral basis. Write $\alpha = 2 + \sum c_i b_i$. Since $b_i \equiv 4$ at the ramified prime, $\sum c_i \equiv 1 \pmod 5$. Thus

$$V = \sum_{i<j} (c_i - c_j)^2 = 5\sum c_i^2 - (\sum c_i)^2 \quad \text{is even and } 4 \pmod 5.$$

The Gram matrix gives $\operatorname{Tr}(\alpha^2) = 20+5V$. The ten pair products have geometric mean $N(\alpha)^{2/5} \geq 1$, so $20 + 5V \leq 10^2 - 20$. Hence $V = 4$. The five coordinates are equal except for a single entry differing by one, and their sum modulo five fixes the sign. Thus $\alpha = 2 + b_i$. But these elements have a conjugate $2 + 2\cos(18\pi/25) + 2\cos(24\pi/25) < 0$.

□

Proof of Theorem 3.1. Rational D gives pseudo-unitarity by [7, Proposition 8.24]. In a nontrivial universal grading Lemma 2.10 gives rank at least $[K : \mathbb{Q}]$ in each component. Thus K is cubic and there are two rank-three components. The neutral component has dimension $D/2$ and trivial universal grading by the same argument. Both its rational character algebras are K by Lemma 2.8, so Lemma 3.3 gives a pseudo-unitary conjugate. This property passes to the whole extension by [1, Proposition 2.8].

For trivial grading and cubic K, an irrational complementary codegree generates K, and its row then accounts for the entire dimension-three complement of the signed algebra. Thus Lemma 3.3 and Proposition 3.4 exhaust the possibilities. The quintic case is Proposition 3.5. □

4. Sphericality in full degree

Theorem 4.1. *Let $\mathcal{C}$ be a complex fusion category of rank six. If $[\mathbb{Q}(\operatorname{FPdim}\mathcal{C}) : \mathbb{Q}] = 6$, then a Galois conjugate of $\mathcal{C}$ is pseudo-unitary. In particular, $\mathcal{C}$ admits a spherical structure.*

Lemma 4.2 (A quadratic norm obstruction)**.** *Let a real quadratic field $F = \mathbb{Q}(\sqrt{d})$, with d squarefree, act by self-adjoint operators on a rational quadratic space V with an orthonormal rational basis. If its dimension over F is odd, then $-1 \in N_{F/\mathbb{Q}}(F^\times)$. In particular no prime* $3 \pmod 4$ *divides d.*

Proof. Self-adjointness writes the pairing as $\operatorname{Tr}_{F/\mathbb{Q}} B$ for a nondegenerate symmetric F-bilinear form: define $B(u, v)$ by $\operatorname{Tr}(aB(u, v)) = \langle au, v\rangle$ for $a \in F$. Diagonalizing B gives rational discriminant $d^{\dim_F V} N_{F/\mathbb{Q}}(\det B)$ modulo squares. This discriminant is one, so d is a norm; rational squares are norms. Dividing by $N(\sqrt{d}) = -d$ gives a norm of -1. Then d is a sum of two rational squares, so its squarefree part has no prime divisor $3 \pmod 4$. □

4.1. **The two reciprocal dimension vectors.** Put $D = \operatorname{FPdim}\mathcal{C}$, $K = \mathbb{Q}(D)$, and transport the category so that its intrinsic dimension E is largest among its conjugates. The ordinary rational algebra is K, since its dimension is six and its FP image contains D. The signed dimension field contains E and the induction dimension E/D, so the signed rational algebra is also K. Every ordinary simple is self-dual by injectivity of the FP map. Cyclotomicity of the codegrees and total reality show that $G = \operatorname{Gal}(K/\mathbb{Q}) = \langle g\rangle$ is cyclic of order six. Write $p_i = g^i(1/D)$, $q_i = g^i(1/E)$, $x_i = p_i/q_i$, and $x = x_0$. Both p, q are positive probability vectors, and p_0, q_0 are their respective minima. Also $x_0 \leq 1$ and $x_0 \leq x_i$ by (2.2).

For the two orders $R, T \subset K$, Lemma 2.6 gives

$$N(D) = \Delta[\mathcal{O}_K : R]^2, \qquad N(E) = \Delta[\mathcal{O}_K : T]^2, \qquad \Delta = \operatorname{disc}(K).$$

Thus the totally positive integer $x = \dim A_{\mathrm{FP}}$ has square norm and invariant ideal, being a quotient of d-numbers. Lemma 2.4 gives positive integers

$$h_j = \sum_i q_i x_{i+j}, \qquad h_0 = 1, \qquad \sum_j h_j = \mathrm{Tr}(x). \tag{S1}$$

It suffices to show that p and q are conjugate, since their least entries are $1/D$ and $1/E$. First we eliminate every proper subfield with the rearrangement used in the cubic case.

Lemma 4.3. *If x belongs to a proper subfield of K, then $E = D$.*

Proof. A rational x is a positive integer at most one. Otherwise let $L = \mathbb{Q}(x)$ and suppose $E < D$. The factorization in Theorem 2.12 writes $E = \delta z$ with $z \in L$, $F = \mathbb{Q}(\delta) \subseteq K_{\mathcal{A}} \subseteq K$, $K = FL$, and $\mathrm{Tr}_{K/L}(1/\delta) = 1$. Since $[F : \mathbb{Q}] \leq 3$ and the group fixing L acts faithfully on F, the degrees of F, L are two and three in some order. Hence $\mathrm{Gal}(K/F)$ induces all embeddings of L. As $1/E$ is least at the distinguished embedding and δ is fixed by this group, $u = 1/z$ is least there too.

Taking the relative trace in (S1) gives $\mathrm{Tr}_L(u) = 1$ and positive integers $\mathrm{Tr}_L(u\rho(x))$ for all $\rho \in \mathrm{Gal}(L/\mathbb{Q})$, equal to one at the identity. If u is rational then $\mathrm{Tr}_L(x) = [L : \mathbb{Q}]$, forcing $x = 1$ by the norm. Otherwise u and x have strict minima at the identity, as $[L : \mathbb{Q}]$ is prime. Lemma 3.2 gives a positive integer smaller than one. □

4.2. **A finite list of conductors.** Let a, b, c be the conductors of the quadratic, cubic and faithful characters of G, respectively. Then $a \geq 5$ is a fundamental discriminant. The primitive cubic conductor $b \geq 7$ is a product of distinct primes $1 \pmod 3$, possibly multiplied by 9. These conditions follow by taking the order-three quotients of the unit groups modulo prime powers. Characters of coprime orders do not cancel on a local unit subgroup, so $c = \mathrm{lcm}(a, b)$. The conductor–discriminant formula gives $\Delta = ab^2c^2$. For the transforms $P_k = \sum_i p_i e^{2\pi i k i/6}$ and $Q_k = \sum_i q_i e^{2\pi i k i/6}$, the conductors are $f_1 = c$, $f_2 = b$, $f_3 = a$. We use the following inequalities from Lemma 2.15, with $f = f_k$:

$$f|P_k|^2 \leq (f, 2), \quad f|Q_k|^2 \leq (f, 4), \quad f(2|P_k|^2 + |Q_k|^2) \leq 4, \quad \sqrt{f}\,|Q_k| + f|P_k|^2 \leq 2. \tag{S2}$$

Write $d = (a, 2)$. Fourier inversion bounds each p_i below by $m = (1 - \sqrt{d/a} - 2/\sqrt{b} - 2\sqrt{d/c})/6$. If $m > 0$, the product of six such coordinates of sum one is at least $m^5(1 - 5m)$, by successively moving all excess over m into one coordinate. Consequently $\Delta m^5(1 - 5m) \leq 1$. For $a \geq 578$ we have $a/d, c/d \geq 289$, $2/\sqrt{7} < 19/25$ and $m > 9/850$; substitution of $\Delta \geq 49 \cdot 578^3$ contradicts the inequality. For $b \geq 121$, $d/a \leq 1/4$, $c/d \geq b$ give $m \geq 1/44$, and the same contradiction follows from $\Delta \geq 5 \cdot 121^4$. The quadratic subfield acts by self-adjoint multiplication on the ordinary algebra K, of dimension three over that subfield. Lemma 4.2 therefore excludes a with a prime factor $3 \pmod 4$.

The 3009 candidate pairs in the resulting rectangular bounds give 94 after the product test and 41 after the norm test:

a	b
$5, 8$	$7, 9, 13, 19, 31, 37, 43$
$13, 17$	$7, 9, 13$
$29, 37, 40, 41$	$7, 9$
$65, 104$	$7, 13$
$53, 61, 73, 85, 89, 97, 101, 109, 136$	7

Each listed b has a unique cubic field K_b. The conductor calculation uses rational upper bounds for $1/\sqrt{n}$.

4.3. **A height bound for the dimension ratio.** Set $r_f = \sqrt{(f,4)/f}$. Fourier inversion, applied to $q_0 + q_i$ and using $2q_i \geq q_0 + q_i$, gives the respective lower bounds

$$6q_{\pm 1} \geq 1 - r_b - \sqrt{3}r_c, \qquad 6q_{\pm 2} \geq 1 - r_a - r_b - r_c, \qquad 6q_3 \geq 1 - 2r_b.$$

For even a we also use $24q_{\pm 2} \geq 4 - 2\sqrt{6/a} - 6/\sqrt{b} - 4\sqrt{3/c}$. Indeed $2p_0 + q_0 + q_{\pm 2} \leq 4q_{\pm 2}$; in its Fourier expansion, use the mixed bound at $k = 1, 3$, and the separate bounds at $k = 2$. If the resulting common lower bound is $\beta > 0$, then $\mathrm{Tr}(x) \leq 1 + (1 - p_0)/\beta < 1 + 1/\beta$. This bounds the trace in 38 fields. For $(8, 13)$ the Fourier certificates give $10^4(q_1, q_2, q_3) > (638, 63, 742)$ coordinatewise. They take nonnegative rational combinations of the minimum conditions and supporting half-spaces of (S2), using $2\,\mathrm{Re}(\bar{z}\sqrt{f}P_k + \bar{u}\sqrt{f}Q_k) \leq 2 + |z|^2$ for $|u| = 1$. Reflection covers q_4, q_5, so $\mathrm{Tr}(x) \leq 160$. Only $(8, 7)$ and $(8, 9)$ still require a height bound.

Lemma 4.4. *For all $i, j \in \mathbb{Z}/6$ one has*

$$p_0 q_{i+j} \leq p_i q_j, \tag{4.1}$$

$$\sum_l \frac{p_{l+i}^2}{p_l} \geq \sum_l \frac{q_{l+i}^2}{q_l}, \tag{4.2}$$

$$k_j = \sum_l \frac{p_l p_{l-j}}{q_l} \in \mathbb{Z}_{>0}. \tag{4.3}$$

Proof. Apply Lemma 2.2 to W_{g^i}, whose ordinary Perron value is p_i/p_0 and whose signed conjugate values are q_{i+j}/q_j. Perron domination gives (4.1); the squared-norm comparison in the two trace forms gives (4.2). For (4.3) apply Lemma 2.4 to $x\, g^{-j}(x)\, g^{-j}(q)/q$ and use $p = xq$. □

Proposition 4.5. *If $(a, b) = (8, 7)$ or $(8, 9)$, then $\mathrm{Tr}_{K/\mathbb{Q}}(E/D) \leq 200$.*

Proof. We show that $q_j > 1/200$ for $j = 2, 4$. It suffices to take $j = 2$ by replacing g by g^{-1}. Suppose $q_2 \leq 1/200$; then $p_0, q_0, q_2 \leq 1/200$. For either $v = p, q$, Fourier inversion and the separate bounds in Lemma 2.15 give

$$v_0 + v_{\pm 1} \geq (1 - 1/\sqrt{b} - \sqrt{3/(2b)})/3, \qquad v_0 + v_3 \geq (1 - 2/\sqrt{b})/3,$$
$$v_0 + v_2 + v_4 \geq (1 - 1/\sqrt{2})/2.$$

For instance the first multiplier vanishes at the quadratic frequency and has magnitude $1, \sqrt{3}$ at the cubic and sextic frequencies after pairing conjugates. As v_0 is minimal, the odd coordinates exceed $1/45$; also $q_4 > 1/50$.

For the remaining ordinary coordinates, put $d = 1/\sqrt{8b} + 1/\sqrt{b} + 1/\sqrt{8} < 87/100$. The absolute multipliers in $3(v_0 + v_{\pm 2}) - 1$ are all one at the three paired frequencies (with the quadratic frequency counted once). Thus $\sum_{k=1}^3 |Q_k| \geq 1 - 3(q_0 + q_2) \geq 97/100$. Cauchy's inequality and the mixed bound yield

$$\sum_{k=1}^3 |P_k| \leq \sqrt{d(2d - \sum_{k=1}^3 |Q_k|)} < 83/100.$$

Applying inversion again to $p_0 + p_{\pm 2}$ proves $p_2, p_4 > 1/50$.

Set $\lambda = p_0/q_0$, $\mu = p_0/q_2$. Then $0 < \mu \leq \lambda \leq 1$. The integer in (4.3) is

$$k_2 = \lambda p_4 + \mu p_2 + \sum_{i \in \{1,3,4,5\}} p_i p_{i-2}/q_i. \tag{4.4}$$

For $1 \le i \le 5$, the column and squared-norm inequalities imply

$$\lambda q_i \le p_i, \qquad \mu q_{i+2} \le p_i,$$

$$\lambda q_i^2 + \mu q_{i+2}^2 \le p_i^2 + p_0 \sum_{l=1}^{5} p_{l+i}^2/p_l. \tag{4.5}$$

These inequalities contradict $k_2 \ge 1$ by the exact interval calculation. Its domain is the product of the two probability simplices with $p_0, q_0, q_2 \in [0, 1/200]$ and all remaining coordinates in $[1/50, 1]$. Use the weaker conductor triple $(56, 7, 8)$ for both fields. On each dyadic box, (S2) supplies twelve supporting half-spaces per Fourier disk; (4.5) bounds the first two terms of (4.4) and outward interval division bounds the other four. The subdivision visits 8 312 559 boxes. Every terminal box violates a constraint or gives $k_2 < 1$.

Thus $q_{\pm 2} > 1/200$. The pair bounds also give $2q_i \ge q_0 + q_i > 2/45$ at each odd i, independently of q_2. Since $x_0 \le 1$, $\mathrm{Tr}(x) < 1 + 200(1 - p_0) < 201$. The trace is integral, as claimed. □

4.4. **The arithmetic conclusion.** It remains to test the ratios x that generate K. A relative discriminant lower bound first eliminates ten conductor pairs. Apply Lemma 2.5 to K/K_b. Its relative discriminant has norm $\delta = ac^2/b^2$ by the tower formula, so $\mathrm{Tr}(x) \ge 3\sqrt{\delta^{1/3} + 4}$.

The upper bounds of 18 at $(37, 9), (41, 9)$, and 25 at $(a, 7)$ for $a = 73, 85, 89, 97, 101, 104, 109, 136$, contradict this inequality.

Listing the ratios. We treat $(13, 13)$ separately below. For the other thirty fields we list the possible x up to conjugacy. In an integral basis e_i let $H_{ij} = \mathrm{Tr}(e_i e_j)$ and $t_i = \mathrm{Tr}(e_i)$. At each allowed trace A the coordinate vector v of x satisfies

$$t^{\mathsf{T}} v = A, \qquad v^{\mathsf{T}} H v \le A^2.$$

Positivity gives the second inequality. Rational square completion enumerates this finite ellipsoid. Retain the x that generate K, have positive conjugates with least conjugate at most one, have square norm, and satisfy the d-number criterion for their minimal polynomial. For $X^d + a_1 X^{d-1} + \cdots + a_d$, the last test is $a_d^j \mid a_j^d$ for every j [19].

Outside the five fields below the trace-lattice enumeration uses a basis of Gram determinant ab^2c^2, hence of the maximal order. At coprime conductors it is the tensor product basis. For $a = 65, 104$ with $b = 13$ take invariants of the diagonal involution on $\mathbb{Q}(\zeta_{13})^+\mathbb{Q}(\sqrt{m})$ with $m = 5, 2$ respectively. Of 4 903 735 vectors, 35 remain, distributed as $27, 7, 1$ at $(5, 13), (13, 7), (29, 7)$.

In the five larger searches write $x = n\pi^j u^2$, with $n \ge 1$ integral, $0 \le j < 3$ and u a unit. The d-number property gives equal valuations over each prime, and the square norm makes the valuation even at the quadratic ramified prime. After removing rational factors it suffices to remove the remaining valuation at the cubic ramified prime using a totally positive generator π of that prime in K_b. Full unit signature rank makes the residual unit a square. The unit enumeration checks the generators and signatures, lists units in the ellipsoid $\mathrm{Tr}(\pi^j u^2) \le A_{\max}$, and tests their multiples.

(a, b)	$A_{\max}$	lattice vectors	primitive ratios
$(5, 7)$	1037	12 273 947	8007
$(5, 9)$	86	5207	128
$(8, 7)$	200	43 959	284
$(8, 9)$	200	33 357	206
$(8, 13)$	160	5973	46

Testing the ratios. Place the least embedding of x at zero. For the 35 ratios above and the 128 at $(5, 9)$, the trace-system check lists the positive integral compositions h of A with $h_0 = 1$ from (S1). In an integral field basis these are rational linear equations on q. The three quadratic bounds in (S2) have rational matrices, since $\cos(2\pi kj/6) \in \mathbb{Q}$; substituting $p = xq$ and completing

squares on the affine solution space excludes every composition. Parseval first bounds $\sum h_i^2$ from $|\widehat{h}_k|^2 = |Q_k|^2|\widehat{x}_k|^2$.

For the other 8543 ratios impose $p_i = x_i q_i$, the normalization and minimum conditions, $h_j \geq 1$ and rational supporting half-spaces of (S2). The complementary-field verifier and the verifier for (8, 7), (8, 9) verify a rational relaxation $A'q \leq d'$ and supplied nonnegative integer weights w with $\min_i(w^{\mathsf{T}}A')_i > w^{\mathsf{T}}d'$. This is impossible for a probability vector.

The real cyclotomic field of conductor thirteen. Finally at $(13, 13)$, Parseval bounds both second moments by $3/13$. The pair bounds give $q_{\pm 1} > 1/25$, $q_{\pm 2} > 1/36$, $q_3 > 1/14$. The six Fourier coefficients of $(0, 25, 36, 14, 36, 25)$ are $(136, -25, -47, 8, -47, -25)$. Hence $A < 1+(136+152/\sqrt{13})/6 < 31$. In the basis $v_j = 2\cos(2\pi j/13)$, (3.1) gives $x = -\sum d_j v_j$ with $1 \leq d_j \leq 9$ and $\sum d_j = A \leq 30$. The conductor-thirteen calculation leaves fifteen primitive ratios, eleven units and two each of norms $64, 169$. For every one of them $x\mathcal{O}_K \subseteq S = \mathbb{Z}[Gx]$. To check this containment, begin with the integer span of $1, Gx$ in the period basis and adjoin products by Gx twice. The resulting lattice contains all six columns of multiplication by x.

Since $\mathrm{Tr}(qS) \subseteq \mathbb{Z}$ by Lemma 2.4, $p = xq$ is in the ordinary trace dual of $\mathcal{O}_K$. Any such vector $z = -\sum_j n_j v_j/13$ with $\mathrm{Tr}(z) = 1$ has integral $n_j = 2 - \mathrm{Tr}(zv_j)$. If $\mathrm{Tr}(z^2) \leq 3/13$, the Gram matrix gives $\sum n_j = 13$ and $\sum n_j^2 \leq 29$. The only possibility is five twos and one three. Thus $p = (2 - v_1)/13$, because p is least at the distinguished embedding. For unit x we have $S = \mathcal{O}_K$ and the same argument gives $q = p$. Otherwise the calculated $\mathrm{Tr}(p/x)$ is $1/2$ at norm 64 and $7/13$ at norm 169, whereas q has trace one.

Conclusion of the proof of Theorem 4.1. Lemma 4.3 gives $E = D$ for a proper-subfield ratio. The conductor and trace arguments exclude every primitive ratio. The chosen conjugate is therefore pseudo-unitary. □

5. Conductor bounds in degree two

The first restriction comes from the action of the neutral component on the other homogeneous components. Once the universal grading is trivial, we can compare the two quadratic dimension characters directly. The conductor bounds at the end of the section leave a small list of discriminants.

5.1. **Nontrivial gradings in the quadratic case.**

Proposition 5.1. *Let $\mathcal{C}$ be a complex fusion category of rank six with a nontrivial faithful finite-group grading. If $\mathbb{Q}(\mathrm{FPdim}(\mathcal{C}))$ is quadratic, then a Galois conjugate of $\mathcal{C}$ is pseudo-unitary.*

Proof. Let $\mathcal{B}$ be the neutral component. Its Perron dimension $D/|G|$ generates the same quadratic field as D [7, Proposition 8.20]. By Lemma 2.10 every component has rank at least two, so $\mathcal{B}$ has rank at most four. It suffices to make $\mathcal{B}$ pseudo-unitary by transport, since this property passes to graded extensions [1, Proposition 2.8].

At ranks two and three $\mathcal{B}$ is spherical by the rank-two classification [18] or by Ostrik's quadratic rank-three theorem [20, Proposition 3.1 and Theorem 5.1]. In rank three the non-self-dual ring is pointed, and the two conjugates of $D/|G|$ give the required quadratic factor in the self-dual case. The ordinary algebra has a unique nonrational character orbit. It contains both dimension characters, since the spherical dimension field contains $D/|G|$ by central induction. Thus $\mathcal{B}$ has a pseudo-unitary conjugate.

At rank four consult the unconditional list [10, Corollary 13.1 and Table 1]. Among the rings with quadratic global Frobenius–Perron dimension, all have only one value of simple Frobenius–Perron dimension greater than one, apart from the Haagerup ring $\mathcal{H}_1$ and the square of the Fibonacci ring. In the one-value case Schopieray gives a pseudo-unitary conjugate of $\mathcal{B}$ [21, Lemma 3.1]. This leaves two rings.

In either case the two exterior simples carry a two-dimensional representation of $K_0(\mathcal{B})$ by nonnegative integral matrices. A self-dual object acts by a symmetric matrix. The Haagerup ring has a self-dual simple of dimension $(1+\sqrt{13})/2$ [9, Table 2]. Its matrix would have trace one and determinant -3, since its eigenvalue has minimal polynomial x^2-x-3. Nonnegative integral diagonal entries summing to one have product zero, forcing the off-diagonal entry to have square three.

For the Fibonacci square choose a spherical section of $\mathcal{B}$ by [10, Theorem 12.3]. Its two Fibonacci generators P, Q act by commuting matrices on each two-dimensional exterior module, ordinary or signed. Each satisfies the irreducible equation $M^2 = M + I$. The centralizer of the P-matrix is therefore the quadratic field it generates: the Q-matrix is P or $I-P$, writing P also for its matrix. In the ordinary module the common Perron eigenvector gives the same positive eigenvalue, hence $Q = P$. In the signed module $Q \equiv P \pmod 2$ now excludes $I-P$. Its nonzero dimension functional gives $\dim P = \dim Q = z$ with $z^2 = z+1$. Thus $\dim(\mathcal{B}) = (1+z^2)^2$ is conjugate to $\mathrm{FPdim}(\mathcal{B})$. □

5.2. **The two character algebras.** Suppose the signed dimension field M is quadratic. Write a prime for its nonidentity embedding and s, t for the masses in Corollary 2.3. At the transport maximizing E, $x = E/D$ and $1/E$ are both least at the distinguished embedding. Thus

$$2s = t\,\mathrm{Tr}(x) + (1/E' - 1/E)(x' - x) \geq t\,\mathrm{Tr}(x), \qquad s = \mathrm{Tr}(1/D), \quad t = \mathrm{Tr}(1/E). \tag{5.1}$$

If $E < D$, the totally positive algebraic integer x has trace at least three by its integral norm; at even discriminant the trace is even, hence at least four.

Proposition 5.2. *Suppose $K = \mathbb{Q}(D)$ has degree two and no Galois conjugate of $\mathcal{C}$ is pseudo-unitary. Its universal grading is trivial. Both quotient algebras are commutative, both dimension characters have full field K, and at least one signed codegree outside the spherical dimension orbit is irrational.*

Proof. Proposition 5.1 settles the graded case. If the universal grading is trivial, Lemma 2.8 identifies both dimension-character fields with K.

Maximize E by transport. The quadratic trace identity (5.1) gives $s \geq 3t/2 > t$ for the two dimension-orbit masses.

If every complementary signed codegree were rational, Corollary 2.3 would give $s \leq t$. Thus at least one is irrational. A noncommutative signed algebra in rank six would have complexification $\mathbb{C}^2 \times M_2(\mathbb{C})$, with its two scalar blocks in the dimension orbit. Its only other codegree would be rational by Galois invariance.

If instead $R_{\mathbb{C}} = \mathbb{C}^2 \times M_2(\mathbb{C})$, let h be its rational matrix-block codegree. The Casimir $\sum b_i b_i^*$ has eigenvalue $2h$ on the matrix block and Perron value D, so $D \geq 2h$. From $1/D + 1/D' + 2/h = 1$ we get $1/D' - 1/D \geq 1 - 3/h$. The square of this gap is at most $(c,2)/c \leq 1/4$ by (2.4), so $h \leq 6$.

Since $R/2R \simeq T/2T$ is commutative, the alternating tensor $C_{ijk} = \tau([b_i, b_j]b_k)$ has even entries, where τ extracts the coefficient of the unit. The sum of their squares is $12h$. Indeed, expanding $\sum_{i,j} \|[b_i, b_j]\|^2$ gives twice the difference of the Casimir square trace $\sum_V n_V^3 f_V$ and the trace of $x \mapsto \sum_i b_i x b_i^*$, equal to $\sum_V n_V f_V$ by matrix units. Only the two-dimensional block contributes.

The regular Perron element annihilates every commutator, so $\sum_k d_k C_{ijk} = 0$ for every pair i, j. Take a triple with $C_{ijk} \neq 0$. Each of its three pairs must belong to another nonzero triple, because every d_k is positive. These three further triples are distinct. Each unordered triple contributes at least $6 \cdot 2^2$ to the sum of squares. Thus $12h \geq 4 \cdot 24$, contradicting $h \leq 6$. □

Lemma 5.3 (Quadratic character rows)**.** *Let a, a' be conjugate linear character rows of either quotient with quadratic character field of discriminant Δ and codegrees f, f'. Write $a_X - a'_X = b_X\sqrt{\Delta}$. Then*

$$b_X \in \mathbb{Z}, \qquad f + f' = |\Delta| \sum_X b_X^2. \tag{5.2}$$

If neither row is the positive dimension row of its quotient, then $f + f' \geq 2|\Delta|$. If $\Delta < 0$, their common codegree is $|\Delta| \sum_{\{X,X^*\}} b_X^2$, summed over the unordered pairs with $X \neq X^*$.

Proof. The integral basis of the quadratic field gives $b_X \in \mathbb{Z}$. Schur orthogonality gives $\|a - a'\|^2 = f + f'$. When both rows are orthogonal to the positive dimension row, their difference has at least two nonzero entries. For $\Delta < 0$, complex conjugation acts by duality, so $b_{X^*} = -b_X$. □

Lemma 5.4. *Let f, f' be conjugate totally positive quadratic d-numbers and write $1/f + 1/f' = a/b$ in lowest terms. For some positive integer m,*

$$ff' = b^2 m, \qquad f + f' = abm, \qquad (1/f - 1/f')^2 = \frac{a^2 - 4/m}{b^2}.$$

If an odd prime p divides a, then $f/f' \equiv -1$ at every prime over p.

Proof. With $N = ff'$ and $L = f + f'$, the unit f/f' gives $L^2/N = 2 + f/f' + f'/f \in \mathbb{Z}$. Since $L/N = a/b$, we get $b^2 \mid N$. The three identities follow. For an odd prime $p \mid a$, the identity $1 + f/f' = fa/b$ gives $f/f' \equiv -1$ at each prime above p. Thus p cannot divide a if the least totally positive unit has odd residue order at one such prime. □

In particular every signed codegree belongs to K, since $E/f = \dim F(A_f)$ lies in K. In the four complementary dimensions an irrational codegree therefore has its full character field of degree two or four. A quartic character field containing K gives two copies of the same codegree pair. If an irrational orbit uses only two characters, its field is K and the remaining two codegrees are another such pair or two rational integers. The last two characters in the rational-codegree case can be rational or can be conjugate over a separate quadratic field. The spherical dimension codegree E may be rational even though its full character field is K.

5.3. **A conductor test for quadratic pairs.** For a codegree f in the real quadratic field of discriminant c, write $t_f = 1/f + 1/f'$ and $\delta_f = 1/f' - 1/f$. Let $U = (a + b\sqrt{c})/2 > 1$ be the least totally positive unit. For irrational f, the larger of f/f' and f'/f is a positive power of U, by the d-number property. Consequently

$$\delta_f^2 \geq \lambda t_f^2, \qquad \lambda = \left(\frac{U-1}{U+1}\right)^2 = \frac{a-2}{a+2}. \tag{5.3}$$

Suppose the signed dimension pair and k further irrational pairs have total reciprocal mass L. Put $q = (c, 4)/c$ and $y_+ = \max(y, 0)$. Counting each pair with its multiplicity, (2.4) and Cauchy's inequality give

$$\begin{cases} \lambda L^2/(k+1) \leq q, & E \notin \mathbb{Q}, \\ \lambda (L - 2(c,2)/c)_+^2/k \leq q, & E \in \mathbb{Q}. \end{cases} \tag{5.4}$$

In the second case orthogonality of the two dimension rows gives $2E = \sum_i (x_i - x_i')^2 \in c\mathbb{Z}$; hence their mass $2/E$ is at most $2(c, 2)/c$.

Proposition 5.5 (Three quadratic codegree pairs)**.** *Let the universal grading be trivial and $K = \mathbb{Q}(D)$ quadratic. If T is commutative with codegrees E, E', H, H', J, J', where H, J are irrational, then $c = 5$ or 8. The pairs may coincide.*

Proof. By Lemma 2.8 the dimension character field is K, and the complementary codegrees generate K. Thus each field factor of $T_{\mathbb{Q}}$ contains a copy of K. Their diagonal action is self-adjoint for $\tau(uv^*)$, since involution on characters acts by complex conjugation. As the signed basis is orthonormal and $\dim_K T_{\mathbb{Q}} = 3$, Lemma 4.2 excludes prime divisors $3 \pmod 4$ of c. An odd $c > 5$ is at least 13. An even $c > 8$ is at least 40: it has the form $8d$ for an odd squarefree $d > 1$ with all prime factors $1 \pmod 4$.

Here $k = 2$ and $L = 1$, so

$$t_E + t_H + t_J = 1, \qquad \delta_E^2 + \delta_H^2 + \delta_J^2 \leq (c, 4)/c. \tag{5.5}$$

For odd $c \geq 13$, $a^2 - cb^2 = 4$ gives $a \geq 5$, hence $\lambda \geq 3/7$. Both left sides of (5.4) exceed $1/13$ already at $c = 13$. For even $c \geq 40$, use $\lambda \geq 1/2$; the corresponding left sides exceed $4/40$ already at 40. They are nondecreasing with c at either parity. □

5.4. **Separation of quadratic codegrees.**

Lemma 5.6 (Separation of the quadratic pairs)**.** *Let K be a real quadratic field whose discriminant c appears in the table below. Suppose both dimension characters generate K, both character algebras are commutative, and outside the dimension pair E, E', the signed codegrees consist of an irrational pair H, H' and either a second copy of this pair or two rational integers. Assume there is no pseudo-unitary Galois conjugate. Transport to $E \geq E'$ and label $H > H'$. Write $s = \mathrm{Tr}(1/D)$, $t = \mathrm{Tr}(1/E)$, $w = \mathrm{Tr}(1/H)$. In the table $U = (A + b\sqrt{c})/2$ is the least totally positive unit greater than one, $V = (1 + U)/d$, and h generates the trace ideal $\mathrm{Tr}(\mathcal{O}_K/V)$.*

c	5	8	12	13	21	24	28	40	60
A	3	6	4	11	5	10	16	38	8
b	1	2	1	3	1	2	3	6	1
d	1	2	1	1	1	2	3	2	1
h	1	2	1	1	1	2	1	2	1

Then

$$\frac{s}{t} \geq \frac{A}{\max(2, d)}, \qquad D/D' \geq U^2,$$

with the sole possible exception $c = 28$, $E = E'$ and $D/D' = H/H' = U$. For $d = 1$, equality in the mass bound forces $E = E'$ and $D = EU$. For $c \neq 28$ we also have $H/H' \geq U^2$.

Proof. The units in the table follow by taking the least positive b for which $a^2 - cb^2 = 4$ has an integral solution with $(a + b\sqrt{c})/2 \in \mathcal{O}_K$. In the standard integral basis V has coprime coordinates, $V/V' = U$, and $V > 2$. Evaluating the trace on that basis gives the last line of the table. The quadratic trace calculations also verify its entries. Every totally positive integral d-number, oriented with its larger conjugate first, has the form mU^k or mVU^k, with $k \geq 0$ and m a positive integer. Indeed its quotient by its conjugate is a power of U. Dividing by the indicated element of the same quotient gives a rational scalar, integral by primitivity.

Apply this observation to $\alpha = E/D$. It is a totally positive algebraic integer by the induction formula, and its ratio to its conjugate is a unit by Lemma 2.1. By (2.2), $\alpha < 1 < \alpha'$. Putting $r = E/E' \geq 1$ gives the identity

$$\frac{s}{t} = \frac{\alpha + r\alpha'}{1 + r} \geq \tfrac{1}{2}\,\mathrm{Tr}(\alpha) > 1.$$

If $\alpha = mU^{-k}$ or $mV'U^{-k}$ with $k \geq 1$, its trace is at least A, with equality only for $\alpha = U^{-1}$. Indeed $VU^k > 2U > A$ in the second family. Equality in the displayed average requires $r = 1$. In either family $D/D' = r\alpha'/\alpha \geq U^2$.

There remains $\alpha = mV' < 1$, which requires $d > 1$. If $r > 1$, then $r \geq U$ and the weighted average gives

$$s/t \geq m(V' + UV)/(1 + U) = mA/d, \qquad D/D' = rU \geq U^2.$$

If $r = 1$, then $D/D' = U$. The simultaneous column of Lemma 2.1 has Perron dimension D/D' and an eigenvalue H/H'. Thus $U \leq H/H' \leq D/D'$, forcing $H/H' = U$.

We now exclude $H/H' = U$ for $c \neq 28$. On the ordinary Perron induction column the signed dimension pair contributes $s > t$, and every rational-codegree orbit contributes at least its reciprocal mass. Indeed its eigenvalues are totally positive algebraic integers by Proposition 5.2, hence their average is at least one. The eigenvalue on a complementary H-row is $\beta = H\,\mathrm{FPdim}(FA_H)/D \in \mathcal{O}_K$, even when its character field is quartic. Comparing unit coefficients with the reciprocal identity therefore gives $\mathrm{Tr}(\beta/H) < \mathrm{Tr}(1/H)$ on at least one

conjugate pair. If $H/H' = U$, then $H = nV$. For $c \neq 28$ the last line of the table gives $\mathrm{Tr}(\beta/V) \geq d = \mathrm{Tr}(1/V)$, a contradiction. □

5.5. **Repeated pairs at conductors five and eight.**

Proposition 5.7 (A repeated complementary pair over $\mathbb{Q}(\sqrt{5})$)**.** *Under the hypotheses of Proposition 5.5, suppose that $K = \mathbb{Q}(\sqrt{5})$ and $J = H$. Then $\mathcal{C}$ has a pseudo-unitary Galois conjugate. In particular this settles the quartic complementary character-field case over $\mathbb{Q}(\sqrt{5})$.*

Proof. Suppose otherwise, and pass to a Galois conjugate for which $E \geq E'$. Put

$$u = \frac{3+\sqrt{5}}{2}, \qquad v = \frac{5+\sqrt{5}}{2}, \qquad t = \mathrm{Tr}(1/E), \quad w = \mathrm{Tr}(1/H), \quad s = \mathrm{Tr}(1/D).$$

Then $t + 2w = 1$. Lemma 5.6 gives $H/H' \geq u^2$, so

$$\delta_H^2 \geq \frac{5}{9} w^2.$$

Because $\delta_E = t - 2/E \geq 0$, the parity bound and Cauchy's inequality now imply

$$\frac{1}{5} \geq \delta_E^2 + 2\delta_H^2 \geq \frac{5}{23}(\delta_E + 2w)^2 = \frac{5}{23}(1 - 2/E)^2.$$

It follows that $E \leq 5(5+\sqrt{23}) < 49$.

Orthogonality of the signed dimension row to its conjugate gives $\mathrm{Tr}(E) = \sum_i (x_i - x_i')^2 \in 5\mathbb{Z}$. Write $E = mP$, $H = nQ$ using the two primitive families of Lemma 5.6, and put $B_P = \mathrm{Tr}(1/P)$. Integrality of E/H gives $\ell = m/n \in \mathbb{Z}$; if P is a unit and Q is in the v family, also $5 \mid \ell$. The mass identity reads

$$m = B_P + 2\ell B_Q, \qquad \ell \mid B_P.$$

Here $B_Q \leq 22$ since $m \leq E < 49$ and $m \leq 45$ for rational $P = 1$ (the other P give $m < 19$). The gap bound gives $Q = u^j$ or vu^j with $j \geq 1$; the corresponding traces up to 22 are $3, 7, 18$ and $2, 5, 13$, by the recurrence $B_{j+2} = 3B_{j+1} - B_j$. For $P = u^k$ with $k \geq 2$, $m > B_P \geq 7$ contradicts $E < 49$; similarly $k \geq 2$ gives $m > B_P \geq 5$ for vu^k. Thus for each $P = 1, u, v, vu$, take the divisors ℓ of B_P and the listed Q. Keeping $E = mP < 49$, $5 \mid m\,\mathrm{Tr}(P)$ and $H = (m/\ell)Q$ integral gives

(E, H)	$(5v, 5vu)$	$(7v, 7u)$	$(11v, 11vu^2)$	$(30, 15u^2)$
$\delta_E^2 + 2\delta_H^2$	$33/125$	$51/245$	$243/605$	$2/5$

Every value exceeds $1/5$. □

Theorem 5.8 (The repeated complementary pair at conductor eight)**.** *Under the hypotheses of Proposition 5.5, suppose $K = \mathbb{Q}(\sqrt{2})$ and $J = H$. Then a Galois conjugate of $\mathcal{C}$ is pseudo-unitary.*

Proof. Assume the contrary. Write $u = 3 + 2\sqrt{2}$, $v = 2 + \sqrt{2}$. In the notation of Lemma 5.6, $t + 2w = 1$. Write $E = mP, H = nQ$, where P, Q are primitive of the form u^i or vu^i, $i \geq 0$. The traces $B_P = \mathrm{Tr}(1/P)$ are all $2 \pmod 4$: in each family they satisfy the recurrence with coefficient six and start with $(2, 6)$ or $(2, 10)$. Integrality of E/H implies $n \mid m$, so $m = B_P + 2(m/n)B_Q \equiv 2 \pmod 4$. Since $8 \mid \mathrm{Tr}(E)$ by row orthogonality, $E = mvu^k$. The same argument at D, using E/D integral, gives $D = Mvu^\ell$, with $M \mid m$, $M \equiv 2 \pmod 4$ and $\ell > k$. Consecutive terms of $(2, 10, 58, \ldots)$ increase by a factor at least five, so $s \geq 5t$.

Every field factor of $T_{\mathbb{Q}}$ contains K, since its codegree is irrational outside the dimension orbit. The discriminant tower formula therefore gives $v_2(\mathrm{disc}\,\mathcal{O}) \geq 3v_2(\mathrm{disc}\,K) = 9$ for the maximal order $\mathcal{O}$. On the other hand

$$[\mathcal{O} : T]^2 |\,\mathrm{disc}\,\mathcal{O}| = N(E)N(H)^2, \qquad v_2(N(E)) = 3, \qquad v_2(N(H)) \leq 3.$$

Here $N(Q) = 1$ or 2 and $n \mid m$. Equality forces T to be maximal at two and $H = nvu^j$. Lemma 5.6 gives $j \geq 1$.

Every rational ordinary codegree g is even: otherwise Lemma 2.6 splits off an unramified factor at two. But $R/2R \simeq T/2T$ has no reduced factor, since T is maximal at two and every field factor contains $\sqrt{2}$, nonzero and square-zero in its maximal order modulo two. As E/g is integral, primitivity gives $g \mid m$, hence $g \equiv 2 \pmod 4$.

If $g = 2$, the reciprocal identity forces the character to be rational. Its row has two nonzero entries $1, \pm 1$, giving a primitive idempotent $(1 \pm X)/2$ and hence $X^2 = 1$ for a nonunit simple X. Its positive lift is likewise an involution in T. Its coordinates in the maximal order are all ± 1, so its class is the unit modulo two by maximality. This contradicts the linear independence of the distinguished basis in $T/2T$.

The formulas for D and H give $D/D', H/H' \geq u^3$. Their squared reciprocal gaps are at least $49/50$ times their squared masses. Suppose first $s < 1/5$. Then $w = (1-t)/2 > 12/25$. For any irrational ordinary pair G, G' the mixed Fourier bound gives

$$\delta_G^2 \leq \tfrac{1}{4} - \delta_H^2 < \tfrac{1}{4} - \tfrac{49}{50}(12/25)^2 < \tfrac{1}{18}.$$

All ordinary codegrees lie in K, and every irrational pair has ratio at least u, hence squared gap at least half its squared mass. It therefore contributes less than $1/3$, counting each pair with its multiplicity. Each rational codegree contributes at most $1/6$. The four complementary characters together contribute at most $2/3$, contradicting $s < 1/5$.

Thus $s \geq 1/5$. Keeping just the D- and H-pairs in the mixed Fourier bound gives

$$\tfrac{1}{2} \geq 2\delta_D^2 + 2\delta_H^2 \geq \tfrac{49}{25}(s^2 + (1 - s/5)^2/4) \geq \tfrac{8281}{15625} > \tfrac{1}{2}.$$

The last quadratic is increasing for $s \geq 1/5$. □

5.6. **Two irrational pairs and two rational codegrees.**

Theorem 5.9 (No signed codegree two in the quadratic profile)**.** *Suppose the ordinary and signed character algebras of a rank-six fusion category are commutative, both dimension-character fields are the same quadratic field K, and the signed codegrees are*

$$E, E', H, H', g_1, g_2,$$

where the H, H' rows form a conjugate pair and the g_i-rows are rational. Then $g_1, g_2 \neq 2$.

Proof. For the induction label P of such a character ψ, set $a_B = 4S_{P,B} = \psi(FB) \in \mathbb{Z}$, $v_B = a_B \dim B$ and $p = \operatorname{Tr}(1/D)$. Orthogonality to JP gives squared norm eight in each monodromy sector. On a J-pair the entries are opposite; its two twists are opposite in the signed sector and equal in the ordinary one. Orthogonality to $1, J$ and the modular identity at $(P,P), (P,1)$ give

$$\sum_{B \text{ in either sector}} v_B = 0, \qquad \sum_B a_B^2 \theta_B = 4a_P, \qquad \sum_B v_B \theta_B = 2E.$$

The signed J-pairs cancel from both twist sums.

In each sector the six induction entries sum to two by the unit-coefficient formula. On the dimension pair the entries are equal positive integers by reciprocity: the two are conjugate, and lie in $\mathbb{Z}$ since P is rational. Their squared norm, including the translates, is at most eight, so both are one. The four remaining entries thus sum to zero with squared norm at most two. In the signed sector the entries on the conjugate pair agree, so they vanish, and the rational entries are $b, -b$ with $a_P = b$. In the ordinary sector a nonzero pair $1, -1$ would exhaust the norm at twist one, forcing $4b = 8$. Thus the ordinary dimension labels and their translates give four units at twist one, and $4b \geq 0$ by taking real parts. If $b = 0$, all remaining ordinary entries in the support have twist -1. The dimension labels and their translates have v-weight $2Ep$, whence the two identities for v give $4Ep = 2E$. But the Perron column has contributions p from the signed dimension pair, $1/2$ from P, and positive terms from the other rows. Thus $b = 1$, as in [11, proof of Theorem 6.2].

The remaining ordinary squared norm four consists of two sign pairs of entries ± 1. Their twists are $\theta, -\theta$ by the first modular identity. Their v-weights V_1, V_2 satisfy

$$V_1 + V_2 = -2Ep, \qquad \theta(V_1 - V_2) = 2E(1-p).$$

Since the weights are real, $\theta = \pm 1$. The twist--1 pair has weight $-E$; its positive-dimensional representative Q has $a_Q = -1$ and $\dim Q = E/2$.

Now $\delta_B = 4S_{Q,B}$ is an algebraic integer since $S_{Q,1} = 1/4$. The modular Galois permutations fix the rational row of P, preserve both monodromy sectors (apply Galois to $SP_J = \mathrm{diag}(-M_B)S$, where M_B is monodromy with J), and preserve twist -1 by the Galois twist rule. Hence they preserve the unique ordinary twist--1 pair in the support of P, namely Q, JQ. Its entries in the row of P force positive sign at Q and negative sign at JQ. Each Galois automorphism thus fixes the signed-sector entries of δ and either fixes or negates all its ordinary-sector entries.

The entry z of δ at the ordinary Perron label satisfies $Dz/2 = \mathrm{FPdim}(Q) > 0$, so $z \in K$. Its conjugate at the other Perron label is $\pm z$, and z^2 is a positive integer. These labels and their translates give $4z^2 \le 8$, the ordinary-sector squared norm. If $z^2 = 2$ they exhaust that sector at twist one; all its entries are real, being rational multiples of z. If $z = 1$, the whole row is integral. Neither Q nor JQ belongs to $I(\mathbf{1})$ because both have twist -1. The ordinary unit coefficient of FQ therefore vanishes, so the ordinary induction entries sum to zero. Outside the Perron pair their sum is -2 and their squared sum at most two. Exactly two are -1, and again the ordinary sector is exhausted at twist one. In either case $\delta_Q = 0$, while signed J-pairs cancel from the modular identity at (Q, Q). This gives $4\delta_Q = \sum_B \delta_B^2 \theta_B = 8$. □

Proposition 5.10 (Conductors with two rational complementary codegrees)**.** *Suppose a rank-six category with trivial universal grading has no pseudo-unitary Galois conjugate and has commutative signed codegrees*

$$E, E', H, H', g_1, g_2,$$

where H is irrational and g_1, g_2 are rational integers. Then its quadratic conductor belongs to

$$\{5, 8, 12, 13, 21, 24, 28, 40, 60\}.$$

Proof. Theorem 5.9 excludes a rational character of codegree two; a conjugate pair of that codegree would consume the full mass. Thus $g_1, g_2 \ge 3$ and $L = t_E + t_H = 1 - 1/g_1 - 1/g_2 \ge 1/3$. In (5.4) take $k = 1$. Since $a^2 - cb^2 = 4$, we have $c\lambda = (\sqrt{c + 4/b^2} - 2/b)^2 \ge (\sqrt{c+4} - 2)^2$. For $L \ge 1/3$ the test gives $c < 35$ at odd conductors and $c < 106$ at even ones. Indeed the irrational case bounds c by $18 + 12\sqrt{2}$ and $72 + 24\sqrt{2}$, respectively; the rational case gives the smaller bounds 30 and 79.

Testing (5.4) on the fundamental discriminants in these ranges with the exact least unit leaves

$$5, 8, 12, 13, 17, 21, 24, 28, 40, 44, 56, 60 \qquad \text{(least-unit sieve)}.$$

For $c = 17, 44, 56$ the prime three is inert. We claim $L \ge 1/2$. Only a codegree-three character needs consideration. If it belonged to a real quadratic orbit, squared row differences would give $3 \ge \mathrm{disc}(F) \ge 5$. In an imaginary orbit the sole dual pair gives $3 = |\mathrm{disc}(F)|b^2$, hence $F = \mathbb{Q}(\sqrt{-3})$. The signed discriminant of $K^2 \times F$ would then make three a rational norm from K, impossible at an inert prime.

The codegree-three character is therefore rational. Lemma 2.7 at three pairs it with the other rational character: the two copies of K are unramified quadratic fields locally and cannot be its totally ramified partner. Their rows ρ, η are consequently congruent modulo three. The row ρ has the unit entry one, two entries ± 1, and all other entries zero. On this support $\rho_i \eta_i \equiv 1 \pmod 3$ and the three products sum to zero. One is therefore at most -2, so $\|\eta\|^2 \ge 1 + 1 + 4 = 6$. The two rows have total reciprocal mass at most $1/3 + 1/6 = 1/2$.

Thus $L \ge 1/2$. At $17, 44, 56$ the least-unit gaps are $16/17, 9/11, 7/8$, respectively. Neither inequality in (5.4) holds at $L = 1/2$. □

Theorem 5.11 (Four quadratic conductors)**.** *Suppose $K = \mathbb{Q}(D)$ has discriminant $13, 21, 40$ or 60. A rank-six fusion category with signed codegrees E, E', H, H', g_1, g_2, with H irrational and g_1, g_2 rational, is spherical. At 40 and 60 it has a pseudo-unitary Galois conjugate.*

Proof. Assume no Galois conjugate is pseudo-unitary. Proposition 5.2 gives the hypotheses of Lemma 5.6. A pair with ratio at least U^2 has squared reciprocal gap at least $1 - 4/A^2$ times its squared reciprocal mass. The ordinary and signed projection bounds of Lemmas 2.14 and 2.15 give respectively $\delta_D^2 \le (c, 2)/c$ and $\delta_H^2 \le (c, 4)/c$, where $\delta_f = |1/f' - 1/f|$. Together with the mass bound these give

c	s at most	w at most	t at most
13	$11/39$	$11/39$	$2/39$
21	$5/21$	$5/21$	$2/21$
40	$19/(60\sqrt{2})$	$19/60$	$1/(60\sqrt{2})$
60	$4/(15\sqrt{2})$	$4/15$	$1/(15\sqrt{2})$

The rational codegrees are at least three by the reciprocal identity and Theorem 5.9. Hence $t + w \ge 1/3$. The last two lines contradict this inequality. In the first two lines equality holds throughout, giving $E = E'$, $D = EU$ and $H/H' = U^2$. Thus $H = nU$, and $w = s$ gives $H = D$. The real signed character of this codegree gives a spherical structure by Lemma 2.9. □

6. The remaining quadratic profiles

For the conductor calculations assume that R and T are commutative, their ordinary and signed dimension characters generate the same real quadratic field $K = \mathbb{Q}(D)$, and the signed codegrees are E, E', H, H', g_1, g_2. Here H is irrational and g_1, g_2 are rational integers at least three. Use the embeddings with $E \ge E'$ after Galois transport and $H > H'$ by labelling its two rows. Put $t = \mathrm{Tr}(1/E)$, $w = \mathrm{Tr}(1/H)$, $s = \mathrm{Tr}(1/D)$, and $\delta_f = |1/f' - 1/f|$. The induction formula gives integral ratios E/f and the simultaneous-column unit gives $H/H' \le D/D'$. If the characters of common rational codegree g form a quadratic pair over F, Lemma 5.3 gives $g \ge \mathrm{disc}(F)$ in the real case and $g = |\mathrm{disc}(F)|b^2$ in the imaginary case, since the involution has exactly one nontrivial two-cycle.

We will use one divisibility for the fields $K = \mathbb{Q}(\sqrt{d})$, $d = 3, 6, 7$. In all three the second primitive family has $V = 3 + \sqrt{d}$. For a quadratic row of codegree mV^eU^k, one has

$$2d/\gcd(3^e, d) \mid m. \tag{6.1}$$

Indeed write $V^eU^k = A + B\sqrt{d}$ in primitive coordinates. The squared distance of the two rows gives $2d \mid mA$; expanding the sum of squares gives $2 \mid mB$. Thus m is even. The rational coordinate of U^k is prime to d by its norm, so $\gcd(A, d) = \gcd(3^e, d)$. For even $d = 6$, A is odd. These facts give the assertion, including for a pair of distinct characters of rational codegree.

Theorem 6.1. *Under these hypotheses, if $K = \mathbb{Q}(\sqrt{6})$ or $K = \mathbb{Q}(\sqrt{7})$, a Galois conjugate of the category is pseudo-unitary.*

Lemma 6.2. *Suppose the signed codegrees are $E, E', H, H', 3, 3$, and the last two characters have absolute value at most one on each basis element. If $s > t$ and a simple object has integral Frobenius–Perron dimension, it is invertible.*

Proof. Apply Lemma 2.11 to the subcategory $\mathcal{A}$ of integral-dimensional simples. Both ordinary Perron conjugates take the value one, giving a lower bound s. If $\mathcal{A}$ contains a simple of dimension at least two, the two short signed characters give zero. The dimension pair gives t, and the H-pair gives either zero or its full mass, since dimensions on $\mathcal{A}$ are rational. Thus the unit coefficient is t or $t + w = 1/3$. The first is too small, the second would give $\mathrm{FPdim}(\mathcal{A}) = 3$, which cannot contain a simple of dimension at least two. □

6.1. **Two rational characters of codegree three.**

Theorem 6.3. *Suppose $\mathcal{C}$ has rank six, R is commutative, and*

$$T_{\mathbb{Q}} \simeq K \times K \times \mathbb{Q} \times \mathbb{Q},$$

where K is a real quadratic field. Suppose the signed spherical dimension character is in the first K-orbit and the Frobenius–Perron character of R has field K. If the two rational characters of T both have formal codegree three, then 3 splits in K. Moreover, writing $E = \dim(\mathcal{C})$ and $D = \mathrm{FPdim}(\mathcal{C})$, one has

$$E \equiv D \equiv -3 \pmod{9\mathcal{O}_K}.$$

We record the estimate for a rational signed row of codegree three that will also be used at $(3,8)$. Write A for its induction label and put $M = 6S$. The row $M_{A,*}$ is integral by the central-character formula. Its two signed dimension entries are one. Its two ordinary Perron entries are equal positive integers, say b, by central symmetry, Galois conjugation and positivity of the Perron dimension. At every induction label C the two unit coefficients give

$$\sum_{V \text{ signed}} 2S_{A_V,C} = \sum_{V \text{ ordinary}} 2S_{A_V,C} = 1. \tag{6.2}$$

For a real vector v supported on such rows, put $y = Mv$ and $z_W = S_{J,W}/S_{1,W}$. The two sign sectors of y each have squared norm $18\|v\|^2$: $v \perp Jv$ and $(SJv)_W = z_W(Sv)_W$. All entries of y are real. In the twist sum the negative sector cancels in J-pairs, since $\theta_{JW} = z_W\theta_W$. The twist identity $S\Theta S = \Theta^{-1}S\Theta^{-1}$ therefore gives

$$6v^tMv \;\geq\; 2 \sum_{z_W=\theta_W=1} y_W^2 - 18\|v\|^2.$$

For $v = e_A$, put $r = M_{A,A}$. The two signed dimension entries and their J-partners give $r^2 \leq 7$. The four ordinary Perron entries, including their translates, give $6r \geq 8b^2 - 18$, forcing $b = 1$. The remaining four ordinary induction entries now sum to one, so their squared norm is at least one. Including their translates improves the bound to $6r \geq -6$.

Apply these facts to the two rational three-rows A, B. At the two labels C, C' of the complementary quadratic pair the entries in each row agree by Galois symmetry; call them a, b. Write $\left(\begin{smallmatrix} r & z \\ z & s \end{smallmatrix}\right)$ for the A, B block of M. We claim

$$a = b = 0. \tag{6.3}$$

The signed row sums and squared norms give $r + z = 1 - 2a$, $s + z = 1 - 2b$ and $2a^2 + r^2 + z^2$, $2b^2 + s^2 + z^2 \leq 7$. Thus $a, b \in \{-1, 0, 1\}$. For $v = e_A + e_B$, the four ordinary Perron entries of Mv are two, and the other four ordinary induction entries sum to two. With their translates the twist-one contribution is at least $16 + 4 = 20$. The twist inequality gives $6(r + 2z + s) \geq 4$, so $a + b \leq 0$. The signed norm gives

$$8 + 2(a + b)^2 + (1 - 2a)^2 + (1 - 2b)^2 \leq 18.$$

This leaves $(a, b) = (0, 0)$ or $\{a, b\} = \{-1, 1\}$. In the latter case the row sums and bounds force one diagonal entry to be -2, contrary to $r, s \geq -1$. This proves the claim.

Now $r = s = 1 - z$. By symmetry the signed class of $F(A - B)$ vanishes on both quadratic factors and has rational values $r - z, z - r$. Its simple coefficients are $(r - z)(\rho_i - \eta_i)/3$. The two rational rows have three entries ± 1, with supports intersecting in precisely two positions by orthogonality. Their difference has an entry ± 1, so $3 \mid r - z$ and $r, z \equiv -1 \bmod 3$.

Proof of Theorem 6.3. Put $T_3 = T \otimes \mathbb{Z}_3$. Its decomposition into complete local rings partitions the field factors of $T_3 \otimes \mathbb{Q}_3$. The two rational characters have distinct reductions, since their supports differ and all nonzero entries are ± 1. They belong to different local factors of T_3. Neither

factor consists of its rational axis alone. Otherwise its identity would be the primitive idempotent of that character, whose unit coefficient $1/3$ is not in $\mathbb{Z}_3$.

By symmetry of the central character formula and (6.3), the signed class of $F(A)$ has value zero on the second quadratic field. Its values on the two rational characters are $r, z \equiv -1 \pmod 3$. A branch of that quadratic field cannot lie in either rational character's local factor, since an element of a local ring cannot have residue zero on one branch and nonzero on another. Each rational character therefore needs a different branch of the first quadratic field. This proves that 3 splits.

At either prime of K above three, the spherical dimension character evaluates $F(A)$ to $E/3$. Its residue must equal that of the rational character in the same local factor, namely -1. Similarly, the spherical dimension character evaluates $F(A_D)$ to E/D, whereas both rational characters take the value one. Thus $E/3 \equiv -1$ and $E/D \equiv 1$ at both primes. Since 3 splits, these congruences give the result. □

In particular this configuration is impossible over $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$. The prime three is ramified in the second field and inert in the other two.

6.2. **Conductor twenty-four.**

Proposition 6.4 (The complementary profiles at conductor twenty-four)**.** *At conductor twenty-four, if no Galois conjugate is pseudo-unitary, the last two characters form an Eisenstein pair of codegree three.*

Proof. Put $U = 5 + 2\sqrt{6}$, $V = 3 + \sqrt{6}$. The scalars in the unit and V families are divisible by twelve and four, respectively, by (6.1).

Lemma 5.6 gives $s \geq 5t$ and squared gaps at least $24/25$ times the squared masses for H and D. The mixed bound gives

$$\frac{1}{6} \geq \frac{24}{25}(w^2 + 2s^2) \geq \frac{16}{17}(w + t)^2.$$

In particular $L = t + w < 7/15$.

As $1/g_1 + 1/g_2 > 8/15$, the only possibilities are $(3, 3)$ and $(3, 4)$. Two rational threes would force three to split by Theorem 6.3, whereas it ramifies. A quadratic orbit must have codegree three: its field is $\mathbb{Q}(\sqrt{-3})$ by Lemma 5.3.

It remains to exclude two rational rows of codegrees three and four.

Here $t + w = 5/12$. If $H = nU$, then $w = 10/n < 5/12$ and $12 \mid n$, so $n \geq 36$ and $t \geq 5/36$. But $s \geq 5t$ gives $2\delta_D^2 \geq 48t^2 > 1/6$. Every other allowed H has $H/H' \geq U^3$ and $\delta_H^2 \geq (242/243)w^2$. The mixed bound then gives

$$\frac{1}{6} \geq 48t^2 + \frac{242}{243}w^2 \geq \frac{5808}{5953}(t + w)^2 = \frac{3025}{17859} > \frac{1}{6}.$$ □

Lemma 6.5 (A cyclic stabilizer obstruction)**.** *Suppose the invertibles of a fusion category form C_3 and fix every noninvertible simple on both sides. Its simple Frobenius–Perron dimensions cannot generate $K = \mathbb{Q}(\sqrt{6})$.*

Proof. Let g generate C_3. A fixed simple gives a one-simple module category over $\langle g \rangle$, so the associator on this group is cohomologically trivial. Write $B = \mathbf{1} \oplus g \oplus g^2$ for its group algebra. Each fixed simple X admits left and right B-actions separately, since $H^2(C_3, \mathbb{C}^\times) = 0$. In particular $X^* \otimes_B X$ has Perron dimension $d_X^2/3$, an algebraic integer [6, Exercises 7.16.8–7.16.9]. Write $d_X = a_X + b_X\sqrt{6}$ with $a_X, b_X \in \mathbb{Z}$. The rational coordinate of $d_X^2/3$ gives $3 \mid a_X$.

Let $l_X : gX \to X$ and $r_X : Xg \to X$ give the two actions. Their interchange on gXg is a scalar κ_X with $\kappa_X^3 = 1$, since the actions are cyclic. If $\kappa_X \neq 1$, take any noninvertible simple Z, put $Y = X^*$ and act on $\mathrm{Hom}(Z, XY)$ by

$$A(f) = (l_X \otimes 1)(1_g \otimes f)l_Z^{-1}, \qquad T(f) = (r_X \otimes 1)(1 \otimes l_Y^{-1})f.$$

The composites differ only by the interchange on gXg, so $AT = \kappa_X^{\pm 1} TA$. Both operators are invertible; determinants give $3 \mid N_{XY}^{Z}$. Every invertible occurs once in XX^* by rigidity. Taking rational coordinates in $d_X^2 = 3 + \sum_Z N_{XY}^{Z} d_Z$ gives $6b_X^2 \equiv 3 \pmod 9$, which is impossible.

Thus $\kappa_X = 1$ for every fixed X. The actions define a B-bimodule on X, of Perron dimension $d_X/3$ in the Morita dual by the same dimension formula. Hence $d_X \in 3\mathcal{O}_K$. Orthogonality of the distinct Perron character and its conjugate gives $3 + \sum_X N_{K/\mathbb{Q}}(d_X) = 0$, with the sum over the noninvertibles. Reduction modulo nine gives the contradiction. □

Theorem 6.6 (The nonreal-three branch at conductor twenty-four)**.** *Let $K = \mathbb{Q}(\sqrt{6})$ and suppose the signed rational algebra is*

$$T_{\mathbb{Q}} = K \times K \times \mathbb{Q}(\sqrt{-3}),$$

with codegrees $E, E', H, H', 3, 3$, where H is irrational. Then $\mathcal{C}$ has a pseudo-unitary Galois conjugate.

Proof. Assume the contrary. Lemma 5.6 gives $D/D' \geq U^2$, where $U = 5 + 2\sqrt{6}$.

The ordinary nonreal pair has codegree three. There is exactly one dual pair X, X^* among the basis simples. The nonreal signed row of norm three has the form

$$(1, \omega, \omega^2, 0, 0, 0), \qquad \omega^3 = 1, \quad \omega \neq 1.$$

Indeed its nonreal entries occupy X, X^* and exhaust the norm remaining after the unit entry. Orthogonality to the positive dimension row selects ω, rather than $-\omega$, and gives $x_X = 1$. Every real signed row then takes the value one on X, X^*. In particular any actual nontrivial invertible must be one of these two basis objects; a self-dual invertible is impossible.

At two the odd-codegree branch is isolated and maximal unramified by Lemma 2.6. Its residue field is $\mathbb{F}_4$, with nontrivial involution. The remaining generic factors are the two ramified quadratic fields $K \otimes \mathbb{Q}_2$; their completed local factors have ranks two or four and residue field $\mathbb{F}_2$. The parity isomorphism $R/2R \simeq T/2T$ preserves the involution. Thus the ordinary nonreal quadratic factor is unramified and maximal at two, and has odd rational codegree g. Its nonreal row is nontrivial modulo two at X. Writing its value there as $(a + b\sqrt{\operatorname{disc}(F)})/2$ gives b odd and

$$g = |\operatorname{disc}(F)| b^2 \equiv 3 \pmod 8.$$

Here the row identity uses the unique dual pair; an imaginary quadratic field unramified and nonsplit at two has discriminant 5 modulo eight.

Any further odd rational codegree would split off an unramified factor by its orbit idempotent and the perfect trace pairing. Such a factor has trivial involution and is absent from the dyadic decomposition. Thus the remaining rational ordinary codegrees are even and at least four. A rational codegree two would force a self-dual invertible; a real complementary quadratic pair of codegree two is excluded by the complementary-pair discriminant bound.

Suppose $g \geq 11$. Every ordinary codegree lies in K by central induction. If the other real ordinary pair has irrational codegree, its character field is K and its mass together with the FP mass is at least $9/11$. The least squared gap in K is $2/3$, so the ordinary Fourier bound would give

$$1/12 \geq \delta_D^2 + \delta_G^2 \geq \tfrac{1}{3}(9/11)^2 = 27/121,$$

which is impossible. Otherwise its two rational codegrees contribute at most $1/2$, and $s \geq 7/22$. But then $2\delta_D^2 \geq (48/25)(7/22)^2 > 1/6$, since D has index at least two.

Consequently $g = 3$.

The ordinary nonreal row is now the same three-supported row. Its orthogonality to the FP row gives $d_X = 1$. Thus $1, X, X^*$ form the actual group C_3 of invertibles. Multiplication by X has spectrum $1, 1, 1, 1, \omega, \omega^2$; as a positive permutation it consists of the three-cycle on these invertibles and fixes all three remaining simples.

Commutativity gives the same right action. Lemma 6.5 now gives the contradiction. □

6.3. **Conductor twenty-eight.**

Proposition 6.7. *If $K = \mathbb{Q}(\sqrt{7})$ and no Galois conjugate is pseudo-unitary, the short rows are rational of codegree three, and $E = mU^k$, $H = nU^j$, $D = MU^\ell$, where $U = 8 + 3\sqrt{7}$, $k, j \geq 1$, $\ell > k$, $M \mid m$, and $E/D \equiv 1 \pmod{3\mathcal{O}_K}$.*

Proof. Put $V = 3 + \sqrt{7}$, so $V/V' = U$ and $N(V) = 2$. The primitive families are those of Lemma 5.6.

The scalars in both primitive families are divisible by fourteen by (6.1).

Since $g_1, g_2 \geq 3$, $t + w \geq 1/3$. For the short pair nV, its squared gap is $(7/9)w^2$; every other irrational pair has squared gap at least $(63/64)w^2$. We use the ordinary, signed and mixed bounds

$$\delta_D^2 \leq \frac{1}{14}, \qquad \delta_H^2 \leq \frac{1}{7}, \qquad \delta_E^2 + \delta_H^2 + 2\delta_D^2 \leq \frac{1}{7}.$$

Put $A_i = \mathrm{Tr}(U^i)$, so $A_0 = 2$, $A_1 = 16$ and $A_{i+2} = 16A_{i+1} - A_i$. We first exclude $H = nV$. Its eigenvalue $\beta = a + b\sqrt{7}$ on the Perron induction column is totally positive and integral. Hence

$$\mathrm{Tr}(\beta/V) = 3a - 7b \geq 2\sqrt{N(\beta)/2} \geq \sqrt{2},$$

so this integral trace is at least two. The dimension pair contributes s on that column, and the rational-codegree orbits contribute at least their reciprocal mass. Since $\mathrm{Tr}(1/V) = 3$, the available deficit is therefore

$$s - t \leq w/3. \tag{6.4}$$

By (5.1), $s \geq 2t$, so $t \leq w/3$. But $14 \mid n$ gives $t + w \leq 4w/3 \leq 2/7 < 1/3$, a contradiction.

Now $H/H' \geq U^2$. This also rules out the exceptional ordinary pair, so $s/t \geq 16/3$ and $D/D' \geq U^2$. The separate Fourier bounds give

$$t + w \leq \frac{1}{14\sqrt{2}} + \frac{8}{21} < \frac{7}{15}.$$

The only possible integer codegree pairs are $(3,3)$ and $(3,4)$. The latter pair is rational and requires $12N(EH)$ to be a square, whereas $N(EH)$ has squareclass one or two.

For $(3,3)$, a quadratic character field would be $\mathbb{Q}(\sqrt{-3})$. Indeed orthogonality of a real quadratic pair gives $\mathrm{disc}(F) \mid 2g$, and no positive quadratic discriminant divides six. An imaginary pair here has $g = |\mathrm{disc}(F)|b^2$. The order discriminant would then require $N(EH)$ to have squareclass three, again impossible. Thus both characters are rational, and their order discriminant forces E, H to belong to the same primitive family.

For later use, Theorem 6.3 gives $E \equiv D \equiv -3 \pmod{9\mathcal{O}_K}$. Neither can be in the V-family: dividing by three, the two residues modulo the primes above three must both be -1, whereas $V = 3 + \sqrt{7}$ has opposite residues and $U \equiv -1$ at both primes. Hence E, H, D belong to the unit family in the rational $(3,3)$ case.

We have reduced all cases to rational codegrees $3, 3$ and $E = mU^k, H = nU^j$. Integrality of E/H gives $q = m/n \in \mathbb{Z}_{>0}$, and the reciprocal identity gives

$$m = 3A_k + 3qA_j, \qquad q \mid 3A_k.$$

Since $A_r \equiv 2 \pmod 7$ and $7 \mid m$, this forces $q \equiv -1 \pmod 7$. If E were rational, then $A_k = 2$, so $q \mid 6$ and $q = 6$. But $n = 3A_j + 1$ would be odd, contradicting $14 \mid n$. Thus $k \geq 1$, while $j \geq 1$ follows from irrationality of H.

The same congruence yields $E/D \equiv 1 \pmod{3\mathcal{O}_K}$, and integrality and $E < D$ give $M \mid m$, $\ell > k$. □

Lemma 6.8 (Two rational-three rows and the dyadic factors)**.** *Suppose a rank-six fusion category has commutative ordinary and signed character algebras, their dimension-character fields equal one quadratic field K ramified at two, and $T_{\mathbb{Q}} = K^2 \times \mathbb{Q}^2$, with the two rational signed characters*

of codegree three. Then every simple is self-dual and the unit is the only invertible simple. The ordinary order at two has exactly two local factors of rank one, and every residue field is $\mathbb{F}_2$. Its rational-odd-codegree character orbits have total degree two. If these are two rational rows, their codegrees are three or seven modulo eight. Every other rational ordinary codegree is even, and no ordinary codegree is two.

Proof. Total reality of the signed characters makes every simple self-dual. The two norm-three rational rows have supports intersecting in exactly two positions, the unit and one nonunit X, where their signs are opposite. An invertible must be nonzero on both rows. If X were invertible, its positive dimension would be one, so in the row with X-entry -1 orthogonality would force the remaining supported simple to have dimension zero.

At two the two short idempotents have odd denominators and split off two copies of $\mathbb{Z}_2$. The remaining generic algebra is K_2^2 with K_2 ramified quadratic. Thus there are exactly two rank-one factors, and all residue fields are $\mathbb{F}_2$. The isomorphism modulo two transfers these properties to the ordinary ring: the idempotents lift uniquely on completion and the rank of each factor is its dimension after reduction.

An ordinary orbit of odd rational codegree is isolated, maximal and unramified at two by Lemma 2.6. With all residue fields $\mathbb{F}_2$ this accounts for one rank-one factor per character. Conversely a rank-one factor cannot have irrational codegree in K, since K does not embed in $\mathbb{Q}_2$, and its denominator must be a unit by perfectness. Thus exactly two characters have odd rational codegree. If rational, their reductions coincide up to permutation with the two short signed rows. Three odd entries give squared norm 3 or 7 (mod 8). Finally a rational row of norm two would have idempotent $(1 \pm X)/2$, implying $X^2 = 1$. A nonrational orbit of norm two would exhaust the reciprocal mass. □

Lemma 6.9 (Integral dimensions from short rows)**.** *Let R be the commutative Grothendieck ring of a rank-six fusion category in which every simple is self-dual and no nonunit simple is invertible. If R has two rational characters of codegree three and also has a rational character of codegree four or six, then some noninvertible simple has integral Frobenius–Perron dimension.*

Proof. The two rows have supports intersecting in the unit and one other simple, with opposite signs there. After reordering, positivity of FP dimensions gives

$$\rho = (1, 1, -1, 0, 0, 0), \qquad \eta = (1, -1, 0, 1, 0, 0),$$

on the basis $1, X, Y, Z, P, Q$, with $d_Y = d_X + 1$ and $d_Z = d_X - 1$. Every other rational row is

$$(1, x, 1 + x, x - 1, u, v), \qquad f = 3 + 3x^2 + u^2 + v^2.$$

For $f = 6$, the only integer possibilities are $x = \pm 1, u = v = 0$. Orthogonality to the FP row gives $3 \pm 3d_X = 0$, which is impossible or makes X invertible.

For $f = 4$, $x = 0$ and (u, v) has squared norm one. Orthogonality to the Perron row gives $d_P = 3$ or $d_Q = 3$. □

Theorem 6.10 (Descent at conductor twenty-eight)**.** *Suppose $K = \mathbb{Q}(\sqrt{7})$. Then $\mathcal{C}$ has a pseudo-unitary Galois conjugate; in particular it is spherical.*

Proof. Assume that there is no pseudo-unitary Galois conjugate, and apply Proposition 6.7. We have

$$T_{\mathbb{Q}} = K^2 \times \mathbb{Q}^2, \qquad K = \mathbb{Q}(\sqrt{7}), \qquad U = 8 + 3\sqrt{7},$$

with signed codegrees $E, E', H, H', 3, 3$, where

$$E = mU^k, \quad H = nU^j, \quad D = MU^\ell, \qquad k, j \geq 1, \quad \ell > k,$$

and $E/D \equiv 1 \pmod{3\mathcal{O}_K}$. By Lemma 6.8 there are no nontrivial invertibles, and all characters are real. Exactly two ordinary characters have odd rational codegree, and all other rational

codegrees are even and at least four. If the odd characters are rational, their codegrees satisfy

$$a, b \equiv 3, 7 \pmod 8. \tag{6.5}$$

If they are conjugate over a quadratic field F, then two splits in F. Thus $\operatorname{disc}(F) \geq 17$, and their common codegree is at least $\operatorname{disc}(F)$ by the row difference bound.

Put $t = \operatorname{Tr}(1/E)$, $s = \operatorname{Tr}(1/D)$, $w = 1/3 - t$, $\lambda = 63/64$. The congruence for E/D gives $(m/M)(-1)^{k-\ell} \equiv 1 \pmod 3$. Thus $m/M \geq 2$ if $\ell = k+1$. The trace ratio A_{k+1}/A_k is at least $127/8$ for $k \geq 1$, so in all cases $s/t \geq 127/4$. The mixed Fourier bound, with all three pairs in the unit family, gives

$$\lambda\big(t^2 + (1/3 - t)^2 + 2s^2\big) \leq 1/7. \tag{6.6}$$

In fact $s < 3/22$. The ordinary bound gives $s < 1/3$, so the left side decreases with t on $0 \leq t \leq 4s/127$. Substituting $t = 4s/127$ gives a quadratic in s which is increasing for $s \geq 3/22$ and already exceeds $1/7$ there.

An irrational complementary ordinary codegree generates K, and its character field has no rank-one factor at two. Since two places must be reserved for odd codegrees, it gives a single quadratic pair G, G'. Unless the two odd codegrees are both three, their total mass is at most $10/21$, by (6.5) or the bound $2/17$ for a nonrational pair. The masses of the D- and G-pairs then sum to at least $11/21$. Their gaps contradict the ordinary bound since $(7/18)(11/21)^2 > 1/14$. If both odd codegrees are three, their characters are rational and the discriminant forces $N(G)$ square. All pairs are then in the unit family. The ordinary and signed pair masses each sum to $1/3$, and Cauchy's inequality in the mixed bound gives $1/7 \geq \lambda/6 = 21/128$. Thus all complementary codegrees are rational.

Their total mass exceeds $6/7$. This excludes a nonrational odd pair, which contributes at most $2/17$ and leaves at most $1/2$. If the two even characters form one quadratic orbit of codegree g, then $g \geq \operatorname{disc}(F) \geq 5$. For $(a, b) \neq (3, 3)$ the mass bound gives $2/g > 6/7 - 10/21$, hence $g = 4$, impossible. For $(a, b) = (3, 3)$ it gives $g < 21/2$, whereas the discriminant squareclass of $28 \operatorname{disc}(F)$ forces $F = K$ and $g \geq 28$. Therefore all four complementary rows are rational.

Write the odd codegrees as $a \leq b$ and the even ones as $c \leq d$. Their product has squareclass seven and their reciprocal sum exceeds $19/22$. Thus $a = 3$. If $b \geq 7$, the mass bound gives $c = 4$. At $d = 4$ the squareclass would require $b = 21h^2$, impossible since $b \equiv 3 \pmod 4$. At $d \geq 6$ the mass bound forces $b = 7, d = 6$, whose product with $3, 4$ has squareclass fourteen. Thus $a = b = 3$. Lemmas 6.9 and 6.2 give $c, d \geq 8$. One is a multiple of seven and therefore at least fourteen. But then

$$1 - s \leq 2/3 + 1/8 + 1/14 = 145/168 < 19/22.$$

□

Proof of Theorem 6.1. At conductor twenty-four Proposition 6.4 reduces to the Eisenstein pair, excluded by Theorem 6.6. Theorem 6.10 treats conductor twenty-eight. □

6.4. **Conductor twelve.**

Theorem 6.11. *Let both dimension characters of a rank-six fusion category have field $K = \mathbb{Q}(\sqrt{3})$ and both character algebras be commutative. Suppose the signed codegrees are E, E', H, H', g, h, where H is irrational and g, h are rational and at least three. Then the category is spherical, or the last two characters are rational with unordered pair $(3, 3), (3, 8), (3, 12)$ or $(4, 4)$, or they form a Gaussian pair of codegree four.*

Proof. Transport to $E \geq E'$ and order $H > H'$. Assume nonsphericity. Write $t = \operatorname{Tr}(1/E)$, $w = \operatorname{Tr}(1/H)$, $s = \operatorname{Tr}(1/D)$ and $L = t + w = 1 - 1/g - 1/h$. Use the two families mU^k, mVU^k of Lemma 5.6, with $U = 2 + \sqrt{3}$, $V = 1 + U$. Equation (6.1) gives $6 \mid m$ in the unit family and $2 \mid m$ in the V family.

The primitive reciprocal traces are A_i, B_i, with initial pairs $(2,4),(1,3)$ and recurrence $z_{i+2} = 4z_{i+1} - z_i$. The mixed conductor-twelve bound reads $\delta_E^2 + \delta_H^2 + 2\delta_D^2 \le 1/3$, where $\delta_f = 1/f' - 1/f$; it also allows twice the squared gaps of any additional ordinary pairs. Lemma 5.6 gives $s \ge 2t$ and $D/D' \ge H/H' \ge U^2$. Thus $\delta_D^2 \ge 3s^2/4$ and $s \le \sqrt{2}/3$.

If $H = nU$, its gap squared is $12/n^2 < 1/3$, so $n \ge 12$ and $w \le 1/3$. This gives $L \le (2+\sqrt{2})/6$. In all other cases $\delta_H^2 \ge 25w^2/27$, so $6t^2 + 25w^2/27 \le 1/3$. Consequently $L^2 \le 187/450$ in every case. For two rational characters the order-discriminant identity makes gh square or six times a square. With $g \le h$, the mass bound gives $g \le 5$ and, for $g = 3,4,5$, respectively $h \le 45, 9, 6$. The resulting list is

$$(3,3),(3,8),(3,12),(3,18),(3,27),(3,32),(4,4),(4,6),(4,9),(5,5).$$

At five the last rational pair would have to occupy a single local factor by Lemma 2.7, since five is inert in K. Its trace $2/5$ is not integral. If the last two characters instead form one quadratic orbit, the common codegree is at most five and the absolute squareclass of the field discriminant is one or six. The real possibility $\mathbb{Q}(\sqrt{6})$ requires codegree at least its discriminant by the quadratic row-difference test. In the imaginary case the unique dual pair gives $h = |\operatorname{disc} F| b^2$ by conjugate-row orthogonality. Among $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-6})$ only the Gaussian pair of norm four remains.

Consider the five rational pairs which still require exclusion. The order-discriminant identity says the primitive families of E, H agree if gh is a square, and differ otherwise. First let $E = m$ be rational. In $(3,27),(4,9)$ the other orbit is $H = nU^j$. For $j = 1$, $t \ge 17/27 - 1/3$ contradicts $6t^2 \le 1/3$. For $j \ge 2$ minimization of $6t^2 + (48/49)(L-t)^2$ gives $(16/19)L^2 > 1/3$. In $(3,18),(3,32),(4,6)$ put $H = nVU^j$. Integrality of E/H gives $m = 6dn$ with d integral. For $L = P/R$ in lowest terms,

$$n = (RB_j + R/(3d))/P.$$

Since n is even, the respective possibilities for d are $1,3$; $1,2,4,8,16$; and $1,2$. The values $B_1 = 3, B_2 = 11$ give no integral n. For $B_j = B \ge 41$, the mass equation gives $n \le (B + 1/3)/L$. In all three cases $L \ge 7/12$, hence

$$\delta_H^2 \ge (7/12)^2 \frac{B^2 - 2/3}{(B+1/3)^2} > 1/3.$$

The last inequality amounts to $3B^2 - 96B - 114 > 0$.

This finishes the rational-E alternatives which require exclusion.

Assume now E irrational. From E/D integral and $D > E$ the same-family exponent increases by at least one; from the V family to the unit family it increases by at least two since $V > U$, and the reverse change multiplies the scalar quotient by at least six and increases the exponent at least once since $V < 6$. The reciprocal trace sequences give, respectively for unit-family and V-family E,

$$\delta_E^2 + 2\delta_D^2 \ge (99/4)t^2 \quad \text{or} \quad 17t^2.$$

Indeed in the former case $s/t \ge 7/2$ and the squared gap fractions are at least $3/4, 48/49$; in the latter $s/t \ge 3$ and they are at least $1/3, 25/27$. The case $H = nU$ contradicts $17t^2 \le 1/3$ since $t \ge 7/12 - 1/3 = 1/4$. Every remaining H has squared gap fraction at least $48/49$ or $25/27$ according to its primitive family. Minimizing the resulting two terms in t gives the following coefficients multiplying L^2:

	H unit family	H in the V family
E unit family	1584/1681	2475/2773
E in the V family	816/881	425/484

For $(3,27),(4,9)$ the diagonal entries exceed $1/(3L^2)$ since $L \ge 17/27$; for $(3,32)$ both off-diagonal entries do likewise at $L = 61/96$. For $(3,18)$ the lower left entry suffices at $L = 11/18$. In the other orientation put $E = mU^k$, $H = nVU^j$. At three the three-axis requires one totally ramified

partner; the eighteen-axis cannot join it as $1/3 + 1/18$ is nonintegral. The simultaneous unit excludes the H branch, whose value reduces to minus one. Thus the three-axis and E form the local factor, implying $1/3 + A_k/m \in \mathbb{Z}_3$. Since $3 \nmid A_k$, we have $3 \parallel m$. Yet $m = 6dn$ and the mass equation is $11m = 18(A_k + 6dB_j)$, a contradiction.

It remains to exclude $(4, 6)$. Here $L = 7/12$. A nonunit invertible would give $t \pm w \pm 1/4 \pm 1/6 = 0$ by its signed unit coefficient. Only $t = 1/12$ is consistent with $17t^2 \le 1/3$, and then $17t^2 + 25w^2/27 > 1/3$. The two short signed rows have different supports modulo two. Indeed their squared norms are four and six and their unit entry is one, so they have respectively four and either two or six odd entries. They belong to different dyadic factors. Each needs a partner, by the nonintegral trace of its rational idempotent. Only the two ramified quadratic branches are available, so the two factors each have rank three and residue field $\mathbb{F}_2$. This description modulo two carries over to R. Every ordinary rational codegree is even: an odd codegree would split off an unramified factor, necessarily of rank one at each prime above two. Codegree two would give an invertible.

All simple objects are self-dual, since $T_{\mathbb{Q}}$ is totally real. By Lemma 8.1, each ordinary four-row is rational. It reduces to the unique residue character with support of size four. Two such rows would give a nonunit invertible: for their idempotents e_1, e_2, the common support makes $1 - 2(e_1 + e_2)$ integral. This element has norm one, zero unit coefficient and Perron value one. Thus among the ordinary characters of rational codegree at most one has codegree four; the rest have codegree at least six.

Put $L = 7/12$ as above. The ratio bounds already proved give $s \ge 3t$, $\delta_D^2 \ge (25/27)s^2$ and $\delta_H^2 \ge (25/27)(L - t)^2$, since $H = nU$ has been excluded. If $s \ge 1/4$, the mixed bound is at least

$$\frac{25}{27}((L - s/3)^2 + 2s^2) \ge \frac{25}{72} > \frac{1}{3},$$

the quadratic being increasing on $[1/4, \sqrt{2}/3]$. Otherwise the ordinary complementary codegrees cannot all be rational, as their total mass would be at most $1/4 + 3/6 = 3/4$. If there is one irrational pair its mass is at least $1 - s - 1/4 - 1/6 = L - s$. If there are two, their masses sum to $1 - s$. Each squared gap is at least a third of its squared mass, and $(1 - s)^2 \ge 2(L - s)^2$ for $0 \le s \le 1/4$. Thus the additional term in the mixed bound is at least $2(L - s)^2/3$. Since $L - t > 1/2$, we obtain

$$\frac{1}{3} \ge \frac{25}{108} + \frac{50}{27}s^2 + \frac{2}{3}(L - s)^2 \ge \frac{25}{108} + \frac{25}{51}L^2 > \frac{1}{3}.$$

This also excludes $(4, 6)$. □

6.5. **Reduction to the conductor-eight profile.**

Theorem 6.12. *Let $\mathcal{C}$ be a complex fusion category of rank six whose global Frobenius–Perron dimension generates a quadratic field. If $\mathcal{C}$ does not admit a spherical structure, this field is $K = \mathbb{Q}(\sqrt{2})$ and its signed character table has formal codegrees*

$$E, E', H, H', 4, 4, \qquad E, H \notin \mathbb{Q}.$$

The two short characters are rational or constitute a Gaussian pair. For a Gaussian pair or two rational rows with the same support, the category has a nonidentity invertible with two fixed points and two free orbits, the ordinary negative rows match the signed short rows, and the other ordinary rows form two irrational quadratic pairs. In this case $\mathrm{Tr}(1/E) < 1/1000$.

Proof. By Proposition 5.2 both algebras are commutative, both dimension characters generate K, and some complementary signed codegree is irrational. Remove the two characters of the dimension orbit. If all four remaining codegrees are irrational, Proposition 5.5 leaves conductors five and eight. A repeated pair is excluded by Proposition 5.7 and Theorem 5.8. This includes a

single quartic character field. Two distinct pairs give the algebra K^3. At conductor five Lemma 7.3 produces the pair excluded by Theorem 7.4. At conductor eight use Theorem 9.8.

There remain one irrational pair and two rational codegrees. Both rational codegrees are at least three. The reciprocal identity excludes one and also a conjugate pair of common codegree two. Theorem 5.9 excludes a rational character of codegree two. Proposition 5.10 now leaves the following conductors:

c	exclusion or remaining profile
$13, 21, 40, 60$	Theorem 5.11
$24, 28$	Theorem 6.1
5	Theorems 7.5 and 7.6
12	$(3,3)$, $(3,8)$, $(3,12)$, $(4,4)$ (Theorem 6.11)
8	$(4,4)$ with irrational E.

At conductor twelve, $(3,3)$ would force three to split by Theorem 6.3. The asymmetric pairs are excluded by Theorems 8.3 and 8.2. For $(4,4)$ use Theorem 8.6 for rational characters with distinct support, and Theorem 8.9 for the other possibilities.

For the last line of the table, Proposition 9.9 gives the finite list. Theorem 9.14 and Propositions 9.13 and 9.12 leave $(4,4)$, and Proposition 9.11 excludes rational E. For the matched characters the asserted conclusions follow from Propositions 9.17 and 9.18, together with Corollaries 9.19 and 9.20. □

7. The quadratic field of conductor five

In the split case with two complementary irrational codegree pairs, the projection estimate forces one pair of reciprocal mass $1/2$. The other arrangement treated here has two rational complementary codegrees. Their contributions to a single induction column impose a congruence. We use the same elementary arithmetic for both arrangements and for the two small conductors treated below.

Lemma 7.1 (Quadratic codegrees and their reciprocal traces)**.** *For the three fields in the table, let u be the least totally positive unit greater than one and let v be as indicated. An oriented totally positive integral d-number $f \geq f'$ has the form $f = mP_{e,i}$, where $P_{e,i} = v^e u^i$, $m \in \mathbb{Z}_{>0}$, $e \in \{0,1\}$ and $i \geq 0$. Write $B_{e,i} = \mathrm{Tr}(1/P_{e,i})$. For fixed e, these integers satisfy $B_{e,i+2} = \mathrm{Tr}(u)B_{e,i+1} - B_{e,i}$. Their initial values are*

$\mathrm{disc}\, K$	u	v	$(B_{0,0}, B_{0,1})$	$(B_{1,0}, B_{1,1})$
5	$(3+\sqrt{5})/2$	$(5+\sqrt{5})/2$	$(2,3)$	$(1,2)$
8	$3+2\sqrt{2}$	$2+\sqrt{2}$	$(2,6)$	$(2,10)$
12	$2+\sqrt{3}$	$3+\sqrt{3}$	$(2,4)$	$(1,3)$

For any shift r, the determinant of two consecutive terms $B_{e,k}B_{f,k+r+1} - B_{e,k+1}B_{f,k+r}$ is independent of k.

Proof. The unit f/f' equals u^{2i+e}. Since $v/v' = u$, division by $P_{e,i}$ gives a positive rational number. The coordinates of v^e in the integral basis are primitive, and multiplication by the unit u^i preserves primitivity. This proves that the rational factor is an integer. The trace recurrence follows from the minimal polynomial of u, and its companion matrix has determinant one. □

Lemma 7.2 (Relative indices in a quadratic recurrence)**.** *In the notation of Lemma 7.1 write $E = mP_{e,k}$, $H = nP_{f,j}$ and $D = MP_{a,\ell} > E$. Set $b = N(v)$, $c = m/M$, $h = m/n$, $r = \ell - k$ and $z = j - k$, with c, h positive integers. Write t, w, s for the reciprocal traces of E, H, D and $L = t + w$. Then $s/t > c^2/(2b)$. If $h \mid qB_{e,k}, qcB_{a,\ell}$ for an integer $q > 0$, then*

$$h \mid qcW, \qquad W = W_{e,a}(r) = B_{e,0}B_{a,r+1} - B_{e,1}B_{a,r} \neq 0.$$

For a lower bound I on k, j, ℓ put $C_g = (v/b)^g$, $x = h(C_f/C_e)u^z$, $y = c(C_a/C_e)u^r$. With $0 < \eta \le (1-u^{-2I})/(1+u^{-2I})$ the three reciprocal gaps obey

$$(\delta_E, \delta_H, \delta_D) \ge \frac{\eta L}{1+x}(1, x, y), \qquad \frac{\eta L y}{1+x} \le s \le \frac{Ly}{\eta(1+x)}.$$

For fixed relative indices, all three trace sequences modulo any integer are exhausted by iterating one period.

Proof. Writing $P = P_{g,i}$ gives $P/b^g \le B_{g,i} \le 2P/b^g$ for $i \ge 0$. Since $P_{a,\ell} > cP_{e,k}$, this bounds $s/t = cB_{a,\ell}/B_{e,k}$ as claimed. Take the constant determinant at k for the divisibility. Its vanishing would make the two sequences proportional. Their coefficients of u^i, u^{-i} would then give $2r + a - e = 0$, hence $r = 0, a = e$, contrary to $D > E$. The exact trace and gap numerators are $C_g u^i(1 \pm u^{-2i-g})$. Dividing by $L = (B_{e,k} + hB_{f,j})/m$ gives the stated bounds. Finally the recurrence is invertible on consecutive pairs modulo every integer, so the initial triple of pairs recurs. □

7.1. **A forced pair of reciprocal mass one half.**

Lemma 7.3. *Assume that R is commutative, both dimension character fields equal $K = \mathbb{Q}(\sqrt{5})$, and $T_{\mathbb{Q}} \simeq K^3$, with irrational codegrees on both complementary factors. If no Galois conjugate is pseudo-unitary, one complementary pair is $5 \pm \sqrt{5}$.*

Proof. Put $u = (3+\sqrt{5})/2$ and $v = (5+\sqrt{5})/2$. Orient each pair by its larger conjugate, and transport the category so that E has this orientation. Every codegree is mu^k or mvu^k, with $m \ge 1$ and $k \ge 0$; irrational unit-family codegrees have $k \ge 1$. Write $t_A = \mathrm{Tr}(1/A)$ and $\delta_A = 1/A' - 1/A$. The separate signed Fourier bound is

$$t_E + t_H + t_J = 1, \qquad \delta_E^2 + \delta_H^2 + \delta_J^2 \le \frac{1}{5}.$$

Equation (5.1) gives $s = \mathrm{Tr}(1/D) \ge 3t_E/2$. The ordinary Perron column has unit coefficient one; the dimension orbit contributes $s > t_E$, so one complementary orbit contributes less than its reciprocal mass.

The shortest orbit nv admits no such deficiency, since $\mathrm{Tr}(\beta/v)$ is a positive integer for every totally positive integral $\beta \in K$. Thus the complementary pairs cannot both be shortest orbits. Every other irrational pair satisfies $\delta_A^2 \ge (5/9)t_A^2$.

We first show that a complementary orbit is $2v$ or $3v$. If both complementary orbits are long, Cauchy's inequality gives

$$\frac{1}{5} \ge \frac{5}{23}(1 - 2/E)^2,$$

and hence $E < 49$. If the shortest orbit is nv with $n \ge 4$, put $w = 1/n \le 1/4$ and let z be the other complementary mass. Since $\delta_E + z = 1 - w - 2/E$, the same inequality gives

$$\frac{1}{5} \ge \frac{5}{14}(1 - w - 2/E)^2 + \frac{w^2}{5}.$$

If $E \ge 80$, the right side is at least $\frac{5}{14}(39/40 - w)^2 + w^2/5$. This function decreases on $0 \le w \le 1/4$ and has value $897/4480 > 1/5$ at $w = 1/4$. Therefore $E < 80$ unless a complementary orbit is $2v$ or $3v$.

For $E < 80$, the squared distance between the two dimension rows gives $5 \mid \mathrm{Tr}(E)$. Enumerate $E = mP_{e,k} < 80$ with this divisibility and $t_E < 1$. There are 49 choices and $m \le 75$. The scalars for H, J divide m, by integrality of $E/H, E/J$. Their exponents are at most four: larger indices have reciprocal trace greater than 75 before scaling. Of these possibilities, exactly ten satisfy the mass identity, the signed Fourier bound and the order-discriminant test $N(EHJ)/125 \in (\mathbb{Z}_{>0})^2$. All ten contain $2v$ or $3v$ (`short_orbit_check.py`).

We next exclude the shortest orbit $H = 3v$ whenever the other complementary orbit is long. The discriminant condition forces E and J to belong to the same primitive family. Suppose both belong to the unit family. The integral central unit of Lemma 2.1 for the nontrivial automorphism of K has values u^{2k}, u, u^{2j} on the three factors. At three the field K is unramified quadratic and $u^2 = -1$ in its residue field. The middle factor therefore has residual polynomial $x^2 + 1$, while the other two have residual values 1 or -1. Polynomial Chinese remaindering isolates the middle factor by an idempotent in $T \otimes \mathbb{Z}_3$. Its coefficient of the unit is $\mathrm{Tr}(1/H) = 1/3$, contradicting integrality of that coefficient.

Suppose instead $E = mvu^k$ and $J = nvu^j$, with $j \geq 1$. Integrality gives $3 \mid m$ and $n \mid m$. Put $t = t_E$ and $z = t_J$, so $t + z = 2/3$. If $k \geq 1$, the two long v-family pairs have squared-gap ratios at least $4/5$. Consequently

$$\frac{1}{5} \geq \frac{4}{5}(t^2 + z^2) + \frac{1}{45} \geq \frac{1}{5}.$$

Equality requires $k = j = 1$ and $t = z = 1/3$, hence $E = J = 6vu$. An ordinary unit-family $D = Mu^\ell$ would require $5 \mid M$ and $M \mid 6$, which is impossible. In the other family $D = Mvu^\ell$, the inequalities $D > E$ and $M \mid 6$ force $\ell \geq 2$. They give $\delta_D^2 \geq 121/(5M^2) \geq 121/180 > 1/5$, contradicting the separate ordinary Fourier bound. If $k = 0$, then $t = 1/m$ and

$$\frac{1}{5} \geq \frac{1}{5}t^2 + \frac{4}{5}(2/3 - t)^2 + \frac{1}{45}.$$

Thus $t \geq (8 - 2\sqrt{6})/15 > 1/5$, so $m < 5$ and $3 \mid m$ give $m = 3$. But $t_J = 1/3$ then implies $n = 3\,\mathrm{Tr}(1/(vu^j)) > 3$, contradicting $n \mid m$. This proves that the forced shortest orbit is $2v$. □

7.2. **Three quadratic character pairs.** Write $\varphi = (1 + \sqrt{5})/2$, $K = \mathbb{Q}(\varphi)$, $u = \varphi^2$ and $v = 2 + \varphi$. Write $E = \dim(\mathcal{C})$ and $D = \mathrm{FPdim}(\mathcal{C})$.

Theorem 7.4. *Let $\mathcal{C}$ have rank six, with ordinary commutative Grothendieck algebra and signed rational algebra K^3. Suppose the ordinary Frobenius–Perron character has field K, the spherical dimension character belongs to the first factor, and a second factor has codegrees $2v, 2v'$. If the last factor has irrational codegrees, then $\mathcal{C}$ is spherical.*

Proof. Set $H = 2v$ and write G, G' for the last pair, oriented by $G > G'$. For $f \in K$ write $t_f = \mathrm{Tr}(1/f)$ and $\delta_f = 1/f' - 1/f$. We record two conductor-five projection bounds

$$\delta_E^2 + \delta_G^2 + \delta_H^2 \leq 1/5, \qquad \delta_D^2 + 1/20 \leq 1/5.$$

The first bound is the signed inequality in (2.4). For the second project the ordinary difference $(e_P - e_{P'} - e_{JP} + e_{JP'})/2$ onto the signed targets, as in Lemma 2.15. The two signed dimension labels and their translates contribute δ_D^2.

Put $\beta = \chi_H(F(P))$ for the signed character of codegree H. Both conjugates of β are positive, by symmetry and the positivity of the ordinary FP character on actual objects. The contribution of this pair to the signed unit coefficient of $F(P)$ is $\mathrm{Tr}(\beta/v)/2 < 1$. As $\mathrm{Tr}((a + b\varphi)/v) = a$, we obtain $\beta = 1 + b\varphi$ with $b = 0$ or 1. At A_H the two rows P, P' have entries $\beta/(2H)$ and its conjugate by the ordinary character formula. The four coordinates at $A_H, A_{H'}$ and their translates contribute $|\beta/H - \beta'/H'|^2 = 1/20$. In both bounds the full $U(2)$-fixed projection has squared norm at most $1/5$.

Assume there is no pseudo-unitary conjugate and maximize E. Write $t = t_E$, $w = t_G$ and $s = t_D$. As in Lemma 7.3, $s \geq 3t/2$. Since $t + w = 1/2$ and the H-pair contributes $1/2$ to the Perron induction column, the G-pair contributes less than w. This excludes $G = nv$: $\mathrm{Tr}(\gamma/v)$ is a positive integer for totally positive integral γ.

For $e \in \{0, 1\}$ set $P_{e,i} = v^e u^i$ and $B_{e,i} = \mathrm{Tr}(1/P_{e,i})$, with $i \in \mathbb{Z}$. Apply Lemma 7.1 to E, G, D. The trace discriminant of T is $N(EHG)$ and differs from 5^3 by a square factor. As $N(H) = 20$,

E and G thus have the same primitive family. Write

$$E = mP_{e,k}, \qquad G = nP_{e,j}, \qquad D = MP_{f,\ell}, \qquad h = m/n, \quad a = m/M.$$

Here $k, \ell \geq 0$, $j \geq 1$, and h, a are positive integers by integrality of E/G and E/D. If $f > e$, then $5 \mid a$. Orthogonality to the conjugate of any K-row of codegree z gives $5 \mid \mathrm{Tr}(z)$ by taking the squared norm of the row difference. Thus a scalar attached to the unit family is divisible by five, since $u \equiv -1 \pmod{\sqrt{5}}$. Lemma 2.2 gives $2\ell + f \geq 2j + e$ by comparing D/D' with G/G'.

Set $B = B_{e,k}$, $L = B_{e,j}$, $C = B_{f,\ell}$. The mass equations provide an integral congruence. The signed G-orbit value at the FP column is $\mathrm{Tr}(\gamma/G) = 1/2 - s$ for $\gamma \in \mathcal{O}_K$. Multiplying by n yields an integer: $1/P_{e,j}$ belongs to the codifferent, since u is a unit and $\mathrm{Tr}(\mathcal{O}_K/v) \subset \mathbb{Z}$. Likewise $n(1/2 - t) = L$. It follows that

$$m = 2(B + hL), \qquad h \mid 2B, \qquad h \mid aC - B. \tag{7.1}$$

This second divisibility of the scalar ratio will bound it uniformly.

We claim that $D = G$. First suppose $k \geq 2$. The initial values and the recurrence give

$$B_{e,i+1}/B_{e,i} \geq 5/2 \quad (i \geq 2), \qquad B_{1,i+1} \geq B_{0,i}, \qquad B_{0,i+2} \geq 7B_{1,i}.$$

Indeed each difference satisfies the same recurrence and starts with nonnegative, nondecreasing values at the indicated index. In the same family $D > E$ forces $r = \ell - k \geq 1$. In passing from the unit family to the v family, $5 \mid a$ and $v < 5$ again give $r \geq 1$. In the opposite direction $u < v$ gives $r \geq 2$. Thus the squared gap fraction of D is at least $45/49$, and the ordinary projection bound gives $s \leq 7/\sqrt{300} < 5/12$.

Also $t \geq 1/10$. Indeed if $j \geq 2$, both signed gap fractions are at least $45/49$, giving $t^2 + (1/2 - t)^2 \leq 49/300$, impossible for $t \leq 1/10$. If $j = 1$, then $w = 3/n \leq 3/10$ in the unit family $(5 \mid n)$, and $w = 2/n \leq 2/5$ in the other family, since $w < 1/2$. It follows that $s/t < 5$. The trace comparisons above now force $f = e$, $a = r = 1$ in $s/t = aB_{f,k+r}/B_{e,k}$.

The determinant $W_{e,e}(1)$ of Lemma 7.2 is five in the unit family and one in the other family. By (7.1) we have $h \mid 2W_{e,e}(1)$ and $h \mid 2B$; since $5 \nmid B_{0,k}$, in either family $h \leq 2$. Dominance gives $j \leq k + 1$. If $j \leq k$, then $t = B/(2(B + hL)) \geq 1/6$ and $s \geq 5t/2 \geq 5/12$. If $j = k + 1$ and $h = 2$, then $w \geq 5/12$ and $\delta_G^2 \geq (45/49)(5/12)^2 > 3/20$. Hence $h = 1$ and $j = k + 1$, proving the claim.

It remains to check $k = 0, 1$, where $B \leq 3$. For $j \geq 1$ the squared gap fraction of G is at least $4/5$, except when $(e, j) = (0, 1)$, where $w \leq 3/10$. Thus $w \leq \sqrt{3}/4$ and $t > 1/16$. Hence $m < 16B \leq 48$, $hL < 7B \leq 21$, and $j \leq 3$. Dominance gives $\delta_D^2 \geq 5s^2/9$, so $C = Ms < 27$ and $\ell \leq 3$. For transparency the remaining scalar checks are as follows. The middle column tests $h \mid 2B$, $m = 2(B + hL) < 16B$, the family divisibilities and $\delta_E^2 + \delta_G^2 \leq 3/20$. The last tests $a \mid m$, $h \mid aC - B$, the family divisibilities, $D > E$, dominance and $\delta_D^2 \leq 3/20$.

(e, k)	(j, h) after the signed test	(j, h, f, ℓ, a)
$(0, 0)$	$(1, 1)$	$(1, 1, 0, 1, 1)$
$(0, 1)$	$(2, 1)$	$(2, 1, 0, 2, 1)$
$(1, 0)$	$(1, 1), (1, 2)$	$(1, 1, 1, 1, 1)$
$(1, 1)$	$(1, 1), (2, 1), (1, 2), (1, 4)$	$(2, 1, 1, 2, 1)$

These substitutions also give $D = G$ (arithmetic check). The real signed character of this codegree gives a spherical structure by Lemma 2.9. □

7.3. **Two irrational pairs and two rational codegrees.** Let $K = \mathbb{Q}(\sqrt{5})$, $u = (3 + \sqrt{5})/2$ and $v = (5 + \sqrt{5})/2$. Write R and T for the ordinary and signed rings.

Theorem 7.5. *Suppose a complex rank-six fusion category has commutative ordinary and signed character algebras. Assume the ordinary Frobenius–Perron and signed spherical dimension characters have field K. If the signed codegrees are E, E', H, H', a, b, with E the intrinsic global*

dimension, E, H irrational in K, and a, b rational integers at least three, then the category admits a spherical structure.

Proof. Write D for the Frobenius–Perron dimension and g' for conjugation in K. Order $a \le b$ and $H > H'$. If there is no pseudo-unitary Galois conjugate, transport to an embedding with $E > E'$; then $E < D$ and $\alpha = E/D$ is a totally positive integer of K with $\alpha < 1 < \alpha'$. Put

$$t = \mathrm{Tr}(1/E), \quad w = \mathrm{Tr}(1/H), \quad s = \mathrm{Tr}(1/D), \quad L = 1 - 1/a - 1/b = t + w,$$

and write $\delta_g = 1/g' - 1/g$ for $g \ge g'$. At conductor five (2.4) gives

$$\delta_E^2 + \delta_H^2 \le 1/5, \qquad \delta_D^2 \le 1/5. \tag{7.2}$$

The central class attached to conjugation has ordinary FP value D/D' and signed H-value H/H' by the induction character formula. Thus $D/D' \ge H/H' > 1$. Also $\mathrm{Tr}(\alpha) \ge 3$ and $\alpha' - \alpha \ge \sqrt{5}$, whence

$$s = \tfrac{1}{2}(t\,\mathrm{Tr}(\alpha) + \delta_E(\alpha' - \alpha)) \ge \tfrac{3}{2}t, \qquad \delta_D = s - 2/D \ge ut - (2 + \sqrt{5})/E. \tag{7.3}$$

Use $P_{e,k} = v^e u^k$, $B_{e,k} = \mathrm{Tr}(1/P_{e,k})$ and the d-number parametrization from Theorem 7.4. Write

$$E = mP_{e,k}, \qquad H = nP_{f,j}, \qquad D = MP_{d,\ell},$$

with $e, f, d \in \{0, 1\}$ and nonnegative indices. The four integral ratios E/H, E/D, E/a and E/b give $a, b, n, M \mid m$, with $5n \mid m$ if $f > e$ and $5M \mid m$ if $d > e$. Every unit-family scalar is divisible by five, by orthogonality of the two rows. For $q = \mathrm{lcm}(a, b)$ the two last signed characters contribute Z/q to the unit coefficient of the ordinary FP induction object. Here Z is an integer between $q/a + q/b$ and $q - 1$: at rational characters the evaluations are positive integers, and for a quadratic orbit of common codegree the trace of the totally positive integral evaluation is at least two. At the first pair the contribution is s; at the second it is $\mathrm{Tr}(\beta/H)$, with β totally positive integral. Thus

$$0 < 1 - Z/q - s = \mathrm{Tr}(\beta/H) < w, \qquad n(1 - Z/q - s) \in \mathbb{Z}. \tag{7.4}$$

Indeed $s > t$ and $1/P_{f,j}$ lies in the codifferent. This excludes $H = nv$, since $\mathrm{Tr}(\beta/v)$ is a positive integer and $\mathrm{Tr}(1/v) = 1$. Thus $2j + f \ge 2$.

For exponent $2j + f = 2$ we have $w = 3/n$, with $5 \mid n$. The value $n = 5$ exhausts the Fourier bound while $\delta_E > 0$, so $w \le 3/10$. For exponent three we have $w = 2/n$ and $\delta_H^2 = 4w^2/5$; likewise $n > 4$, hence $w \le 2/5$. For exponent at least four, $H/H' \ge u^4$ gives $\delta_H^2 \ge 45w^2/49$. Consequently $w \le 7/15$ in all cases. Except at exponent two, $\delta_H^2 \ge 4w^2/5$.

Large rational codegrees. Suppose $1/a + 1/b \le 9/25$, so $L \ge 16/25$. If $E \ge 1000$, (7.3) implies $t < 173/1000$, and then $w > 467/1000$. This contradicts $w \le 7/15$. Hence $E < 1000$. Now $m < 1000$, $k, j \le 7$ by $E \ge u^k$, $B_{f,j} < n \le m$, and $B_{f,8} \ge 1597$. Test $1 \le m < 1000$, $0 \le k \le 7$, $1 \le j \le 7$, $n, a, b \mid m$ against $E < 1000$, the scalar divisibilities, $L = t + w \ge 16/25$, and the signed inequality in (7.2). The three surviving rows $(e, k, m; f, j, n; a, b)$ are

$$(0, 2, 20; 0, 1, 10; 4, 10), \quad (1, 1, 6; 1, 1, 6; 6, 6), \quad (1, 2, 20; 1, 1, 5; 4, 10).$$

For distinct rational axes the absolute regular trace discriminant $N(EH)ab$ is a square, the two quadratic field discriminants multiplying to 25. This excludes both rows with $a = 4, b = 10$. In the middle row the unit family for D would require $5 \mid M \mid 6$. In the other family $D > E$ forces $\ell \ge 2$, and then $\delta_D^2 \ge (25 - 4/5)/6^2 > 1/5$.

Small rational codegrees. We may now assume $1/a + 1/b > 9/25$, so either $a = 3$ and $b \le 37$, or $a = 4$ and $b \le 9$, or $a = 5$ and $b \le 6$. For two rational rows the discriminant condition requires ab to have squareclass 1 or 5. If $a = 3$, its partner at three is the rational b-axis by Lemma 2.7, since K is inert there. The congruent rows have squared distance $3 + b$, divisible by nine.

If the complement is quadratic, $a = b$ and its field F has $|\operatorname{disc}(F)|$ of squareclass 1 or 5 since the absolute order discriminant is $N(EH)a^2$. Moreover $|\operatorname{disc}(F)| \mid 2a$ by taking the squared norm of the difference of the two conjugate rows. This excludes $a = 3$; for $a = 4$ it gives $F = \mathbb{Q}(i)$. The complete list of profiles is therefore

$$(3, 15), \quad (4, 4), \quad (4, 5), \quad (4, 9), \quad (5, 5).$$

In particular $L \geq 1/2$. The discriminant forces $e+f$ odd for $(3, 15), (4, 5)$ and even for $(4, 4), (4, 9)$.

The finite recurrence. Put $h = m/n$, $c = m/M$, $r = \ell - k$, $z = j - k$ and write $B_0 = B_{e,k}$, $B_1 = B_{f,j}$, $B_2 = B_{d,\ell}$. We have

$$m = (B_0 + hB_1)/L, \qquad m(q - Z) - qcB_2 \in qh\mathbb{Z}_{>0}. \tag{7.5}$$

The second assertion is (7.4). Thus $h \mid qB_0, qcB_2$. From $w \leq 7/15$, $L \geq 1/2$ and $\delta_D^2 \geq 5s^2/9$ we have $t \geq 1/30$, $s \leq 3/5$ and $s/t \leq 18$. Lemma 7.2 gives $c \leq 13$, $1/v \leq u^r < 180v$ and $-1 \leq r \leq 6$. The initial pairs give $0 < |W| \leq 720$ on the allowed range, so $h \mid qcW$ and $h \leq 36 \cdot 13 \cdot 720$. Since $1/4 \leq L/t - 1 = hB_1/B_0 < 30$ and $u^z/3 < B_1/B_0 < 3u^z$, we may take $-15 \leq z \leq 4$.

The finite recurrence check enumerates these relative indices with the family divisibilities, $2z + f \leq 2r + d$ and $v^{d-e}u^r > c$. Equality $(f, z, h) = (d, r, c)$ gives a spherical section since $D = H$. For all other tuples and $0 \leq k < 36$, test $2k + e \geq 1$, $j, \ell \geq 1$, the scalar divisibilities, (7.5) and (7.2). None survives.

For $k \geq 36$ take $I = 21$, $\eta = 999999/1000000$ and x, y of Lemma 7.2. The two Fourier tests and positivity of the H contribution require

$$5\eta^2L^2(1 + x^2) \leq (1 + x)^2, \qquad 5\eta^2L^2y^2 \leq (1 + x)^2, \qquad 1 - Z/q > \eta Ly/(1 + x).$$

The first two inequalities leave $0, 16, 6, 0, 0$ tuples for the five profiles in order. Writing $L = N/P$ in lowest terms, the modulus $N \operatorname{lcm}(qh, 5h, 5c, q)$ determines integrality and the required residues of $m = P(B_0 + hB_1)/N$. The largest recurrence periods are 120 and 750; no state satisfies the divisibilities and (7.5) for an allowed Z. □

7.4. **Rational intrinsic dimension.**

Theorem 7.6. *Theorem 7.5 also holds when E is rational and the signed dimension character still has field K.*

Proof. Use the notation of that theorem, with $E = m$ a multiple of five and $t = 2/m$. Assume $D > E$. The bound $\delta_H^2 \leq 1/5$ now allows equality. As before, H has exponent $2j + f \geq 2$. The exponent-two and exponent-three families still give $w \leq 7/15$ except when $H = 5u$ or $H = 4vu$, of masses $3/5$ and $1/2$, respectively. Hence $w \leq 3/5$ always. If $L \geq 16/25$ then $t \geq 1/25$, giving $m \leq 50$. Otherwise the same discriminant and local argument gives the five previously listed (a, b) and $L \geq 1/2$. Unless H is exceptional, $t \geq 1/30$, so $m \leq 60$. At the two exceptional values, the only possibilities for L are $1/2, 11/20, 3/5, 23/36$. Direct substitution in $m = 2/(L - w)$, requiring a positive integer, gives $m \leq 40$. Thus $m \leq 60$ in every case.

For $m = 5, 10, \ldots, 60$ all scalar divisors and the rational codegrees divide m, and $j, \ell \leq 4$ since both primitive traces at index five exceed 60. Put aside $D = H$, which gives a spherical section. Test $D > m$, $D/D' \geq H/H'$, $s \geq 3/m$, the mass equation $2/m + w = L$, the discriminant and local restrictions, and the two Fourier bounds of the preceding theorem. One further projection, as in Theorem 7.4, gives

$$\delta_D^2 + n^{-2}(\operatorname{Tr}(\gamma/v^f)^2 - 4N(\gamma)/5^f) \leq 1/5, \qquad \gamma = \beta/u^j = x + y\varphi \in \mathcal{O}_K.$$

Here $t_0 = n(1 - s - Z/q) > 0$ equals $2x + y$ if $f = 0$ and x if $f = 1$; positive norm gives $|y| \leq 2t_0$. This leaves seven numerical profiles.

To reconstruct the character rows for each profile, write a quadratic row of codegree $c + d\varphi$ as $A + B\varphi$. Orthogonality to its conjugate gives three finite sphere conditions:

$$B^2 = (2c + d)/5, \quad A^2 = c - B^2, \quad 2A \cdot B = d - B^2, \qquad A_0 = 1,\ B_0 = 0.$$

Enumerate the two mutually orthogonal signed pairs with nonvanishing dimension row, and the Perron rows with $d_i \geq \max(1, |d_i'|)$. The signed pairs leave a plane with projector P. For unequal complementary codegrees g, h, a rational row r of norm g determines the other row as $h(P_{0i} - r_i/g)_i$. For equal codegrees their means are $C_i = gP_{0i}/2$ and their centered products are $gP_{ij}/2 - C_iC_j$. These determine the contributions to the signed table for either real or complex conjugate rows. Keep integral tables T dominated by the Perron row and satisfying

$$d_id_j - \sum_k |T_{ij}^k| d_k \in 2 \sum_{k \neq 0} \mathbb{Z}_{\geq 0} d_k.$$

The row computation leaves only $(E, H, g, h, D) = (20, 4vu, 5, 5, 20u^2)$ and $(60, 15u^2, 4, 4, 60u^3)$, each with two signed matches.

For $E = 60$, put $d = 1 + 2\varphi$, so $d^2 = 1 + 4d$. Order the simples as $1, g, Y, A, B, V$ with dimensions $1, 1, 4d, d, d, 2 + 6d$. In both signed matches the coefficient of V in Y^2 is -2. The unique simple Y of its dimension is self-dual and fixed by g. Hence $1, g$ occur once each in its ordinary square. If l denotes the multiplicity of V, comparison of constant coordinates in the basis $1, d$ gives $16 = 2 + 2l$, since $\mathrm{FPdim}(Y^2) = 16 + 64d$. Thus $l = 7$, contradicting the parity of the signed coefficient.

For $E = 20$ the Perron row is $d = (1, \varphi, \varphi, 1 + 3\varphi, 1 + \varphi, 3 + 4\varphi)$. Let W be the simultaneous column of Lemma 2.1 for the nontrivial automorphism of K. In the signed order it is $(1, 1, 1, 1, 1, 2)$ in both tables (witness calculation). Indeed W acts as f/f' on a row of codegree f, so character orthogonality recovers its coefficients. More explicitly, the rational codegrees give eigenvalue one, and the H-pair has $H = 12 + 16\varphi$, $N(H) = 80$. Thus the coefficient of b_i in $W - 1$ is $(1/H' - 1/H)(\chi_H(b_i^*) - \chi_H'(b_i^*))$, the φ-coefficient of $\chi_H(b_i^*)$. Since the two orders agree modulo two, evaluating this vector on d must give D/D' modulo $2\mathcal{O}_K$. The evaluation is $9 + 14\varphi$, whereas $D/D' = u^4 = 13 + 21\varphi$. □

8. The quadratic field of conductor twelve

The two rational codegrees left by the conductor reduction are $(3, 3)$, $(3, 8)$, $(3, 12)$ and $(4, 4)$. Theorem 6.3 excludes the first pair since three ramifies in K. For the middle two, a projection estimate bounds the reciprocal mass of the Perron orbit. The last pair is governed by the supports of its two rows and reduction at two.

8.1. **Rational codegrees three and twelve.**

Lemma 8.1. *Let R be the commutative Grothendieck ring of a complex fusion category of rank six. A real character of formal codegree four or six is rational.*

Proof. Let F be the field of a real row of codegree $h = 4$ or 6. It is totally real and abelian by cyclotomicity [7, Theorem 8.51]. On its factor the coefficient-of-unit pairing is $\mathrm{Tr}_F(xy)/h$, so $h\mathfrak{D}_F^{-1}$ pairs integrally with R. Perfectness puts it in $R \cap F \subseteq \mathcal{O}_F$. Hence

$$v_p(\mathrm{disc}\, F) \leq [F : \mathbb{Q}] v_p(h).$$

For $h = 4$ the field lies in the real cyclotomic 2-power tower. This tower is cyclic, with unique quartic subfield $\mathbb{Q}(\zeta_{16})^+$ of discriminant $8 \cdot 16^2 = 2^{11}$. The tower discriminant formula would give $v_2(\mathrm{disc}\, F)/[F : \mathbb{Q}] \geq 11/4 > 2$ if $[F : \mathbb{Q}] \geq 4$. Thus $F \subseteq \mathbb{Q}(\sqrt{2})$.

For $h = 6$ the conductor–discriminant formula bounds the average conductor exponent by one at both two and three. At two, each ramified character has exponent at least two, so nontrivial inertia has order two and every ramified character has exponent two. At three, wild inertia would

give exponent at least two for at least two thirds of the characters. Thus the conductor divides twelve, and $F \subseteq \mathbb{Q}(\zeta_{12})^+ = \mathbb{Q}(\sqrt{3})$.

In either quadratic alternative write $c = 8$ or 12 respectively. By row orthogonality, the squared distance between the two conjugate rows is $2h = c\sum_i b_i^2$, where b_i are the integral coefficients of their differences in units of $\sqrt{c}$. Since $c = 2h$, exactly one coordinate differs. This contradicts orthogonality of their difference to the positive Perron row. Indeed the orbit cannot contain that row: $D = h \leq 6$ would force every simple to be invertible, and the Perron row would be rational. □

Theorem 8.2. *Let $\mathcal{C}$ be a complex rank-six fusion category with commutative ordinary and signed Grothendieck algebras. Suppose its ordinary Frobenius–Perron and signed spherical dimension characters have field $K = \mathbb{Q}(\sqrt{3})$. Suppose the signed rational algebra is $K \times K \times \mathbb{Q} \times \mathbb{Q}$, with rational characters of formal codegrees three and twelve. Then $\mathcal{C}$ admits a spherical structure.*

Proof. Use $U = 2 + \sqrt{3}$, $V = 3 + \sqrt{3}$, $B_{e,i}$ from Lemma 7.1, and $A_i = B_{0,i}$. Assume nonsphericity, orient $E \geq E'$, $H \geq H'$, and write t, w, s for the reciprocal traces of E, H, D. Then $t + w = 7/12$. For $\delta_f = 1/f' - 1/f$ we use

$$\sum_{\text{signed pairs}} \delta_f^2 + 2 \sum_{\text{ordinary pairs}} \delta_f^2 \leq 1/3. \tag{8.1}$$

At three the rational rows have distinct residues, since their squared distance is 15. Each needs another branch and there are just two, both ramified quadratic. The simultaneous unit has value one on the rational rows, while on a branch of type VU^i its value U^{2i+1} reduces to -1. Thus $E = mU^k$, $H = nU^j$. The identity of each local factor has integral trace $1/a + \mathrm{Tr}(1/f)$, with $a = 3$ or 12 and $f = mU^k$ or nU^j. Since $3 \nmid A_i$, the scalar has three-adic valuation one. Multiply the trace by three and reduce modulo the ramified prime: $3/a + 2/(f/3) = 0$ in $\mathbb{F}_3$. Consequently $m/3 \equiv (-1)^k$ and $n/3 \equiv (-1)^j \bmod 3$.

Write $D = MV^dU^\ell$, $c = m/M$, $h = m/n$, $r = \ell - k$, $z = j - k$. Integrality gives $c, h \in \mathbb{Z}_{>0}$ and $6 \mid c$ if $d = 1$. Also $r \geq 1$ by $D > E$, and $\ell \geq j$ by $D/D' \geq H/H'$. The complementary pair is irrational since otherwise Corollary 2.3 would give $s \leq t$, contrary to (5.1). In particular $j \geq 1$.

We claim $j \geq 3$ and $t > 1/180$. For $j = 1$, $6 \mid n$ by (6.1) and $\delta_H^2 = 12/n^2 < 1/3$, so $w \leq 1/3$. Then $s \geq 2t \geq 1/2$ by (5.1), contrary to $2\delta_D^2 \geq 3s^2/2$. For $j \geq 2$, the signed dominance and the recurrence give $\delta_D^2 \geq 48s^2/49$, $s/t \geq 7/2$: in the unit family use $A_{k+1}/A_k \geq 7/2$ for $k \geq 1$ and $\ell \geq 2$ for $k = 0$; in the other use $6B_{1,k+1} \geq 9A_k$. If $j = 2$, then $192/n^2 < 1/3$ gives $n \geq 30$ and $t \geq 7/60$. The mixed bound fails since $24t^2 + (48/49)(7/12 - t)^2 > 1/3$ on this interval. For $j \geq 3$, $\delta_H^2 \geq 675w^2/676$ gives $w < 26/45$, proving the claim.

Only relative indices remain. The two unit coefficients give

$$7m = 12(A_k + hA_j), \qquad m(12 - Z) - 12cB_{d,\ell} \in 12h\mathbb{Z},$$

where $Z/12$ is the rational complementary contribution on the Perron induction class; the H contribution belongs to $\mathbb{Z}/n$. Since $6h \mid m$, we obtain $h \mid 2A_k, 2cB_{d,\ell}$, so $h \mid 2cW_{0,d}(r)$ by Lemma 7.2. Also $h \equiv (-1)^z \bmod 3$. Since $2\delta_D^2 \geq (96/49)s^2$, we have $s < 1/2$ and $s/t < 90$. The same lemma gives $c \leq 32$. From $U^r \leq 2(6/V)^d s/t$ we get $r \leq 4$, hence $|W_{0,d}(r)| \leq 672$, $h \leq 43008$. Finally $1 < hA_j/A_k < 104$ and $U^z/2 \leq A_j/A_k \leq 2U^z$ give $-11 \leq z \leq 4$.

At $D = H$ apply Lemma 2.9. For the other tuples in these bounds the exact recurrence check uses the mass identity, scalar divisibilities and (8.1) at $k = 0, 1, 2$; none survives. For $k \geq 3$ all three indices are at least three. Put $x = hU^z$, $y = c(V/6)^dU^r$, $\eta = 999/1000$. Lemma 7.2 and the mixed bound give

$$147\eta^2(1 + x^2 + 2y^2) \leq 144(1 + x)^2.$$

Only $(d, r, c, h, z) = (0, 1, 1, 8, 1)$ survives. But then $7 \mid A_k + 8A_{k+1}$, whose residues modulo seven cycle through $6, 4, 3, 1, 1, 3, 4, 6$. This final contradiction proves the theorem. □

8.2. **Rational codegrees three and eight.**

Theorem 8.3. *The conclusion of Theorem 8.2 also holds when the two rational signed codegrees are three and eight.*

Proof. Use $K, U, V, E, H, D, t, w, s, A_i, B_i$ as in the preceding theorem and again assume nonsphericality. Here $t + w = L = 13/24$. Again (5.1) gives $s > t$, so H is irrational. The discriminant of the signed maximal order is a square. The signed trace discriminant has squareclass $24N(E)N(H)$, hence exactly one of E, H belongs to the family of multiples of VU^j.

The local factor of the codegree-three row. Denote its induction label by A and the eight-label by B. As in the proof of Theorem 6.3, the row of $M = 6S$ at A has entries one on both dimension pairs. Let a be its common entry on the H-pair and put $r = M_{A,A}$, $z = M_{A,B}$. The signed row sum and norm give

$$r + z = 1 - 2a, \qquad 2a^2 + r^2 + z^2 \le 7.$$

The rational eight-row takes the integral value $8z/3$ on $F(A)$. Thus $z = 0$ and (a, r) is $(0, 1)$ or $(1, -1)$.

At three the rational eight-axis splits off by its idempotent. The central element of simultaneous character values f/f' separates the V-family branch, with residue -1, from the rational three-axis. Since the latter cannot split off by itself, its local factor consists of that axis and the unit-family branch, of codegree $f = m_0U^i$. The integral trace of its identity is $1/3 + A_i/m_0$. As $3 \nmid A_i$, it follows that $3 \parallel m_0$. Writing $\mathfrak{p}$ for the prime of K over three, multiplication by three and reduction further give $1 + 2/(f/3) = 0$ in its residue field. Hence $f/3 \equiv 1 \pmod{\mathfrak{p}}$. If $f = H$, the signed values of $F(A)$ on the two paired branches are r and $aH/3$, so $r \equiv a \pmod 3$. Both possibilities above contradict this. Thus $f = E$. The signed character values of $F(A_D)$ on the paired branches are 1 and E/D, so $E/D \equiv 1 \pmod{\mathfrak{p}}$. This forces D to be in the unit family like E, since a V-family element has odd valuation at $\mathfrak{p}$. We have therefore proved

$$E = mU^k, \quad H = nVU^j, \quad D = MU^\ell, \qquad 3 \parallel m, \quad 6n \mid m, \quad M \mid m.$$

Here E/H and E/D are integral by induction. The coefficient of $\sqrt{3}$ in VU^j is odd, while that of a sum of squares in $\mathbb{Z}[\sqrt{3}]$ is even; thus n is even. We have $\ell \ge k + 1$ because $D > E$, and $\ell \ge j + 1$ because $D/D' \ge H/H' = U^{2j+1}$. Lemma 5.6 gives $H/H' \ge U^2$, hence $j \ge 1$.

Now $s/t \ge 7/2$: for $k \ge 1$ this follows from $A_{k+1}/A_k \ge 7/2$, and when $k = 0$ use $\ell \ge 2$. If $j = 1$, the mixed inequality gives $25/(3n^2) = \delta_H^2 < 1/3$, so $n > 5$. Since $3 \nmid n$ and $2 \mid n$, this gives $n \ge 8$, hence $w \le 3/8$, $t \ge 1/6$. But $2\delta_D^2 \ge (96/49)s^2 \ge 24t^2$, a contradiction. We have obtained

$$j \ge 2, \qquad \delta_H^2 \ge (361/363)w^2, \quad \delta_D^2 \ge (48/49)s^2, \quad s \le 7/(12\sqrt{2}) < 5/12. \tag{8.2}$$

Invertibles and dyadic local factors. On a positive lift of an invertible all character values are signs. A nonunit invertible's zero unit coefficient gives $t + \lambda w + e/3 + h_0/8 = 0$ with signs λ, e, h_0. As $L > 1/3 + 1/8$, one needs $\lambda = -1$. The four possible t are $1/24, 1/6, 3/8, 1/2$, of which only $1/24$ satisfies $24t^2 \le 1/3$. It forces $e = h_0 = 1$. Thus multiplication by the positive lift acts as a permutation with eigenvalue -1 exactly on the two H-rows. It is an involution since every ordinary simple is self-dual and the positive lift respects duality. Two transpositions occur, one interchanging the unit and this invertible. Accordingly the H-row has the form $(1, -1, z, -z, 0, 0)$; orthogonality to its conjugate gives $N(z) = -1$. That norm equation is impossible in $\mathbb{Z}[\sqrt{3}]$ modulo three. Thus all nonunit FP dimensions exceed one.

At two the rational three-axis is isolated. The other axes constitute a single local factor. Indeed, the eight-axis cannot be isolated, having nonintegral coefficient trace $1/8$. An isolated H-axis would have regular trace discriminant $N(H)$ of odd valuation, impossible since the field discriminant has even valuation. For the E-axis note that

$$13m/24 = A_k + (m/n)B_j \in \mathbb{Z},$$

so $24 \mid m$. Any order on an isolated K-axis has the form $A = \mathbb{Z}_2 + 2^v\sqrt{3}\,\mathbb{Z}_2$ for $v \geq 0$, since projection to the irrational coordinate gives a submodule of $\mathbb{Z}_2$. Its trace dual is $(2^{v+1}\sqrt{3})^{-1}A$, by direct computation or the derivative formula, and is $E^{-1}A$ by perfectness. Thus $mU^k/(2^{v+1}\sqrt{3}) \in A^\times$, giving $v = v_2(m) - 1 \geq 2$. Writing $U^k = c_k + d_k\sqrt{3}$ now gives $4 \mid c_k$, impossible since $c_0 = 1, c_1 = 2, c_{k+2} = 4c_{k+1} - c_k$. This establishes the claimed local decomposition.

Transfer the factor ranks across $R/2R \simeq T/2T$. As in Lemma 6.8, the rank-one factor is the unique odd rational-codegree row, of norm $a \equiv 3 \bmod 4$. All even-codegree rational rows of R reduce to the unique residue character of the other local factor, the reduction of the eight-row of T. A norm-eight integral vector of length six with unit coordinate one has exactly four odd entries. Hence these rational ordinary codegrees are divisible by four. Two rational four-rows would have the same support. If e_1, e_2 are their primitive idempotents, $u = 1 - 2(e_1 + e_2)$ is integral by the row idempotent formulas and the common support. It satisfies $u^2 = 1$, has unit coefficient zero and ordinary FP value one. Since $\|u\|^2 = 1$, it is a basis element of FP dimension one, a contradiction. Thus there is at most one rational ordinary four-row.

The ordinary spectrum. If an ordinary complementary pair had irrational codegrees of mass z, one would have $z \geq 5/12 - s > 0$ and squared gap at least $z^2/3$ as before. By (8.1) and (8.2),

$$1/3 \;\geq\; (361/363)(13/24 - 2s/7)^2 + (96/49)s^2 + (2/3)(5/12 - s)^2.$$

The minimum exceeds $1/3$ by $222697/41601600$, a contradiction. Also $s < 1/4$, since the sum of the first two indicated terms is increasing on $s \geq 1/4$ and at $1/4$ exceeds $1/3$ by $92425/10245312$.

Thus the four ordinary complementary codegrees are rational. Lemma 8.1 makes the four-rows rational and excludes quadratic six-orbits.

Since $1 - s > 3/4$ and at most one complementary character has codegree four, the odd codegree must be $a = 3$: for $a \geq 7$ the sum is at most $1/7 + 1/4 + 2/6 < 3/4$. A four-row is necessary, since all three even rows would otherwise have norms at least eight: Lemma 8.1 makes a six-row integral, whereas integral even rows here have norm divisible by four.

If the two remaining characters form a quadratic orbit of common rational codegree g, the regular trace discriminant has squareclass $N(D) \cdot 3 \cdot 4g^2 \sim 3$, while the maximal order has squareclass $3\,\mathrm{disc}(F)$, impossible for a real quadratic field F. Thus all four characters are rational. Write their codegrees as $3, 4, c, d$, where $8 \leq c \leq d$ are multiples of four. The strict inequality $1/3 + 1/4 + 1/c + 1/d > 3/4$ forces $c = 8$ and $d < 24$. The discriminant squareclass requires $8d$ to be a square, so $d = 8$.

The four ordinary rows. The four-row and both eight-rows have the same four odd positions S. An eight-row is ± 1 on S with just one further entry ± 2. The latter entries must be in different positions: otherwise orthogonality forces the two rows to agree on S (their unit entries are one) and have opposite even entries. Their difference would be supported at one position, contrary to orthogonality to the Perron row.

Negate basis elements, keeping the unit fixed, so the four-row is one on S and both even entries are positive. This preserves the integrality of the multiplication table. On S the three rows are now pairwise orthogonal sign vectors. Permuting the three nonunit positions gives the first three rows below. The three-row meets S in the unit and exactly one other position, by orthogonality to the four-row. Up to interchanging the eight-rows its third entry lies at the first even position. Orthogonality then fixes the entire row:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 2 & 0 \\ 1 & -1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

The Perron row in this signed basis must consequently be $(1, y-2, y, 1-2y, 1-y, -1-y)$. Put $v = 2y - 1$. Its squared norm is $D = 2(v^2 + 3)$, and its orthogonality to the conjugate row gives

$vv' = -3$. At the fifth basis element, of Perron value $(1-v)/2$, the unit coefficient of its cube is the integer

$$\mathrm{Tr}_K\left(\frac{(1-v)^3}{8D}\right) + \frac{2^3}{8} - \frac{1}{3} = \frac{14 - \mathrm{Tr}_K(v)}{16}.$$

The equality follows from $v' = -3/v$. Write $v = A + B\sqrt{3}$. Then $A \equiv 7 \bmod 8$; the norm equation gives $3B^2 = A^2 + 3 \equiv 4 \bmod 16$, which is impossible. □

8.3. **Two rational four-rows with different supports.** The same calculation at two serves both remaining conductors. We use the notation $u, v, B_{e,k}$ of Lemma 7.1.

Proposition 8.4. *Suppose $T_{\mathbb{Q}} = K^2 \times \mathbb{Q}^2$, with K of discriminant $c = 8$ or 12. Suppose the two rational characters have codegree four and different supports. At two the order has two local factors of rank three, each containing one rational and one quadratic branch. For either quadratic branch write $f = mv^e u^k$ for its oriented codegree and $\mathbb{Z}_2 + 2^b\mathcal{O}_{K,2}$ for its projected order. Then*

c	e	$v_2(m)$	b
8	1	3	1
12	0	3 (k *even*), 4 (k *odd*)	$v_2(m) - 2$.

At conductor twelve, $3 \mid m$ as well.

Proof. Each rational row has exactly four entries ± 1, counting its unit entry. Different supports distinguish the residue characters. Neither rational branch can form a local factor by itself, since its idempotent has unit coefficient $1/4$. Both quadratic branches are ramified, so each rational branch takes precisely one.

Write A for one local factor and B for its quadratic projection. Perfectness of the coefficient-of-unit pairing gives $A \cap (\mathbb{Q}_2 \times 0) = 4\mathbb{Z}_2 \times 0$: the rational projection is $\mathbb{Z}_2$ and its trace weight is $1/4$. Thus $[\mathbb{Z}_2 \times B : A] = 4$. Computing the regular discriminant first by index and then by formal codegrees gives

$$4 + v_2(c) + 2b = 2 + v_2(N(f)).$$

At conductor eight this forces $e = 1$, at conductor twelve $e = 0$, and in both cases $v_2(m) = 2 + b$. The local identity has integral unit coefficient $1/4 + B_{e,k}/m$, so $b = v_2(B_{e,k})$. The trace recurrence gives the table. At conductor twelve the quadratic row-difference test also gives $3 \mid m$ as in Theorem 8.2. □

Lemma 8.5. *In Proposition 8.4, let t, w be the reciprocal masses of the signed quadratic branches. A nonunit invertible forces $t = w = 1/4$. In the ordinary ring every integral character has norm divisible by four, and no orbit has odd common rational codegree. If the unit is the only invertible, there are at most two ordinary rational four-rows.*

Proof. The signed four-rows intersect in precisely two coordinates: their intersection contains the unit, has even size by orthogonality, and has size less than four. At the other coordinate their signs are opposite. An invertible must belong to both supports. All signed characters are real, so at an invertible each quadratic pair has a common sign. The zero unit coefficient of a nonunit invertible now gives $\pm t \pm w = 0$, and $t + w = 1/2$.

The parity isomorphism identifies the two residue characters and the ranks of their local factors in the ordinary and signed orders. Every integral ordinary row therefore has four odd entries. An orbit of odd rational codegree is an isolated maximal unramified factor by Lemma 2.6. Its residue fields are all $\mathbb{F}_2$, so it would have a local factor of rank one. Finally two rational four-rows with equal support give a nonunit invertible by $1 - 2(e_1 + e_2)$, as in Theorem 8.3. There are only two residue characters, hence only two possible supports. □

Theorem 8.6. *Under the hypotheses of Proposition 8.4, suppose the ordinary Frobenius–Perron field is also $K = \mathbb{Q}(\sqrt{3})$ and the ordinary character algebra is commutative. Then the category is spherical.*

Proof. Use U, V, A_k, B_k, t, w, s as in Theorem 8.2. Suppose the category is nonspherical and choose $E \geq E'$. Write $E = mU^k$ and $H = nU^j$ for the quadratic signed codegrees by Proposition 8.4. We have $t + w = 1/2$. If H is rational the positivity argument on the ordinary Frobenius–Perron induction label in Theorem 8.2 gives $s \leq t$, contrary to strict nonspherical dominance. Thus $j \geq 1$. The integrality of E/D and $D > E$ give $s \geq 2t$, as in that theorem. If $j = 1$, then $48 \mid n$ by the dyadic proposition. Hence $w \leq 1/12$, $s \geq 5/6$ and $2\delta_D^2 \geq (3/2)s^2 > 1/3$, contradicting the mixed projection bound. Therefore $j \geq 2$. Central dominance $D/D' \geq H/H'$ and the same unit computations as in Theorem 8.2 give

$$s \geq 7t/2, \qquad \delta_H^2 \geq 48w^2/49, \qquad \delta_D^2 \geq 48s^2/49.$$

Put $F(s) = (48/49)(1/2 - 2s/7)^2 + (96/49)s^2$. Then the mixed bound gives $F(s) \leq 1/3$. This quadratic is increasing on $[7/24, 1]$ and $F(7/24) > 1/3$, hence $s < 7/24$.

By Lemma 8.5, a nonunit invertible would give $t = w = 1/4$, contradicting $s \geq 7t/2$ and $s < 7/24$. Hence there are at most two rational four-rows. Integral rows have norm divisible by four, and odd common rational codegrees are excluded.

Lemma 8.1 also excludes a real quadratic orbit of common codegree four or six.

Consider an irrational ordinary complementary pair of mass z. Every irrational codegree lies in K and its squared gap is at least $z^2/3$. If there are two such pairs counting multiplicity, their combined mixed contribution is at least $(1 - s)^2/3$, but $F(s) + (1 - s)^2/3 \geq 8268/17089 > 1/3$. With just one pair, the remaining two rows of rational codegree have reciprocal mass at most $3/8$, unless both are rational four-rows. Indeed an integral row has norm divisible by four and a quadratic orbit of common norm has even codegree at least eight. Without two four-rows we get $F(s) + (2/3)(5/8 - s)^2 \geq 7617/19490 > 1/3$. In the two-four-row case the two distinct dyadic local factors each contain one rational row and one ramified quadratic branch K. In the factor containing the irrational complementary pair, the regular trace discriminant has valuation $2 + v_2(N(h))$, where h is its oriented codegree. The maximal order has discriminant valuation two, so $v_2(N(h))$ is even. Thus h belongs to the U-family, for which the squared gap is at least $3z^2/4$. Since $z = 1/2 - s$, we obtain the final contradiction $F(s) + (3/2)(1/2 - s)^2 \geq 204/515 > 1/3$. All these displayed lower bounds follow by completing the square.

The four ordinary complementary codegrees are now all rational, even, with reciprocal sum $1 - s > 17/24$. At least one is four, since four rows of codegree at least six contribute at most $2/3$. Every four-row is rational by Lemma 8.1.

If there were just one four-row, the mass would require a six-row. It would be integral by Lemma 8.1, contradicting divisibility of its norm by four.

There are consequently two rational four-rows. If the remaining two rows are rational their codegrees are multiples of four at least eight, and their reciprocal sum exceeds $5/24$. This forces both codegrees to be eight. The product of the codegrees has squareclass $N(D)$, either one or six, whereas the character algebra $K \times \mathbb{Q}^4$ has discriminant squareclass three. If the remaining rows form a quadratic orbit their common even codegree is at least eight and the same reciprocal sum forces eight. Its quadratic discriminant c divides 16 and satisfies $2c \leq 16$, so the field is $\mathbb{Q}(\sqrt{2})$. Each of the two rational four-rows lies in a different rank-three dyadic factor. The factor containing $\mathbb{Q}_2(\sqrt{2})$ has regular discriminant valuation $v_2(4 \cdot 8^2) = 8$, while its maximal order has valuation three. These valuations must differ by twice an index valuation. This excludes the last possibility. □

8.4. **An involution fixing a near-group subcategory.**

Proposition 8.7. *Let $\mathcal{C}$ be a rank-six complex fusion category whose sphericalization has commutative Grothendieck ring. Suppose all ordinary Frobenius–Perron dimensions lie in $K = \mathbb{Q}(\sqrt{3})$. Suppose a nonidentity invertible g is an involution with action $(1, g)(X, Y)$ on the simples and fixed points P, Q. If $1, g, P$ form a fusion subcategory with $P^2 = 1 + g + 2P$, then $\mathcal{C}$ is spherical.*

Proof. Write $p = 1 + \sqrt{3}$, $x = \mathrm{FPdim}(X) = \mathrm{FPdim}(Y)$ and $q = \mathrm{FPdim}(Q)$. The span of X, Y, Q is a module for $1, g, P$ by reciprocity. Since $P = P^*$ and $gP = P$, the matrix of P on this span is

$$\begin{pmatrix} a & a & c \\ a & a & c \\ c & c & d \end{pmatrix} \quad (a, c, d \in \mathbb{Z}_{\geq 0}).$$

On the symmetric subspace its eigenvalues solve $z^2 - 2z - 2 = 0$. This quadratic is irreducible over $\mathbb{Q}$, so the trace and determinant there give $2a + d = 2$ and $2ad - 2c^2 = -2$. Thus $c = 1$ and (a, d) equals $(0, 2)$ or $(1, 0)$.

In the first case $PX = PY = Q$, so $q = px$ and Q is self-dual: X^* belongs to $\{X, Y\}$, since duality permutes the g-orbits and fixes P and the unit. The multiplicities of $1, g, P$ in XX^* are $1, 0, 0$ by reciprocity. Therefore $x^2 = 1 + (r + sp)x$ for nonnegative integers r, s. This makes x an algebraic unit. No element of $\mathbb{Z}[\sqrt{3}]$ has norm -1, modulo three. Thus $x - 1/x = x - x'$ has rational trace zero. The nonnegative coefficients of $r + sp$ force $r = s = 0$ and $x = 1$. The four objects $1, g, X, Y$ are precisely the invertibles.

In the second case $PQ = X + Y$, so $x = pq/2$. Here again $Q = Q^*$. The multiplicities of $1, g, P$ in Q^2 are $1, 1, 0$. The equality $gQ = Q$ gives equal multiplicities on X, Y. It follows that

$$q - 2/q = hp + k, \qquad h, k \in \mathbb{Z}_{\geq 0}.$$

In particular q divides two in $\mathbb{Z}[\sqrt{3}]$, and $q > 1$ since Q has nontrivial stabilizer. The ramification $(2) = (1 + \sqrt{3})^2$ as ideals and the positive unit group $\langle U \rangle$, $U = 2 + \sqrt{3}$, give $q = U^n, (1 + \sqrt{3})U^n$ or $2U^n$, with $n \geq 0$. For the unit option $q - 2/q$ has negative trace. For the middle option $N(q) = -2$, so $q - 2/q = q + q'$ is rational and $h = 0$. For the last option write $U^n = A + B\sqrt{3}$. Then $h = 3B$ and $k = A - 3B$. For $n \geq 1$ we have $B \geq 1$ and $A^2 = 3B^2 + 1 < 9B^2$, which is impossible. At $n = 0$ again $h = 0$. Therefore $Q^2 = 1 + g + kQ$ and $1, g, Q$ form a fusion subcategory.

It remains to lift the category. The subcategory on $1, g, P$ has a spherical tensor section of the sphericalization by the corank-one sphericity theorem [4, Theorem 4.5]. The second subcategory, on $1, g, X, Y$ in the first case and $1, g, Q$ in the second case, likewise has a spherical tensor section, positive on its invertible objects in the pointed case. All these sections choose the lift of g of dimension one: in the rank-three subcategories it fixes a simple of nonzero spherical dimension. These two subcategories generate $\mathcal{C}$, by $PX = Q$ or $PQ = X + Y$, and their lifted images have the same intersection $\langle g \rangle$. For commuting Grothendieck subrings the regular Perron sums satisfy

$$R_{\mathcal{A}} R_{\mathcal{B}} = \mathrm{FPdim}(\mathcal{A} \cap \mathcal{B})\, R_{\mathcal{A} \vee \mathcal{B}}.$$

Indeed their product is a common Perron vector for multiplication by the simples of $\mathcal{A}, \mathcal{B}$, and the unit coefficient determines its scalar. Apply this identity downstairs and upstairs, where commutativity is a hypothesis. The lifted join has the same Perron dimension as $\mathcal{C}$ and maps dominantly to it, hence equivalently. Its spherical structure descends. □

8.5. **The folded codegree-four pairs at conductor twelve.**

Lemma 8.8. *Suppose the ordinary and signed character algebras are commutative of rank six, their respective Frobenius–Perron and spherical dimension fields are $K = \mathbb{Q}(\sqrt{3})$, and the signed rational algebra has two K factors. Assume the other two signed characters have common codegree four and are either rational with the same support, or Gaussian conjugates. If no Galois conjugate is pseudo-unitary, write $t = \mathrm{Tr}_K(1/E)$ and $s = \mathrm{Tr}_K(1/D)$ and orient $E \geq E'$. Then $t > 1/128$.*

Moreover, either $s > 2/9$ or the ordinary complementary irrational pairs contribute at least $(2/3)(1/2 - s)^2$ to the mixed squared-gap bound.

Proof. Write $w = 1/2 - t$ for the other signed mass, of codegree H. The two short signed rows can be written $(1, -1, \pm z, \mp z, 0, 0)$ on $1, g, X, Y, P, Q$, with $z = 1$ or i. For rational rows with equal supports this follows by adding and subtracting their two orthogonality relations with the positive row. For the Gaussian pair, X, Y are the unique dual pair, and norm four and positivity force the stated row. In either case the four real signed rows take value one at g and agree at X, Y. Thus $g^2 = 1$ in the signed ring. Its multiplication matrix is integral orthogonal and preserves the strictly positive spherical dimension vector, so it is a permutation matrix. Its minus-one space has dimension two and contains the two differences $1 - g, X - Y$. Consequently its permutation is $(1, g)(X, Y)$.

Equation (5.1) gives $s \geq 2t$. The complementary four-rows contribute at least $1/2$ on the ordinary Perron column. Thus H is irrational by Corollary 2.3; and $H/H' \geq U^2$ by Lemma 5.6, where $U = 2 + \sqrt{3}$ and $V = 3 + \sqrt{3}$. Central dominance gives $D/D' \geq H/H'$,

so $\delta_D^2 \geq 3s^2/4$ and $s < 1/2$ by the mixed inequality. If X or Y were invertible its spherical value would be one, its H-values equal signs, and the four-row pair would contribute zero to its unit coefficient. Then $t = w = 1/4$, contradicting $s < 1/2$. Every nonunit invertible must have nonzero value on the short rows, so g is the only possible one.

At two, every signed branch is ramified or rational, with residue field $\mathbb{F}_2$. The two short rational rows have the same reduction, so their rational branches share a dyadic local factor. Their joint projected order consists of pairs congruent modulo two: the unit and the value pair $(1, -1)$ at X generate this order. They cannot form an isolated local factor, whose regular discriminant would have valuation $v_2(4^2) = 4$ by the perfect codegree trace, whereas their projected order has discriminant valuation two. In the Gaussian case the projection is the full $\mathbb{Z}_2[i]$, again of discriminant valuation two and so not isolated. There are therefore at most two dyadic local factors, all of rank at least two and residue degree one. This carries over to the ordinary order via the parity of the structure constants and lifting idempotents. Lemma 2.6 excludes odd rational codegrees: their unramified maximal factors with residue field $\mathbb{F}_2$ would have rank one.

All rational-codegree ordinary rows have norm at least four. In the absence of nonunit invertibles, a rational two-row is impossible by the idempotent formula, and an orbit of common codegree two cannot be nonrational (each nonunit coordinate is a totally real or complex algebraic integer of house at most one, hence zero or a root of unity; there is room for just one nonzero entry, and duality and orthogonality to the FP row force it to be -1). If g is invertible its ordinary action agrees with the signed permutation on positive lifts. The two characters nontrivial at g have rows $(1, -1, \lambda, -\lambda, 0, 0)$. In the Gaussian case $Y = X^*$, so λ is nonzero purely imaginary. In the real case all simples are self-dual, so λ is real. Evaluating X^2 gives $\lambda^2 = \pm 1 + n\lambda$ with integral n, the constant being minus one in the Gaussian case and plus one in the real case by reciprocity. The first case forces $n = 0, \lambda = \pm i$. In the second, if $n \neq 0$ the roots are quadratic conjugates of norm -1 and the codegrees $4 + 2n\lambda$ lie in K by central induction, which is impossible modulo three. Thus $\lambda = \pm 1$. Every remaining ordinary row takes value one on g and cannot be a two-row by orthogonality to FP.

If there is one irrational ordinary pair of mass z, the other two rational-codegree rows contribute at most $1/2$. Hence $z \geq 1/2 - s$ and the extra mixed contribution is at least $(2/3)(1/2 - s)^2$. In the case of two irrational pairs (including multiplicities) their combined contribution is at least $(1 - s)^2/3 \geq (2/3)(1/2 - s)^2$. Suppose henceforth that all complementary codegrees are rational and $s \leq 2/9$. Their reciprocal sum is at least $7/9$, so at least two rows have codegree four. Real four-rows and six-rows are rational by Lemma 8.1.

Any nonreal ordinary row has imaginary quadratic character field F: there is only one nonreal pair, and character fields are abelian. A nonreal pair of common codegree h has $h = |\operatorname{disc}(F)|b^2$,

$b \in \mathbb{Z}_{>0}$, by orthogonality to its conjugate, since X, Y are the unique dual pair. Hence it cannot have codegree six. At codegree four its field is Gaussian and its row has exactly one nonunit real entry, equal to -1 on an invertible by orthogonality to FP. This forces g to be an actual invertible by the preceding discussion.

In the absence of a nonunit invertible there are therefore at least two integral ordinary four-rows. They have different supports, since the usual idempotent $1 - 2(e_1 + e_2)$ from two four-rows with equal support would be a nonunit invertible. Their distinct supports exhaust the possible dyadic residue rows, so all integral ordinary rows have norm divisible by four, and there are at most two ordinary four-rows. The remaining rows have even norms at least eight: the only other possibility, a nonrational pair of common codegree six, was just excluded. Their total mass is at most $3/4$, a contradiction.

It remains to consider actual g. The two g-negative rows are the matched four-rows above. The other ordinary rows are all real and have shape $(1, 1, z, z, b, c)$. If neither remaining row has codegree four, the reciprocal sum requires $(6, 6)$ or $(6, 8)$ for their two codegrees. Both are integral rows by the quadratic orbit exclusion at six. At $(6, 6)$ the product of the ordinary codegrees has absolute squareclass $N(D)$, one or six. The maximal order has absolute discriminant squareclass three since the matched anti-invariant algebra is rational split or Gaussian. At $(6, 8)$ one would have $s = 5/24$. By Lemma 5.4, $U^3 \equiv 1 \pmod 5$ excludes this numerator, since no power D/D' of U can equal -1 modulo five as $1 + D/D' = Ds$ would require.

If there is a third four-row it has shape $(1, 1, 0, 0, -1, 1)$ after relabelling. The alternative supported at X, Y would force these objects invertible, and the two tail signs cannot be equal since both fixed objects have dimension greater than one. Thus $d_P = d_Q + 2$. The last row must be integral and has shape $(1, 1, z, z, b + 2, b)$ by orthogonality. Its even codegree $h \leq 36$ has squareclass two or three by the same discriminant comparison, hence $h = 8, 12, 18$ or 32. Since $h = 4 + 2z^2 + 2(b + 1)^2$, the last two would represent seven or fourteen as a sum of two squares. At $h = 12$, either $z = \pm 2, b = -1$ forces $d_X = 1$, or $z = 0, b = 1, -3$ contradicts $d_P = d_Q + 2$ and $d_Q > 1$. At $h = 8$ the two possibilities give the FP rows $(1, 1, x+1, x+1, x, x-2)$ and $(1, 1, x - 1, x - 1, x + 2, x)$. Orthogonality with the conjugate yields $N(x) = -2$, and $D = 4(x^2 + 2) = 4x(x - x')$. At the fixed simple of dimension x the coefficient of its own simple in its square is $\mathrm{Tr}(x^3/D) \pm 3/4 = (\mathrm{Tr}(x) \pm 3)/4$. This contradicts the even trace in $\mathbb{Z}[\sqrt{3}]$. Thus $s > 2/9$ whenever the ordinary complement has only rational codegrees.

We now bound w. If $2j + e = 2$, the mixed inequality excludes $n = 6$ by $\delta_H^2 = 12/n^2$ and the unit-family multiple-of-six condition, so $w = 4/n \leq 1/3$. For $2j + e = 3$ we have $\delta_H^2 = 25/(3n^2)$ and $2 \mid n$ by the even irrational coefficient of a sum of squares in K. Since $w = 3/n < 1/2$, $n \geq 8$ and $w \leq 3/8$. For exponent at least four, both H and D have squared gap at least $48/49$ times their squared reciprocal mass. If $s > 2/9$, the mixed inequality gives $w^2 < 49/144 - 2(2/9)^2 < (63/128)^2$. With an extra irrational ordinary contribution it gives $\delta_H^2 \leq 1/3 - \min_s[(3/2)s^2 + (2/3)(1/2 - s)^2] = 17/78$. Thus again $w^2 \leq (49/48)(17/78) < (63/128)^2$. In every case $t = 1/2 - w > 1/128$. □

Theorem 8.9. *Suppose a rank-six complex fusion category has commutative ordinary and signed character algebras, and both dimension characters have field $K = \mathbb{Q}(\sqrt{3})$. Suppose its signed rational character algebra is $K^2 \times F$, with the dimension character in the first quadratic factor. Each of the two complementary characters of F has formal codegree four. If $F = \mathbb{Q}(i)$, or if $F = \mathbb{Q}^2$ and its characters have the same support, then the category admits a spherical structure.*

Proof. Suppose otherwise. Use $U = 2 + \sqrt{3}$, $V = 3 + \sqrt{3}$, and E, H, D, t, w, s as in Theorem 8.2, with $E \geq E'$ and $H \geq H'$. We have $t + w = 1/2$, $D > E$, $s \geq 2t$, and $D/D' \geq H/H'$. The inequality $s \geq 2t$ also follows from the trace rearrangement since $\mathrm{Tr}(E/D) \geq 4$ for a totally positive nonrational integer of K with least conjugate less than one. The other signed K-factor

has irrational codegree: if H were rational, all complementary signed orbits would contribute at least their reciprocal mass on the ordinary FP induction column, contradicting $s > t$.

By Lemma 8.8 the only possible nonunit invertible is the second coordinate g in the short rows, whose action, if it is invertible, has two free orbits and two fixed simples.

We may take $D/D' > H/H'$. For if equality holds, the real signed H-character attains the ordinary Perron bound on the central conjugation witness, whose FP dimension D/D' is irrational. Its equality subcategory is spherical. If proper it has field K (it contains that witness) and rank at most four by Lemma 2.10. The pivotal classifications [18, 20, 5] leave the rank-three E_6 ring, the C_3 Tambara–Yamagami ring and the rank-four C_3 near-group ring of parameter six. Our invertible-object restriction excludes both C_3 cases. The E_6 case requires the actual involution g and implies sphericality by Proposition 8.7. If the equality subcategory is the whole category the spherical section already exists.

Write $P_{e,i} = V^e U^i$ and $B_{e,i} = \mathrm{Tr}(1/P_{e,i})$, with $e \in \{0, 1\}$. The d-number and row-orthogonality calculations from Theorem 8.2 give unit-family scalars divisible by six and V-family scalars divisible by two (the coefficient of $\sqrt{3}$ in a sum of squares is even, and that in VU^i is odd). The absolute order discriminant is $16N(EH)$; the absolute maximal-order discriminant is a square, also for $F = \mathbb{Q}(i)$. Hence E, H are in the same family. Put

$$E = mP_{e,k}, \qquad H = nP_{e,j}, \qquad D = MP_{d,\ell}, \qquad h = m/n, \quad c = m/M.$$

The quantities h, c are positive integers, and $6 \mid c$ if $d > e$, by integrality of $E/H, E/D$. Moreover $4 \mid m$ by $E/4 \in \mathcal{O}_K$. At the FP induction label the remaining two signed codegrees give $Z/4$ with $Z = 2$ or 3, by integral positivity or by taking the trace of a totally positive pair. Its H-mass is positive, strictly smaller than w, and belongs to $\mathbb{Z}/n$, since $\mathrm{Tr}((a+b\sqrt{3})/V) = a-b$. In particular $H \neq nV$ and $j \geq 1$, so $D/D' \geq H/H' \geq U^2$.

A uniform recurrence calculation. We use the preceding bound $t \geq 1/128$ and the alternatives $s > 2/9$ or an additional ordinary contribution at least $(2/3)(1/2 - s)^2$ in the mixed Fourier inequality. Set $r = \ell - k$ and $z = j - k$, and abbreviate $b_0 = B_{e,k}$, $b_1 = B_{e,j}$, $b_2 = B_{d,\ell}$. The two column mass identities give

$$m = 2(b_0 + hb_1), \qquad m(4 - Z) - 4cb_2 \in 4h\mathbb{Z}_{>0}. \tag{8.3}$$

In particular $h \mid 4b_0, 4cb_2$. The ordinary Fourier bound and $D/D' \geq U^2$ give $s \leq \sqrt{2}/3$, hence $t \leq s/2 < 1/4$ and $s/t < 61$. Lemma 7.2 gives $c \leq 27$, $1/V \leq U^r < 12 \cdot 61V$, so $-1 \leq r \leq 6$. The initial recurrence pairs give $0 < |W| \leq 9360$ on the allowed range, with $h \mid 4cW$. Finally $1 < hb_1/b_0 = 1/(2t) - 1 < 64$ and $U^z/2 \leq b_1/b_0 \leq 2U^z$ give $-14 \leq z \leq 3$.

The relative-index check tests these tuples with $6 \mid c$ if $d > e$, $V^{d-e}U^r > c$ and $2z + e < 2r + d$. At $0 \leq k < 28$, no tuple with $j, \ell \geq 1$ satisfies (8.3), the scalar divisibilities, $s \geq 2t$ and both Fourier conditions

$$G = \delta_E^2 + \delta_H^2 + 2\delta_D^2 \leq 1/3, \qquad s > 2/9 \quad \text{or} \quad G + \tfrac{2}{3}(1/2 - s)^2 \leq 1/3.$$

For $k \geq 28$ take $I = 14$ and $\eta = 999999/1000000$ in Lemma 7.2. The resulting bounds for the gaps and s, with $s < 1/2$, leave

$$(e, d, r, c, h, z) = (0, 0, 1, 1, 6, 0),\ (0, 0, 1, 1, 24, -1),\ (0, 0, 1, 3, 24, 0),\ (1, 1, 1, 3, 24, 0).$$

Modulo $\mathrm{lcm}(4h, 6h, 6c)$ their periods are $12, 24, 24, 144$. None satisfies (8.3) and the scalar divisibilities. □

9. The quadratic field of conductor eight

When every complementary signed codegree is irrational, the two projection estimates again force one orbit of reciprocal mass $1/2$. When the complement has rational codegrees, the same estimates leave a pair of codegree four. Its two rows determine the induction objects needed for the final descent.

9.1. **Three quadratic character pairs.** Let $K = \mathbb{Q}(\sqrt{2})$, $u = 3 + 2\sqrt{2}$ and $v = 2 + \sqrt{2}$.

Lemma 9.1 (Three ordinary four-rows)**.** *Suppose the ordinary algebra is commutative with Perron field K and the signed rational algebra is K^3. If its four ordinary complementary codegrees are even and at least four, then*

$$s = \mathrm{Tr}(1/D) < 21/100 \quad \Longrightarrow \quad \textit{three ordinary codegrees are four.}$$

They force an actual involution fixing precisely two simples, and a signed v-family pair of reciprocal mass $1/2$. If $s < 1/6$, the ordinary codegrees are $4, 4, 4, 8$ and $D = 8A_\ell u^\ell$, where $A_\ell = u^\ell + u^{-\ell}$.

Proof. Sorting the denominators with reciprocal sum $1 - s > 79/100$ leaves $(4, 4, 4, d)$, $(4, 4, 6, 6)$ or $(4, 4, 6, 8)$, with largest denominator at most 24. The signed algebra is totally real, so all simples are self-dual. All four rows are rational by Lemma 8.1 and Galois invariance. Their product times $N(D)$ has squareclass two, with $N(D)$ a square or twice a square. This excludes $(4, 4, 6, 8)$. At $(4, 4, 6, 6)$ the two six-rows are distinct modulo three by squared distance twelve. Each needs a partner, whereas the four-axes split off and three is inert in K. Finally at $s < 1/6$ we get $4 < d < 12$, and the same squareclass test gives $d = 8$, $N(D)$ square and $s = 1/8$.

Each rational codegree-four row has four entries ± 1, including the unit. By orthogonality the two-element zero sets of two such rows coincide or are disjoint. There are three such sets on five nonunit coordinates, so two coincide. Their sum has support the unit and a nonunit g, with coefficients $2, -2$ by orthogonality to the Perron row. Thus $\mathrm{FPdim}(g) = 1$ and $1 - g$ is twice the sum of the two primitive idempotents. Hence g is an involution with a two-dimensional minus-eigenspace. Consequently g swaps $1, g$ and X, gX for one additional simple X, and fixes the other two. The same is true on the positive signed basis. The two signed characters with $g = -1$ therefore form a full K-orbit and have rows $(1, -1, q, -q, 0, 0)$. On the two-dimensional space with basis $1 - g, X - gX$, signed multiplication by X has matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & d_0 \end{smallmatrix}\right)$ by self-duality and symmetry. Thus $qq' = -1$; q cannot be rational since the characters are distinct and their values on X determine them. The quotient of their codegrees $2(1 + q^2)$ is q^2 up to inversion, an odd power of u, since the norm-negative units of K are the signed odd powers of $1 + \sqrt{2}$. This orbit is therefore in the v-family and has reciprocal mass $1/2$. □

Theorem 9.2. *Suppose $\mathcal{C}$ has rank six, its ordinary character algebra is commutative, its signed rational algebra is K^3, and the signed dimension and ordinary Frobenius–Perron characters both have field K. Suppose both complementary signed codegrees are irrational. Then $\mathcal{C}$ is spherical, or its two complementary orbits can be represented by*

$$H = 8 + 4\sqrt{2}, \qquad G = nv^e w^j, \qquad e \in \{0, 1\}, \quad j \geq 2, \quad n \in \mathbb{Z}_{>0}.$$

In the latter case $s = \mathrm{Tr}(1/D) < 1/3$.

Proof. Argue under the assumption that $\mathcal{C}$ is nonspherical and orient $E \geq E'$ by Galois transport. As in the preceding section write $t = \mathrm{Tr}(1/E)$, $s = \mathrm{Tr}(1/D)$, and $\delta_f = 1/f' - 1/f$. The orbit masses are t, w, z on the signed side, with $t + w + z = 1$. All codegrees lie in K by the dimension formula $E/f = \dim F(A_f)$. For a conjugate ordinary pair with irrational codegrees L, L' include one term δ_L^2, counting repeated pairs separately. The modular projection inequality is

$$\sum_{\text{signed pairs}} \delta_f^2 + 2 \sum_{\text{ordinary pairs}} \delta_f^2 \leq 1/2. \tag{9.1}$$

This is (2.4) at discriminant eight. Lemma 2.1 gives $D/D' \geq f/f'$ by spectral domination. Equation (5.1) gives $s \geq 2t$.

Write $f = mv^e u^k$ as in Lemma 7.1, and put $A_k = B_{0,k}$, $B_k = B_{1,k}$. The squared distance between a quadratic row and its conjugate gives $8 \mid \mathrm{Tr}(f)$. Since $\mathrm{Tr}(u^k) \equiv 2 \pmod 4$ and

$\mathrm{Tr}(vu^k) \equiv 4 \pmod 8$, we have $4 \mid m$ for $e = 0$ and $2 \mid m$ for $e = 1$. Both reciprocal trace sequences are $2 \pmod 4$.

Suppose first that both complementary signed exponents are at least two. Their squared gap ratios $\delta_f^2/\mathrm{Tr}(1/f)^2$ and that of D are then at least $c = 288/289$, since $f/f' \geq u^4$. We also have $s/t \geq 10/3$. In the same primitive family the ordinary exponent increases by at least one; the successive trace ratios are at least 5 except $A_1/A_0 = 3$, which is excluded by $D/D' \geq u^4$. From unit-family $E = mu^k$ to $D = Mvu^\ell$ the integer m/M is even and $\ell \geq k$, giving $s/t \geq 2B_k/A_k \geq 10/3$ for $k \geq 1$. From $E = mvu^k$ to $D = Mu^\ell$ one has $\ell \geq k+1$ and $s/t \geq A_{k+1}/B_k \geq 17/5$ for $k \geq 1$. In either cross-family case $k = 0$ and $D/D' \geq u^4$ give the same asserted bound. Hence, if $s \geq 1/6$, the left side of (9.1) is at least

$$c((1 - 3s/10)^2/2 + 2s^2) > 1/2.$$

The quadratic is increasing for $s \geq 1/6$ and exceeds $1/2$ already at that endpoint. Thus $s < 1/6$ and $t < 1/20$.

Every rational ordinary codegree is even and at least four. Indeed the completed signed order at two has residue fields $\mathbb{F}_2$ and has no local factor of rank one, because its normalization has three ramified quadratic branches. Both facts transfer by $R/2R \simeq T/2T$, lifting idempotents over $\mathbb{Z}_2$. By Lemma 2.6, an orbit of odd rational codegree splits off maximally and unramified at two. Its residue fields have degree one, which would supply a rank-one factor.

Codegree two on at least two characters exhausts the reciprocal mass. A single character of codegree two has row $(1, -1, 0, \ldots, 0)$ by orthogonality to positive dimensions; call the second basis element g. The idempotent $(1 - g)/2$ is primitive, implying $g^2 = 1$ and a one-dimensional minus-eigenspace for multiplication by g. The ordinary and positive signed lifts of an actual invertible act by the same permutation, since the positive signed dimension row forces all its nonzero signs. This would give a minus-eigenspace of odd dimension in K^3, which is impossible. Here and below all simples are self-dual since $T_{\mathbb{Q}}$ is totally real.

If exactly two of the four remaining ordinary codegrees are irrational, their pair has mass at least $1/2 - s > 1/3$. Its contribution to the left side of (9.1) is at least $(1/2 - s)^2$, since the minimal quadratic squared gap ratio is $1/2$. With four irrational codegrees their contribution is at least $(1 - s)^2/2$. Either contribution exceeds $1/9$, whereas the two signed complementary orbits already supply $c(1 - t)^2/2 > c(19/20)^2/2$. This contradicts (9.1).

Lemma 9.1 now gives $s = 1/8$, $D = Mu^\ell$ with $M = 8A_\ell$, and a signed v-family orbit of mass $1/2$.

If $s/t \geq 5$, the signed complementary orbits and D already contribute at least $c((1 - 1/40)^2/2 + 2/64) > 1/2$. Thus $s/t < 5$. The preceding exponent comparisons with unit-family D force $E = mvu^k$ with $m = M$, $\ell = k + 1$: a larger scalar quotient or exponent gap gives $s/t \geq 6$. The signed-order discriminant squareclass requires an odd number of v-families among the three signed factors. Since both E and the half-mass orbit are of that type and $t < 1/20$, the remaining factor is $G = nvu^p$ as well. Set $q_0 = m/n \in \mathbb{Z}_{>0}$. Its mass and the half-mass identity give

$$q_0 B_p = 4A_{k+1} - B_k, \qquad q_0 \mid m = 8A_{k+1}.$$

Since $B_p, B_k \equiv 2 \pmod 4$, q_0 is odd and divides $\gcd(A_{k+1}, B_k)$. This gcd divides $|B_k A_{k+2} - B_{k+1}A_{k+1}| = |2 \cdot 34 - 10 \cdot 6| = 8$. Thus $q_0 = 1$ and $B_p > B_k$, so $p > k$. On the other hand $D/D' \geq G/G'$ gives $2\ell \geq 2p + 1$, equivalently $p \leq k$. This contradiction proves that a signed complementary exponent is at most one.

Choose such an orbit H with mass w, giving preference to nv if it occurs. The two complementary orbits cannot both have this form: their contributions at the ordinary FP induction column are at least their reciprocal masses since for integral totally positive b one has $\mathrm{Tr}(b/v) \in 2\mathbb{Z}_{>0}$. This contradicts $s > t$. The other orbit G thus satisfies $G/G' \geq u^2$. The same holds for D, giving

squared gap ratios at least $8/9$. Dropping all other terms from (9.1) yields

$$\tfrac{1}{2} \geq \tfrac{8}{9}((1-w-t)^2+8t^2) \geq \tfrac{64}{81}(1-w)^2.$$

Hence $w > 1/5$. An exponent comparison improves $s \geq 2t$ to $s \geq 3t$: with $D/D' \geq u^2$ the only two additional exponent-zero exceptions to the comparisons above are $A_1/A_0 = A_1/B_0 = 3$. Retaining the gap of H yields

$$\delta_H^2 + \tfrac{16}{19}(1-w)^2 \leq 1/2$$

after minimizing over t. The possible forms of H are nv, nu, nvu, with $w = 2/n, 6/n, 10/n$, respectively. Applying the scalar divisibilities, $w > 1/5$, and this last inequality gives

H	n
nv	$4, 6$
nu	$12, 16, 20$
nvu	$18, 20, 22, 24, 26, 28, 30, 32.$

If both complementary exponents are at most one, our choice of H leaves $G = nu$ or nvu. Since $w \leq 5/9$ and $t \leq s/3 \leq 1/(4\sqrt{2})$, its mass exceeds $1/10$ and $n < 100$. The bound $\delta_H^2 + \delta_G^2 + 16t^2 \leq 1/2$ leaves eight unordered pairs, all with $t \geq 1/36$. Their complementary scalars have least common multiple $N \leq 220$. Both Nt and Ns are integral by the signed mass and FP induction identities, since $\mathrm{Tr}(\mathcal{O}_K/(v^e u^j)) \subseteq 2\mathbb{Z}$.

Write $E = mv^e u^k$, $D = Mv^f u^\ell$, $a = m/M$, $r = \ell - k$. Now $s/t \leq 36 \cdot 3/(4\sqrt{2}) < 20$ and Lemma 7.2 gives $a^2 < 4s/t < 80$. The leading coefficients of the primitive traces give $s/t \geq au^r/v$, so $0 \leq r \leq 2$. Because m divides both $NB_{e,k}$ and $NaB_{f,\ell}$, it divides $NaW_{e,f}(r)$. Here $0 < |W_{e,f}(r)| \leq 328$, whence $m \leq 220 \cdot 8 \cdot 328$ and $B_{e,k} = mt$ forces $k \leq 6$.

In these finite ranges the divisibility and Fourier comparisons leave exactly

(H, G)	E	D	s
$(12u, 22vu)$	$132u$	$66vu$	$5/33$
$(16u, 4v)$	16	$16u$	$3/8$
$(4v, 22vu)$	$44v$	$44vu$	$5/22$
$(20vu, 22vu)$	$220vu$	$220u^2$	$17/110$

The file `silver_small_check.py` verifies these reductions by integer arithmetic.

The second row of the last table has D as a signed codegree, so gives sphericality. In the other three rows the available remaining upper bound for the contributions of the ordinary complementary orbits is, respectively, $31/1089, 17/242, 7/2420$. Each is less than $(1/2 - s)^2$, the minimum contribution of an irrational ordinary pair; two such pairs give an even larger bound. Thus the ordinary complements are rational. The masses $5/33$ and $17/110$ are both less than $1/6$ and different from $1/8$, contrary to Lemma 9.1.

For $s = 5/22$ the rational ordinary codegrees divide 44 by integrality of E/f, so the only possible tuple is $(4, 4, 4, 44)$; its discriminant squareclass is 22, whereas the maximal order gives two. This excludes all eight pairs.

The other complementary orbit G consequently has exponent at least two. We again have $s/t \geq 10/3$, and δ_G^2/z^2, δ_D^2/s^2 at least $c = 288/289$. In particular

$$L(s) := \delta_H^2 + c((1 - w - 3s/10)^2 + 2s^2) \leq 1/2.$$

Minimizing the quadratic over $s \geq 0$ gives $\delta_H^2 + (57600/60401)(1-w)^2 \leq 1/2$, reducing the preceding table to $4v, 6v, 12u, 16u, 18vu, 20vu, 22vu, 24vu$. For all of these except $4v$, the ordinary complementary codegrees are rational. Indeed $2cs^2 + \delta_H^2 \leq 1/2$ and $\delta_H^2 \geq 1/18$ give $s < 1/2$. An irrational ordinary complement then requires $L(s) + (1/2 - s)^2 \leq 1/2$, since two quadratic pairs give at least $(1-s)^2/2 \geq (1/2 - s)^2$. For the seven values the minimum of this last quadratic exceeds $29/50$. These comparisons are direct substitutions of w and δ_H^2.

For $16u$ and the four nvu cases, $L(s) > 1/2$ on $s \geq 1/6$: the quadratic is increasing from its value at $1/6$. Thus $s = 1/8$ and Lemma 9.1 supplies a signed v-family half-mass orbit. Since $t \leq 3/80$, the masses of $18vu, 22vu, 24vu$ preclude a half-mass orbit. At $H = 16u$, $L(1/8) > 1/2$. At $H = 20vu$ we have $w = 1/2$ and $\delta_H^2 = 49/200$. If $s/t \geq 5$, then the mixed sum is at least $49/200 + c((19/40)^2 + 2/64) > 1/2$. Thus $s/t < 5$ and the preceding Casoratian argument for $s = 1/8$ applies verbatim to the other v-family orbit, contradicting its exponent bound.

It remains to rule out $H = 6v$ and $12u$. In these two cases $L(21/100) > 1/2$ and L is increasing thereafter, hence $s < 21/100$. Lemma 9.1 now forces a signed v-family orbit of mass $1/2$. For $H = 6v$ none of the three masses is $1/2$, since $w = 1/3$ and $t \leq 3s/10$. For $H = 12u$ the half-mass orbit is in the unit family.

This leaves $H = 4v$, of squared gap $1/8$. The long signed pair has mass $1/2 - t$. Using $s \geq 10t/3$ in the mixed bound gives

$$\frac{1}{2} \geq \frac{1}{8} + \frac{288}{289}((1/2 - 3s/10)^2 + 2s^2).$$

The right side increases for $s \geq 1/3$ and already exceeds $1/2$ at $1/3$. Thus $s < 1/3$. □

Lemma 9.3. *In the conductor-eight setting, let a real signed complementary character of codegree U satisfy $U/U' = D/D' > 1$. Then $\mathcal{C}$ is spherical or contains a spherical rank-four subcategory with Perron dimensions $(1, 1, 1+\sqrt{2}, 1+\sqrt{2})$. Its nonunit invertible g has positive lift in every spherical section of this subcategory. If the signed algebra is K^3, then $\mathcal{C}$ is spherical.*

Proof. The character attains the Perron bound on the conjugate dimension induction object $Z = F(B_\tau)$. By Lemma 2.9 and equality in the triangle inequality, its constituents generate a spherical subcategory. Its Perron field is K, since it contains Z of irrational Perron dimension D/D'. If proper, it has rank at most four by Lemma 2.10. The spherical classifications leave the Ising ring and the asserted rank-four rings [18, 20, 5]. In the Ising ring the rational character $(1, -1, 0)$ takes Z to one: it extends to the full algebra, where the values on Z are positive units. Since the unit coefficient is one, $Z = 1 + bX$, $b \geq 0$. Its Perron dimension $1 + b\sqrt{2}$ has norm one, forcing $b = 0$, a contradiction.

In rank four, g interchanges the other two nonunit simples Y, Z_0. If a spherical section gave g dimension -1, their dimensions would be opposite. This is impossible when $Y^* = Z_0$. In the self-dual ring the two characters with this sign are $(1, -1, 1, -1)$ and $(1, -1, -1, 1)$, so the intrinsic dimension would be four. Its quotient by the Perron dimension $8 + 4\sqrt{2}$ has norm $1/2$, contrary to integrality by induction in the spherical center.

Finally suppose the signed algebra is K^3. Then every simple is self-dual. The two displayed characters extend to the signed algebra, and each restriction occurs on a full quadratic orbit. Thus the positive lift of g would have at least four negative eigenvalues. It permutes six basis elements and can have at most three transpositions. □

Proposition 9.4. *In any nonspherical case of Theorem 9.2, all four ordinary complementary codegrees are rational integers.*

Proof. Use the orbit $H = 4v$, and write $E = mv^e u^k$, $G = nv^e u^j$, $D = Mv^f u^\ell$. The primitive families of E, G agree by the discriminant. The inequalities $2\ell + f \geq 2j + e$ and Lemma 9.3 give strict inequality. Set $a = m/M$, $h = m/n$, $r = \ell - k$, $b = j - k$. The positive FP induction column has H-orbit contribution $\mathrm{Tr}(\beta/v)/4 = 1/2$: its numerator is a positive even integer strictly smaller than four. Put $C = B_{f,\ell}$, $B = B_{e,k}$, $L = B_{e,j}$. Since $\mathrm{Tr}(\mathcal{O}_K/(v^e u^j)) \subset \mathbb{Z}$, the G-orbit contributions on this and the unit column give $n(1/2 - s), n(1/2 - t) \in \mathbb{Z}$. Thus

$$m = 2(B + hL), \quad h \mid 2B, \quad h \mid aC - B.$$

In particular $8 \nmid h$.

If an ordinary complementary codegree is irrational, its additional mixed contribution is at least $(1/2 - s)^2$ as above. We claim $s/t < 5$. Otherwise the mixed bound and $288/289 > 49/50$ would require

$$\frac{1}{8} + \frac{49}{50}((1/2 - s/5)^2 + 2s^2) + (1/2 - s)^2 \le \frac{1}{2}.$$

The quadratic on the left has minimum $1/2 + 1/1304$.

Now $s/t = aB_{f,k+r}/B_{e,k}$. The only choices with $s/t < 5$ are

$$(e, f, r, a) = (0, 1, 0, 2) \quad \text{or} \quad (1, 0, 1, 1).$$

Indeed $v^{f-e}u^r > a$ by $D > E$. In the same family one must advance at least one index, and the successive trace ratios are at least five except $B_{0,1}/B_{0,0} = 3$, which cannot occur since $j \ge 2$ and $2j + e < 2(k + r) + f$. From the unit family to the v-family, a is even, $B_{1,k} \ge B_{0,k}$, and advancing one index gives a ratio at least six including a. Without an advance $k \ge j \ge 2$ and $B_{1,k}/B_{0,k} \ge 5/3$, forcing $a = 2$. In the opposite direction $r \ge 1$, with $B_{0,k+1}/B_{1,k} \ge 3$, so $r = a = 1$.

Both choices give $j \le k$ and $|W_{e,f}(r)| = 8$. By Lemma 7.2, $h \mid 16a$; since $8 \nmid h$, we have $h \le 4$. Thus $t = B/[2(B + hL)] \ge 1/10$ because $L = B_{e,j} \le B$. But $s \ge 10t/3$ then contradicts $s < 1/3$. □

Lemma 9.5. *Suppose the only invertible is the unit, the Perron field is $K = \mathbb{Q}(\sqrt{2})$, and the four ordinary complementary codegrees are even integers at least four. Assume their four-rows and six-rows are rational. Then $s \ge 1/4$. If the dyadic factors have ranks six or four and two, all with residue field $\mathbb{F}_2$, then $s \ge 5/14$.*

Proof. Three four-rows force an invertible. Above mass $3/4$ the only profiles are therefore $(4, 4, 6, d)$, $d = 6, 8, 10$. At the inert prime three a six-row needs a totally ramified partner by Lemma 2.7. The prime-to-three axes split off, and two six-rows are incongruent by squared distance twelve. This proves $s \ge 1/4$.

Now impose the dyadic hypotheses and suppose the mass exceeds $9/14$. Without two four-rows the sorted norms must be $(6, 6, 6, 6)$, $(4, 6, 6, d)$ with $d \le 16$, or $(4, 6, 8, 8)$. In each case the same argument at three leaves too few partners for the six-rows.

Two four-rows have different supports, since a matched pair gives an invertible. They exhaust the dyadic residue characters, so all integral rows have norm divisible by four. In particular there are exactly two four-rows: their distinct supports, after deletion of the unit, have disjoint two-element complements in five positions. A four-row in the rank-two factor needs a rational local partner, as the Perron field is ramified. If the last two rows formed a quadratic field F, their common codegree would contribute a square to the order discriminant. Thus $2|\operatorname{disc}(F)|$ has the squareclass of $N(D)$, which is one or two. The only possible fields are $\mathbb{Q}(i), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-2})$, none split at two. Hence the two last rows are rational.

The four-row's partner has norm $d \equiv 12 \pmod{16}$ by Lemma 2.6. At $d = 12$ the difference of the two rows is divisible by four and has squared norm sixteen. Its sole nonzero entry contradicts orthogonality to the Perron row. Thus $d \ge 28$. Since $1/2 + 1/28 + 1/12 < 9/14$, the remaining norm is eight; then $d < 56$ gives $d = 28$ or 44. Neither $4^2 \cdot 8dN(D)$ has squareclass two, as required by the ordinary algebra $K \times \mathbb{Q}^4$. □

Proposition 9.6. *In any nonspherical case of Theorem 9.2, there is exactly one nontrivial invertible object g. Its permutation on the simple classes has two transpositions and two fixed points.*

Proof. Write the entries of the $H = 4v$ signed character as $a_i + b_i\sqrt{2}$. The codegree and conjugate-row orthogonality give $\sum a_i^2 = 4$, $\sum b_i^2 = 2$, $\sum a_i b_i = 2$. Since the unit entry is one, the three

other nonzero a_i are signs. The two nonzero b_i equal the corresponding a_i. Orthogonality of both rows to the positive dimension character now forces their shape, up to permutation,

$$(1, -1, 1+\sqrt{2}, -(1+\sqrt{2}), 0, 0).$$

Every character outside this pair takes the same values on the first two entries, and equal values on the next two. Thus the second simple g satisfies $g^2 = 1$ in T, and signed left multiplication by g is the two-transposition permutation. Indeed this matrix is symmetric and integral with square one, and the positive dimension character rules out negative permutation signs. If there is any nontrivial ordinary invertible, its positive lift takes value $+1$ on the spherical dimension orbit, and the orbits where it takes value -1 must have total reciprocal mass $1/2$ by its zero unit coefficient. Only the H-orbit has this property outside the physical orbit. Therefore that invertible must be g.

Suppose g is not invertible. All four ordinary complementary codegrees are even and at least four, and their four- and six-rows are rational by Lemma 8.1.

The algebra $T/2T$ has a local factor of rank at least four. Indeed the ideal generated by $g - 1$ has basis the two disjoint pair differences, and is square-zero. It is indecomposable: its integral version maps under the H-character isomorphically onto $2\mathbb{Z}[\sqrt{2}]$, so reduction gives the regular module of the local algebra $\mathcal{O}_K/2\mathcal{O}_K$. The two differences remain independent after reduction. Consequently the ideal lies in one local factor. Its coefficient pairing vanishes there, whereas the pairing on the entire factor is perfect. The rank is therefore at least four. All these statements about local ranks transfer to R by parity.

The signed ramified quadratic branches exclude rank-one factors; all residue fields are $\mathbb{F}_2$. Lemma 9.5 now contradicts $s < 1/3$ from Theorem 9.2. □

Lemma 9.7. *In the conductor-eight setting, suppose the ordinary algebra is commutative, its Perron field is $K = \mathbb{Q}(\sqrt{2})$, and its only nonunit invertible acts by $(1, g)(X, Y)$ with fixed simples P, Q. Suppose both negative rows have codegree four and common support, with values ± 1 or $\pm i$ at X, Y. At two assume the order is local or has factors of ranks four and two, with both negative rows in the rank-four factor. If the two remaining ordinary characters outside the Perron pair are real and have even rational codegrees at least four, then* $\mathrm{Tr}(1/D) \geq 5/14$.

Proof. Write $s = \mathrm{Tr}(1/D)$ and $a \leq b$ for the last two codegrees. The two positive rows have the form $(1, 1, z, z, p, q)$. If $s < 5/14$, then

$$1/a + 1/b = 1/2 - s > 1/7. \tag{9.2}$$

There is no third four-row. Such a row would be rational by Lemma 8.1. In the shape $(1, 1, z, z, p, q)$ it cannot have $z = \pm 1$ by orthogonality to FP and $d_X > 1$. Its support thus differs from that of the matched short pair. The latter pair cannot occupy a rank-two dyadic factor, by the dyadic hypothesis. The third four therefore lies in the rank-two factor with one rational partner of codegree b; the FP field is ramified at two and cannot supply a rank-one branch. The two rows in this factor are congruent modulo four by perfect trace duality and projection to its two axes. Thus $b \equiv 12 \pmod{16}$ by the squared row distance. The absolute discriminant identity gives b of squareclass one or two, neither compatible with that residue modulo sixteen.

Thus $a = 6, 8, 10$ or 12. For distinct a, b the two folded rational-codegree rows are rational and

$$abN(D) \text{ has squareclass two.} \tag{9.3}$$

The short split algebra or Gaussian field contributes square absolute discriminant, while K contributes squareclass two. If $a = b$ on a quadratic field F, its positive discriminant c satisfies $c \mid 2a$, $c \leq a$ by the real row-difference test, and $N(D)$ has squareclass $2c$.

At $a = 6$ both rows are rational by Lemma 8.1. The first has norm two modulo four and hence represents a different dyadic residue row from that of the matched four-pair. It lies in the rank-two factor and must pair with the other rational row, so $b \equiv 2 \pmod 4$. At three the

inert K-branch cannot pair with the six-row by Lemma 2.7; the codegree-four summand splits off. Hence the rational rows are congruent modulo three and $9 \mid 6+b$. But (9.3) forces $b = 6z^2$ or $3z^2$. The first fails the modulo-nine test and b even in the second case requires $4 \mid b$.

At $a = 10$, (9.2) gives $10 \le b \le 22$. If both rows are rational, the same argument at the inert prime five gives $25 \mid 10 + b$, impossible for even b in this range. If they form a quadratic orbit, $c \mid 20$ and $c \le 10$ force $c = 5$, inconsistent with $N(D)$ having squareclass $2c$. At $a = 12$, the same mass bound gives $b = 12, 14, 16$, with $14, 16$ excluded by (9.3). Two rational twelve-rows would have to pair at three by the same local argument, whereas nine does not divide their squared distance 24. For a quadratic pair $c = 8$ or 12. Only 8 is compatible with the discriminant, and then $D = Mu^\ell$ has square norm. But $s = 1/3$ gives $M = 3(u^\ell + u^{-\ell})$, of two-adic valuation one, whereas $4 \mid M$.

Finally let $a = 8$. A quadratic pair requires $b = 8, c = 8$. For its two rows write the first as $(1, 1, z, z, p, q)$ with $z, p, q \in \mathbb{Z}[\sqrt{2}]$. Their squared difference 16 gives

$$2[\sqrt{2}](z)^2 + [\sqrt{2}](p)^2 + [\sqrt{2}](q)^2 = 2,$$

where each bracket denotes the coefficient of $\sqrt{2}$ before squaring. If $[\sqrt{2}](z) \neq 0$ the row difference is supported on X, Y with equal entries, contrary to FP orthogonality. Otherwise the last two coefficients are opposite signs and $d_P = d_Q$. Write $p = A + \sqrt{2}, q = C - \sqrt{2}$ after conjugation. The rational norm equation gives $A = C$, $z \in \mathbb{Z}$ and $2z^2 + 2A^2 = 2$. FP orthogonality would require $d_X = 1$ or $d_P = 1$.

If the eight-row is rational it has form $(1, 1, z, z, 2e, 0)$ after interchanging P, Q, where z, e are signs. Its reduction agrees with the two negative rows. In a dyadic decomposition of ranks four and two the negative pair and the eight-row occupy three dimensions in the rank-four factor. The last denominator is rational, so that factor must also contain its axis, leaving the FP branch alone. Its idempotent has dyadically integral unit coefficient $s = 3/8 - 1/b$. Hence $b = 8B$ with $B \equiv 3 \pmod 8$. By (9.2), $b < 56$, so $b = 24$, excluded by (9.3). We can thus assume the ordinary dyadic order local. Its remaining row reduces to the same short row, so $4 \mid b$. The discriminant identity and $8 \le b < 56$ give $b = 8, 16, 32, 36$. Write its coordinates as $(1, 1, z', z', -e(1 + zz'), C)$, using orthogonality to the eight-row. Its norm is

$$b = 3 + 3(zz')^2 + 2zz' + C^2,$$

where z' is odd by reduction at two. At $b = 16, 32$ there is no solution. At $b = 36$ the only possibility $zz' = 3, C = 0$ gives $d_X = 3/5$ by the two FP orthogonality equations. At $b = 8$ both rows cannot be nonzero at the same fixed simple, since their two equations would force $d_X = 1$. The other possibility has $zz' = -1, C = \pm 2$. The two fixed dimensions must then be $x - 1, x + 1$ with $x = d_X = d_Y$. Conjugate-row orthogonality gives $N(x) = -1$ and $D = 4(x^2+1) = 4x(x-x')$. At the fixed simple of dimension $x+1$, the two eight-rows contribute ± 1 to its self-square coefficient and the negative rows contribute zero. The FP pair contributes

$$\mathrm{Tr}((x + 1)^3/D) = (\mathrm{Tr}(x) + 4)/4,$$

since $1/x = -x'$. Writing $x = A + B\sqrt{2}$, the equation $A^2 - 2B^2 = -1$ makes A odd. Thus this coefficient is nonintegral. All rational alternatives are excluded. □

Theorem 9.8. *Let $K = \mathbb{Q}(\sqrt{2})$. Suppose a rank-six fusion category has commutative ordinary character algebra, signed rational algebra K^3, and both its ordinary Frobenius–Perron and spherical dimension characters have field K. If each complementary signed character has irrational formal codegree, the category admits a spherical structure.*

Proof. Assume otherwise. By Theorem 9.2 and Propositions 9.4 and 9.6, the signed short orbit has codegree $4v = 8 + 4\sqrt{2}$, all four ordinary complementary codegrees are rational even integers, and the unique nontrivial invertible g acts on simples by $(1, g)(X, Y)$ with two fixed points P, Q. The two g-negative ordinary characters have rows $(1, -1, y, -y, 0, 0)$. On the span of $1 - g, X - Y$

multiplication by X has matrix $\left(\begin{smallmatrix}0&1\\1&n\end{smallmatrix}\right)$, by self-duality. Since $2(1+y^2)$ is rational for both roots and their product is -1, necessarily $n=0$: otherwise y is rational, and the two rational algebraic units must be $1,-1$. Thus both negative rows are rational four-rows. All other ordinary rows have shape $(1,1,z,z,b,c)$.

We can now apply Lemma 9.7. The proof of Proposition 9.6 gives a dyadic factor of rank at least four. There are no rank-one factors, since the signed order has three ramified quadratic branches. In the two-factor case the negative rows cannot occupy the rank-two factor: their projection $\{(a,b) : a \equiv b \pmod 2\}$ has discriminant valuation two, whereas $4\cdot 4$ has valuation four. Thus $s \geq 5/14$.

This contradicts $s < 1/3$ from Theorem 9.2. □

9.2. **Two rational complementary codegrees.**

Proposition 9.9. *Suppose that a rank-six fusion category has commutative ordinary and signed character algebras and common dimension-character field $K = \mathbb{Q}(\sqrt{2})$. Suppose the signed codegrees are E, E', H, H', g, h, with H irrational and g, h rational integers at least three. If the category has no pseudo-unitary Galois conjugate, then either the last two characters form a Gaussian pair with $g = h = 4$, or they are rational and, upon interchanging them,*

g	h
4	$4, 8, 9, 16, 18, 25, 32, 36, 49, 50$
7	7.

Proof. Write $u = 3+2\sqrt{2}$, $v = 2+\sqrt{2}$ and orient E, H by their larger conjugate. Let $t = \mathrm{Tr}(1/E)$, $w = \mathrm{Tr}(1/H)$, $s = \mathrm{Tr}(1/D)$ and $L = t + w = 1 - 1/g - 1/h$. For $f \geq f'$ put $\delta_f = 1/f' - 1/f$. From (9.1) and Lemmas 2.1 and 5.6 we obtain

$$\delta_E^2 + \delta_H^2 + 2\delta_D^2 \leq \tfrac{1}{2}, \qquad D/D' \geq H/H' \geq u^2, \qquad s \geq 3t.$$

Write an oriented codegree as $nv^e u^j$ as before. Its ratio to its conjugate is u^{2j+e}, and $8 \mid \mathrm{Tr}(f)$ for a quadratic row by orthogonality. Thus $4 \mid n$ if $e = 0$, and $2 \mid n$ if $e = 1$.

If $H/H' \geq u^3$, the squared gap divided by squared mass is at least $49/50$ for both D and H. Hence

$$\tfrac{1}{2} \geq (49/50)(18t^2 + w^2), \qquad L^2 \leq 475/882.$$

If $H/H' = u^2$, write $H = nu$. Now $4 \mid n$ and $\delta_H^2 = 32/n^2 < 1/2$, so $n \geq 12$, and $w = 6/n \leq 1/2$. The squared gap ratio is $8/9$ at u^2, so $16t^2 + (8/9)w^2 \leq 1/2$. The upper bound $w + \frac{1}{4}\sqrt{1/2 - (8/9)w^2}$ for L is increasing on $[0, 1/2]$ and is less than $2/3$ at the endpoint. Thus $L^2 \leq 475/882$ always.

We show there is no rational signed character ρ of codegree three. Three is inert in K. The local partner Lemma 2.7 puts ρ and the other rational character η of codegree h in their own two-branch factor at three, with congruent character rows modulo three. The coefficient trace of the identity of this factor is integral over $\mathbb{Z}_3$, hence $1/3 + 1/h \in \mathbb{Z}_3$ and $h/3 \equiv -1 \pmod 3$. The absolute discriminant of the signed order in $K^2 \times \mathbb{Q}^2$ gives that $N(E)N(H)3h$ is a square. The squareclasses of the two norms are one or two, so $h = 3a^2$ or $6a^2$. The congruence forces $h = 6a^2$ with $3 \nmid a$. Consequently E, H belong to opposite primitive families. The simultaneous column W has components $E/E', H/H', 1, 1$, allowing reversal of orientation in the second component. The residue of u has order four in $\mathbb{F}_9$. Thus W has a residue generating $\mathbb{F}_9$ on the unique v-family branch, and a residue in $\mathbb{F}_3$ on every other branch. The v-family branch splits integrally at three: the two factors of the characteristic polynomial on the disjoint residue sets are coprime over $\mathbb{Z}_3$. Its order is maximal since it contains W, and $1, W$ span its residue extension. Its coefficient functional is $\mathrm{Tr}_K(-/f)$ for the corresponding codegree f; as K is unramified at three, integrality forces f to be a 3-adic unit. On the other hand $E/3$ is an algebraic integer, the dimension of an induction label. It follows that H is the v-family codegree and is a 3-adic unit.

For each signed induction label B write $m_B = \rho(FB) \in \mathbb{Z}$. On the signed dimension labels the two entries are one, since $F(B_1) = 1$ and the rational row takes W to one. The entries on H, H' agree by rationality and Galois symmetry; call them c. Symmetry gives $\chi_H(FA_\rho) = (H/3)c$, an algebraic integer, so $3 \mid c$. Write $m = m_{A_\rho}$ and $n = m_{A_\eta}$. The same symmetry gives $\eta(FA_\rho) = (h/3)n \equiv -n \pmod 3$. Since $\eta \equiv \rho$, this makes $m + n \equiv 0 \pmod 3$. But the signed unit coefficient of FA_ρ is one, and symmetry expresses it as $(2 + 2c + m + n)/3$. Its numerator is two modulo three, a contradiction.

It remains to list the complements. A rational pair $g \le h$ has $g, h \ge 4$, and gh is a square or twice a square by the discriminant. In $(1 - 1/g - 1/h)^2 \le 475/882$ we have $g \le 7$. For $g = 4, 5, 6, 7$ this inequality bounds h by $61, 15, 10, 8$ respectively. The squareclasses leave exactly the table and $(5, 5), (5, 10), (6, 6)$. At the inert prime five, the first two pairs have integral combined trace by the local partner lemma, whereas neither $2/5$ nor $3/10$ is integral there. In the third case use the same lemma at the inert prime three: $1/6 + 1/6$ is nonintegral. For a quadratic complementary field F with a common rational codegree g, the same bound gives $g \le 7$; the absolute discriminant of F has squareclass one or two. A real such quadratic field is $\mathbb{Q}(\sqrt{2})$ and the complementary codegree is at least its discriminant eight by character orthogonality. In the imaginary case the single nonself-dual pair gives $g = |\operatorname{disc}(F)|b^2$ by row orthogonality. Hence $F = \mathbb{Q}(i)$ and $g = 4$. □

9.3. **Two invertibles in a rank-four subcategory.**

Proposition 9.10. *Suppose a complex rank-six fusion category has commutative sphericalization and all ordinary Frobenius–Perron dimensions in $K = \mathbb{Q}(\sqrt{2})$. Suppose tensoring by an involutive invertible g interchanges $1, g$ and Y, Z and fixes P, Q. If $1, g, Y, Z$ form a fusion subcategory $\mathcal{B}$ admitting a spherical structure and satisfying* $\operatorname{FPdim}(Y) = \operatorname{FPdim}(Z) = \xi = 1 + \sqrt{2}$*, then the category admits a spherical structure.*

Proof. The quadratic relation in the Grothendieck ring of $\mathcal{B}$ is $Y^2 = \epsilon + Y + Z$, where ϵ is 1 or g. Indeed exactly one invertible occurs by rigidity, and two copies of just one of Y, Z are impossible. In the self-dual case they would form a rank-two fusion subcategory on that simple with product $X^2 = 1 + 2X$, excluded by the rank-two classification. In the nonself-dual case the eigenvalues of Y in the $g = -1$ component are purely imaginary, since $Y^* = Z = gY$; a relation $\lambda^2 = -1 \pm 2\lambda$ is impossible there.

Multiplication by Y preserves the span of P, Q by reciprocity. Its matrix is symmetric since Y^* is Y or gY and g is trivial on this span. Its minimal polynomial is $x^2 - 2x - 1$, irreducible over $\mathbb{Q}$. The trace and determinant give, after interchanging P, Q, the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Thus $YP = Q$ and the dimensions of these objects are $p, \xi p$. In particular P, Q are self-dual, since duality preserves the fixed objects and their dimensions differ. In P^2 reciprocity gives the multiplicities $1, 1, 0, 0$ at $1, g, Y, Z$. Hence

$$p - 2/p = r + s\xi, \qquad r, s \in \mathbb{Z}_{\ge 0}.$$

In particular $p \mid 2$. Since $(2) = (\sqrt{2})^2$ as ideals, and the units are $\pm\xi^k$, we have $p = \xi^k, \sqrt{2}\xi^k$ or $2\xi^k$ for $k \ge 0$ by $p \ge 1$. Also

$$\operatorname{FPdim}(\mathcal{C})/\operatorname{FPdim}(\mathcal{B}) = 1 + p^2/2$$

is a d-number in the sense that its quotient by its conjugate is a unit: both global FP dimensions have this property. Write $u = \xi^2$, the generator of the totally positive units. For $k \ge 1$ the quotient of $2 + u^k$ by its conjugate lies strictly between u^{k-1} and u^k, while the quotient of $1 + 2u^k$ by its conjugate lies strictly between u^k and u^{k+1}. Indeed division by u^k gives $(1+2u^{-k})/(2+u^{-k})$ or its reciprocal. These exclude $p = \xi^k, 2\xi^k$ for $k \ge 1$. A fixed object cannot be invertible, so $p \ne 1$. At $p = 2$ the equation above gives $s = 0$. In the remaining family $N(p) = \pm 2$, so $p - 2/p$ is $p + p'$ or $p - p'$; in the first case it is rational, while in the second it has trace zero. Since r, s are nonnegative, either case gives $s = 0$. Consequently $1, g, P$ form a rank-three fusion

subcategory with $P^2 = 1 + g + rP$. It has a spherical tensor section, by the corank-one theorem [4, Theorem 4.5], and its section takes g to its positive lift since $gP = P$.

By Lemma 9.3 and its rank-four argument, every spherical section of $\mathcal{B}$ takes g to its positive lift. Thus the two lifted subcategories agree on g.

Their products furnish one lift of each simple: $YP = Q$, and the products of P with the selected lifts of $Y, Z = gY$ agree since $gP = P$. The span is tensor-closed because the sphericalization ring is commutative and the two original lifted subcategories are tensor-closed. Forgetting gives a tensor equivalence and the spherical structure descends. □

Proposition 9.11. *Under the hypotheses of Proposition 9.9, suppose both complementary rational codegrees are four and E is rational. Then the category is spherical.*

Proof. The absolute discriminant of the maximal signed order is a square for the split or Gaussian complement. Thus $H = nu^j$ is in the unit family, and $4 \mid n$. We have $j \geq 1$ by the positive-induction argument, $h = E/n \in \mathbb{Z}$, and the signed reciprocal mass gives $E = 4 + 2hA_j$, where $A_j = u^j + u^{-j} \equiv 2 \pmod 4$. Hence $h \mid 4$. Were h even the last equation would give $v_2(E) = 2$, contrary to $4h \mid E$. Thus $h = 1$ and $n = E$. Put $w = A_j/(4 + 2A_j) \geq 3/8$. On the positive FP induction column the two rational codegrees contribute at least $1/2$, so $s < 1/2$. Write $D = Mv^d u^\ell$ with $v = 2 + \sqrt{2}$. By integrality of E/D, $c = E/M$ is an integer, even if $d = 1$. By central dominance $\ell \geq j$. Therefore in the unit family $s/w = cA_\ell/A_j < 4/3$ gives $c = 1, \ell = j$; the successive trace ratios from index one exceed $4/3$. But then $D = H$, which yields a spherical section by signed domination. In the other family $s/w = c\,\mathrm{Tr}(1/(vu^\ell))/A_j \geq 2$, since the reciprocal traces in this family equal or exceed the unit-family traces at the same index. This contradiction completes the proof. □

9.4. **Two short rational profiles.**

Proposition 9.12. *In the setting of Proposition 9.9, the signed rational codegrees $(4, 16)$ force sphericality. For $(4, 9)$, a nonspherical category would have $h = 6$ or 9 and $b = 0$, or $h = 36$ and $b = -1$, where $h = m/n$ is the ratio of the signed scalars and $b = j - k$ the difference of their exponents.*

Proof. Let $L = t + w$ be the total quadratic mass. We first record the ordinary contribution to the mixed bound, writing

$$L = 11/16 \quad \text{or} \quad 23/36, \qquad \kappa = 1/4, \qquad G_0 = \delta_E^2 + \delta_H^2 + 2\delta_D^2.$$

In both cases

$$G_0 \leq 1/2, \qquad s \geq \kappa \quad \text{or} \quad G_0 + (1/2 - s)^2 \leq 1/2. \tag{9.4}$$

Indeed there is no nonunit invertible. Its signed unit coefficient would give $t \pm w \pm 1/4 \pm 1/q = 0$, $q = 16, 9$. Since $t \leq \sqrt{9/32}/3$, this is impossible at $q = 16$ and forces $(t, w) = (5/36, 1/2)$ at $q = 9$. In the latter case $(8/9)(w^2 + 2(3t)^2) > 1/2$. We have used $s \geq 3t$ and the squared gap ratios at H, D. Lemma 9.3 now gives $D/D' > H/H'$.

The dyadic orders have only degree-one residue fields. Their rank-one factors are exactly the odd-codegree axes: the two quadratic branches ramify, and an even rational axis cannot be isolated. Thus at $q = 16$ all rational ordinary codegrees are even and at least four. For an all-rational complement, Lemma 9.5 gives $s \geq 1/4$.

At $q = 9$ there is exactly one isolated odd row. Its reduction agrees with the signed nine-row, so its norm is $1 \pmod 4$ and is at least five. For an all-rational complement with two four-rows, their distinct reductions place them in factors of ranks two and three. The partner of the four-row in rank two has norm $d \equiv 12 \pmod{16}$ by Lemma 2.6. At $d = 12$ the rational row difference has squared norm sixteen and is divisible by four, contrary to orthogonality to the Perron row. Hence $d \geq 28$ and $s \geq 1 - 1/5 - 1/2 - 1/28 > 1/4$. With at most one four-row the only exception to

$s \geq 1/4$ would be $(5,4,6,6)$. The rational six-rows have distinct reductions at three by squared distance twelve, and neither can pair with the inert Perron field by Lemma 2.7. Finally an irrational ordinary pair contributes at least $(1/2-s)^2$ when $s<\kappa$, since the rational companions have reciprocal mass at most $1/2$. Two irrational pairs contribute at least $(1-s)^2/2$. This proves (9.4).

The case $H=nu$ would give $w=6/n$, $4 \mid n$, $n \geq 12$. Since $t \leq \sqrt{9/32}/3$, only $q=9, n=12$ is possible, giving the excluded pair $(t,w)=(5/36,1/2)$. Thus the squared gap ratios at H, D are at least $c=49/50$. We claim

$$s/t < 15.$$

Otherwise $G_0 \geq c((L-s/15)^2+2s^2)$. On $s \geq \kappa$ this quadratic increases from a value greater than $1/2$ for both values of L. Adding $(1/2-s)^2$ gives a quadratic with minimum greater than $1/2$ in both cases, contradicting (9.4) also for $s<\kappa$.

Write

$$E = mv^e u^k, \qquad H = (m/h)v^e u^{k+b}, \qquad D = (m/a)v^f u^{k+r}.$$

The signed discriminant gives the same family for E, H. Set $B=B_{e,k}$, $G=B_{e,k+b}$, $C=B_{f,k+r}$ and $L=P/Q$ in lowest terms. The unit and Perron columns give

$$Pm = Q(B+hG), \qquad h \mid QB,\ QaC.$$

For the last divisibility multiply the Perron column by Qm/h: the rational terms and the H-trace are then integral. Strict dominance gives $2(k+b)+e < 2(k+r)+f$.

Only six choices remain for the family and scalar of D:

$$(f,r,a) = (e,1,1),\ (e,1,2),\ (1-e,e,2-e), \qquad e=0,1.$$

Indeed $a < v^{f-e}u^r$ and $aC/B<15$. An advance of two indices already gives $C/B \geq 17$ in either family. At one index in the same family the trace ratio is at least five, except at $e=k=0$, which violates strict dominance. From $e=0$ to $f=1$ the scalar a is even, and any advance gives $aC/B \geq 2B_{1,2}/B_{0,1} > 15$ for $k \geq 1$. At $k=0, r=1$ strict dominance would force $H=nu$. These comparisons follow directly from the two-term trace formula.

In all six cases $b \leq 0$ and $|W_{e,f}(r)|$ is $8, 16$ or 32. Lemma 7.2 gives $h \mid QaW$. Since $B \equiv 2 \pmod 4$ and $h \mid QB$, we obtain $h \mid 2Q$. If h is even, $v_2(m) = 1+v_2(Q)$ and m/h is a multiple of $4/(1+e)$, so $h \cdot 4/(1+e) \mid 2Q$. From $1 < w/t = hG/B \leq 2hu^b$ and $2Q \leq 72$, it follows that $b=-2,-1$ or 0.

These short lists can be checked at every exponent at once. Put $x=hu^b$, $y=a(v/2)^{f-e}u^r$ and $\eta = 989/1000$. All three ratio exponents are at least three: $H=nu$ is excluded, $k \geq k+b$ and $D/D' > H/H'$. Thus the correction to each leading trace or gap term has relative size at most u^{-3}. Since $\eta < (1-u^{-3})/(1+u^{-3})$, Lemma 7.2 and its two-term formulas give

$$G_0 \geq (\eta L)^2 \frac{1+x^2+2y^2}{(1+x)^2}, \qquad s \leq \frac{Ly}{\eta(1+x)}.$$

Substitute these bounds in (9.4). At $Q=16$ the 81 choices leave none. At $Q=36$ the 135 choices leave $(h,b)=(6,0),(9,0)$ for either mixed family and $(36,-1)$ for $e=1, f=0$. The exact comparisons use rational endpoints for $3+2\sqrt{2}$. □

Proposition 9.13. *The pair of rational signed codegrees $(4,9)$ also forces sphericality.*

Proof. The signed mass equation requires $23 \mid B+hG$, the trace of $\alpha = (1+hu^{-b})/(v^e u^k)$. A trace-zero element of $\mathbb{Z}[\sqrt{2}]/23$ has norm -2 times a square. As two is a square modulo 23, this requires $-N(1+hu^{-b})$ to be a square. For $(h,b)=(6,0),(9,0)$ the norm is the nonzero square $(1+h)^2$. For $(36,-1)$ it is $1+36^2+6\cdot 36 \equiv 18 \pmod{23}$, also a nonzero square. But -1 is a nonsquare modulo 23. □

9.5. **The eight remaining codegree pairs.**

Theorem 9.14. *A category satisfying the hypotheses of Proposition 9.9 is spherical whenever its two rational complementary characters have one of the following pairs of formal codegrees:*

$$(4,8),\ (4,18),\ (4,25),\ (4,32),\ (4,36),\ (4,49),\ (4,50),\ (7,7).$$

Proof. Suppose the category is not spherical. Then it has no pseudo-unitary Galois conjugate, since Galois transport preserves sphericality. Write $L = 1 - 1/g_1 - 1/g_2$, and keep the notation of Proposition 9.9. Except at $(4,8)$, there is no nonunit invertible. Such an object would have order two by total reality, and its positive lift would have dimension one. Its character values on each field are constant signs. Since $s \le \sqrt{9/32}$ and $s \ge 3t$, the gap $w - t = L - 2t$ is at least $25/36 - 2\sqrt{9/32}/3 > 11/36$, greater than the total rational mass. The coefficient of the unit of such a lift cannot vanish. By Lemma 9.3, equality $H/H' = D/D'$ would supply a spherical rank-four subcategory and a nonunit invertible g. This leaves only $(4,8)$. Here the spherical section takes g to its positive lift, and the saturating H-character takes this lift to one. Thus both signed quadratic pairs take g to one. Its two transpositions within the subcategory already span a two-dimensional negative eigenspace, so the other two simples are fixed and the two rational characters span that eigenspace. They vanish on the fixed simples and restrict to the two rational minus-characters of the self-dual rank-four ring. Both then have codegree four, contrary to $(4,8)$. Hence $D/D' > H/H'$ in all cases.

We recall the necessary bounds on $t = \mathrm{Tr}(1/E)$. For the six profiles with $L = 1 - 1/g_1 - 1/g_2 > 17/24$, one has $t \ge L - 17/24$. Indeed $H = nu$ gives $w \le 1/2$ as above; at $H = nvu$ the gap $7\sqrt{2}/n$ leaves $n > 14$, hence $w \le 5/8$. All later orbits have gap ratio at least $\sqrt{288/289}$, giving $w \le 17/24$. For $(4,18)$ the even signed rational axes and the ramified branches give no dyadic factor of rank one. The same holds at $(4,8)$, where $t < 1/8$ also rules out an invertible, since $w - t > 3/8$. As in Proposition 9.12, every rational ordinary codegree is then even and at least four, and the ordinary four- and six-rows are rational. Lemma 9.5 gives $s \ge 1/4$ for an all-rational complement. Otherwise, for $s < 1/4$ the additional ordinary contribution is at least $(1/2 - s)^2$.

Suppose $t \le 1/100$. At $(4,18)$ we have $w > 2/3$, excluding $H = nu$, and the squared gap ratios at H, D are at least $c = 49/50$. At $(4,8)$ we have $123/200 \le w < 5/8$, excluding both $H = nu$ and $H = nvu$, since $10/n$ skips this interval for even n. Here take $c = 288/289$. In both cases $w \ge L - 1/100$ and the mixed bound would require

$$c(L - 1/100)^2 + \min\{c/8,\ c/(2(1+2c))\} \le 1/2.$$

The two terms in the minimum bound respectively $2cs^2$ on $s \ge 1/4$ and $2cs^2 + (1/2 - s)^2$ on the line. Substitution of the two values of L, c proves $t > 1/100$.

Write δ for the asserted positive lower bound on t.

In this proof $B_{e,k} = \mathrm{Tr}(1/(v^e u^k))$ and $d_{e,k} = (1/(v'^e u^{-k}) - 1/(v^e u^k))/\sqrt{2}$. The pairs $(B_{e,k}, d_{e,k})$ start at $(2,0)$ and $(2,1)$ for $e = 0, 1$ and advance by $(x,y) \mapsto (3x + 4y, 2x + 3y)$. Write

$$E = mv^e u^k, \quad D = (m/a)v^f u^{k+r}, \quad H = (m/h)v^i u^{k+b}.$$

Here a, h are positive integers by integrality of $E/D, E/H$ at both conjugate branches, with a even if $f > e$ and h even if $i > e$. The squareclass of $g_1 g_2$ gives $i = e$ if this product is a square, and $i = 1 - e$ otherwise. The inequalities $D > E$, $D/D' \ge E/E'$ and nonsaturation yield

$$a < v^{f-e} u^r, \qquad r \ge 0, \qquad 2r + f > 2b + i.$$

Put $B = B_{e,k}$, $C = B_{f,k+r}$, $G = B_{i,k+b}$, and write $L = P/Q$ in lowest terms. Then

$$m = (B + hG)Q/P, \qquad h \mid QB, \qquad h \mid Q_0 aC, \qquad Q_0 = \mathrm{lcm}(g_1, g_2).$$

The second follows from $h \mid m$. For the third, the signed trace of the ordinary FP induction column is s plus the two rational terms and the trace of an integral H-evaluation divided by H.

Multiplication by $Q_0(m/h)$ makes the latter three terms integral. Since $Q \mid Q_0$, Lemma 7.2 gives $h \mid aQ_0W_{e,f}(r)$. By the leading terms of the traces, $s < 9/16$, $t \geq \delta$, and $t \leq s/3$ give

$$a < 2\sqrt{9/(16\delta)}, \qquad au^r \leq 4(9/(16\delta)), \qquad \tfrac{1}{2} \leq hu^b \leq 4/\delta.$$

Here $\delta \geq 1/600$ and $Q_0 \leq 196$. The first two bounds give $0 \leq r \leq 5$, hence $0 < |W_{e,f}(r)| \leq 64952$; the third now gives $-16 \leq b \leq 6$.

There are two local simplifications. For $(4, 25)$ the two primitive families agree. If they are both v-families, the residues of $E/E', H/H'$ are not one at five since u has order six modulo five. The simultaneous column therefore splits the rational axes from the quadratic branches. The codegree-four idempotent is integral at five, so the codegree-25 rational factor would be isolated, contrary to its nonintegral trace. For $(4, 18)$ the families are opposite. At three the unique v-family branch is maximal, as proved in Proposition 9.9. Since $E/18$ is integral this gives $e = 0$. Moreover k is even: otherwise W has residue -1 on this branch and generates $\mathbb{F}_9$ on the v-branch, isolating the codegree-18 axis after the codegree-four idempotent splits.

For $k < 32$ test the scalar divisibilities and mixed bound directly. For $k \geq 32$ all three indices are at least 16. Put $x = h(v/2)^{i-e}u^b$, $y = a(v/2)^{f-e}u^r$ and $\eta = 999999/1000000$. Lemma 7.2 gives the necessary bound

$$2\eta^2L^2(1 + x^2 + 2y^2) \leq (1 + x)^2$$

and upper bounds $L/(\eta(1 + x))$, $Ly/(\eta(1 + x))$ on t, s. Keep $t \geq \delta$. At $(4, 8)$, if $t < 1/8$, require also $s \geq 1/4$ or the extra $(1/2 - s)^2$ contribution. The remaining conditions are $h \mid Q_0aC$ and divisibility of m by $g_1, g_2, 4/(1 + e), 4a/(1 + f), 4h/(1 + i)$. Testing one full recurrence period at each prime power and combining the exponents by the Chinese remainder theorem excludes every tuple, as verified in `silver_large_pairs_check.py`. □

Proposition 9.15. *Assume the setting of Proposition 9.9 with two rational signed four-rows of distinct supports. If the category has no pseudo-unitary Galois conjugate, its only invertible object is the unit. Each signed quadratic codegree is mvu^k, $8 \parallel m$, and both quadratic branch projections at two equal $\mathbb{Z}_2 + 2\sqrt{2}\mathbb{Z}_2$. If the four ordinary complementary codegrees are all rational, $s \geq 3/8$. If $t \leq 1/1000$, the ordinary complement consists of two rational four-rows of distinct supports and an irrational quadratic codegree pair G, G' having $G = nvu^j$, $j \geq 1$, $8 \parallel n$. Its mass is $1/2 - s$.*

Proof. Proposition 8.4 gives the assertion on the quadratic branches. Lemma 8.5 excludes a nonunit invertible: $t = w = 1/4$ would give $s \geq 3/4$ and $2\delta_D^2 \geq (16/9)s^2 > 1/2$ by Proposition 9.9. The same lemma excludes odd common rational codegrees, gives $4 \mid h$ for every integral ordinary row of norm h, and permits at most two rational four-rows.

Lemma 8.1 also shows that each four-row is rational and excludes a quadratic six-orbit.

Suppose all remaining ordinary codegrees are rational. All four-rows and six-rows are integral by Lemma 8.1. An integral row has norm divisible by four, so no six-row occurs. With at most one four-row the four masses sum to at most $1/4 + 3/8 = 5/8$, giving $s \geq 3/8$.

With two four-rows, each local factor contains one of them. The ramified FP branch therefore lies in one with a four-row, so the preceding dyadic calculation forces $N(D)$ to have squareclass two. If the other two rows are rational, write their norms as c, d. The global discriminant gives cd a square. On the local factor carrying these three rational branches, $1/4 + 1/c + 1/d \in \mathbb{Z}_2$. If $v_2(c) \neq v_2(d)$ this is impossible as c, d are divisible by four. At equal valuations their odd parts are congruent modulo eight since cd is a square, so $v_2(1/c + 1/d) = 1 - v_2(c)$. We obtain $v_2(c) = v_2(d) = 3$. Writing $c = 8a, d = 8b$, the same trace condition gives $a \equiv b \equiv 3 \pmod 4$, hence $c, d \geq 24$. For a quadratic pair of common rational codegree c the same argument on $1/4 + 2/c$ gives $c \geq 24$. Thus in the two-four-row case $s \geq 5/12$.

Now assume $t \leq 1/1000$, so $499/1000 \leq w < 1/2$. The short v-family exponent $j = 0$ for H was excluded in Proposition 9.9. The next exponent has $w = 10/m$, impossible for $8 \parallel m$ in this interval. Thus $H/H' \geq u^5$ and by dominance the same holds for D/D'; their squared gap ratios

are at least $c_0 = 1681/1682$. The all-rational case is impossible, as $c_0((499/1000)^2+2(3/8)^2) > 1/2$. With two irrational ordinary complementary pairs their mixed contribution is at least $(1-s)^2/2$, and $c_0(499/1000)^2 + 2c_0s^2 + (1-s)^2/2 > 1/2$. With a single irrational pair of mass z, unless the remaining ordinary rows are two rational four-rows their reciprocal sum is at most $3/8$. Thus $z \geq 5/8 - s$, and the same contradiction follows from $c_0(499/1000)^2 + 2c_0s^2 + (5/8-s)^2 > 1/2$ for $s \leq 5/8$ (the bound on D already gives $s < 5/8$). The two four-rows pair dyadically with the two ramified K branches, so the irrational ordinary complement has $G = nvu^j$, $8 \parallel n$, by the local calculation above. If $j = 0$ and $n = 8$, the projected row entries are $a_i + 2b_i\sqrt{2}$ with $a_i, b_i \in \mathbb{Z}$. Its norm $8v$ and orthogonality to its conjugate give $\sum b_i^2 = 1$. The difference from its conjugate would have just one nonzero coordinate, contradicting orthogonality to FP. For every other $j = 0$ value we have $n \geq 24$, $z = 2/n \leq 1/12$, $s \geq 5/12$, impossible as before. □

Corollary 9.16. *In the nonspherical case of Proposition 9.15 one has $t < 1/1000$ and the ordinary complementary exponent satisfies $j \geq 2$. In particular both the ordinary and signed character algebras have two K factors and two rational four-rows of distinct supports.*

Proof. Write $E = mvu^k, H = nvu^j, D = Mv^du^\ell$. Since there is no nonunit invertible, Lemma 9.3 gives strict dominance $2j+1 < 2\ell+d$. Suppose $t \geq 1/1000$ and set $h = m/n$, $c = m/M$, $z = j-k$, $r = \ell - k$; c, h are positive integers by central induction.

Let $B_{e,i} = \mathrm{Tr}(1/(v^eu^i))$, with recurrence $B_{e,i+2} = 6B_{e,i+1} - B_{e,i}$ and initial pairs $(2,6),(2,10)$. At a v-family branch $B_{1,i} \equiv 2 \pmod 8$. The dyadic integrality $1/4 + B_{1,i}/m \in \mathbb{Z}_2$ and $8 \parallel m$ therefore sharpen to $m \equiv 24 \pmod{32}$, and likewise for n. The central FP induction class on the signed pair of rational rows contributes $Z/4$, where $Z = 2$ or 3 by total positivity. Its contribution on the H-rows is $1 - Z/4 - s$, positive, and $n(1 - Z/4 - s) \in 2\mathbb{Z}$ since $\mathrm{Tr}(\mathcal{O}_K/(vu^j)) \subset 2\mathbb{Z}$. Consequently, writing $b = B_{1,k}, g = B_{1,k+z}, o = B_{d,k+r}$,

$$m = 2(b + hg), \qquad s = co/m, \qquad 0 < m(4 - Z) - 4co \in 8h\mathbb{Z}.$$

In particular $h \mid 2b$ and $h \mid 4co$. Hence $h \mid 4cW_{1,d}(r)$, with $W_{1,d}(r) \neq 0$ by Lemma 7.2. The leading terms give $c(v/2)^{d-1}u^r < 1200$ from $s/t < 600$. Together with $v^{d-1}u^r > c$ this gives $c^2 < 1200 \cdot 2^{d-1}$, $c \leq 34$, $1 \leq r \leq 4$. Now $4c|W_{1,d}(r)| < 5^{14}$ and $1 < w/t < 500$, so $-15 \leq z \leq 3$.

If $s < 3/8$ there is an irrational ordinary pair. Its mixed contribution is at least $(1/2-s)^2$, or at least $(5/8-s)^2$ if $d = 0$: in the latter case the two remaining rational-codegree rows cannot both be four, by the dyadic computation applied to D. With two irrational pairs the bound $(1-s)^2/2$ is at least either displayed bound. These expressions are used with positive parts when necessary. For each d, r, c, take the positive divisors $h \equiv 1 \pmod 4$ of $4c|W_{1,d}(r)|$; retain z satisfying strict central dominance. For $0 \leq k < 32$, test directly the preceding congruence, scalar divisibilities, $t \geq 1/1000$, $s \geq 3t$ and the mixed inequality. For $k \geq 32$ all indices are at least 17. The two-term formulas of Lemma 7.2 bound each mass and gap to a relative factor in $[1 - 10^{-6}, 1/(1-10^{-6})]$ of its mass at $k = 32$: every exponential correction is below 5^{-34}. Use lower bounds on the gaps and the upper bound on s in the ordinary deficit. Neither range contains a surviving tuple among the 9908 tested in `silver_four_distinct_check.py`. Thus $t < 1/1000$.

Proposition 9.15 now applies. If the ordinary irrational pair had $j = 1$, its mass would be $z = 10/n$ with $n \equiv 24 \pmod{32}$, so $z = 5/12$ or $z \leq 5/28$. For $c_0 = 1681/1682$ the mixed inequality would give

$$c_0(499/1000)^2 + 2c_0(1/2 - z)^2 + 2(49/50)z^2 \leq 1/2.$$

The quadratic exceeds $1/2$ at $5/12$ and on $[0, 5/28]$. Thus $j \geq 2$. □

Proposition 9.17. *Under the hypotheses of Proposition 9.9, suppose that the complementary signed characters have codegree four and are Gaussian conjugates or rational with common support. If the category is not spherical, choose $E \geq E'$. Then $\mathrm{Tr}(1/E) < 1/1000$.*

Proof. The two rows have the forms $(1,-1,\pm1,\mp1,0,0)$ or $(1,-1,\pm i,\mp i,0,0)$, exactly as in Lemma 8.8. The signed involution at their second coordinate acts by the permutation $(1,g)(X,Y)$. It is the only possible nontrivial actual invertible: at X,Y a putative invertible would give $t=w=1/4$ by cancellation in its signed unit coefficient, contradicting $s\geq 3t$ and $s<1/2$. This last upper bound follows on the ordinary FP induction column because the short pair contributes at least $1/2$ in the signed quotient. Indeed its two evaluations sum to an even integer: the sum of their rows is twice the difference of two coordinate functionals. Positivity and the unit coefficient bound force this sum to be exactly two, so the short pair contributes precisely $1/2$.

If $D/D'=H/H'$, Lemma 9.3 supplies the spherical rank-four subcategory, with nonunit invertible necessarily g. Proposition 9.10 then gives a spherical section on all of $\mathcal{C}$. Thus $D/D'>H/H'$.

Use $P_{e,k}=v^eu^k$ and $B_{e,k}=\mathrm{Tr}(1/P_{e,k})$, $u=3+2\sqrt{2}$, $v=2+\sqrt{2}$. As in Proposition 9.11, the square discriminant gives $E=mP_{e,k}$ and $H=nP_{e,j}$ in the same family. Put $D=MP_{d,\ell}$, $h=m/n$, $c=m/M$, $r=\ell-k$, $z=j-k$. Write $b=B_{e,k}$, $G=B_{e,j}$, $C=B_{d,\ell}$. The unit column and the ordinary FP column give

$$m=2(b+hG),\qquad (cC-b)/(2h)\in\mathbb{Z}.$$

The second equality uses $n(1/2-s),nw\in 2\mathbb{Z}$, since $\mathrm{Tr}(\mathcal{O}_K/(v^eu^j))\subset 2\mathbb{Z}$. Thus $h\mid 2b,2cC$, so $h\mid 2cW$, where $W=W_{e,d}(r)$ in Lemma 7.2. All primitive traces are two modulo four. If h is even then $v_2(m)=2$, so h is odd for $e=0$ and is never divisible by four for $e=1$.

Suppose $t\geq 1/1000$. Then $s/t<500$, so $c\leq 45$, $-1\leq r\leq 6$ by $s/t\geq cP_{d,\ell}/(4P_{e,k})>c^2/4$ and $V^{d-e}u^r>c$, with $V=v$. The initial trace pairs $(2,6),(2,10)$ and recurrence $B_{i+2}=6B_{i+1}-B_i$ give $0<|W|<2\cdot 10^6$ in this range when $v^{d-e}u^r>c$. From $2<hG/b=1/(2t)-1\leq 499$ and $u^z/2\leq G/b\leq 2u^z$ we may take $-14\leq z\leq 3$.

The same finite verification as in Theorem 8.9 is given in `folded_silver_check.py`. For reproducibility, enumerate (e,d,r,c,h,z) in the stated ranges with $h\mid 2cW$, the h-parity conditions, $2\mid c$ for $d>e$, $v^{d-e}u^r>c$, and $2z+e<2r+d$. For $0\leq k<28$ require $j,\ell\geq 1$, $4\mid m$, scalar multiples of four in the unit family and of two in the other family, $2h\mid cC-b$, $3t\leq s<1/2$, $t\geq 1/1000$, and the mixed gap bound. At $k\geq 28$ use $x=hu^z$, $y=c(v/2)^{d-e}u^r$, $T=2(1+x)$ and $\eta=999999/1000000$. The three gaps are at least η/T times $1,x,y$ and $s\geq \eta y/T$, $t\leq 1/(\eta T)$ by Lemma 7.2. Test the resulting necessary bounds and iterate the six recurrence residues modulo $\mathrm{lcm}(4h,4c)$ to decide the congruences. Among 228239 parameter tuples there are nine small matches. Twenty-two tuples pass the tail inequalities, with largest congruence period 42. After congruence tests all matches have $(e,d,r,c,h,z)=(1,1,3,1,10,2)$, with $k\geq 1$ (at $k=0$ the scalar divisibility fails).

In that tuple $H,D>10^5$, $t<15/10000$, and $2905/10000<s<2906/10000$. The ratios $t=B_{1,k}/[2(B_{1,k}+10B_{1,k+2})]$ and $s=B_{1,k+3}/[2(B_{1,k}+10B_{1,k+2})]$ lie between their values at $k=1$ and their limits: each is a linear fractional function of u^{-2k}. These endpoint inequalities follow from $s_1=197/678$, $t_1=1/678$ and the limits $u^3/[2(1+10u^2)]$, $1/[2(1+10u^2)]$. Consequently $\delta_H^2+2\delta_D^2>(4984/10000)^2+2(2904/10000)^2>417/1000$.

All rational ordinary codegrees are even and at least four. The dyadic signed order has no rank-one factor and residue fields $\mathbb{F}_2$ by the argument of Lemma 8.8, valid also for $\mathbb{Q}(\sqrt{2})$. The parity and perfect-trace argument there rules out odd common rational codegrees. A rational two-row requires an invertible, necessarily g; its value on g is -1. If such g exists its two negative rows have shape $(1,-1,\lambda,-\lambda,0,0)$, and $\lambda\neq 0$ because $\lambda^2=\pm1+n\lambda$. A nonrational pair of codegree two alone exhausts the reciprocal mass.

Two ordinary complementary irrational pairs would contribute at least $(1-s)^2/2>83/1000$. For one pair of mass z_0, unless both rational companions have codegree four, $z_0\geq 7/12-s$ and its contribution is at least $z_0^2>83/1000$. Thus both rational companions must be four and $2094/10000<z_0=1/2-s<2095/10000$. An exponent at least three in the ordinary

pair contributes at least $2(49/50)z_0^2 > 83/1000$. The shorter possibilities have mass $2/n$ or $6/n$, neither in the displayed interval for any admissible integer scalar n (the latter is a multiple of four). Finally four even rational codegrees at least four cannot have reciprocal sum in $(7094/10000, 7095/10000)$. The smallest must be four, the next four or six, and the only candidate first three are $(4,4,6),(4,4,8),(4,6,6)$; substitution gives no even fourth denominator. This contradicts every ordinary decomposition. □

Proposition 9.18. *Under the hypotheses of Proposition 9.17, suppose $\mathcal{C}$ is not spherical. Then $s = \mathrm{Tr}(1/D) < 356/1000$, and the signed involution is an actual invertible. Its ordinary action interchanges two pairs of simples and fixes the other two.*

Proof. Put $t = \mathrm{Tr}(1/E) < 1/1000$ and $w = 1/2 - t$. For ratio exponents one, two and three the possible masses of the H-pair are $2/n, 6/n, 10/n$, respectively. None belongs to $(499/1000, 1/2)$ for integral n. Thus both H and D have squared reciprocal gap at least $288/289$ times their squared mass. With $c_0 = 288/289$ the mixed bound gives

$$c_0((499/1000)^2 + 2s^2) < 1/2, \qquad s < 356/1000.$$

Suppose now that the signed involution is not an actual invertible. All ordinary rational codegrees are even and at least four as before. At two, the signed short factor cannot be isolated (in the rational case its two branches share a common reduction): on the projection its regular discriminant valuation is two, whereas its isolated perfect-trace value would be four. The two K-branches are ramified. Thus the only possible dyadic local ranks are six or four and two, and all residue fields are $\mathbb{F}_2$. The ordinary order has the same local ranks and residue fields by reduction modulo two.

Two irrational ordinary complementary pairs contribute at least $(1-s)^2/2$. With just one such pair the two smaller rational denominators cannot be $(4,4)$ or $(4,6)$. A nonrational pair of codegree four is Gaussian, forcing an actual invertible by its real part, while two rational four-rows with common support do so by their idempotents. In the other cases the two rational rows have distinct reductions (norm four and norm six cannot be congruent modulo four). Each ramified quadratic branch then has to join one of them, since neither rational axis splits off at two. This would give dyadic ranks three and three. Therefore the irrational pair has mass at least $5/8 - s$ and contributes at least $(5/8 - s)^2$. Both alternatives contradict the mixed bound: the minima over real s of $c_0((499/1000)^2 + 2s^2) + (1-s)^2/2$ and $c_0((499/1000)^2 + 2s^2) + (5/8 - s)^2$ both exceed $1/2$.

Suppose all four ordinary complementary codegrees are rational. By Lemma 8.1, every real four-row or six-row is integral. A nonreal four-row forces a nonunit invertible as above. A nonreal six-row would give $6 = |\operatorname{disc}(F)|b^2$ for its imaginary quadratic character field, since there is only one dual pair of simples. No such discriminant exists.

Lemma 9.5 now gives $s \geq 5/14 > 356/1000$, the desired contradiction. □

Corollary 9.19. *In every nonspherical case of Proposition 9.18, both ordinary characters taking g to -1 have formal codegree four. Their ordinary rows agree with the signed short pair up to ordering.*

Proof. They have shape $(1, -1, \lambda, -\lambda, 0, 0)$; in the Gaussian case $Y = X^*$ and the equation $\lambda^2 = -1 + a\lambda$ forces $\lambda = \pm i$. In the real case $\lambda^2 = 1 + a\lambda$. If $a \neq 0$, the two roots are conjugate quadratic units of norm -1. The second formal codegree $2(1+\lambda^2) = 4 + 2a\lambda$ lies in K, so the roots belong to K. Up to signs their larger absolute value is at least $1 + \sqrt{2}$, since every unit has absolute value a power of $1 + \sqrt{2}$. Their total reciprocal mass is $1/2$ and the squared gap is at least $1/8$. By the previous proposition $w > 499/1000$, and the squared gaps at H, D exceed or equal $c_0 = 288/289$ times their respective squared masses. Thus

$$1/2 \geq c_0((499/1000)^2 + 2s^2) + 1/4,$$

which gives $s < 1/25$. The two remaining ordinary characters have mass $1/2 - s > 23/50$. If irrational, they contribute at least $(1/2 - s)^2$ to the same mixed inequality, an impossibility. If rational, their codegrees are even and at least four and must both equal four, since $1/4 + 1/6 < 23/50$. That would leave $s = 0$. Hence $a = 0$ and the negative roots are $1, -1$. □

Corollary 9.20. *In the nonspherical alternative of Corollary 9.19, the two remaining ordinary rows have irrational conjugate codegrees.*

Proof. They take the same value at X, Y and are real, since these are the only possibly non-self-dual simples. The dyadic decomposition and the bound $s < 356/1000 < 5/14$ are given in the proof of Proposition 9.18. In the two-factor case the negative pair cannot occupy the rank-two factor: on its projected order the discriminant has valuation two, whereas two codegrees four would give four. Lemma 9.7 now excludes rational codegrees. □

10. Completing the quadratic case

The two remaining short characters have codegree four. Pell divisibility rules out different supports, and also the case in which the supports agree and the order is local at two. In the remaining case, bimodules over $\mathbf{1} \oplus g$ supply two new ordinary codegrees with reciprocal sum $s = \mathrm{Tr}(1/D)$. A free involution of the simple bimodules imposes the parity that rules out rational codegrees.

10.1. **The dyadic orders and their Pell traces.** Distinct supports and the matched local case will lead to the same contradiction. We first determine which quadratic branches attach to the short rows at two in the matched case.

Proposition 10.1. *In the nonspherical case of Corollary 9.20, write each of the two conjugate pairs of positive-character codegrees in a given ring as $mv^e u^k, nv^e u^j$, with $v = 2+\sqrt{2}$, $u = 3+2\sqrt{2}$. If its completed dyadic order is local then*

$$v_2(m) = v_2(n) \in \{3, 4\} \quad (e = 0), \qquad v_2(m) = v_2(n) = 3 \quad (e = 1).$$

Otherwise both signed and ordinary rings have local factors of ranks four and two, both have $e = 1$, and the rank-two factor in each is the maximal quadratic order. Its codegree scalar has valuation one and its character takes every simple to one modulo $\sqrt{2}$. The rank-four factor contains the signed physical pair or the ordinary Perron pair, respectively, and its quadratic codegree scalar has valuation two.

Proof. The two split or Gaussian negative rows have a common residual character and cannot form a dyadic factor by Proposition 9.18. The completed local ranks agree between rings; each four-dimensional factor, when present, therefore contains the two negative rows. The squareclasses of the absolute discriminants of the rational algebras are squares, so the two positive characters belong to the same primitive family in either ring. An isolated quadratic factor is an order in $K_2 = \mathbb{Q}_2(\sqrt{2})$ and hence has discriminant valuation $3 + 2a$ for $a \geq 0$. If its codegree is $F = nv^e u^j$, perfect trace duality gives $3 + 2a = v_2 N(F) = 2v_2(n) + e$. Its integral idempotent has unit coefficient $\mathrm{Tr}(1/F)$, of valuation $1 - v_2(n)$. Thus $a = 0, e = 1, v_2(n) = 1$. Since the two positive pairs have total mass $1/2$, the other scalar has valuation two.

Represent one of the positive rows in its four coordinates as $(1, A + \sqrt{2}B)$, where $A, B \in \mathbb{Z}^3$ and the first entry in each triple corresponds to the other free pair. Put $[x, y] = 2x_1y_1 + x_2y_2 + x_3y_3$. Writing $F = c + d\sqrt{2}$, orthogonality to its conjugate gives

$$2 + [A, A] = 2[B, B] = c/2, \qquad [A, B] = d/2.$$

For another pair, represented by C, D, the four cross equations are

$$2 + [A, C] = [B, C] = [A, D] = [B, D] = 0.$$

An attached row, in the factor containing the negative pair, has $A = (a, 2p, 2q)$ with a odd. Suppose first that both pairs are attached and write $C = (a', 2p', 2q')$. The identity

$$[B, B] = 1 + a^2 + 2(p^2 + q^2)$$

shows that B_2, B_3 have the same parity t. Put $x = p + q \pmod 2$. Reduction modulo four then gives $B_1 \equiv 1 + x + t \pmod 2$. Write y, t' for the analogous parities of the other pair. Dividing $[B, C] = [A, D] = 0$ by two gives

$$1 + x = t(1 + y), \qquad 1 + y = t'(1 + x) \quad \text{in } \mathbb{F}_2.$$

Thus $x = y$. If $x = 1$, exactly one of p, q is odd, so $[B, B] \equiv 4 \pmod 8$. If $x = 0$, then $t = t' = 1$ and B_1 is even. Thus $[B, B] \equiv 2 \pmod 8$, and $[A, B] \equiv 2(p + q) = 0 \pmod 4$. The same conclusions hold for the second pair. Hence both rational parts $c = 4[B, B]$ have the same valuation, either three or four. For $e = 0$ this equals $v_2(m)$. For $e = 1$ we have $v_2(c) = 1 + v_2(m)$ and $v_2(d) = v_2(m)$. Valuation three for c is impossible in that family, since $d = 2[A, B]$ would be divisible by eight.

For an isolated pair C, D we have already proved $e = 1$ and $v_2(n) = 1$. Thus $[D, D]$ and $[C, D]$ are odd, and $[C, C] = 2[D, D] - 2 \equiv 0 \pmod 4$. The latter congruence forces C_1, C_2, C_3 to have the same parity; $[C, D]$ odd forces all three to be odd.

It remains to identify the isolated pair in each ring. The signed physical scalar is a multiple of four since $E/4$ is an induction dimension and $v^e u^k$ is primitive. Thus its pair is attached. For an ordinary fixed simple P its Perron dimension p satisfies $p^2/2 \in \mathcal{O}_K$. Indeed P admits a right module structure over the algebra $1 \oplus g$. The pointed subcategory has trivial associator class since its action preserves the simple P, and a module structure on P can be normalized since $H^2(C_2, \mathbb{C}^\times) = 0$. The dimension of $P \otimes_{1\oplus g} P^*$ is $p^2/2$, by the Frobenius–Perron relative tensor product formula. Hence the rational part of p is even. This rules out the isolated residual pattern at the Perron row and proves the result. □

Theorem 10.2. *If a nonspherical conductor-eight category has two complementary signed characters of formal codegree four, their supports agree and the dyadic order splits into factors of ranks four and two.*

Proof. Suppose that the supports differ, or that they agree and the order is local at two. Write $u = 3 + 2\sqrt{2}$, $v = 2 + \sqrt{2}$, and put

$$E = mv^e u^k, \quad H = nv^e u^j, \quad D = Mv^d u^\ell, \quad G = Nv^d u^i.$$

In both cases m, n, M, N are divisible by eight and $h = m/n$ is odd. For different supports this follows from Proposition 9.15 and Corollary 9.16; for the local order it follows from Proposition 10.1. The same reductions give $t = \mathrm{Tr}(1/E) < 1/1000$ and $2\ell + d > 2j + e$ (strict dominance over H). Write $w = 1/2 - t$, $s = \mathrm{Tr}(1/D)$, $z = 1/2 - s$. We have $s < w$: on the distinct supports the mixed bound with $H/H', D/D' \geq u^5$ gives $2(1681/1682)s^2 \leq 1/2 - (1681/1682)w^2$; on the matched supports $s < 356/1000$ by Proposition 9.18.

One common divisor. Put $\xi = 1 + \sqrt{2}$ and define the odd integers

$$C_p = \frac{\xi^p + \xi^{-p}}{d_p}, \qquad d_p = \begin{cases} 2, & p \text{ even}, \\ 2\sqrt{2}, & p \text{ odd}. \end{cases}$$

Indeed these are Q_p for even p and P_p for odd p, where $\xi^p = Q_p + P_p\sqrt{2}$. They are nondecreasing for $p \geq 0$, since $Q_{2n} \leq P_{2n+1} = P_{2n} + Q_{2n} \leq Q_{2n+2}$. The reciprocal trace of $v^e u^k$ is $B_{e,k} = 2C_{2k+e}$. Write $K_0 = 2k + e$, $J = 2j + e$, $L = 2\ell + d$ and reduce $h/c = h_0/a$ to lowest terms, where $c = m/M$ is integral by central induction. Then

$$\frac{s}{w} = \frac{aC_L}{h_0 C_J}.$$

Consequently $M > n$ and $h_0 > 1$, since $L > J$ and $s < w$.

The signed mass identity $m = 2(B_{e,k} + hB_{e,j})$ gives $h \mid 2B_{e,k}$. On the ordinary Perron induction column the short signed rows contribute $Z/4$ with $Z \in \mathbb{Z}$, and the H-orbit contributes $1 - Z/4 - s$. Multiplication by n makes this last trace integral, since $\mathrm{Tr}(\mathcal{O}_K/(v^e u^j)) \subset \mathbb{Z}$. Thus $h \mid 4cB_{d,\ell}$. Oddness of h_0 gives $h_0 \mid C_{K_0}, C_L$, or equivalently $u^{K_0} = u^L = -1$ modulo $h_0\mathbb{Z}[\sqrt{2}]$. In any residue field of odd characteristic dividing h_0, the order of u has two-adic valuation $1 + v_2(K_0) = 1 + v_2(L)$. In particular $K_0 > 0$, $d = e$, and for $g = \gcd(K_0, L)$ both $K_0/g, L/g$ are odd. Bézout's identity gives $u^g = -1$ modulo h_0, hence $h_0 \mid C_g$. Now $D > E$ and $c \geq 1$ imply $r = \ell - k > 0$. Since $2g \mid L - K_0$, we have $g \leq r$.

Rounding to an odd integer. Let $b = \ell - j > 0$ and $\omega = s - 1/4$. Expanding (9.1) with $\delta_f^2 = \mathrm{Tr}(1/f)^2 - 4/N_K(f)$ gives

$$4\omega^2 \leq t - 2t^2 + 4/m^2 + 4/n^2 + 8/M^2 + 8/N^2.$$

Here $mt = B_{e,k}$ and $nt = n/2 - B_{e,j}$ are positive even integers; moreover $M > n$. If $\omega \neq 0$, then $N\omega = N/4 - B_{d,i}$ is a nonzero even integer. In either case $2\omega^2 \leq t + 2t^2$, so $|\omega| < 0.71\sqrt{t}$ and $s < 0.273$.

Set $d'_j = u^{-2j-e}$. The closed formula for $B_{e,j} = nw$ gives $B_{e,j}^2 d'_j \leq 4$, hence $d'_j \leq (2t/w)^2 < 17t^2$. Using the same formula on both sides of the mass ratio yields

$$\frac{2au^b}{h_0} = \frac{2s}{w}\frac{1 + d'_j}{1 + d'_\ell}, \qquad \left|\frac{2au^b}{h_0} - 1\right| < 4\sqrt{t}, qquadtu^r = \frac{s}{c}\frac{1 + d'_k}{1 + d'_\ell} < 1.$$

Indeed $d'_\ell < d'_j$ and $|2s/w - 1| \leq (2|\omega| + t)/w < 3\sqrt{t}$. In particular $h_0 > u^b$, so $h_0 \mid C_g \leq \xi^g$ forces $g > 2b$. The even integer $2aC_g/h_0$ differs from $C_g u^{-b}$ by less than $4u^{-b}$, since $g \leq r$ and $\sqrt{t}\,\xi^r < 1$. But $C_g u^{-b}$ differs from the odd integer C_{g-2b} by at most $1/(2\xi)$, by the definition of C_g. The total error is at most $4/u + 1/(2\xi) < 1$, a contradiction. □

10.2. **The split case and Morita duality.** In the split case the two isolated quadratic factors constrain the induction of the unit after Morita duality. The two new linear characters will have reciprocal codegrees summing to s. We first need an ordinary lower bound on this mass.

Lemma 10.3. *In the split matched case, $G/G' \geq u^3$ and $s = \mathrm{Tr}(1/D) > 1/6$.*

Proof. First, only $\mathbf{1}, g$ have integral Perron dimension. For the integral subcategory of Perron dimension n, evaluate its regular idempotent (Lemma 2.11) in each order. The short characters take g to -1, the dimension pair agrees on the subcategory, and the last pair occurs together or not at all. The two unit coefficients give $1/n \in \{s, 1/2\} \cap \{t, 1/2\}$. Since $t < s < 1/2$, we get $n = 2$.

Here $G = Nvu^i$. If $i = 0$, then $2/N = 1/2 - s$ and $s < 356/1000$ give $4 < N < 14$. Write the row as $(1, 1, x, x, y, z)$ with $(x, y, z) = A + \sqrt{2}B$. The entries of A are odd by Proposition 10.1. In the form $[A, B] = 2A_1B_1 + A_2B_2 + A_3B_3$, the norm and conjugate orthogonality give $[A, A] = N - 2$, $[A, B] = [B, B] = N/2$. Thus $N \equiv 6 \pmod 8$, so $N = 6$ and every A_i is a sign. The last equations give $|B_1| = 1$, one further sign among B_2, B_3, and $A_i = B_i$ on their support. Orthogonality of their difference to the Perron row says the signed Perron dimensions on this support sum to zero (with the first coordinate counted twice). Orthogonality of their mean then leaves two unit entries and a single signed fixed dimension. That dimension would be two, a contradiction.

Consequently the squared gaps of G, H, D are at least $49/50$ times their squared masses. Put $w = 1/2 - t > 499/1000$, $z = 1/2 - s$. At $s \leq 1/6$ we have $s^2 + z^2 \geq 5/36$. Then $(49/50)(w^2 + 2s^2 + 2z^2) > 1/2$, contradicting (9.1). □

The isolated signed quadratic factor now locates every copy of g in the induction of the unit.

Lemma 10.4. *In the split matched case g occurs once in FA for each of the four induction labels of codegrees H, H', G, G', and nowhere else. Here g denotes the positive lift of the involution.*

Proof. Write $m_A = [FA : g]$. The sum $FI(\mathbf{1}) = \sum_Y YY^*$ contains four copies of g, one for each fixed lift, and no copy of εg, which reverses the sign of the spherical dimension. The two signed short rows have sum $(2, -2, 0, 0, 0, 0)$. As $[FA : \mathbf{1}] = 1$ and $[FA : \varepsilon] = 0$, they contribute $(1 - m_A)/2$ to the unit coefficient of FA in the signed order.

If A has codegree f in either sector, the signed dimension pair contributes $\mathrm{Tr}(1/f)$, since $\dim FA = E/f$ and its conjugate row takes the conjugate value. The remaining signed pair is the isolated H-factor of Proposition 10.1. Its idempotent is integral at two. Projecting FA to this factor therefore gives

$$(1 + m_A)/2 - \mathrm{Tr}(1/f) \in \mathbb{Z}_2.$$

For $f = H, H', G, G'$ the trace is 2-adically integral. Indeed the isolated factors in the two orders have integral idempotents, of unit coefficients $\mathrm{Tr}(1/H)$ and $\mathrm{Tr}(1/G)$. Each of the four multiplicities is thus odd. Their sum exhausts the four copies of g. □

Proposition 10.5. *In the split matched case the Morita dual ${}_B\mathcal{C}_B$ for $B = \mathbf{1} \oplus g$ has rank twelve and a freely acting invertible involution. The irreducible representations of its Grothendieck algebra have dimensions and formal codegrees as follows.*

dimension	1	1	2	2	1	1
codegree	D	D'	G	G'	a	b.

The pair $\{a, b\} \subset K$ is Galois-stable, $1/a + 1/b = s$, and the two corresponding labels in the center of the sphericalization occur in the dual unit induction outside the original list.

Proof. The action on a fixed simple trivializes the associator class of $\langle g \rangle$. Choose the group algebra multiplication on B in this gauge. The positive lift of g has dimension one, so the pivotal structure on $\langle g \rangle$ is the ordinary one.

Put $\mathcal{D} = {}_B\mathcal{S}_B$ and transport its unit induction L to $Z(\mathcal{S})$. For the original induction labels we claim

$$[L : A_\chi] = (1 + \chi(g))/2 + [FA_\chi : g]. \tag{10.1}$$

Under the center equivalence Y has underlying bimodule $FY \otimes B$. A right-linear map to B is a pair of maps $f_a : FY \to g^a$, $a = 0, 1$. Write $m_{a,b} : g^a \otimes g^b \to g^{a+b}$ for the chosen multiplication. Left linearity at g says exactly

$$m_{a,1}(f_a \otimes 1_g) = m_{1,a}(1_g \otimes f_a)c_{Y,g},$$

where $c_{Y,g} : FY \otimes g \to g \otimes FY$ is the half-braiding. Using $g^* = g$ and $\dim g = 1$, this is the fixed-space equation for the rotation $E^{(a,1)}_{Y,g}$ of [17, Definition 2.4]. For $Y = A_\chi$ the first space is a line. The rotation on it has trace

$$\sum_W S_{Y,W}[FW : g] = \chi(FI(g))/(2f_\chi) = \chi(g)$$

by [17, (5.4)], since $FI(g) = gFI(\mathbf{1})$ in the commutative ring and $\chi(FI(\mathbf{1})) = 2f_\chi$.

On the second space the rotation is precomposition with $\theta_Y^{-1} = 1$ [17, Lemma 2.7]. This proves (10.1).

The preceding lemma now gives four constituents of L with multiplicity one and four with multiplicity two, hence $\mathrm{rank}(\mathcal{D}) = \dim \mathrm{End}(L) \geq 20$. The left and right B-actions make a bimodule an equivariant object for the two translations by g. Thus its simple objects on each orbit are indexed by the irreducible projective representations of the stabilizer [2, Corollary 2.13]. Each of the four size-two orbits contributes two simples. Each of the four fixed lifts contributes one or four, according as the two stabilizer generators anticommute or commute. The central sign pairs opposite lifts with the same cocycle, so the rank is $12, 18$ or 24. Thus the rank is 24 and every

stabilizer representation is one-dimensional. Each orbit downstairs has two lifts: multiplication by the positive lift of g preserves spherical dimension. Its stabilizer maps lift uniquely on either orbit and have the same composition scalars. Each simple bimodule downstairs therefore has two lifts, giving rank twelve. Twisting the left B-action by the automorphism $g \mapsto -g$ is tensoring on the left by the correspondingly twisted regular bimodule. It tensors the stabilizer representation by $(i,j) \mapsto (-1)^i$ on the left-right translation group C_2^2. This sign is nontrivial on each stabilizer: these are the diagonal C_2 on the two-point orbits and all of C_2^2 on fixed points. Thus it changes every one-dimensional projective representation and acts freely.

The haploid normalized special symmetric algebra B gives $\mathcal{D}$ its spherical structure [14, §3.10, after (70)]. Closing the bimodule trace with the unit and counit in [13, (2.16)] multiplies its scalar by $\dim_{\mathcal{S}} B = 2$. Thus

$$\dim_{\mathcal{D}}(V) = \dim_{\mathcal{S}}(V)/2. \tag{10.2}$$

It follows that the center equivalence preserves spherical dimensions and hence twists: the underlying bimodule of Y is $FY \otimes B$. Also $\dim \mathcal{D} = 2E$ by [7, Theorem 2.15].

The central sign J acts by $\varepsilon \otimes B$, which interchanges the two lifts of each simple bimodule. Forgetting lifts respects relative tensor products, since it preserves the separability projector. Thus the positive quotient is $K_0({}_B\mathcal{C}_B)_{\mathbb{C}}$, of dimension twelve. An irreducible representation of dimension n and codegree f there has full codegree $2f$ by summing over both lifts. Its label Y occurs in L with multiplicity n and has dimension E/f. The scalar action of the sign is $S_{J,Y}/S_{1,Y}$, so the old labels keep their sectors.

Lemma 10.4 and (10.1) give multiplicities $1, 1, 2, 2$ for the inherited ordinary labels D, D', G, G'. Their matrix blocks occupy ten dimensions, so precisely two new linear characters remain. Denote their labels by Y_1, Y_2 and codegrees by a, b. The unit coefficient gives $1/a+1/b = 1-s-2(1/2-s) = s$. The discarded pairs are Galois-stable, hence so is $\{a, b\}$. Since $\dim Y_i = \dim_{\mathcal{S}} FY_i$, the dimension-character field in Proposition 5.2 contains $E/\dim Y_i$. □

Theorem 10.6. *Every complex fusion category of rank six whose global Frobenius–Perron dimension generates $\mathbb{Q}(\sqrt{2})$ admits a spherical structure.*

Proof. By Theorems 6.12 and 10.2, only the split matched case remains. Use the two new labels of Proposition 10.5, and write $z = 1/2 - s$, $w = 1/2 - t$.

The two remaining codegrees are rational. If they are irrational then they are distinct conjugate totally positive d-numbers, with

$$(1/a - 1/b)^2 \geq s^2/2$$

by the smallest totally positive unit $u = 3 + 2\sqrt{2}$. The four labels Y_i, JY_i have twist one: Y_i occurs in dual unit induction and its sign is ordinary. The original labels occurring in the dual induction are precisely those already counted in (10.1). Thus these two new labels are outside the original list, and negative dimensions distinguish the translates from both lists of induction labels.

At Y_i, the signed dimension-orbit row indexed by σ has entry $\sigma(\dim_{\mathcal{S}} FY_i)/(2\sigma(E))$, hence $1/(2\sigma(a))$ or $1/(2\sigma(b))$. Retaining these four twist-one coordinates in the projection of (9.1) gives

$$\tfrac{1}{2} \;\geq\; \delta_H^2 + 2\delta_D^2 + 2\delta_G^2 + 2(1/a - 1/b)^2.$$

By the gap estimates of Lemma 10.3, the right side is at least

$$(49/50)w^2 + (74/25)s^2 + (49/25)z^2 \;\geq\; (49/50)(49/100)^2 + \frac{74 \cdot 49}{100(74+49)} \;>\; 1/2.$$

The middle inequality minimizes the last two terms at fixed $s + z = 1/2$. Thus a, b are rational integers.

The rational codegrees contradict the Pell norm. The free involution in Proposition 10.5 pairs the entries of either linear character with equal absolute values. Its codegree is therefore twice an algebraic integer, so a, b are even. Also $a, b \mid m$, since $E/a, E/b$ are integral and $E = mvu^k$

with vu^k primitive. In the split profile $v_2(m) = v_2(M) = 2$, $v_2(N) = 1$. Writing P_j for the Pell numbers as above gives

$$s = 2P_{2\ell+1}/M, \qquad z = 2P_{2i+1}/N.$$

Pell numbers of odd index are odd. Hence $v_2(s) = -1$ and $v_2(z) = 0$. Of the possibilities $v_2(a), v_2(b) \in \{1, 2\}$, only $(2, 2)$ can give $v_2(1/a + 1/b) = -1$. Since $s > 1/6$ by Lemma 10.3, the smaller denominator must be four. Write the other as 4β with β odd. Now $z = (\beta - 1)/(4\beta)$ has valuation zero, so $\beta \equiv 5 \pmod 8$. The reduced numerator of $s = (\beta + 1)/(4\beta)$ is $(\beta + 1)/2 \equiv 3 \pmod 4$. But it divides $P_{2\ell+1}$, whose odd prime divisors all admit a square root of -1 by $Q_{2\ell+1}^2 - 2P_{2\ell+1}^2 = -1$. This contradiction completes the proof. □

11. The dimension ratio in degree four

Write $K = \mathbb{Q}(D)$ and assume $[K : \mathbb{Q}] = 4$. The ordinary and signed dimension characters both generate K: their fields contain D by Lemma 2.1 and have degree at most six. Each rational character algebra is therefore the product of K and a two-dimensional algebra.

Theorem 11.1. *Let $\mathcal{C}$ have rank six. Suppose $K = \mathbb{Q}(D)$ is quartic. If E/D belongs to a proper subfield of K, then $\mathcal{C}$ has a pseudo-unitary Galois conjugate.*

Proof. A rational ratio is an integer in $(0, 1]$, and Corollary 2.13 treats cyclic K. Suppose then that K is biquadratic, $E < D$ and $G = \mathbb{Q}(E/D)$ is quadratic. For the equality subcategory $\mathcal{A}$ let $D_0 = \mathrm{FPdim}(\mathcal{A})$ and $F = K_{\mathcal{A}}$. Theorem 2.12 and Lemma 2.10 give a second quadratic field $F \neq G$ and a factorization $D = \delta w$, $E = \delta z$ with $\delta \in F$ and $w, z \in G$.

It suffices to show $s = \mathrm{Tr}_K(1/D) \leq t = \mathrm{Tr}_K(1/E)$. Indeed, transport if necessary by the nontrivial automorphism fixing F, so $z \geq z'$. This fixes δ and leaves D, w, s, t unchanged. The new ratio $x = z/w$ has $x \leq x'$ by (2.2). The two-term identity (5.1), with weights $1/z, 1/z'$, gives

$$\mathrm{Tr}_G(x) \leq 2\,\mathrm{Tr}_G(x/z)/\,\mathrm{Tr}_G(1/z) = 2s/t \leq 2. \tag{11.1}$$

Thus $x = 1$ by its positive integral norm.

Over $\mathbb{R}$, consider in each quotient the regular idempotent of $\mathcal{A}$, $e = D_0^{-1} \sum_{X \in \mathcal{A}} d_X b_X$. Its rank is the number of indecomposable $\mathcal{A}$-components: it projects to the Perron line in each component, and the signed action agrees with the ordinary one by Lemma 2.10. In either quartic factor precisely the two embeddings fixing F take e to one. If there is a single outside component, these exhaust its rank. Taking the unit coefficient in both sectors gives

$$1/D_0 = 1/D + 1/\gamma D = 1/E + 1/\gamma E$$

for the nontrivial γ fixing F. Tracing from F gives $s = t$.

Otherwise all three components have rank two by Lemma 2.10. Then $\mathcal{A}$ is Fibonacci by [18], with generator of dimension $\varphi = (1 + \sqrt{5})/2$ and $D_0 = 2 + \varphi$. For the nontrivial τ fixing G, the simultaneous column is supported on positive Fibonacci lifts and has unit coefficient one. Thus $D/\tau D = 1 + m\varphi$ with $m \in \mathbb{Z}_{>0}$, since D generates K. Its norm $1 + m - m^2$ equals one, so $m = 1$ and $\delta = 1 + D/\tau D = D_0$ in Theorem 2.12.

The Fibonacci generator embeds F in the two-dimensional signed complement. Its character with positive Fibonacci dimension is the third constituent selected by e. If its codegree is h, the unit coefficient of e gives $1/h = 1/D_0 - 1/E - 1/\gamma E = (1 - t)/D_0$, since $\mathrm{Tr}_F(1/D_0) = 1$. Let $\beta \in \mathbb{Z}[\varphi]$ be the complementary value of the ordinary Perron induction class. It is totally positive, and $\mathrm{Tr}_F(\beta/D_0)$ is a positive integer: it is its unit coefficient in the Fibonacci order. Hence the unit coefficient of the induction class gives $1 - s = (1 - t)\,\mathrm{Tr}_F(\beta/D_0) \geq 1 - t$, as required. □

Lemma 11.2 (A Galois difference in the trace dual)**.** *Let $x = E/D$ and let τ fix the character fields of the signed complement. Then $(\tau x - x, 0)$ belongs to the signed order. If x has quartic minimal polynomial P, then $(\tau x - x)P'(x)/E \in \mathbb{Z}[x]$.*

Proof. Subtract the signed class of the Perron induction label from its integral translate in Corollary 2.3. Multiplication by any polynomial in that class gives $\mathrm{Tr}_K((\tau x - x)\mathbb{Z}[x]/E) \subseteq \mathbb{Z}$. For $\deg r < 4$, Lagrange interpolation gives $\mathrm{Tr}_K(r(x)/P'(x)) = [X^3]r(X)$. On the power basis this coefficient pairing is integral with anti-diagonal ones and zeros for total degree below three. Thus the trace dual is $P'(x)^{-1}\mathbb{Z}[x]$ (see also [15]). □

12. Reduction to three quartic fields

Theorem 12.1. *Let $\mathcal{C}$ be a complex fusion category of rank six, and suppose its global Frobenius–Perron dimension generates a quartic field K. If*

$$K \notin \{\mathbb{Q}(\sqrt{2},\sqrt{3}),\ \mathbb{Q}(\sqrt{2},\sqrt{5}),\ \mathbb{Q}(\zeta_{16}+\zeta_{16}^{-1})\},$$

then a Galois conjugate of $\mathcal{C}$ is pseudo-unitary. In particular $\mathcal{C}$ admits a spherical structure.

The dimension orbit accounts for four of the six characters. We first consider the case in which the other two signed codegrees are irrational conjugates. For rational codegrees, Lemma 12.15 restricts the reciprocal masses, and Theorem 12.17 reduces the problem to twenty-four fields. The last step combines these two reductions.

12.1. **An irrational pair of complementary codegrees.** Write $G = \mathrm{Gal}(K/\mathbb{Q})$, $p_g = g(1/D)$, $q_g = g(1/E)$ and $\alpha_g = p_g/q_g$. Write $s = \sum p_g$, $t = \sum q_g$, and $\widehat{y}(\eta) = \sum_g \eta(g) y_g$. The two residual signed codegrees, when distinct and irrational, are quadratic conjugates f, f'. Their field F lies in K by Lemma 2.1, and $t = 1 - 1/f - 1/f'$. We transport to maximize E. Then q_1, p_1, α_1 are each the smallest entry of their respective orbit by (2.2) and Perron domination. In particular $\alpha_1 \le 1$ and $s < 1$.

Lemma 12.2. *Let c be the conductor of a nontrivial character η of G. Then*

$$|\widehat{q}(\eta)|^2 \le \frac{\gcd(c,4)}{c}, \qquad |\widehat{p}(\eta)|^2 \le \frac{\gcd(c,2)}{c}.$$

For a quadratic complementary pair f, f' in either sector add $2(1/f - 1/f')^2$ to the corresponding left side at the character cutting out its field.

Proof. In (2.3), each complementary row contributes $\frac{1}{4}|2/f - 2/f'|^2$ at this character. The four dimension rows contribute $|\widehat{p}|^2$ or $|\widehat{q}|^2$. □

Lemma 12.3 (From Fourier coefficients to a trace bound)**.** *At an involution j put $M_j = |\widehat{q}(\eta)|$ for the nontrivial character trivial at j. At a cyclic generator put $M_j = \sqrt{2}|\widehat{q}(\eta)|$ for a faithful character. If $M = \max_{j\neq 1} M_j < t$, then*

$$\mathrm{Tr}_K(E/D) < 1 + 4s/(t - M).$$

Proof. Fourier inversion gives $2(q_1 + q_j) = t + \widehat{q}(\eta)$ at an involution. At a cyclic generator with $\eta(j) = \mathrm{i}$, the last term becomes $\mathrm{Re}((1-\mathrm{i})\widehat{q}(\eta))$. Thus each $q_j \ge (t - M_j)/4$ for $j \neq 1$ by minimality of q_1. Now sum p_j/q_j, using $p_1/q_1 \le 1$ and $\sum_{j\neq 1} p_j < s$. □

Lemma 12.4. *In either sector a distinct quadratic complementary pair satisfies $1/f + 1/f' \le 1/2$.*

Proof. The character field $F = \mathbb{Q}(f)$ is real quadratic, as the two codegrees are totally positive. Put $c = \mathrm{disc}(F)$, $L = f + f'$ and $N = ff'$. The d-number theorem for formal codegrees [19], applied to $2f$, shows that f/f' is a unit, hence $L^2/N = f/f' + f'/f + 2$ is an integer. The positive dimension row belongs to the quartic orbit. Thus (5.2) gives $c \mid L$ and $L \ge 2c$.

Lemma 12.2 implies in either sector

$$2\left(1/f - 1/f'\right)^2 \le \gcd(c,4)/c. \tag{12.1}$$

Put $u = L/N$, $h = L^2/N$ and $\kappa = (c,4)/2$. Then $cu^2 \le \kappa + 4cu/L \le \kappa + 2u$. If $u > 1/2$, this forces $c = 5$ or 8. Orient $f > f'$ and take the least totally positive unit $U > 1$ of the quadratic field. For some $k \ge 1$,

$$h = 2 + U^k + U^{-k}, \qquad u = h/L, \qquad (1/f - 1/f')^2 = u^2(1 - 4/h).$$

At $c = 5$, $u > 1/2$ forces $h < 20/3$ and hence $h = 5$ by the unit table in Lemma 5.6. But $L \ge 10$ then gives $u \le 1/2$. At $c = 8$ put $b = 2L - N$. The bound $(1/f - 1/f')^2 \le 1/4$ gives $(N + b)^2 \le N^2 + 16N$, so $1 \le b \le 7$. Since $N \mid b^2$, $4h = N + 2b + b^2/N \le (b + 1)^2 \le 64$. Only $h = 8$ occurs for this quadratic field. Again $L \ge 16$ gives $u \le 1/2$. □

Lemma 12.5 (Short quadratic codegrees)**.** *Let f, f' be distinct quadratic conjugate codegrees complementary to the quartic dimension orbit, in either sector. Put $c = \operatorname{disc}\mathbb{Q}(f)$, $L = f + f'$, $N = ff'$ and $u = L/N$. The table gives all possibilities in the indicated ranges, including all ordinary fields at $u = 1/2$. An empty entry in the last column excludes that range.*

sector	c	*range*	*h to test*	(L, N)
signed	5	$u > 1/3$	$5, 9, 20$	$(10, 20)$
	8	$N \le 98$	8	$(16, 32), (24, 72)$
	12	$N \le 98$	$6, 16$	$(24, 96), (36, 81)$
	12	$u > 4/9$	$6, 16$	
	13	$u \ge 1/4$	$\varnothing$	
	24	$u \ge 29/100$	$12, 100$	
	28	$u \ge 27/100$	18	
ordinary	8	$37/100 \le u \le 1/2$	$8, 36$	$(16, 32), (96, 256)$
	$12, 24$	$u \ge 37/100$	6 $(c = 12)$	
	5	$u = 1/2$	5	$(10, 20)$
	8	$u = 1/2$	8	$(16, 32)$

Proof. The row-difference calculation (5.2) applies in either sector. Take $\kappa = (c,4)/2$ in the signed sector and $\kappa = (c,2)/2$ in the ordinary one by Lemma 12.2. As above, $h = 2 + U^k + U^{-k}$ and $cu^2 \le \kappa + 2u$. Thus $u \ge r > 0$ requires $c \le (\kappa + 2r)/r^2$ and, if $cr^2 > \kappa$, $h \le 4/(1 - \kappa/(cr^2))$. A bound $N \le B$ gives $h \le 4 + B\kappa/c$, since $(1/f - 1/f')^2 = (h - 4)/N$. These bounds and the unit traces of Lemma 5.6 give the fourth column. For each entry take

$$c \mid L, \quad L \ge 2c, \quad h \mid L^2, \quad \kappa L^2 \ge ch(h - 4), \quad u = h/L, \quad N = L^2/h.$$

Use $u \le 1/2$ in the signed sector. For example, at $c = 24$, $h = 12$ gives $L \ge 48 > 12/(29/100)$, while $h = 100$ requires $339 < L < 345$, containing no multiple of 24. □

Lemma 12.6. *Suppose the signed rational algebra in rank six is $K \times F$, where the dimension field K is quartic and the complementary pair has distinct quadratic codegrees $f, f' \in F$. Write $c = \operatorname{disc}(F)$, $u = 1/f + 1/f'$, $v = 1/f - 1/f'$, $Q = \widehat{(1/E)}(\chi)$ for the character χ cutting out F, and $M = \sqrt{\gcd(c,4)/c}$. There exists a trace-zero $z \in F$ such that $f(1/2 - u + z) \in \mathcal{O}_F$ and*

$$\left\| \begin{pmatrix} Q & \sqrt{2}\, v \\ \sqrt{2}\, v & 2z \end{pmatrix} \right\|_{\mathrm{op}} \le M.$$

In particular,

$$|Q| \le \sqrt{2}/3 \quad (f, f' = 8 \pm 4\sqrt{2}), \qquad |Q| \le \sqrt{5}/10 \quad (f, f' = 5 \pm \sqrt{5}).$$

Proof. Let A, B be the signed induction constituents indexed by the complementary characters. At every central simple Y the center character formula shows that $S_{A,Y} \in F$ and that conjugation over $\mathbb{Q}$ sends $S_{A,Y}$ to $S_{B,Y}$. By symmetry $S_{A,A} = S_{B,B}$ and $z = S_{A,A} - S_{A,B}$ has trace zero. Summing the six signed induction entries in the column of A gives $1/2$ by its signed unit

coefficient. Four dimension labels contribute u, so $S_{A,A}+S_{A,B}=1/2-u$. The character value $2fS_{A,A}=f(1/2-u+z)$ is an algebraic integer.

For the four dimension labels B_σ put $v_1=\frac{1}{2}\sum_\sigma \chi(\sigma)e_{B_\sigma}$; put $v_2=(e_A-e_B)/\sqrt{2}$. Their S-compression in this orthonormal basis is

$$\begin{pmatrix} Q/2 & v/\sqrt{2} \\ v/\sqrt{2} & z \end{pmatrix}.$$

Indeed $S_{B_\sigma,A}=1/(2\sigma(f))$, giving the off-diagonal entries, and the other two entries follow from the dimension-column formula and the definition of z. Let P_J be tensoring by the central sign on the simple basis. Both $(v_i-P_Jv_i)/\sqrt{2}$ belong to the T^2-fixed subspace and transform by the quadratic character χ under the diagonal torus: Galois permutes the two complementary rows without signs, since their character formulas are conjugate. Between signed induction labels multiplication by J in either entry reverses the sign of S, so the compression in these two new vectors is twice the preceding matrix. The $U(2)$-fixed projection estimate of Lemma 2.14, applied to their entire span, gives the asserted norm bound.

In both explicit pairs $u=1/2$ and fz is integral. At $f=8+4\sqrt{2}$, write $z=r\sqrt{2}$. Then $8r\in\mathbb{Z}$, and the second column norm gives $|z|\le 1/4$; therefore $|z|\le\sqrt{2}/8$. Positive semidefiniteness of MI plus and minus the displayed symmetric matrix yields

$$|Q|\le M-\frac{2v^2}{M+2|z|}.$$

Here $M=1/\sqrt{2}$, $v^2=1/8$, so the right side is at most $\sqrt{2}/3$. At $f=5+\sqrt{5}$, $z=r\sqrt{5}$ has $10r\in\mathbb{Z}$, and $v^2=1/20$. The second column norm gives $|2z|\le 1/\sqrt{10}$, so $z=0$. Thus $|Q|\le 1/\sqrt{5}-(1/10)\sqrt{5}=\sqrt{5}/10$. □

Proposition 12.7. *Let $\mathcal{C}$ have rank six, let $K=\mathbb{Q}(D)$ be quartic, and suppose the complementary signed codegrees are distinct quadratic conjugates with field F. Make a Galois transport so that E has its largest conjugate as spherical dimension. Assume both of the following:*

1. *Each quadratic subfield of K of discriminant 8 or 12 is F (if such a subfield exists).*
2. *If K is cyclic, its conductor is different from 16 and 20.*

Then $\mathrm{Tr}_{K/\mathbb{Q}}(E/D)\le 94$. In particular there are only finitely many possible such dimension fields with E/D primitive.

Proof. Put $q=1/E$, $t=\mathrm{Tr}_K(q)$ and $\alpha=E/D$. Write f,f' for the complementary codegrees and put $c=\mathrm{disc}(F)$, $L=f+f'$, $N=ff'$, $u=L/N=1-t$, and $v=1/f-1/f'$. By Lemma 12.4, $t\ge 1/2$. For the character η cutting out F, Lemma 12.2 gives $|\widehat{q}(\eta)|^2+2v^2\le \gcd(c,4)/c$. Since $v^2=u^2-4/N$ and $(1-u)^2+2u^2\ge 2/3$,

$$t-|\widehat{q}(\eta)|\ge \tfrac{1}{2}(2/3-\gcd(c,4)/c-8/N)$$

when the right side is positive. The denominator in obtaining this bound is $t+|\widehat{q}(\eta)|\le 2$. For $N\ge 99$ this gives a lower bound $1/12-4/99>4/94$.

For $N\le 98$ and $c=8,12$ use Lemma 12.5 (also checked in `verify_quartic_bounds.py`). Substituting $t=1-L/N$ and $v^2=(L^2-4N)/N^2$ gives $t-|\widehat{q}|>4/94$ except at $(8,16,32)$. There the matrix bound of Lemma 12.6 gives $|\widehat{q}|\le\sqrt{2}/3$, so the gap exceeds $2/94$. Moreover $s=\mathrm{Tr}_K(1/D)=1/2$ in this case. Indeed if $\beta=m+n\sqrt{2}$ is the complementary value of the signed induction class of the ordinary FP character, its unit coefficient gives $1-s=\mathrm{Tr}(\beta/f)=(m-n)/2$. Since $0<s<1$, equality follows.

For every other involution j, the quadratic character of kernel $\{1,j\}$ has conductor a distinct from $8,12$, so the separate estimate gives $t-|\widehat{q}|\ge 1/2-1/\sqrt{5}>4/94$. In a cyclic quartic field the two adjacent pair sums satisfy $2(q_0+q_j)\ge t-\sqrt{2\gcd(J,4)/J}$ by Fourier inversion, where J is the faithful conductor. For odd $J\ge 15$ and even $J\ge 40$ this again exceeds $4/94$. The possible

cyclic field conductors below forty that are divisible by four are $16, 20$: indeed the unit groups modulo $4, 8, 12, 24$ have exponent at most two, and modulo $28, 36$ have exponent dividing six; modulo 32 the even characters factor through the cyclic group of order eight, and its order-four characters factor through modulus sixteen. (Odd or twice-odd conductors in this range satisfy $\gcd(J, 4)/J \leq 1/15$.)

Apart from $8 \pm 4\sqrt{2}$, all pair gaps $t - M_j$ exceed $4/94$, using $1/2 - 1/\sqrt{5}$ also when the complementary conductor differs from $8, 12$. At $8 \pm 4\sqrt{2}$ they exceed $2/94$ and $s = 1/2$. Lemma 12.3 now gives $\mathrm{Tr}_K(\alpha) < 95$. Bounded trace bounds all elementary symmetric functions of the positive conjugates. □

Theorem 12.8. *Let a rank-six fusion category have quartic field $K = \mathbb{Q}(D)$ and two distinct irrational complementary signed codegrees. Write $a \leq b \leq c$ for the conductors of the three nontrivial characters of K. Transport the category so that E is its largest Galois conjugate, and put $A = \mathrm{Tr}_K(E/D)$. If $b \geq 33$, then $A \leq 94$. If E/D generates K, then $A \geq 2\sqrt{\sqrt{bc/a} + 4}$.*

Proof. Let c_0 be the discriminant of the complementary pair. When $b \geq 33$ and $c_0 \leq 32$, that pair gives the only character of conductor at most 32, so Proposition 12.7 applies. When $c_0 > 32$ the inequality $c_0 u^2 \leq (c_0, 4)/2 + 2u$ of Lemma 12.5 gives $u \leq 1/4$ (the least even discriminant here is 40). Every pair amplitude is at most $1/\sqrt{2}$. Since $3/4 - 1/\sqrt{2} > 4/94$, Lemma 12.3 gives $A \leq 94$. For the quadratic subfield F of least conductor, the conductor–discriminant formula gives $N_{F/\mathbb{Q}}(\mathfrak{d}_{K/F}) = bc/a$. If $x = E/D$ generates K, then $K = F(x)$ and the totally positive integer x has norm at least one. Lemma 2.5 gives $A = \mathrm{Tr}_K(x) \geq 2\sqrt{\sqrt{bc/a} + 4}$.

□

Corollary 12.9. *Let F be the field generated by two distinct signed complementary codegrees. If its discriminant c_0 is different from $5, 8, 12$, then $t = \mathrm{Tr}_K(1/E) > 71/100$. Under the same hypothesis, unless $c_0 = 24, 28$ one has $t \geq 3/4$.*

Proof. The bound $c_0 u^2 \leq (c_0, 4)/2 + 2u$ gives $u = 1 - t \leq 1/4$ for odd $c_0 \geq 17$ or even $c_0 \geq 40$. For the remaining $c_0 = 13, 24, 28$ use Lemma 12.5. □

Lemma 12.10. *Suppose the two distinct signed complementary codegrees f, f' generate $\mathbb{Q}(\sqrt{5})$. Then either $t = \mathrm{Tr}_K(1/E) \geq 2/3$ or the pair is $5 \pm \sqrt{5}$ and $\mathrm{Tr}_K(1/D) = t = 1/2$. In the latter case, with unnormalized $p_i = i(1/D)$ and $q_i = i(1/E)$, the moments*

$$h_j = \sum_i q_i \alpha_{i+j}, \qquad K_j = \sum_i p_i p_{i+j}/q_i$$

are at least $1/2$; $2K_j$ is odd when j exchanges f, f'.

Proof. Lemma 12.5 gives the stated alternative. For the exceptional pair orient $f = 5 + \sqrt{5}$ and write the complementary evaluation of the ordinary FP column as $m + n(1 + \sqrt{5})/2$. Its contribution to the unit coefficient is then $m/2 = 1 - \mathrm{Tr}_K(1/D)$, which belongs to $(0, 1)$. Hence $m = 1$ and both masses equal $1/2$.

Put $\phi = (1 + \sqrt{5})/2$ and let β be the complementary value of the ordinary Perron induction class V. Positivity at both embeddings gives $\beta = 1$ or $\phi^2 = f/f'$. By Corollary 2.3 the integral translate of V has complementary value $j(\beta)$, and the raw induction class V_j has value $fj(\beta/f)$. Thus the complementary contributions to the unit coefficients defining h_j and K_j are respectively $\mathrm{Tr}_F(j(\beta)/f)$ and $\mathrm{Tr}_F(\beta j(\beta/f))$, where $F = \mathbb{Q}(\sqrt{5})$. Both are $1/2$ or 1; the second is $1/2$ whenever j exchanges f, f', since $N_F(\beta) = 1$. Subtract from the integral unit coefficients and use positivity of both moments. □

Lemma 12.11 (Coordinates for the trace bounds)**.** *Let x be a totally positive integer of trace A in a quartic field K. If K is biquadratic with quadratic discriminants a, b, c, then*

$$4x = A + i\sqrt{a} + j\sqrt{b} + k\sqrt{c}, \qquad i, j, k \in \mathbb{Z}, \qquad ai^2, bj^2, ck^2 < A^2.$$

If K is cyclic and contains $\sqrt{2}$, then $K = \mathbb{Q}(y_+)$, $y_\pm = \sqrt{m(2 \pm \sqrt{2})}$, for odd squarefree $m > 0$. Its conductor is $16m$. In the integral basis $1, \sqrt{2}, y_+, y_-$ the coordinates (r, i, j, k) satisfy

$$4r = A, \qquad 4r^2 + 8i^2 + 8m(j^2 + k^2) < A^2.$$

Proof. In the biquadratic case the three relative traces give the integral coefficients, and positivity bounds each signed sum of conjugates by A. For a cyclic quartic the parameterization and integral basis are those of Hudson and Williams [12, pp. 145–146, Theorem (i)] with $D = 2$, $B = C = 1$. The trace Gram matrix has diagonal $4, 8, 8m, 8m$ and zero off-diagonal entries. Its determinant is $8(16m)^2$, so the conductor–discriminant formula gives $16m$. Finally $\mathrm{Tr}(x^2) < A^2$ by positivity. □

Theorem 12.12. *Suppose $\mathbb{Q}(D) = K$ is quartic and the two complementary signed codegrees are distinct and irrational. A Galois conjugate is pseudo-unitary in either of the following cases:*

1. *K has no quadratic subfield of discriminant $5, 8$ or 12.*
2. *K is biquadratic, contains $\sqrt{5}$, and contains neither $\sqrt{2}$ nor $\sqrt{3}$.*

Proof. By Theorem 11.1 we may take $x = E/D$ primitive after transport maximizing E. Write $A = \mathrm{Tr}(x)$, $s = \sum p_i < 1$, $t = \sum q_i$, and $u = 1 - t$. We apply Lemma 12.3 with the separate Fourier bounds of Lemma 12.2.

In (1) the complementary quadratic field has discriminant at least thirteen, so Corollary 12.9 gives $t > 71/100$. On a quadratic character take $M = 1/\sqrt{6}$ (the smallest even fundamental discriminant above twelve is twenty-four). In a cyclic field the adjacent pair bound is at most $1/\sqrt{5}$: for odd J one has $J \geq 13$, and the even quartic conductors below forty can only be $16, 20$, whose quadratic subfields are excluded here. It follows that $A \leq 14$ in a biquadratic field and $A \leq 16$ in a cyclic field.

Write $P = X^4 - AX^3 + BX^2 - CX + H$ for the minimal polynomial. For these small heights one may enumerate the polynomials directly. Positivity gives $0 < B \leq 3A^2/8$, $0 < C \leq A^3/16$, $256H \leq A^4$, and the d-number divisibilities give $H \mid A^4$, $H^2 \mid B^4$, $H^3 \mid C^4$. Keep irreducible polynomials with four positive real roots, one less than one, and abelian Galois group. The cubic resolvent is

$$Y^3 - BY^2 + (AC - 4H)Y - (A^2 - 4B)H - C^2.$$

In the biquadratic case its roots r are rational integers. The pair sums of the four roots give the three signed differences of complementary pair sums. Their squares are $A^2 - 4(B - r)$. The nonzero differences generate the quadratic subfields present in x and hence generate K. In the cyclic case the discriminant of P generates the quadratic subfield on taking its square root, since a generator of order four acts by an odd permutation. The following five polynomials remain. In the cyclic rows the entry in the right column is the quadratic conductor.

A	B	C	H	quadratic conductors
9	19	9	1	13
12	45	54	1	28, 44, 77
13	40	13	1	17
14	38	28	4	21, 60, 140
15	60	72	9	13

For each polynomial fix its least root at the identity and allow every ordering of the other three. Impose $p_i = x_i q_i$, the minimum conditions, $s, t \leq 1$, $t \geq 71/100$ and the Fourier inequalities of Lemma 2.15. In the cyclic cases the odd quadratic conductor d can replace the faithful conductor: the latter is an odd multiple of d or a multiple of $4d$.

In (2) all characters except the one of conductor five have separate amplitude at most $1/\sqrt{6}$. If the complementary character field differs from $\mathbb{Q}(\sqrt{5})$ then $t > 71/100$. Otherwise Lemma 12.10 gives either $t \geq 16/25$ or $f, f' = 5 \pm \sqrt{5}$. In the former case $M = 1/\sqrt{5}$ gives $A \leq 21$. For the exceptional pair $t = s = 1/2$ and Lemma 12.6 improves its own signed Fourier amplitude to $\sqrt{5}/10$. Taking $M = 1/\sqrt{6}$ gives $A \leq 22$.

In the biquadratic coordinates of Lemma 12.11, primitivity requires two nonzero coefficients, so the smaller discriminant other than five is $< 22^2$. The resulting 120 fields give ten primitive ratios. Test each in both mass ranges, using the sharper conductor-five amplitude at half mass.

Here and in the remaining quartic tests the Fourier disks and their mixed bounds have rational supporting half-spaces. The root intervals are rounded outwards. If the resulting system is $Bq \leq d$ with $q \geq 0$, $\sum q_i \leq 1$, an integer vector $w \geq 0$ excludes it when

$$\min\{0, \min_i (B^{\mathsf{T}} w)_i\} > d^{\mathsf{T}} w. \tag{12.2}$$

Indeed $(B^{\mathsf{T}} w)^{\mathsf{T}} q$ is at least the left-hand side. The polynomial check verifies the 30 orderings in (1), and the biquadratic check verifies the 20 mass and ratio combinations in (2). □

Theorem 12.13. *Let $\mathcal{C}$ have rank six and quartic dimension field $K = \mathbb{Q}(D)$, with two distinct irrational complementary signed codegrees. If K contains $\sqrt{2}$ or $\sqrt{3}$, a Galois conjugate is pseudo-unitary except possibly when K is one of*

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}), \quad \mathbb{Q}(\sqrt{2}, \sqrt{5}), \quad \mathbb{Q}(\sqrt{3}, \sqrt{5}), \quad \mathbb{Q}(\zeta_{16})^{+}.$$

Proof. Write $x = E/D$ after transport maximizing E, and put $p_g = 1/g(D)$, $q_g = 1/g(E)$, $s = \sum p_g$, $t = \sum q_g$. By Theorem 11.1 it suffices to exclude primitive x. Let F be the complementary field of discriminant c_0, $u = 1 - t = 1/f + 1/f'$, $L = f + f'$, $N = ff'$.

The trace bounds. Outside the four stated fields, K contains just one of $\sqrt{2}, \sqrt{3}$ and does not contain $\sqrt{5}$. Thus $c_0 = 5, 24$ are impossible: $\sqrt{6}$ together with either $\sqrt{2}$ or $\sqrt{3}$ generates both. At $c_0 = 8, 12$ apply Proposition 12.7. All other c_0 except 28 give $t \geq 3/4$ by Corollary 12.9. Every pair amplitude is at most $1/\sqrt{2}$, so the trace is at most 94. At $c_0 = 28$ one has $t > 73/100$. The field is then $\mathbb{Q}(\sqrt{3}, \sqrt{7})$ or $\mathbb{Q}(\sqrt{2}, \sqrt{7})$; the maximal amplitudes $1/\sqrt{3}, 1/\sqrt{2}$ give $A \leq 94, 176$ respectively. For the latter field and for cyclic conductor 48 we also use

$$t - |\widehat{q}(\chi_8)| \geq 2/94. \tag{Q1}$$

This follows from Proposition 12.7 when $c_0 = 8$, and from $73/100 - 1/\sqrt{2} > 2/94$ otherwise.

Enumerating fields and ratios together. By Lemma 12.11 at least two radical coefficients of a primitive biquadratic ratio are nonzero. The second smallest quadratic discriminant is therefore $< A^2$. For $A \leq 94$, enumerate $\mathbb{Q}(\sqrt{d}, \sqrt{n})$ with $d = 2, 3$ and squarefree $1 < n < 94^2$, including the field with cap 176. This gives 4 358 conductor triples, nine with second conductor below 33. A real cyclic quartic cannot contain $\sqrt{3}$, since the quadratic character at three is nontrivial on -1 and cannot be a square. With quadratic subfield $\mathbb{Q}(\sqrt{2})$, the cyclic coordinates of the same lemma give $8m < A^2$ by primitivity. There are 446 odd squarefree $m > 1$ to test, with $m = 1$ one of the stated exceptions.

The separating inequalities. In the preceding boxes retain primitive totally positive integral d-numbers with least conjugate below one. If $P = X^4 - AX^3 + BX^2 - CX + H$ is their polynomial, the divisibilities are $H \mid A^4$, $H^2 \mid B^4$, $H^3 \mid C^4$. The enumeration leaves 662 biquadratic and 74 cyclic polynomials. For each impose $p_g = g(x) q_g$, $1/2 \leq t \leq 1$, $s \leq 1$, the two coordinate minima and the separate and mixed Fourier inequalities, with (Q1) in its two fields. Every system has integer weights satisfying (12.2). □

12.2. **Rational complementary codegrees in quartic fields.**

Lemma 12.14 (Codegree three at an inert prime)**.** *If every prime of the quartic dimension field K above three has even residue degree, neither sector has a complementary character of codegree three.*

Proof. A rational row ρ of codegree three needs a totally ramified local partner by Lemma 2.7. It must be the other rational row η, say of codegree h. The two rows agree modulo three and the unit coefficient of their common local idempotent gives $1/3 + 1/h \in \mathbb{Z}_3$, hence $h/3 \equiv -1 \bmod 3$.

Write $m = \rho(FA_\rho)$ and $n = \rho(FA_\eta)$. Central symmetry gives $\eta(FA_\rho) = (h/3)n$, so $m + n \equiv 0 \bmod 3$. The four dimension labels of the chosen sector have the same integral value b on ρ: it is one on signed simultaneous columns, and on ordinary columns Galois permutes their normalized rows while fixing ρ. In the ordinary case $b = 3\,\mathrm{FPdim}(FA_\rho)/D > 0$ by symmetry. The row $6S_{A_\rho,-}$ has squared norm 36 and includes the vacuum entry one besides these four entries, so $b \leq 2$. In either case the unit coefficient of FA_ρ, computed by symmetry, gives $4b + m + n = 3$, contradicting $3 \nmid b$.

A quadratic orbit of common codegree three has field $\mathbb{Q}(\sqrt{-3})$ by Lemma 5.3. At three both the discriminant of K and any norm from K have even valuation, since every residue degree is even. The quadratic discriminant contributes one and the two codegrees contribute two, contrary to Lemma 2.6. □

Lemma 12.15. *In rank six suppose $K = \mathbb{Q}(D)$ is quartic and the two signed complementary codegrees are rational. Put $s = \mathrm{Tr}_K(1/D)$ and $t = \mathrm{Tr}_K(1/E)$. Then $s \leq t$, $t \geq 1/2$, and either $s \geq 1/2$, or $s = 1/3$ with $t \geq 2/3$. The latter case requires two ordinary codegrees three. At $s = t$ the complementary signed evaluations of every forgotten ordinary dimension induction label are one.*

Proof. Corollary 2.3 gives $s \leq t$ and the equality assertion; the raw induction columns have the same complementary evaluations as their integral Galois translates because the simultaneous column evaluates at one. It remains to bound s and t.

Neither algebra has a complementary codegree two. A signed character of codegree two must be rational, as a second conjugate of the same codegree would exhaust the mass. Its central row has four entries $1/4$ on the signed dimension orbit and four $\beta/4$ on the ordinary orbit, with $\beta \in \mathbb{Z}_{>0}$. The distinct J-partners have the same magnitudes. Their squared norm is already at least one, precluding any other support. Yet the sum on all six signed induction labels is $1/2$ by the coefficient of the unit, while the four signed dimension entries alone sum to one. For an ordinary rational codegree two, the orthogonal character row is $(1, -1, 0, 0, 0, 0)$ up to permutation: its norm is two and orthogonality to FP forces the negative sign and the second FP dimension to be one. The corresponding ordinary basis element g is $1 - 2e_\chi$, hence an involutive invertible with a one-dimensional negative eigenspace. Its positive spherical lift acts by the same permutation since it sends the positive lift of each simple to the positive lift of its unique product. Thus in the signed order $(1 - g)/2$ is the primitive idempotent for a rational character, with unit coefficient $1/2$, giving the excluded signed codegree. A common ordinary codegree two already exhausts the mass. Codegrees below two on rational rows are excluded by positivity of the residual mass.

A pair of rational characters cannot have codegrees $(3, 4)$ or $(3, 5)$, in either algebra. For the corresponding quartic codegree $H = D$ or E, Lemma 2.6 gives $N(H)gh = |\,\mathrm{disc}(K)|I^2$, where I is the order index.

At three the quartic field is tame, and its inertia index is at most two: tame inertia in an abelian extension at three has order dividing two. Hence the field discriminant has even valuation, and the d-number property makes $v_3(N(H))$ a multiple of $4/e$, which is also even. Both proposed products have odd valuation.

We record a trace inequality valid for the two complementary sectors. For the signed complementary labels A, B put $v = e_A + e_B$ and $\Phi(x) = 6x^2 - 8x + 3$. The vector Sv is real, since v is

invariant under duality. For any induction label of either sector with codegree f, the sum of its six signed entries is $1/2$ by adjunction and the unit coefficient. Its four signed dimension entries sum to $\mathrm{Tr}_K(1/f)/2$ by the center formula. Consequently Sv has value $(1-t)/2$ at each of the four signed dimension labels and its two complementary signed values sum to $2t-1$; the analogous ordinary values have s in place of t. In the complementary sum use $\sum_{\text{two } f} \mathrm{Tr}_K(1/f) = 4(1-t)$ or $4(1-s)$, respectively. Including the J-partners shows that the squared norms on these signed and ordinary coordinates are $M_s \geq \Phi(t)$ and $M_o \geq \Phi(s)$ by Cauchy's inequality. In the quadratic form $(Sv)^t T(Sv)$ the signed pairs cancel, the ordinary pairs contribute M_o, and the remaining real part is at least $-(2 - M_s - M_o)$. The twist fixes v and the center has central charge one, so $v^t STSv = v^t Sv = 2t - 1$. Therefore

$$\Phi(t) + 2\Phi(s) \leq 2t + 1.$$

If the ordinary complementary codegrees are rational, reciprocal mass greater than half thus leaves only $(3, 3)$ and $s = 1/3$, also allowing a quadratic character pair of common codegree three. The displayed inequality at $s = 1/3$ reads $(3t-2)(t-1) \leq 0$, so $t \geq 2/3$.

Distinct quadratic ordinary codegrees have $s \geq 1/2$ by Lemma 12.4. Finally $t < 1/2$ would give signed codegrees $(3, 3)$, hence $s = t = 1/3$ by $s \leq t$, contrary to the preceding bound $t \geq 2/3$. □

Lemma 12.16. *Suppose an odd prime p is inert in a quadratic subfield of K. If two rational complementary characters in either sector have codegrees g, h of odd p-adic valuation, their rows are congruent modulo p. In particular $p^2 \mid g + h$.*

Proof. Distinct residue characters belong to distinct factors of the completed order. Neither factor can consist of a rational branch alone, since its idempotent has nonintegral unit coefficient $1/g$ or $1/h$. Each therefore needs a branch of K. Their residue degrees are even by inertness, so degree four allows exactly two unramified quadratic branches. On each factor the unit-coefficient form is unimodular. Passing to the regular trace multiplies its discriminant by its rational codegree and the norm from an unramified quadratic field of the dimension codegree. This gives odd valuation, whereas every order in an unramified algebra has even discriminant valuation. The rows are therefore congruent. Their squared distance is $g + h$ by orthogonality. □

Theorem 12.17. *Let a rank-six category have quartic dimension field $K = \mathbb{Q}(D)$ and rational complementary signed codegrees. A Galois conjugate is pseudo-unitary outside the following list of twenty-four fields:*

K	*parameters*
$\mathbb{Q}(\sqrt{2}, \sqrt{d})$	$d = 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 33, 37, 41$
$\mathbb{Q}(\sqrt{3}, \sqrt{d})$	$d = 2, 5, 7, 10, 11, 13$
cyclic quartic of conductor J	$J = 16, 20, 48, 80, 112.$

The first two rows have one common field. The cyclic fields in this list have quadratic subfield $\mathbb{Q}(\sqrt{2})$ except at $J = 20$, where it is $\mathbb{Q}(\sqrt{5})$; each specification is unique.

Proof. By Theorem 11.1 take $x = E/D$ primitive at the conjugate maximizing E. Put $p_i = i(1/D)$, $q_i = i(1/E)$, $s = \sum p_i$, $t = \sum q_i$ and $A = \mathrm{Tr}(x)$. We use Lemma 12.15, in particular $1/3 \leq s \leq t < 1$, $t \geq 1/2$ and $s + t \geq 1$. The two-orbit compression gives $\sqrt{c}|Q| + c|P|^2 \leq 2$ at every nontrivial character of conductor c, together with the separate bounds $c|P|^2 \leq (c, 2)$ and $c|Q|^2 \leq (c, 4)$. Here and below the identity coordinates of p, q, x are minima, with $p_0 \leq q_0$.

If $\sqrt{2} \in K$ then $s \geq 1/2$: three is inert in $\mathbb{Q}(\sqrt{2})$, so Lemma 12.14 excludes $(3, 3)$. *Heights.* For a pair $\{0, j\}$ with quadratic annihilating character of conductor c, let $y = q_0 + q_j$ and let B be the sum of the ordinary Fourier amplitude bounds for the other two characters. Since $p_0 \leq q_0 \leq y/2$, Fourier inversion gives $|Q| \geq t - 2y$ and $|P| \geq s - 2y - B$. A tangent to the mixed bound, with

parameter $r > 0$, now gives

$$y \geq \frac{\sqrt{c}\,t + 2cr(s-B) - cr^2 - 2}{2\sqrt{c} + 4cr}. \tag{R1}$$

The separate bound gives $2y \geq t - \sqrt{(c,4)/c}$. For an adjacent pair in a cyclic field of faithful conductor J replace the subtracted term by $\sqrt{2(J,4)/J}$. Thus, apart from the quadratic characters of conductors $8, 12$ and adjacent pairs at $J = 16, 20$, $2y \geq t - 1/\sqrt{5}$. There are no other even faithful conductors below 40 and the least odd one is at least 13. If every pair has this bound, $q_j \geq y/2$ gives $A < 1 + 4s/(t - 1/\sqrt{5}) \leq 21 + 8\sqrt{5} < 39$.

For the exceptional quadratic characters outside fields containing $\sqrt{5}$, the mass lemma and the preceding local argument give $s \geq 1/2$ except possibly at $c = 12$, $s = 1/3$, $t \geq 2/3$; the latter case gives $A \leq 15$ at this pair. For $s \geq 1/2$ set $r = 1/2 - B$ and

$$\ell = \frac{\sqrt{c}/2 + cr^2 - 2}{2\sqrt{c} + 4cr}.$$

When $r, \ell > 0$, (R1) and $t \geq s \geq 1/2$ give $y \geq 2s\ell$. Thus the trace cap at this pair is $A < 1 + 1/\ell$. The maximum cap over all three pairs suffices. Here is the complete infinite-range check. For $\mathbb{Q}(\sqrt{2}, \sqrt{d})$ take d odd squarefree; $B = 3/(2\sqrt{d})$ for $d \equiv 1 \pmod 4$ and $B = (1/\sqrt{2} + 1/2)/\sqrt{d}$ otherwise. For $\mathbb{Q}(\sqrt{3}, \sqrt{d})$ take d squarefree prime to three; $B = \sqrt{(c_d, 2)/c_d} + \sqrt{(c_{3d}, 2)/c_{3d}} \leq 3/(2\sqrt{d})$, where $c_n = \operatorname{disc} \mathbb{Q}(\sqrt{n})$. Compute (R1) up to $d = 54, 34$ respectively; beyond these bounds substitute $3/(2\sqrt{55})$, $3/(2\sqrt{35})$ and use monotonicity. In cyclic fields with quadratic subfield $\mathbb{Q}(\sqrt{2})$ the faithful conductor is $16m$ for odd squarefree m, and $B = 2\sqrt{2/J}$. The conductor comparison checks $J \leq 176$ and then uses monotonicity. This gives exactly the displayed twenty-four exceptions; the caps elsewhere in these three families are respectively $264, 437, 206$. When the second smallest character conductor is at least 1000, both other amplitudes at an exceptional character total at most $2\sqrt{2/1000}$, giving $\ell > 1/38$ and $A \leq 38$.

The finite field inventory. Write $a \leq b \leq c$ for the three conductors. The relative discriminant bound from Lemma 2.5 gives $A \geq 2\sqrt{\sqrt{bc/a} + 4}$ and so $b \leq \sqrt{a}(A^2/4 - 4)$. For $a > 16$ every pair has effective separate amplitude at most $2/\sqrt{a}$ (for a cyclic adjacent pair use $J \geq a$ when J is odd and $J \geq 2a$ when $4 \mid J$). Thus $A < 1 + 2/(1/2 - 2/\sqrt{a})$. At $a \geq 88$ this gives $A < 8$, whereas the relative bound gives $A > 7$. For the positive fundamental $a < 88$ and $b \geq 1000$ test $5 \leq A \leq 38$, imposing $a(A-5)^2 < 16(A-1)^2$ if $a > 16$, and then the bound on b. This gives $b \leq 1471$. The field inventory may also use $ab \leq 25007$ outside the stated exceptions: if $a = 8, 12$ and $b < 1000$ use $b \leq 999$; otherwise the same bound $A \leq 38$ applies unless K is already listed. In the biquadratic case loop over the two smallest fundamental discriminants and multiply their quadratic characters to recover the third. Keep the field-specific height and relative discriminant tests above.

For a cyclic field write J for its conductor and d for the radicand of the quadratic subfield. The local components of a primitive even quartic character enumerate (a, J): the odd part of J is squarefree, $v_2(J) \in \{0, 2, 3, 4\}$, and the odd local orders are two or four. The quadratic conductor a is the product of the odd primes of local order four, times eight if $v_2(J) = 4$. The local parities multiply to one.

Enumerating the integral ratios. Use Lemma 12.11 for the 710 biquadratic fields. For each cyclic pair choose $h, k > 0$ with $d = h^2 + k^2$. All cyclic quartics with this quadratic subfield are rational twists of the same field:

$$K = \mathbb{Q}(\sqrt{d}, Y), \quad Y^2 = m(d + h\sqrt{d}), \quad Z^2 = m(d - h\sqrt{d}).$$

Indeed the quadratic radicand has norm d times a square [12, pp. 145–146]. Its quotient by $d + h\sqrt{d}$ has square norm r^2 and equals a rational multiple of $(1 + z/r)^2$ if the quotient is z and

$r > 0$. At odd primes outside d, the two factors $d \pm h\sqrt{d}$ have even valuations since their product is dk^2 and they are relatively prime there. The squarefree twist m therefore has the same odd prime factors as J outside d. Trying the divisors of $2d$ for the remaining primes, up to the twist by d, covers every field.

The trace Gram diagonal of $1, \sqrt{d}, Y, Z$ is $(4, 4d, 4md, 4md)$, giving index $I = \sqrt{256m^2d^3/(aJ^2)}$ in the maximal order. Requiring I integral leaves 232 presentations. Relative traces write $x = (A + u\sqrt{a})/4 + (vY + wZ)/I$. Positivity bounds the lattice by $au^2 < A^2$ and $16md(v^2 + w^2) < I^2(A^2 + au^2)$.

In both enumerations retain primitive classes with monic integral polynomial $X^4 - AX^3 + BX^2 - CX + H$, four positive roots, least root below one, and $H \mid A^4$, $H^2 \mid B^4$, $H^3 \mid C^4$. The biquadratic norm must be a square or twice a square: odd inertia indices divide two and the ideal is invariant. A dyadic odd norm valuation requires ramification index four. The respective counts are 96 433 626 and 7 581 015 canonical tuples, leaving 258 and 281 ratios or outer presentations.

For the final exclusions Corollary 2.3 gives $h_j = \sum_i q_i x_{i+j} \geq s$.

Use the probability variables $z_i = (p_i + q_i)/(s + t)$; then $q_i/(s + t) = z_i/(1 + x_i)$, $p_i/(s + t) = x_i z_i/(1 + x_i)$ and $1/4 \leq s/(s + t) \leq 1/2$. The separate and mixed Fourier estimates continue to hold after division by $s + t \geq 1$. For each candidate impose these inequalities, the two minimum conditions, and $h_j \geq s$. Outward rational root intervals and supporting half-spaces give a system $Bz \leq d$. The companion `quartic_rational_frontier/verify_rational_frontier.py` rebuilds both inventories, enumerates the boxes and checks nonnegative integer weights w with $\min_i (B^{\mathsf{T}} w)_i > d^{\mathsf{T}} w$ for all $258 + 281$ candidates. This contradicts $\sum z_i = 1$ and proves the theorem. □

12.3. **The exceptional fields with rational complementary codegrees.**

Theorem 12.18. *Let $K = \mathbb{Q}(D)$ have one of the following ten values:*

$$\mathbb{Q}(\sqrt{2}, \sqrt{d}), \quad d \in \{15, 19, 21, 23, 29, 33, 37, 41\}, \qquad \mathbb{Q}\left(\sqrt{m(2 + \sqrt{2})}\right), \quad m \in \{5, 7\}.$$

Suppose the two ordinary and the two signed complementary codegrees are all 4. After Galois transport maximizing E,

$$\mathrm{Tr}_K(E/D) \leq 19.$$

Proof. Normalize $p_i = i(2/D)$ and $q_i = i(2/E)$ to have sum one. Label the involution fixing $\sqrt{2}$ by $j = 2$ and the remaining nonidentity elements by $1, 3$. Put $x_i = p_i/q_i$. At common codegree four the complements contribute equally to the ordinary and signed squared norms of the simultaneous column. Lemmas 2.2 and 2.4 therefore give

$$\sum_l q_{l+i}^2/q_l \leq \sum_l p_{l+i}^2/p_l, \qquad K_j = \sum_l q_l x_l x_{l+j} \in 2\mathbb{Z} + 1. \tag{12.3}$$

The normalized Fourier caps at conductor c are

$$c|P|^2 \leq 4(c, 2), \qquad c|Q|^2 \leq 4(c, 4), \qquad c|P|^2 + 2\sqrt{c}\,|Q| \leq 8.$$

Here and in the next height estimate we use the following test for a small q_j. Put $a = p_0/q_0$, $b = p_0/q_j$; then $0 < b \leq a \leq 1$. For $i = 1, 3$ the spectral bound on the columns and their squared-norm bound imply

$$aq_i,\ bq_{i+j} \leq p_i,$$

$$aq_i^2 + bq_{i+j}^2 \leq p_i^2 + p_0 \left(\delta + \sum_{l \neq 0} p_{l+i}^2/p_l\right).$$

Here $\delta = 0$ in (12.3); more generally one adds δ when the complementary terms give that correction to the first inequality. On a box in p, q these are linear bounds on a, b and bound the positive moment

$$K_j = \sum_l p_l p_{l+j}/q_l = (a+b)p_j + p_1 p_3(1/q_1 + 1/q_3).$$

Bisect the box after applying the Fourier constraints. Discard a box when its upper bound for K_j is below one or when its trace upper bound $1 + \sum_{i\neq 0} \sup(p_i)/\inf(q_i)$ is below the proposed trace. The trace test is used only for positive denominators.

Among the eight biquadratic fields the two other conductors are bounded below by $(60, 120)$ when the first is divisible by four, and by $(21, 168)$ when it is odd. In the two cyclic fields the faithful conductor is at least 80. Within each group the separate and mixed Fourier caps decrease, so it suffices to check the three least conductor tuples. For $i = 1, 3$ the separate bounds give $q_i > 9/100$: use $4q_i \geq 2(q_0 + q_i) \geq 1 - |Q|$ in the biquadratic fields, and $1 - \sqrt{2}|Q|$ for a cyclic adjacent pair. If the trace were at least 20, we would have $p_0 \leq q_0 \leq q_j < 1/19$, since $x_0 \leq 1$. The three interval checks exhaust precisely these initial boxes using the two-variable bound above. This proves the assertion. □

Corollary 12.19. *A rank-six fusion category in the ten fields of Theorem 12.18 with both complementary codegree pairs equal to $(4, 4)$ has a pseudo-unitary Galois conjugate.*

Proof. By Theorem 11.1 take $x = E/D$ primitive. Write $P = X^4 - AX^3 + BX^2 - CX + H$ for its polynomial. By Lemma 2.4, A, B, C are even and H is odd. In the trace boxes of Lemma 12.11 retain the positive primitive d-numbers with these parities and least root below one. The lattice enumeration finds no polynomial with $A \leq 19$ (already its seven candidates up to trace 39 start at $A = 20$). □

For the neighboring fields we will need the weights at somewhat larger traces. Given a candidate P, let x_i be its four roots in increasing order, with q_i permuted accordingly, and put $m_r = \sum q_i x_i^r$. At equal half mass $m_0 = m_1 = 1$ and every m_r is a positive odd integer by Lemma 2.4. Thus $3 \leq m_2 < A$ and $m_2^2 \leq m_3 < Am_2$. For each possible pair (m_2, m_3) one has

$$q_i = \frac{x_i^3 + (1-A)x_i^2 + (B - A + m_2)x_i + m_3 - Am_2 + B - C}{P'(x_i)}. \tag{12.4}$$

Indeed the numerator is the polynomial part of $P(X)\sum_{r=0}^{3} m_r X^{-r-1}$, and the residues of its quotient by P are the weights. Retain positive weights with q_0 least and all $\sum_i q_i x_{i+j}$ odd in the correct Galois ordering.

Theorem 12.20. *Let $\mathcal{C}$ be a rank-six fusion category with quartic dimension field K and rational signed complementary codegrees. If*

$$K = \mathbb{Q}(\sqrt{3}, \sqrt{d}), \qquad d = 5, 7, 10, 11, 13,$$

then $\mathcal{C}$ admits a pseudo-unitary Galois conjugate. At signed mass $t = 1/2$ the same conclusion holds for $K = \mathbb{Q}(\sqrt{2}, \sqrt{d})$, $d = 7, 11, 13, 17$, and for $K = \mathbb{Q}(\sqrt{3(2+\sqrt{2})})$.

Proof. Use the three mass ranges of Lemma 12.15 and the integral shifts of Corollary 2.3. At $t = 1/2$ we have $s = t$, and all complementary signed evaluations of the ordinary columns are one.

Use the normalized probabilities $p_i = i(2/D)$, $q_i = i(2/E)$. Their shifted two-column moments are positive odd integers. If the ordinary complementary codegrees are rational, both are fixed under Galois and the simultaneous-column Gram inequalities hold without change. Otherwise their quadratic codegrees have $u = 1/2$. The last two lines of Lemma 12.5 give $f, f' = n \pm \sqrt{n(n-4)}$ with $(c, n) = (5, 5), (8, 8)$.

If the automorphism fixes f the Gram inequality still holds as stated. If it exchanges f, f', its ordinary complementary squared-norm contribution is $f/f'^2 + f'/f^2 = (n-3)/2$ and its signed contribution is $1/2$. Thus in every case

$$\sum_l q_{l+i}^2/q_l \le \sum_l p_{l+i}^2/p_l + \delta, \qquad \delta = 0 \text{ or } n-4.$$

Take j fixing the named quadratic subfield, of conductor 8 or 12. For $i = 1, 3$ the other two pair sums give $q_i > 1/64$: their lower bounds for $2(q_0 + q_i)$ are $1 - 2\sqrt{(c,4)/c}$ in a biquadratic field and $1 - \sqrt{32/48}$ for a cyclic adjacent pair. Apply the small-q_j test in Theorem 12.18 to $p_0, q_0, q_j \le 1/64$, with the preceding correction δ.

For all five fields containing $\sqrt{2}$ an irrational ordinary pair must have $(c, n) = (8, 8)$. Then $cO_{\chi_8} = 2$ in (2.3), which forces $T_{\chi_8} = 0$ and gives $q_0 + q_j = 1/2$, hence $q_j \ge 1/4$ directly. For fields containing $\sqrt{3}$ there is no irrational ordinary pair unless the field contains $\sqrt{5}$, where $n - 4 = 1$. Thus $\delta = 0$ suffices except in $\mathbb{Q}(\sqrt{3}, \sqrt{5})$, where one suffices. The interval calculation permits larger corrections on several rows.

The interval calculation gives $K_j < 1$ throughout these boxes, hence $q_j > 1/64$ and $A < 65$. Apply the same lattice enumeration and (12.4) with $A \le 64$.

Exactly one tuple survives them, at $(8, 17, 136)$ with $(A, B, C, H) = (24, 78, 24, 1)$. For its possible $m_2 = 7, m_3 = 111$ one weight exceeds $7/10$. But the conductor-136 signed Fourier amplitude is at most $4/\sqrt{136} < 2/5$; positivity of the other three weights gives $2 \max q_i - 1 \le 4/\sqrt{136}$. This contradiction excludes the last tuple.

Suppose $t > 1/2$ in one of the five fields containing $\sqrt{3}$. A quadratic common codegree gives $t \ge 3/5$. For rational rows of codegrees $g \le h$, gh is a square or twice a square: at each odd prime the inertia index divides two, so $N(E)$ has even valuation, as does the biquadratic discriminant. Apply Lemma 2.6. For $t < 3/5$ we must have $g = 3, 4$. The only pair with $1/2 < t < 3/5$ in these square classes is $(3, 12)$, giving $t = 7/12$. At $d = 5, 11$ three is inert in $\mathbb{Q}(\sqrt{d})$, and Lemma 12.14 excludes this possibility. All nontrivial signed Fourier amplitudes in these five fields are at most $1/\sqrt{3}$. In unnormalized variables $q_i = i(1/E)$, $p_i = i(1/D)$ it follows that $q_i \ge (t - 1/\sqrt{3})/4$ for $i \ne 0$. Using $s \le t$ gives $A < 1 + 4t/(t - 1/\sqrt{3})$. Its integer cap is 106 at $t \ge 3/5$ and 390 at $t \ge 7/12$. The same biquadratic lattice enumeration, now without the half-mass parity condition, gives respectively $168, 321, 108, 16, 139$ candidates in the order $d = 5, 7, 10, 11, 13$. For each candidate impose the coordinate minima, $1/3 \le s \le t \le 1$, the stated lower bound on t, and $\sum q_i x_{i+j} \ge s$ at each shift. If $\sqrt{5} \notin K$ also impose $s + 2t \ge 5/3$ from the mass strata. Use the separate and mixed Fourier inequalities of Lemma 2.15, including the joint projection in directions (u, v), $-3 \le u, v \le 3$. Substitution of $p_i = x_i q_i$ gives linear inequalities in the four weights. The 752 integer certificates in `quartic_more/verify.py` exclude every primitive ratio. □

Theorem 12.21. *Suppose a rank-six fusion category has rational signed complementary codegrees. It has a pseudo-unitary Galois conjugate if its quartic dimension field is one of the following fifteen fields:*

$$\mathbb{Q}(\sqrt{2}, \sqrt{d}), \qquad d = 7, 11, 13, 15, 17, 19, 21, 23, 29, 33, 37, 41, \quad \mathbb{Q}\left(\sqrt{m(2+\sqrt{2})}\right), \quad m = 3, 5, 7.$$

Proof. We prove two mass gaps, now using both the ordinary and the signed algebras. In any displayed field K, all odd valuations of the norm of a d-number are even, since the ramification index is at most two at odd primes. For the cyclic fields this follows since the quadratic subfield is unramified there. Also the field discriminant has only even odd-prime valuations. The regular-trace discriminant of the integral character order with rational complementary axes of codegrees g, h therefore shows that gh is a square or twice a square. For a quadratic complementary factor F with common rational codegree h, the same identity shows that $|\operatorname{disc} F|$ has even odd-prime valuations. Thus $F = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-2})$. Lemma 5.3 gives $h \ge 8$ in the real case. In

the imaginary case duality exchanges just two characters and hence just two basis elements, by invariance of trace. The same lemma gives $h = |\operatorname{disc} F|b^2$. A quadratic pair with common rational codegree has $h = 4$ or $h \geq 8$.

Three and five are inert in $\mathbb{Q}(\sqrt{2})$. Lemma 12.14 excludes codegree three; Lemma 12.16 excludes $(5,5),(5,10)$ at five and $(6,6)$ at three. In either sector a rational-codegree mass is thus $1/2$, with pair $(4,4)$, or at least $5/8$: the square classes above leave only $(3,12),(5,5)$ in the intervening interval, and mass below $1/2$ would require codegree three. For a distinct irrational ordinary pair let $u = 1 - s$ and $c = \operatorname{disc}\mathbb{Q}(f)$. If $1/2 \leq s \leq 5/8$, then $u \geq 3/8$ and $c \leq (1+2u)/u^2 \leq 112/9$. Among the quadratic subfields here this leaves $c = 8$. Both pairs in Lemma 12.5 saturate $c(1/f - 1/f')^2 = 1$; their masses u are $1/2, 3/8$. Thus $cO_{\chi_8} = 2$ and the mixed bound in (2.3) gives $\widehat{q}(\chi_8) = 0$.

We have thus shown in either sector that the mass equals $1/2$ or is at least $5/8$. At half mass rational complementary codegrees must be $(4,4)$ since $(3,6)$ was excluded.

The five half-mass fields in Theorem 12.20 are already settled. For the other ten at half mass, ordinary rational codegrees give Corollary 12.19. An irrational ordinary pair saturates at conductor eight, giving normalized $q_0 + q_j = 1/2$ for the involution fixing $\sqrt{2}$. Also $q_1, q_3 > 9/100$ as in Theorem 12.18, and $q_j \geq 1/4$ gives $A < 1 + 100/9 < 13$. The empty trace-19 inventory of Corollary 12.19 requires only the signed equal-mass parity and the trace cap.

Suppose $t > 1/2$, and use unnormalized $p_i = i(1/D)$, $q_i = i(1/E)$. If $s \geq 5/8$, then either $t \geq 23/36$ or $t = s = 5/8$. Indeed for rational rows $g, h \geq 4$ with gh square or twice a square there is no mass in $(5/8, 23/36)$, and a common codegree in this interval is impossible. Write $y = q_0 + q_j$ for the conductor-eight pair and let B be the sum of the other two ordinary Fourier amplitudes, or twice the faithful amplitude in the cyclic case. At this pair use the tangent estimate (R1) with $c = 8$.

For $s \geq s_0$, $t \geq t_0$ appearing below, use $r = s_0 - B > 0$. The right side divided by s is minimized at $s = s_0, t = t_0$ since $\sqrt{8}\,t_0 - 16rB - 8r^2 - 2 < 0$. Once this minimum ℓ is positive, $q_j \geq s\ell/2$. The other coordinates satisfy $q_i \geq (t - a_i)/4$ where a_i is the quadratic amplitude $\sqrt{\gcd(c,4)/c}$ or the adjacent cyclic amplitude $\sqrt{8/J}$. These give $A < 1 + \max(2/\ell, \max_i 4t/(t - a_i))$. Apply the formula with $(s_0, t_0) = (5/8, 5/8)$, except at $d = 7$, where we use $(5/8, 23/36)$ and treat $s = t = 5/8$ below. If instead $s = 1/2$, $t \geq 5/7$, apply it with $(1/2, 5/7)$. The resulting integer caps fit in the final table below.

It remains to treat $s = 1/2$, $5/8 \leq t < 5/7$, and the one case $s = t = 5/8$ at $d = 7$. In the first range the signed pair has separate rational characters. If their signed evaluations on V are b, e, then $b/g + e/h = 1/2$. For $4 \leq g \leq h$ the range gives $g \leq 6$. At $g = 4$ positivity forces $b = 1$ and $h = 4e < 28$; the square classes leave $(h, e) = (8, 2), (16, 4)$. At $g = 5, 6$ only the three excluded pairs $(5,5),(5,10),(6,6)$ satisfy the bounds and square classes. For $K_j = \sum_l p_l p_{l+j}/q_l$, the unit coefficient gives $K_j + b^2/g + e^2/h \in \mathbb{Z}$; hence $K_j \geq 1/4, 3/4$ respectively. At $s = t = 5/8$, Lemma 2.4 gives $K_j \geq 5/8$. If the ordinary pair is irrational, the saturation proved above already bounds every coordinate. Otherwise its simultaneous columns have both complementary evaluations one, as do the signed ones. The unsigned Gram bound thus becomes

$$\sum_l q_{l+i}^2/q_l \leq \sum_l p_{l+i}^2/p_l + t - s.$$

The interval bounds give $q_j > 1/256$. For $(4,8)$ they use $(s, t, K_{\min}, \delta) = (1/2, 5/8, 1/4, 1/8)$, and for $(4,16)$ its weaker enclosing range $(1/2, [5/8, 11/16], 1/2, 1/4)$. At $d = 7$ also use $(5/8, 5/8, 5/8, 0)$. Apply the two-variable argument of Theorem 12.18 to the unnormalized masses. There is one extra linear constraint: by the shifted first moment,

$$\frac{b}{a}p_j + aq_j + q_k p_m/q_m + q_m p_k/q_k \in s + \mathbb{Z}_{\geq 0}, \qquad a = p_0/q_0, \quad b = p_0/q_j.$$

Here b denotes the variable of the two-variable program, distinct from the integer character evaluation above. Interval evaluation of the last three terms bounds b/a above and below using the permitted integers; the resulting linear bounds are included before taking the objective maximum. The checks visit at most 17239 boxes per row. The other two coordinates exceed $1/256$ by their separate Fourier bounds. Thus $A \le 128$ at $s = 1/2$ or $A \le 160$ at $s = 5/8$.

The final integer caps and counts of primitive d-number candidates are as follows:

d	7	11	13	15	17	19	21	23	29	33	37	41	$m=3$	$m=5$	$m=7$
$A \le$	397	128	230	128	128	128	128	128	128	128	128	128	128	128	128
#	283	9	49	40	7	5	81	9	2	4	0	0	29	12	26

Use the mass constraints $1/2 \le s \le t$, $5/8 \le t \le 1$, the nonnegative integral shifted differences, coordinate minima, and the separate and mixed Fourier inequalities as in Theorem 12.20. The file `quartic_more/two_nonhalf.py` verifies positive integer dual gaps for each of the 556 candidates. It generates the cyclic fields in the integral basis from Corollary 12.19, valid also at $m = 3$. This eliminates every primitive ratio and proves the theorem. □

12.4. **The remaining fields containing the golden ratio.**

Theorem 12.22. *Let K be a real cyclic quartic field with quadratic subfield $\mathbb{Q}(\sqrt{5})$, and suppose $K \ne \mathbb{Q}(\zeta_{20})^+$. If $K = \mathbb{Q}(D)$ and the signed complementary pair has distinct irrational codegrees, then the category has a pseudo-unitary Galois conjugate.*

Proof. The pair has field $\mathbb{Q}(\sqrt{5})$. With the notation of Theorem 12.12, either $t \ge 16/25$, or $s = t = 1/2$ and the signed amplitude at conductor five is at most $\sqrt{5}/10$. The conductor J is a multiple of five, and if even it is a multiple of twenty. A real quartic field cannot have conductor five. For $J \ne 20, 40$, the adjacent-pair amplitude is at most $\sqrt{2/15}$, since an odd conductor is at least fifteen and an even one is at least sixty. Thus the three pair bounds again give $A \le 22$. For $J = 40$ one may use the weaker bound $1/\sqrt{5}$ for all three amplitudes. It gives $A \le 21$ in the first case and $A \le 38$ in the second.

The quadratic resolvent test makes this enumeration small. There is an integer root r of the cubic resolvent for which $A^2 - 4(B - r) = 5v^2$, $v \in \mathbb{Z}$: the two pairs are the orbits of the involution, and the difference of their sums is an anti-invariant integer in $\mathbb{Q}(\sqrt{5})$. For each A, H, B, v, putting $r = B - (A^2 - 5v^2)/4$ in the resolvent gives a quadratic equation for C. At trace at most 22, the positivity, primitivity and d-number tests give 53 cyclic polynomials with the required quadratic subfield. At trace at most 38 a second list retains only those for which $\operatorname{disc}(P)/(5 \cdot 40^2)$ is an integer square; it has 62 entries. These are necessary conductor conditions, and overlap of the two searches is harmless.

Except for two polynomials, the linear constraints of Theorem 12.12 exclude every entry of both lists. With the minimum root at zero there are three choices for the opposite root, and interchange of the two adjacent roots preserves the bounds. For the first list the constants of odd conductor fifteen weaken both separate bounds and the mixed bound at the faithful character, since J is odd and at least fifteen or a multiple of twenty at least sixty. Use the actual constants of conductor forty for the second list. The two choices for the signed mass give 678 rational separating certificates, with three more for the short-pair branch of the last case below.

The polynomial $X^4 - 8X^3 + 19X^2 - 12X + 1$ has root $2 - (\zeta_{20} + \zeta_{20}^{-1})$ and generates the field excluded in the statement. For the remaining polynomial put

$$P = X^4 - 7X^3 + 14X^2 - 8X + 1, \qquad x = E/D.$$

Choose $b_i = \zeta_{15}^{2^i} + \zeta_{15}^{-2^i}$, $i \in \mathbb{Z}/4$. Then $x = 2 - b_0$ at its least embedding. The b_i have trace one and circulant Gram matrix with first row $(9, -1, -6, -1)$. Its determinant $1125 = 5 \cdot 15^2$ is the field discriminant, so they form an integral basis. The involution τ sends b_0 to b_2 and fixes the

complementary field. Reduction modulo P gives

$$(\tau x - x)P'(x)(x^3 - 7x^2 + 13x - 4) = 15.$$

These period identities and Lemma 11.2 give $z = 15/E = \sum c_i b_i$ with integral c_i.

Write $T = \sum c_i$, $U = c_0 - c_1 + c_2 - c_3$, $V = c_0 - c_2$, $W = c_1 - c_3$. The Gram eigenvalues at the quadratic and faithful characters are $5, 15$, respectively. The squared Fourier amplitudes of z there are $5U^2$ and $15(V^2 + W^2)$. The separate bounds for $q = z/15$ therefore give $|U| \le 3$ and $V^2 + W^2 \le 1$. The half-mass case already passed the separating test, so $T = 15t \ge 15(16/25)$ and $T \ge 10$. The same Gram matrix gives

$$15s = \mathrm{Tr}(xz) = (7T - 5U - 30V)/4 < 15.$$

Hence $V > 0$: at $T = 10$ use also $U \equiv T \pmod 2$, and at $T \ge 11$ use $U \le 3$. This gives $V = 1, W = 0$. But then $z - \tau z = b_0 - b_2 > 0$, contradicting minimality of q.

□

Theorem 12.23. *A complex fusion category of rank six with $\mathbb{Q}(D) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$ has a pseudo-unitary Galois conjugate.*

Proof. The rational complementary case is Theorem 12.20. For an irrational complementary pair let $c_0 = 5, 12$ or 60 be its discriminant. If $c_0 = 60$, then $t \ge 3/4$ by the proof of Corollary 12.9. All three quadratic amplitudes are at most $1/\sqrt{3}$, so $A \le 24$ by the pair bound. For $c_0 = 12$ we have $A \le 94$ by Proposition 12.7 and $t \ge 5/9 > 11/20$ by Lemma 12.5.

At $c_0 = 5$ a pair other than $5 \pm \sqrt{5}$ gives $t \ge 16/25$, and the same pair bound gives $A \le 64$. For the short pair, Lemma 12.10 gives $s = t = 1/2$ and the positive odd normalized moment of Theorem 12.18, since the involution fixing $\sqrt{3}$ exchanges the two complementary codegrees. Set $\varphi = (1 + \sqrt{5})/2$.

The signed complementary contribution to the squared norm of the simultaneous column is $1/2$ for an automorphism fixing the pair, and $\mathrm{Tr}(\varphi^4/f) = 1$ if it interchanges it. Both are at least the $1/2$ subtracted in the Gram inequality of Theorem 12.20. The ordinary correction is therefore at most one, by the same calculation there. The already verified row $(12, 5, 60)$ of that theorem's height calculation yields $A \le 64$; its inputs are the half-mass equalities, the Fourier bounds, a Gram correction at most 16, and $K_j \ge 1$.

The biquadratic coefficient enumeration at trace 94 produces 143 primitive d-number candidates. For each test the four alternatives just described, retaining the corresponding trace cap, the lower bound for t, and in the short case $s = t = 1/2$ and $|\widehat{q}(\chi_5)| \le \sqrt{5}/10$. Every test has a strict rational separating certificate as in Theorem 12.12. The verifier is `generic_quartic/three_five.py`. The proper ratios are covered by Theorem 11.1. □

Theorem 12.24. *Every rank-six fusion category with dimension field $K = \mathbb{Q}(\zeta_{20} + \zeta_{20}^{-1})$ has a pseudo-unitary Galois conjugate.*

Proof. Let g generate its cyclic group in the order $0, 1, 2, 3$ and write $p_i = g^i(1/D)$, $q_i = g^i(1/E)$ after transport maximizing E. The quartic and quadratic character conductors are 20 and 5. In particular $q_2 \ge (t - 1/\sqrt{5})/4$ and $q_1, q_3 \ge (t - \sqrt{2/5})/4$.

Here $s \ge 1/2$ in both signed cases. Indeed the ordinary part of the proof of Lemma 12.15 leaves only $(3, 3)$ below half mass, and three is inert in $\mathbb{Q}(\sqrt{5})$. Apply Lemma 12.14.

For rational signed codegrees we have $s \le t$. If $t \ge 23/36$ then Lemma 12.3 gives $A \le 398$. Otherwise $1/2 \le t < 23/36$. For separate rational rows of codegrees g_0, h, $g_0 h$ is a square or five times a square: only five has inertia index four and the field discriminant is $5 \cdot 20^2$. For the same reason a quadratic common-codegree field has square class $-1, 5$ or -5, by Lemma 2.6. A common codegree in this mass range is four or five. For separate rows $4 \le g_0 \le h$, the square classes leave $(4, 4), (4, 5), (5, 5)$. At five K has a single prime, so two distinct residue rows cannot

both be assigned a quartic branch. Thus in the last pair the rows agree modulo five, but their squared distance is ten. This excludes $(5,5)$.

In each remaining rational case $s = t$. Indeed the complementary values of the ordinary induction class are totally positive integers on separate rational axes, or conjugate algebraic integers on a common-codegree orbit. Their weighted sum $1 - s$ is at most $1/2$. At $(4,4)$ and $(4,5)$ this forces both values to be one. At common five their sum is at most $5/2$, hence two by positivity and integrality of their norm. The three rational low masses are $1/2, 11/20, 3/5$.

For distinct signed codegrees Lemma 12.10 gives either $t \geq 2/3$, whence $A \leq 117$, or the short golden pair, with $s = t = 1/2$. In the former case the involution g^2 fixes the signed complementary field, so $\sum_i q_i x_{i+2} \geq s$ by Lemma 11.2.

In all four low cases $K_j = \sum_i p_i p_{i-j}/q_i \geq t$. For rational codegrees this follows from Lemma 2.4; for the short golden pair use Lemma 12.10, which also gives all three shifted first-moment bounds. Its signed quadratic amplitude is at most $\sqrt{5}/10$ by Lemma 12.6. We claim $A \leq 77$. The opposite weight q_2 already exceeds $1/128$; by reflection, consider $p_0, q_0, q_1 \leq 1/128$. With $a = p_0/q_0,\ b = p_0/q_1$ the spectral inequalities are $0 \leq b \leq a \leq 1$, $aq_i, bq_{i+1} \leq p_i$, and

$$K_1 = ap_3 + bp_1 + p_1p_2/q_2 + p_2p_3/q_3.$$

At half mass the ordinary irrational pair can only have $(c,n) = (5,5)$ by Lemma 12.5, so its ordinary Gram contribution is at most $(n-3)/2 = 1$. The signed contribution is at least $1/2$ also for the short golden pair, as in Theorem 12.23. Thus the unnormalized Gram correction is at most $1/2$. The interval calculation with these bounds and the separate and joint Fourier inequalities gives $K_1 < t$ on every such box. Hence $A < 1 + 128s < 78$.

For $y = \zeta_{20} + \zeta_{20}^{-1}$ one has $y^4 - 5y^2 + 5 = 0$ and $\mathcal{O}_K = \mathbb{Z}[y]$: write each cyclotomic integer as $u(y) + v(y)\zeta_{20}$ using $\zeta_{20}^2 = y\zeta_{20} - 1$ and take complex conjugation. Write $x = a + by + cy^2 + dy^3$. The inverse trace Gram matrix of this basis has diagonal $(3/2, 1, 1/5, 1/10)$, so $\mathrm{Tr}(x^2) < A^2$ bounds $b^2, 5c^2, 10d^2 < A^2$. Conjugating to have $0 < x < 1$ at a fixed embedding puts a in an interval of length one. The norm polynomial of each lattice point is tested for positivity and the d-number divisibilities, leaving 721 polynomials at $A \leq 398$. For each ratio impose the coordinate minima and Fourier bounds with $p_i = x_i q_i$. In the high rational case use $t \geq 23/36$, $s \leq t$ and all three shifted inequalities; in the high irrational case use $t \geq 2/3$, $s \geq 1/2$ and the shift by two. Both have $s + t \geq 1$. At the four low profiles use their mass equalities and $\sum_i q_i x_{i+j} \geq s$, $\sum_i q_i x_i x_{i+j} \geq t$ for every shift, with the improved golden amplitude or the rational half-mass parity where applicable. The lattice enumeration, rational inequalities and integer certificates exclude all 721 high rational and 183 high irrational candidates. At the rational low masses, Lemma 2.4 gives $P(X) \equiv (X-1)^4$ modulo $2, 20, 5$, leaving $24, 3, 12$ respective candidates. The short golden pair has 108. Thus the four low cases use 147 certificates. □

Proof of Theorem 12.1. Theorem 11.1 treats proper dimension ratios. For a primitive ratio, two cases exhaust the two complementary signed characters. Rational codegrees leave twenty-four fields by Theorem 12.17; Theorems 12.20, 12.21 and 12.24 exclude twenty-one. For irrational codegrees, Theorems 12.12 and 12.13 leave the three stated fields, $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ and the cyclic quartics containing $\sqrt{5}$. The latter two cases are covered by Theorems 12.23, 12.22 and 12.24. □

13. The quartic fields of conductors sixteen and twenty-four

Theorem 13.1. *Every complex fusion category of rank six whose global Frobenius–Perron dimension generates $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ or $\mathbb{Q}(\zeta_{16})^+$ admits a spherical structure.*

Failure of sphericality would force both complementary pairs to have codegrees $4, 4$. The dimension orbit would then carry half the reciprocal mass. Its odd integral moments determine a short list of possibilities over $\mathbb{R}$; reduction modulo small primes excludes each of them.

13.1. **Small complementary codegrees.**

Lemma 13.2. *Suppose $K = \mathbb{Q}(D)$ is quartic with one prime over two, and the two complementary ordinary characters are rational of codegree four. There is an invertible g of order two acting by two transpositions. If the signed complementary characters are conjugate over a real quadratic field, their rows have the form $(1, -1, y, -y, 0, 0)$ and its conjugate, with $yy' = -1$. Both quartic dimension orbits then have reciprocal mass $1/2$.*

Proof. Neither rational branch splits off at two, since its unit coefficient would be $1/4$, and their union cannot split off either, since its coefficient would be $1/2$. Thus the two rows agree modulo two. Each has four entries ± 1. By orthogonality their sum is supported at $1, g$ with entries $2, \pm 2$. Orthogonality to the Perron row gives $d_g = 1$ and the entry -2. Multiplication by g has eigenvalues 1 four times and -1 twice, so g is an involution with two transpositions. Its positive lift has the same permutation and is 1 on the quartic orbit. The two conjugate characters take the same sign on g, necessarily -1 by the zero unit coefficient. Their rows therefore vanish at the two fixed points and alternate on each transposition. Orthogonality gives $yy' = -1$ and hence $1/(2 + 2y^2) + 1/(2 + 2y'^2) = 1/2$. □

13.1.1. *The pair $8 \pm 4\sqrt{2}$ in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.*

Lemma 13.3. *At either conductor, suppose the signed complementary codegrees are $f = 4(2+\sqrt{2})$ and f'. Then $s = t = 1/2$. Put $q = 2/E$, $x = E/D$ and $L(B) = \mathrm{Tr}_K(qB)$. Every monomial in conjugates of x has positive integral L-value, and each $L(\sigma x)$ is odd. The squared-norm inequality in (12.3) holds.*

Proof. Write $F = \mathbb{Q}(\sqrt{2})$ and $\beta = a + b\sqrt{2}$ for the complementary value of the signed Perron induction class. Its contribution to the unit coefficient is $\mathrm{Tr}_F(\beta/f) = (a - b)/2$. It lies strictly between zero and one, so $a - b = 1$ and $s = 1/2 = t$. Total positivity then gives $b = 0, 1, 2$. By Corollary 2.3 the integral translates have coordinates $(\sigma x, \sigma\beta)$. For any product with coordinates (B, γ), its doubled unit coefficient is $L(B) + 2\,\mathrm{Tr}_F(\gamma/f)$. The second summand is integral for every $\gamma \in \mathcal{O}_F$, and is odd for either conjugate of each of the three possible β. This proves the moment assertions.

By Lemma 12.5 the ordinary complementary codegrees are rational or are the same pair f, f': neither field contains $\sqrt{5}$. A rational pair contributes $1/2$ to every simultaneous-column squared norm. The signed pair contributes $1/2$ when fixed and $f/f'^2 + f'/f^2 \geq 1/2$ when exchanged. Subtract the complementary contributions in Lemma 2.2. □

Theorem 13.4. *A rank-six complex fusion category with quartic dimension field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and complementary signed codegrees $8 \pm 4\sqrt{2}$ admits a spherical structure.*

Proof. It suffices to exclude the primitive ratio $x = E/D$ by Theorem 11.1. Use the normalized probabilities $p_i = i(2/D)$, $q_i = i(2/E)$ as in Theorem 12.18; their sums are one by Lemma 13.3. Write P_c, Q_c for their character transforms. Keeping the signed complementary pair in the conductor-eight projection gives $2P_8^2 + Q_8^2 \leq 1$ by (2.3); also $|Q_8| \leq 2\sqrt{2}/3$ by Lemma 12.6. Let j fix $\sqrt{3}$. The two-variable test of Theorem 12.18 applies with $K_j = L(xj(x)) \geq 1$ by Lemma 13.3. If $q_j \leq 1/320$, the strengthened conductor-eight bounds give $K_j < 1$ in all 1077 interval boxes. The bounds on Q_8, Q_{24} give $q_i > 1/320$ for the other nonidentity indices. Since $x_0 \leq 1$ and $\sum_i p_i = 1$, $\mathrm{Tr}(x) \leq 320$. The biquadratic enumeration of Theorem 12.13 at this cap gives 444 primitive totally positive d-number polynomials with least root below one. Impose the preceding Fourier bounds on $p_i = x_i q_i$, $\sum p_i = \sum q_i = 1$, with p_0, q_0 least. The separating weights exclude 443 polynomials.

The single remaining minimum embedding is

$$x = (52 - 18\sqrt{8} + 14\sqrt{12} - 10\sqrt{24})/4.$$

Put $M_l = L(lx)$. By Lemma 13.3, these are positive odd integers and $M' = L(x^2)$ is integral. Moreover $M_0 = 1$ and $\sum_l M_l = 52$. In conductor order $8, 12, 24$ put $T_c = \sum_i \chi_c(i) M_i$. Since $T_c = Q_c \widehat{x}_c$, Fourier inversion gives

$$4M' = \mathrm{Tr}(x^2) + \tfrac{148}{3} T_8 + \tfrac{362}{7} T_{12} + \tfrac{256}{5} T_{24}.$$

The coefficients are $\widehat{x^2}_c / \widehat{x}_c$. Testing positive odd M_1, M_2 with $M_3 = 51 - M_1 - M_2 > 0$ leaves just $(1, 25, 25), (7, 39, 5), (25, 11, 15)$ by integrality. In each case

$$2(q_0 - q_2) = Q_8 + Q_{24} = -T_8/(18\sqrt{8}) - T_{24}/(10\sqrt{24}) > 0,$$

contrary to minimality of q_0. This excludes all primitive ratios. □

Lemma 13.5. *Suppose the quartic dimension field is $K = \mathbb{Q}(\zeta_c)^+$, $c = 16, 24$. Let $p_i = i(1/D)$, $q_i = i(1/E)$, $t = \sum q_i$, after maximizing E by transport. If $t \geq 71/100$, then $A = \sum p_i/q_i$ is less than 740 at $c = 16$ and 758 at $c = 24$.*

Proof. We have $p_0/q_0 \leq 1$, $q_0 \leq q_i$, and $\sum p_i < 1$. If every q_i, $i \neq 0$, exceeds $1/400$ then $A < 401$. Otherwise choose $j \neq 0$ with $q_j \leq 1/400$. The pair imbalance $|2(q_0 + q_j) - t|$ is at least $7/10$. At conductor 24 this is a Fourier modulus $|Q_d|$, and the separate bounds force $d = 8$. At conductor 16 it is $|Q_2|$ for the opposite pair or at most $\sqrt{2}|Q_1|$ for an adjacent pair (cyclic indices). In each case the separate and mixed bounds of Lemma 2.15 give

$$q_j \geq (t - 1/\sqrt{2})/4, \qquad p_j < (1 + 51/1000)/2.$$

Indeed the corresponding imbalance for p is bounded by $|P_8|$, $|P_2|$ or $\sqrt{2}|P_1|$, respectively, each below $51/1000$.

At $c = 24$ the other two nonidentity entries of q are at least $(t - 1/\sqrt{3})/4$ by the separate pair bounds. At $c = 16$ they are greater than $1/10$: their sum is at least .705; for an opposite pair their difference is at most $|Q_1| \leq 1/2$, and for an adjacent pair each is at most $1/2 + 1/400$, by comparison with the opposite small entry. Thus A is less than

$$1 + \frac{2(1 + 51/1000)}{71/100 - 1/\sqrt{2}} + \begin{cases} 10, & c = 16, \\ 4/(71/100 - 1/\sqrt{3}), & c = 24. \end{cases}$$

This gives the stated bounds. □

Theorem 13.6. *Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ be the quartic dimension field of a rank-six fusion category. If $\mathrm{Tr}_K(1/E) \geq 71/100$, the category admits a spherical structure. This includes all cases where the two complementary signed codegrees are distinct conjugates in $\mathbb{Q}(\sqrt{6})$.*

Proof. Let $p_i = i(1/D)$, $q_i = i(1/E)$ and $s = \sum p_i$, $t = \sum q_i$. Lemma 13.5 gives $\mathrm{Tr}(E/D) < 758$.

Use the cap 760 in the same biquadratic integer enumeration as above. It gives 1061 primitive polynomials. The complete list is excluded by $q_i, p_i \geq 0$, $q_0 \leq q_i$, $p_0 \leq p_i$, $p_i = x_i q_i$, $s, t \leq 1$, $t \geq 71/100$, and the separate and coupled Fourier estimates at conductors $8, 12, 24$. The verifier `generic_quartic/short24/large_mass.py` regenerates the list and checks integer weights $w \geq 0$ for each outward-rounded system $Rq \leq b$, with $\min(0, \min_i(w^{\mathsf{T}} R)_i) > w^{\mathsf{T}} b$. The weights are in `generic_quartic/short24/large_mass_duals.json`. Corollary 12.9 proves $t > 71/100$ for quadratic complementary field of discriminant twenty-four in its explicit bounds on the complementary mass. Finally apply Theorem 11.1. □

13.2. **The reciprocal mass of the quartic factor.**

13.2.1. *Quadratic codegrees and small weights.*

Lemma 13.7. *Let $K=\mathbb{Q}(\zeta_{16})^+$ and suppose that the signed complementary codegrees f,f' are distinct and irrational. Write $s=\mathrm{Tr}_K(1/D)$, $t=\mathrm{Tr}_K(1/E)$, and $v=1/f-1/f'$. Then $s\geq 1/2$. For the Fourier coefficients of $1/D,1/E$ at the faithful and quadratic characters one has*

$$4|Q_1|+16|P_1|^2\leq 2,\qquad 2P_2^2+Q_2^2+2v^2\leq 1/2.$$

If the two complementary ordinary characters are rational of codegree four, there is an involutive invertible acting by two transpositions. In this case $s=t=1/2$ and $f=2+2y^2$, where $y\in\mathbb{Z}[\sqrt{2}]$ has norm -1.

Proof. Write $F=\mathbb{Q}(\sqrt{2})$ for the unique quadratic subfield and put $u=1/f+1/f'=1-t$. An ordinary character of codegree two would force a rational signed character of codegree two, by the invertible-object argument of Lemma 12.15. The signed algebra $K\times F$ has no rational character. A complementary codegree three is excluded by Lemma 12.14. Hence rational ordinary complementary codegrees leave $s\geq 1/2$. For a distinct irrational ordinary pair the same bound $s\geq 1/2$ is Lemma 12.4. The two Fourier inequalities are (2.3) and the mixed bound of Lemma 2.15, retaining the additional signed quadratic pair at conductor eight. The last assertion is Lemma 13.2, since two is totally ramified in K. □

Lemma 13.8. *Let $K=\mathbb{Q}(\sqrt{2},\sqrt{3})$. The norm of every totally positive d-number in K is a rational square. In either character algebra of a rank-six category with dimension field K, the quadratic field of a pair of distinct complementary codegrees, if present, is $\mathbb{Q}(\sqrt{2})$ and the norm of its codegree is twice a square.*

Proof. For an invariant ideal the norm valuation at a prime of ramification index e is a multiple of $4/e$. Only two can contribute an odd valuation, so the positive norm is a square or twice a square. The latter is impossible: it would be a norm from $\mathbb{Q}(\sqrt{3})$, but $a^2-3b^2=2c^2$ has no nonzero integral solution by descent modulo three.

For distinct irrational complementary codegrees f,f' write $F=\mathbb{Q}(f)=\mathbb{Q}(\sqrt{d})$, with $d=2,3$ or 6. The rational algebra of the corresponding quotient is $K\times F$. By its order discriminant identity, $N_K(H)ff'$ has the square class of $\mathrm{disc}(K)\,\mathrm{disc}(F)$, where $H=D$ or E. The first part and the square discriminant of the biquadratic field imply $N_F(f)=dh^2$ for some $h\in\mathbb{Q}^\times$. Consequently $f/(h\sqrt{d})$ has norm -1. This is impossible for $d=3$ or 6: clearing denominators in a putative norm equation gives $a^2+c^2=db^2$, which forces $3\mid a,c$ and then $3\mid b$. This leaves $d=2$. □

Lemma 13.9 (A small weight in the presence of a quadratic pair)**.** *Suppose the signed complementary codegrees f,f' are distinct and lie in $\mathbb{Q}(\sqrt{2})$. At either conductor the unnormalized weights $q_i=i(1/E)$ satisfy $t-|Q_8|>2/94$. Now exclude the pair $8\pm4\sqrt{2}$ and take q_0 least. At conductor 24 let j fix $\sqrt{3}$. At conductor 16 let j generate the cyclic group and assume $t<71/100$. Then*

c	q_j *at most*	*consequence*
24	1/188	$s<2/3$, $(1/f-1/f')^2>5/36$
16	1/1000	$s<5/8$.

Proof. Put $u=1-t$, $v=1/f-1/f'$ and $N=ff'$, with $f>f'$. The projection gives $Q_8^2+2v^2\leq 1/2$. For $N\geq 99$, $t^2+2u^2\geq 2/3$ and $v^2=u^2-4/N$ give $t-|Q_8|\geq 1/12-4/N>2/94$. For $N\leq 98$ the signed row of Lemma 12.5 gives $(f+f',N)=(24,72)$ or $(16,32)$. In the former case $t-|Q_8|\geq 2/3-\sqrt{7/18}>2/94$; in the latter use $|Q_8|\leq\sqrt{2}/3$ from Lemma 12.6 and $t=1/2$.

The unit $f/f'=\eta^n$, $\eta=3+2\sqrt{2}$, gives $v^2/u^2=((\eta^n-1)/(\eta^n+1))^2$. For $n=1$ one has $f=k(2+\sqrt{2})$ with k integral, and $8\mid f+f'=4k$ makes k even. Thus $u=2/k$. For $n\geq 2$ we have $v^2\geq 8u^2/9$, and for $n\geq 3$ we have $v^2\geq 49u^2/50$.

At conductor 24, Lemma 13.8 gives $ff' = 2h^2$. The square root $f/(h\sqrt{2})$ of f/f' is a unit of norm -1 in $\mathbb{Q}(\sqrt{2})$, hence n is odd. Write $e = 1/188$ and suppose $q_j \le e$. Then $t \le 1/\sqrt{3}+4e < 3/5$ by the conductor-twelve bound, while $t \ge 1/2$. This rules out $n = 1$ except for the excluded $k = 4$. In particular $v^2 \ge 49u^2/50 > 5/36$. The identities $P_{12} = 2(p_0 + p_j) - s$ and $P_8 + P_{24} = 2(p_0 - p_j)$ give $s \le 4e + |P_8| + |P_{24}| + |P_{12}|$. The separate and mixed bounds give $|P_{24}| \le 1/\sqrt{12}$ and

$$|P_8|^2 \le 1/4 - v^2, \qquad |P_{12}|^2 \le 1/6 - (t - 4e)\sqrt{3}/6.$$

Use $(a+b)^2 \le (4/3)(a^2+3b^2)$ for these last two moduli. Completing the square in $u = 1-t$ gives

$$s \le 4e + 1/\sqrt{12} + \sqrt{\tfrac{4}{3}\left(\tfrac{3}{4} - (1-4e)\sqrt{3}/2 + 75/392\right)} < 2/3.$$

At conductor 16 put $e = 1/1000$ and suppose $q_j \le e$. The adjacent imbalance gives $\sqrt{2}|Q_1| \ge t - 4e$. From $s \le 4e + 2|P_1| + |P_2|$ and the two Fourier inequalities of Lemma 13.7 we obtain

$$s \le 4e + \sqrt{1/2 - (t-4e)/\sqrt{2}} + \sqrt{1/4 - v^2}. \tag{13.1}$$

If $n = 1$, the mass range leaves only $k = 6$, so $t = 2/3$ and $v^2 = 1/18$; the two radicals are less than .179 and .441. Otherwise $(a+b)^2 \le 2(a^2+b^2)$ and $v^2 \ge 8u^2/9$ give

$$s \le 4e + \sqrt{2(3/4 - (1-4e)/\sqrt{2} + 9/64)} < 5/8.$$

All strict numerical bounds follow by squaring with $1/\sqrt{2} > .7071$, $\sqrt{3}/2 > .86602$ and $1/\sqrt{12} < .28868$. □

13.2.2. *Conductor twenty-four with a quadratic complementary field.*

Theorem 13.10. *Let $K = \mathbb{Q}(\sqrt{2},\sqrt{3})$ be the dimension field, with distinct complementary signed codegrees f, f' in $\mathbb{Q}(\sqrt{2})$. Then the category admits a spherical structure.*

Proof. Write $p_i = 1/i(D)$, $q_i = 1/i(E)$, with sums s, t. Put $N = ff'$, $L = f + f'$, $v = 1/f - 1/f'$, so $v^2 = (1-t)^2 - 4/N$. The short pair $L = 16, N = 32$ is covered by Theorem 13.4.

If both ordinary characters are rational of codegree four, Lemma 13.2 gives $s = t = 1/2$ and $f = 2 + 2y^2$ for a norm-minus-one unit $y \in \mathbb{Z}[\sqrt{2}]$.

The fundamental unit is $1+\sqrt{2}$. Apart from the short pair, $|y|$ or $|y'|$ is at least $(1+\sqrt{2})^3$; hence $N = 4(2 + y^2 + y'^2) \ge 800$ and $v^2 \ge 49/200$.

In this paragraph normalize p, q by doubling both. The conductor-eight projection estimate used in Theorem 13.4 gives $2P_8^2 + Q_8^2 \le 2 - 8v^2 \le 1/25$. The simultaneous-column norm estimate there still holds: the ordinary pair contributes $1/2$ and the signed contribution is at least $1/2$. The complementary contribution to the signed coefficient of the product of the Perron column and its integral Galois translate at j fixing $\sqrt{3}$ is $N_F(\beta)/2$, as j interchanges f, f'. Thus the same positive integral moment and the same height and separation certificates apply. Their last ratio has both conjugates on the positive χ_8 half less than $2/5$, as read from its displayed radical expression. This gives an ordinary mass there at most $(2/5)(1+1/5)/2 = 6/25$, while $|P_8| \le 1/\sqrt{50}$ requires mass exceeding $2/5$.

Return to unnormalized coordinates. We claim that every remaining ordinary complement has $s \ge 63/100$. Each rational row has even codegree: on the four signed dimension labels its normalized entries are one, on the two signed complementary labels they are equal integers, and their sum is the codegree. For split rational characters of codegrees $g \le h$ the discriminant identity makes gh a square by Lemma 13.8. Unless $(g,h) = (4,4)$, already treated, these even denominators give $s \ge 2/3$: for $g = 4$ the next even square is $h = 16$, and for $g \ge 6$ the reciprocal sum is at most $1/3$. Codegrees two and three are excluded by Lemmas 12.15 and 12.14. A nonsplit character pair of common rational codegree is impossible: all simples are self-dual since the signed algebra is totally real, whereas the discriminant identity would make the positive quadratic field discriminant a square.

A distinct irrational ordinary pair lies in $\mathbb{Q}(\sqrt{2})$ by Lemma 13.8, and $1-s \le 1/2$ by Lemma 12.4. If $1-s \ge 37/100$, both possibilities in Lemma 12.5 saturate the conductor-eight ordinary bound, contradicting $v \ne 0$ in the mixed projection. This proves the claim. Moreover, writing w for the ordinary reciprocal gap, its unit ratio gives $w^2 \ge (1-s)^2/2$. Equation (2.3) gives $4w^2+2v^2 \le 1/2$. Thus $v^2 > 5/36$ forces $s > 2/3$ in this last case as well.

Lemma 13.9 now gives $q_j > 1/188$ for j fixing $\sqrt{3}$. For the other two nonidentity indices use $t-|Q_8| > 2/94$ from the same lemma and $t-|Q_{24}| \ge 1/2-1/\sqrt{6} > 2/94$. Since q_0 is least, both weights are at least $1/188$. Thus $\mathrm{Tr}(x) < 189$.

It remains to test primitive ratios through trace 188 by Theorem 11.1. Put $p_i = x_i q_i$ and impose $s \ge 63/100$, $t \ge 1/2$, the two common minima and the separate and mixed Fourier bounds. In particular

$$2P_8^2 + Q_8^2 + 2(1-t-1/12)^2 \le 1/2 + 1/72,$$

since $L \ge 24$ gives $8/N \le (1-t)/3$. The shift fixing $\sqrt{2}$ also satisfies $\mathrm{Tr}(q\sigma x) \ge s$ by Corollary 2.3, with $q = 1/E$. The exact enumeration gives 247 ratios; nonnegative rational combinations of the outward-rounded inequalities exclude each of them. □

13.2.3. *Large reciprocal mass at conductor sixteen.*

Theorem 13.11. *Let $K = \mathbb{Q}(\zeta_{16})^+$ be the global Frobenius–Perron dimension field of a complex fusion category of rank six. If $t = \mathrm{Tr}_K(1/E) \ge 71/100$, a Galois conjugate is pseudo-unitary.*

Proof. With the notation of Lemma 13.5, $A = \mathrm{Tr}(E/D) < 740$. Take σ a cyclic generator, put $s = \sum p_i$ and write P_k, Q_k for the cyclic transforms. By Theorem 11.1 we may assume $x = E/D$ primitive.

At cap 740 the cyclic enumeration yields 695 ratios. More explicitly write $y = \sqrt{2+\sqrt{2}}$, $z = \sqrt{2-\sqrt{2}}$. The integral basis $1, \sqrt{2}, y, z$ follows from $\mathcal{O}_K = \mathbb{Z}[y]$, since $y^2 = 2+\sqrt{2}$ and $y^3 = 3y+z$. The generator σ sends y to z and z to $-y$. The presentation input for the cyclic enumerator is $(8, 16, 2, 1, 1, 1, 1, 740)$. Use the inequalities for the separate and coupled projections, the two coordinate minima, $s \le 1$ and $71/100 \le t \le 1$. Inserting $p_i = x_i q_i$ gives a strict rational separating certificate for all but one polynomial. The remaining ratio has $(A, u, v, w) = (460, 162, 106, 44)$ in the presentation $A/4 + u\sqrt{8}/4 + vy + wz$, up to conjugacy.

Here $s \ge 1/2$ by Lemmas 12.15 and 12.14 for rational signed complements and by the ordinary-pair argument of Lemma 13.7 otherwise. The involution $\tau = \sigma^2$ fixes the complementary codegrees, so Corollary 2.3 gives $\mathrm{Tr}(q\tau(x)) \ge s$. Four additional separating certificates for this one ratio yield

$$t < 711/1000, \qquad 63/100 < s < 67/100, \qquad 2(q_0+q_2) < 2/94.$$

The last inequality is incompatible with a distinct irrational signed pair in $\mathbb{Q}(\sqrt{2})$ by Lemma 13.9, since $t-|Q_2| \le 2(q_0+q_2)$. In the rational case the complementary codegrees are integers at least four. In the interval $71/100 \le t < 711/1000$, the reciprocal mass $1-t$ forces the pair $(4, 25)$: the smaller denominator is at most six, and five and six permit no integral partner in this interval. The two signed evaluations a, b of the ordinary FP induction column are then positive integers, satisfying $s = 1 - a/4 - b/25$. The displayed interval forces $a = 1$, and contains no number of the form $3/4 - b/25$. The verifier `generic_quartic/sixteen.py` checks the finite enumerations and all 698 strict certificates. □

13.2.4. *The nonhalf rational pairs at both conductors.*

Lemma 13.12. *Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, with $U_1 = 3+2\sqrt{2}$, $U_2 = 2+\sqrt{3}$, $U_3 = 5+2\sqrt{6}$. Every totally positive integral d-number in K has a unique expression $nz\prod_{i=1}^3 U_i^{k_i}$, with $n \in \mathbb{Z}_{>0}$, $k_i \in \mathbb{Z}$,*

and $4z = A + a\sqrt{8} + b\sqrt{12} + c\sqrt{24}$ *from the following table:*

A	a	b	c	$N(z)$
4	0	0	0	1
8	−2	0	0	4
8	−1	0	−1	1
12	−3	−2	1	9
12	0	−2	0	36
12	0	0	−2	9
12	2	−2	−2	4
24	−6	4	−4	36

Proof. First remove the largest positive rational integer dividing z. The remaining ideal has norm $2^e 9^f$, with $0 \le e < 4$ and $0 \le f < 2$: the ramification indices at two and three are four and two, and the residue degree at three is two. Lemma 13.8 makes e even.

In any totally real biquadratic field, let z be a totally positive d-number of norm m^2. Its relative norm to each quadratic subfield L_i has invariant ideal of norm m^2, hence ideal (m): the norm determines the common valuation above every prime. If U_i generates the totally positive units of L_i, it follows that

$$N_{K/L_i}(z) = mU_i^{r_i}, \qquad z^2 = m\prod_i U_i^{r_i}. \tag{13.2}$$

The second equality follows by multiplying the three relative norms.

Here $m = 2^\alpha 3^\beta$ with $\alpha, \beta \in \{0, 1\}$, and the quadratic Pell equations give the three units in the statement. Multiplying by U_i changes only r_i, by two, so take $r_i \in \{-1, 0\}$. The positive square roots of U_i are $1 + \sqrt{2}$, $(\sqrt{2} + \sqrt{6})/2$ and $\sqrt{2} + \sqrt{3}$. Their signs at the four embeddings are those of $\sqrt{2}, \sqrt{6}, \sqrt{3}$, respectively. Thus total positivity of $\sqrt{m}\prod_i(\sqrt{U_i})^{r_i}$ is equivalent to

$$r_1 + r_2 \equiv \alpha, \qquad r_2 + r_3 \equiv \beta \pmod 2.$$

The two solutions for each m give the eight entries displayed. They are integral in the basis $1, \sqrt{2}, \sqrt{3}, (\sqrt{2} + \sqrt{6})/2$, which in the table coordinates means $4 \mid A$, $2 \mid b$ and $a \equiv c \pmod 2$. Their norms and the unique primes over two and three give invariant ideals.

If $\mathrm{Tr}(z) \le C$ then both embeddings of each relative norm are at most C^2, and their product is at least one. Equation (13.2) therefore bounds the three exponents in the stated decomposition by $|k_i| \le \log C / \log U_i + 1/2$. The enumeration in `quartic24_fast.py` uses this box, the eight seeds and positive rational integers. It keeps one representative per Galois orbit with a conjugate below one. □

Theorem 13.13. *Let* $K = \mathbb{Q}(\zeta_c)^+$ *with* $c = 16$ *or* 24 *be the dimension field of a rank-six fusion category. If its signed complementary codegrees are* $(4, h)$ *with* $h = 8, 9, 16$ *or* 18*, the category has a pseudo-unitary Galois conjugate.*

Proof. By Theorem 11.1 we may assume $x = E/D$ primitive after transport maximizing E. Write h for the second codegree. The dyadic completion of K is a field. Thus if h is even, the rational rows have the same mod-two reduction: otherwise both local factors would need the unique field branch, as their idempotents have nonintegral trace. The four-row has four odd entries, so $4 \mid h$, excluding eighteen.

Put $s = \mathrm{Tr}_K(1/D)$, $t = \mathrm{Tr}_K(1/E)$. Here $s \ge 1/2$ by Lemma 12.15: $s = 1/3$ gives ordinary complementary codegrees three, excluded by Lemma 12.14. The two rational evaluations of the signed FP column are positive integers, so the first is one, and if the second is b then $s = 3/4 - b/h$. Congruence of the two rows makes b odd at even h. The only alternatives to $b = 1$ are $(h, b) = (9, 2), (16, 3)$. Their ordinary complementary masses are $17/36, 7/16$ respectively. Neither is the sum of reciprocals of two integers ≥ 4: the smaller would be four and the other

would be $9/2, 16/3$. Ordinary codegrees two and three have already been excluded. Thus an irrational ordinary pair of mass $a/d = 17/36$ or $7/16$ would be required. Lemma 5.4 gives a squared reciprocal gap at least $(a^2-4)/d^2 > 1/8$ in both cases. This exceeds the ordinary projection bound at each possible conductor $8, 12, 24$. Therefore $b = 1$ and $s = t \geq 5/8$.

Normalize $p_i = i(1/D)/s$ and $q_i = i(1/E)/s$ to have sum one. By Lemma 2.4, the shifted first and mixed moments $L(jx), L(xj(x))$ are at least one, where $L(v) = \mathrm{Tr}(qv)$. For each character of conductor d, the normalized Fourier coefficients satisfy

$$ds^2|P|^2 \leq (d,2), \qquad ds^2|Q|^2 \leq (d,4), \qquad ds^2|P|^2 + \sqrt{d}\, s|Q| \leq 2.$$

Use the weaker common value $s = 5/8$ in these bounds.

A small coordinate q_j, $j \neq 0$, would contradict a mixed moment. Write the group additively and put $a = p_0/q_0$, $b' = p_0/q_j$. Simultaneous-column dominance gives $0 < b' \leq a \leq 1$ and $aq_i \leq p_i$, $b'q_i \leq p_{i-j}$. Reindexing the four terms of the appropriate mixed moment gives

$$1 \leq ap_{-j} + b'p_j + \sum_{i \notin \{0,j\}} p_i p_{i-j}/q_i. \tag{13.3}$$

The other two denominators have uniform lower bounds. At conductor 24, pair inversion and minimality of q_0 give $q_i \geq (1 - 8/(5\sqrt{3}))/4 > 1/60$ outside the kernel of the conductor-eight character. Thus only its nonidentity element needs testing. At conductor 16, if $q_j \leq 1/100$ then the other two coordinates sum to at least .98. The normalized faithful inequality $|Q_1| \leq .8$ bounds their difference for an opposite pair and each coordinate by .81 for an adjacent pair. Both therefore exceed $1/20$.

The following two finite tests complete the argument in each field. First impose $q_j \leq \delta$, the probability and Fourier bounds, and the column inequalities above. Bisection of the three remaining p intervals and two q intervals, rounding outwards to $2^{-20}\mathbb{Z}$, makes the right side of (13.3) strictly less than one on every feasible box. The resulting lower bounds on q_i for $i \neq 0$, together with $x_0 \leq 1$, give the indicated integral trace caps. Then enumerate the primitive ratios at that cap. For fixed x substitute $p_i = x_i q_i$ in the probability, minimum, Fourier and moment inequalities. Nonnegative rational combinations of the outward linear bounds contradict $\sum q_i = 1$ for every candidate.

c	δ	boxes	$\mathrm{Tr}(x)$ cap	primitive ratios
24	1/50	3117	60	62
16	1/100	125 + 1379	100	86

The complete checks are linked separately for conductor twenty-four (separating weights) and sixteen (separating weights). □

13.3. **Reduction to common complementary codegrees four.**

Corollary 13.14. *If $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and the complementary signed codegrees are distinct irrational conjugates, the category admits a spherical structure.*

Proof. Combine Lemma 13.8 with Theorem 13.10. □

Proposition 13.15. *At either conductor, if both signed complementary codegrees are four and the ordinary complementary codegrees are distinct irrational conjugates, some Galois conjugate is pseudo-unitary.*

Proof. We may assume $x = E/D$ primitive by Theorem 11.1. Lemmas 12.15 and 12.14 give $s = t = 1/2$. The ordinary pair must therefore be $8 \pm 4\sqrt{2}$ by Lemma 12.5. Its reciprocal gap saturates the conductor-eight projection. Thus $P_8 = Q_8 = 0$ by (2.3).

Double p, q to give them sum one and let j fix $\sqrt{2}$. For each i the pair i, ij has total weight $1/2$ in both vectors. Hence $p_i = x_i q_i$ determines the weights:

$$q_i = \frac{x_{ij} - 1}{2(x_{ij} - x_i)}.$$

Primitivity ensures a nonzero denominator. It remains to check positivity and minimality of q_0 at the least conjugate of x. For finiteness we bound the other three q_i from below.

At conductor 24 let b, c fix $\sqrt{3}, \sqrt{6}$, respectively. We have $q_j \geq 1/4$ and $q_c \geq (1-\sqrt{2/3})/4 > 1/64$. If $q_b \leq 1/64$, then $p_0 \leq q_0 \leq q_b$, $|Q_{12}| \geq 15/16$, and $P_{12} + P_{24} = 4p_0 - 1$. The Fourier bounds would give

$$15/16 \leq |P_{12}| + |P_{24}| \leq \sqrt{(8 - 2\sqrt{12}\,(15/16))/12} + 1/\sqrt{3} < 15/16.$$

Thus $\mathrm{Tr}(x) \leq 64$. At conductor 16 write $j = 2$ in cyclic indices. If q_1 or q_3 were at most .01, the opposite half sums would give $|Q_1| \geq .48\sqrt{2}$, $|P_1| \geq .48$, contrary to $4|P_1|^2 + 2|Q_1| \leq 2$. Since $q_2 \geq 1/4$, $\mathrm{Tr}(x) \leq 100$.

Substitute the primitive ratios at these caps into the displayed formula. The two exact checks give

conductor	ratios	$q > 0$	$q > 0$, q_0 least
24	69	13	0
16	86	20	0.

□

Lemma 13.16. *At either conductor, suppose the two signed complementary codegrees are rational and $t < 71/100$. Then they are both four, or the category has a pseudo-unitary Galois conjugate.*

Proof. Codegrees two and three are excluded by Lemmas 12.15 and 12.14. For two rational characters of codegrees $g \leq h$, the discriminant identity gives gh square or twice a square. Indeed $N_K(E)$ is a square at conductor 24 by Lemma 13.8; at conductor 16 its odd valuations are multiples of four because the ideal (E) is Galois invariant and unramified at odd primes. The two quartic discriminants have squareclasses one and two. Since $1/g + 1/h > 29/100$, the pairs are

$$(4,4), (4,8), (4,9), (4,16), (4,18), (5,5), (5,10), (6,6).$$

For $g = 5, 6$ take $p = 5, 3$ respectively. The quartic branches contain the unramified quadratic field at p, so the two rows coincide modulo p by Lemma 2.7; their squared distance $g + h$ is not divisible by p^2. Theorem 13.13 handles $(4, h)$ for $h > 4$.

For a quadratic character orbit the common codegree g is $4, 5$ or 6. A real field of discriminant d needs $d \mid 2g$ and $d \leq g$ by Lemma 5.3. Only $d = g = 5$ survives, contrary to the discriminant identity at five. In the imaginary case there is one non-self-dual pair, since duality has a one-dimensional negative eigenspace in $K \times F$. Hence $g = |d|b^2$ by the same lemma, so $g = 4$. □

Theorem 13.17. *If $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, failure of sphericality forces both pairs of complementary formal codegrees to be $4, 4$.*

Proof. Corollary 13.14 and Theorem 13.6 reduce the signed complement to rational codegrees and $t < 71/100$. Apply Lemma 13.16. The signed pair is thus $4, 4$ and $s = 1/2$. An irrational ordinary complementary pair is excluded by Proposition 13.15. The remaining ordinary codegrees are both at least four and their reciprocals sum to $1/2$, so both equal four. □

13.3.1. *Reduction to two pairs of codegree four at conductor sixteen.*

Theorem 13.18. *Let $K = \mathbb{Q}(\zeta_{16})^+$ be the global Frobenius–Perron dimension field of a complex fusion category of rank six. Unless both ordinary and signed complementary codegrees are $4, 4$, the category has a pseudo-unitary Galois conjugate.*

Proof. Choose the embedding maximizing E and take $x = E/D$ primitive by Theorem 11.1. Throughout, $p_i = \sigma^i(1/D)$ and $q_i = \sigma^i(1/E)$ with respective sums s, t, where σ generates the cyclic Galois group and indices run modulo four. We may assume $t < 71/100$ by Theorem 13.11. Write P_k, Q_k for the Fourier transforms at frequencies $k = 1, 2$ of conductors $16, 8$. We have $s \geq 1/2$, by Lemmas 12.15 and 12.14 in the rational case and by the ordinary-pair argument in Lemma 13.7 otherwise. Also $t \geq 1/2$. If $x_i = p_i/q_i$, the separate, mixed and simultaneous-column bounds are

$$c|P_k|^2 \leq (c, 2), \quad c|Q_k|^2 \leq (c, 4), \quad c|P_k|^2 + \sqrt{c}|Q_k| \leq 2, \qquad p_0 q_{l+i} \leq p_i q_l.$$

Here $(c, d) = \gcd(c, d)$. We distinguish the two possibilities for the signed complement.

Rational formal codegrees. Lemma 13.16 leaves two signed codegrees four, so $s = t = 1/2$. Rational ordinary codegrees, all at least four, then also equal four. Proposition 13.15 excludes the distinct irrational ordinary case.

Distinct irrational signed codegrees. Return to the unnormalized vectors and set $u = 1 - t$, $v = 1/f - 1/f'$, $L = f + f'$, $N = ff'$. This field has just one quadratic subfield, $\mathbb{Q}(\sqrt{2})$. Lemmas 13.9 and 13.7 give

$$t - |Q_2| \geq 2/94, \qquad 2P_2^2 + Q_2^2 + 2v^2 \leq 1/2.$$

As q_0 is least, $q_2 \geq 1/188$. Let $e = 1/1000$. For a nonshort signed pair, if $q_1 \leq e$ or $q_3 \leq e$, Lemma 13.9 gives $s < 5/8$.

For $s < 5/8$ the ordinary pair must consist of two rational four-rows. Indeed split rational characters have even codegrees (the equal entries at the two signed complementary labels give $g = 4 + 2k$), and their product is a square or twice a square. Reciprocal sum greater than $3/8$ then leaves just $(4, 4)$. The signed algebra is totally real, so all simples are self-dual. A quadratic ordinary pair of common rational codegree four or five is excluded by the squared row-difference test, using also the order-discriminant identity at five. If the ordinary pair is irrational, its field is $\mathbb{Q}(\sqrt{2})$. Lemma 12.5 leaves only the ordinary codegrees $8 \pm 4\sqrt{2}$ at reciprocal mass greater than $3/8$. Their projection equality contradicts $v \neq 0$.

With two rational ordinary four-rows there is an actual invertible with two transpositions, by the proof of Lemma 13.7. Hence $s = t = 1/2$ and $f = 2 + 2y^2$ for $y \in \mathbb{Z}[\sqrt{2}]$ with $yy' = -1$. Outside the short pair, $N \geq 800$ and $v^2 \geq 49/200$. Thus $|P_2| \leq 1/\sqrt{200}$. Under the assumed collapse, $p_2 - p_0 \geq (s - |P_2|)/2 - 2e > 21/100$, and $|Q_1| \geq (t - 4e)/\sqrt{2}$. These bounds violate $4|Q_1| + 16|P_1|^2 \leq 2$. This proves the claim for every nonshort pair and gives $\mathrm{Tr}_K(x) \leq 1000$.

For the short pair, Lemma 13.3 gives $s = t = 1/2$, Gram dominance, positive integral mixed moments and odd first moments. Normalize p, q to sum one. We have $2P_2^2 + Q_2^2 \leq 1$, $|Q_2| \leq 2\sqrt{2}/3$ and $4|P_1|^2 + 2|Q_1| \leq 2$. If $q_1 \leq e$, then $p_0, q_0 \leq e$ and $q_2, q_3 > 1/40$ by the quadratic Fourier bound.

For clarity we describe the interval contradiction certifying $q_1 > e$. With $a = p_0/q_0$, $b = p_0/q_1$, impose $0 \leq b \leq a \leq 1$ and

$$aq_i \leq p_i, \quad bq_{i+1} \leq p_i, \quad aq_i^2 + bq_{i+1}^2 \leq p_i^2 + p_0 \sum_{l \neq 0} p_{l+i}^2/p_l \qquad (i = 1, 2, 3).$$

The shifted moment is $(b/a)p_1 + q_1 p_2/q_2 + q_2 p_3/q_3 + q_3 a$. Outward interval bounds for its last three terms and the odd integral values in its range give two further linear inequalities on a, b. On each box, two-variable linear duality then bounds $ap_3 + bp_1$ in (13.3). The verifier `field_checks/sixteen/short_height.py` rules out all 697 boxes by Fourier or shifted-moment infeasibility or by $k_{-1} < 1$. The same holds after reversing the generator; the quadratic pair bound gives $q_2 > e$. The trace cap 1000 follows in this case as well.

Primitive ratios through trace 1000. Work again with unnormalized masses. In addition to the projection bounds, $1/2 \leq s$, $1/2 \leq t < 71/100$, and the two lower bounds $2(q_0 + q_2), 2(q_1 + q_3) \geq$

$2/94$, the involution $\tau = \sigma^2$ fixes the complementary character field. Thus $\sum q_i(x_{i+2} - x_i) \geq 0$, by Corollary 2.3. Since f/f' is a nontrivial totally positive unit, $v^2 \geq u^2/2$. At the 915 ratios furnished by the cyclic enumerator, these inequalities and $2P_2^2 + Q_2^2 + u^2 \leq 1/2$ leave twelve. For those twelve use the following four cases.

(1) At the short pair both masses are $1/2$, all normalized first moments are at least one, and the normalized mixed moments are at least one. For even shifts the latter follows from integrality of $2\,\mathrm{Tr}(\beta^2/f)$ for the three possible β. Here $2P_2^2 + Q_2^2 \leq 1/4$.
(2) At an ordinary four-pair and a longer signed unit y, both masses are $1/2$ and $2P_2^2 + Q_2^2 \leq 1/100$. All normalized first moments are positive integers: multiply each shifted FP column divided by its simultaneous unit by $1 + g$ to isolate the dimension factor.
(3) In the remaining cases $s \geq 5/8$ by the ordinary-pair argument. If $f/f' = 3 + 2\sqrt{2}$, then $f = m(2 + \sqrt{2})$ with m integral and even by $8 \mid L$. The mass interval leaves only $m = 4, 6$. The first was treated above; the second gives $t = 2/3$, $2P_2^2 + Q_2^2 \leq 1/2 - 1/9$.
(4) For the remaining powers of the fundamental positive unit, $v^2 \geq 8u^2/9$, giving $2P_2^2 + Q_2^2 + 16u^2/9 \leq 1/2$.

Using rational supporting half-spaces of these ellipsoids excludes every ratio except one in the last case. The sole polynomial and a representative are

$$X^4 - 24X^3 + 36X^2 - 16X + 2, \qquad x = 6 + 4\sqrt{2} + 5y + 2z,$$

with $y = \sqrt{2 + \sqrt{2}}$, $z = \sqrt{2 - \sqrt{2}}$. The same tests give $51/100 < t < 7/10$ for this ratio. The verifier `field_checks/sixteen/irr_verify.py` checks the 952 strict certificates by outward radical enclosures and integer arithmetic.

The remaining obstruction is a trace duality argument that will also be used at conductor forty. Let $S = \mathbb{Z}[Gx]$, and suppose $\tau \in G$ fixes the complementary evaluations of the FP column. By Corollary 2.3 its translated columns have coordinates $(gx, g\beta)$. Their differences and products show that the signed order contains $(J, 0)$, where

$$J = \sum_{g \in G} S\, g(\tau x - x), \qquad \mathrm{Tr}_K(q_0 J) \subseteq \mathbb{Z}. \tag{13.4}$$

Here $[\mathcal{O}_K : \mathbb{Z}[x]] = 17$ by the polynomial discriminant, while $1, x, \sigma x, \tau x$ have determinant 232 in the basis $1, \sqrt{2}, y, z$. Hence $S = \mathcal{O}_K$. Now $\tau x - x = -2(5y + 2z)$ has norm 32, since $(5y + 2z)^2 = 58 + 41\sqrt{2}$. Both its ideal and the different (of norm 2^{11}) are supported at the unique dyadic prime, so their product is $16\mathcal{O}_K$. Equation (13.4) gives $16q_0 \in \mathcal{O}_K$. All integral traces are multiples of four in the displayed basis, forcing $t \in \frac{1}{4}\mathbb{Z}$, contrary to $51/100 < t < 7/10$. □

13.4. **The common four-codegree case.**

Lemma 13.19 (A lower bound for the trace)**.** *At either conductor, suppose both signed complementary codegrees equal four. Unless a Galois conjugate is pseudo-unitary, transport maximizing E gives $\mathrm{Tr}_K(E/D) > 10^6$.*

Proof. By Lemmas 12.15 and 12.14, $s = t = 1/2$. Put $x = E/D$, $q = 2/E$, $R_x = \mathbb{Z}[Gx]$, with $G = \mathrm{Gal}(K/\mathbb{Q})$. Theorem 11.1 and Lemma 2.4 allow us to take x primitive with characteristic polynomial $(X - 1)^4$ modulo two. The functional $L(v) = \mathrm{Tr}(qv)$ is integral on R_x, odd on its monomials, and $L(1) = L(x) = 1$. The least conjugate of x is below one and minimizes both q and qx.

Bounding the denominator. Write $q = z/T$ with z a primitive integral lattice vector and $T = \mathrm{Tr}(z)$. Then $T \mid M$ whenever Mq is integral. The following bounds come from the trace dual of explicit elements of R_x.

At conductor 24, write $4x = A + \sum c_i\sqrt{d_i}$ for $(d_1, d_2, d_3) = (8, 12, 24)$. Character sums give $c_i\sqrt{d_i}$ in R_x. Their pairwise products give $g_i\sqrt{d_i} \in R_x$, where $g_i = \gcd(c_i, \sqrt{d_j d_k/d_i}\, c_j c_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. Each g_i is nonzero by primitivity. Pairing with q gives $T \mid 4\,\mathrm{lcm}_i(d_i g_i)$.

At conductor 16, take $y = \sqrt{2+\sqrt{2}}$ and $w = \sqrt{2-\sqrt{2}}$ with $x = a + b\sqrt{2} + cy + dw$. The conjugate sums and differences give $4b\sqrt{2}$, $2(cy + dw)$ and $2(cw - dy)$ in R_x. Products and integer combinations give $H\sqrt{2}$, $2Ry$ and $2Rw$, where $H = 4\gcd(b, c^2 - d^2 + 2cd, c^2 - d^2 - 2cd)$ and $R = c^2 + d^2$. The integral basis $1, \sqrt{2}, y, w$ has trace Gram diagonal $(4, 8, 8, 8)$, so $T \mid \mathrm{lcm}(4, 8H, 16R)$.

Enumerating the two integral d-numbers. At conductor 24 use Lemma 13.12. At conductor 16 the stripped ideal, after division by its largest rational integer factor, has norm $1, 2, 4$ or 8: the only ramified prime is two, and it is totally ramified. Use the positive units $U = (1+\sqrt{2})^2$, $V = (1+y)^2$, $W = (1+w)^2$. Their logarithms span the trace-zero hyperplane: $\log U$ alternates in cyclic order and the opposite half-differences of $\log V, \log W$ are $(c_0, d_0), (d_0, -c_0)$ with $c_0, d_0 > 1$. Every embedding of $U^{\pm 1}$ is below six and of $V^{\pm 1}, W^{\pm 1}$ below nineteen. Rounding all three logarithmic coordinates to the nearest integer therefore puts a representative of every stripped orbit at trace at most $4 \cdot 8^{1/4}\sqrt{6 \cdot 19^2} < 320$. The integral basis enumeration at this bound retains the local trace minima under multiplication by the three units and their inverses. Every orbit has such a minimum in the box. Reducing these minima gives one representative per norm:

$$1, \qquad 2 - y, \qquad 2 - \sqrt{2}, \qquad 4 - 2\sqrt{2} - y + w.$$

For the exponent search at trace C, the centered log vector lies in the simplex with vertices the permutations of $(-3, 1, 1, 1)\log C$. The inverse half-difference matrix bounds the V, W coordinates by $2\max(c_0, d_0)\log C/(c_0^2 + d_0^2) < \log C$. Their alternating components are half of $\log U$ up to sign, giving $2\log C$ for the U coordinate. Add the same bounds for the representative at $C = 320$.

List x through trace 10^6, then the stripped z at the trace divisors just obtained. Impose $L(x) = 1$, oddness of $L(\sigma x)$, the common minimum, oddness of $L(x^i)$ for $2 \le i \le 5$, and the Fourier bounds, in that order. The exact enumeration gives

c	x	(x, z)	$L(x) = 1$	shifts	minima	powers	Fourier
24	113412	23270270	3311	2615	3	1	0
16	38115	5032847	1597	1031	1	1	0

□

13.4.1. *Odd trace identities.* At either conductor, the nonspherical alternative has two complementary codegrees four in each sector and $\mathrm{Tr}(E/D) > 10^6$. We use the normalized quantities $p = 2/D$, $q = 2/E$, $x = p/q$ and $L(v) = \mathrm{Tr}(qv)$. The normalization gives $L(1) = L(x) = 1$. At conductor sixteen the signed complement is split or $\mathbb{Q}(i)$. Lemma 5.3 excludes a real quadratic pair of common codegree four, and its imaginary analogue gives $|\operatorname{disc} F| = 4$. In either case the absolute discriminant of the complement is a square. The order discriminant formula then gives

$$N_K(q) \in 2\mathbb{Q}^{\times 2}, \qquad \mathbb{Q}(q) = K, \tag{13.5}$$

since $\mathrm{disc}(K) = 2^{11}$ and an element of a proper subfield of a cyclic quartic field has square norm.

Lemma 13.20. *For either quartic field let j be an involution, let q and $p = qx$ be the two probability vectors, and put $L(v) = \mathrm{Tr}(qv)$. Use the notation $b = q_j$, $z = p_j$, $a = x_0$, $r = q_0/q_j$, $s = q_m/q_l \le 1$ for the two other indices l, m. Assume that $b \le 1/250000$, $0 < r < .003$, and that the entries of*

$$(h, k, M, v, N) = (L(jx), L(xj(x)), L((jx)^2), L(W^2), L(jx/W)), \qquad W = j(q)/q,$$

are odd positive integers. In the two indicated ranges the remaining two moments are uniquely determined.

h	k	z	s	a	M	v
3	1	$[.228, .332]$	$[.138, .185]$	$(0, .270]$	11	5
7	3	$[.228, .233]$	$[.054, .058]$	$(0, .409]$	53	17

Proof. Write $A = x_l$, $B = x_m$, $S = s + s^{-1}$, $t = r/b$ and

$$C = q_l + q_m = 1 - b(1+r), \qquad D_1 = q_l A + q_m B = 1 - z - arb, \qquad H = q_l B + q_m A = h - zr - ba.$$

Then $CAB = k - za(1+r)$. Grouping the four trace terms into the two orbits of j gives

$$M = H(H + D_1)/C - CAB + z^2 t + ba^2,$$
$$v = C(S-1) + 1/t + br^2,$$
$$N = HS - D_1 + zr^2 + a/t.$$

For example, $q_l B^2 + q_m A^2 = H(A+B) - CAB$ and $A + B = (H + D_1)/C$. These equalities isolate the only unbounded quantity, t. Substituting the bounds in either row yields

$$\big|M - [h(h+1) - k - z(h-a) + z^2 t]\big| < .02,$$
$$\big|v - (S - 1 + 1/t)\big| < .001,$$
$$\big|N - hv - (h-1) - z - (a-h)/t\big| < .02.$$

For example, put $e = 1 - C$, $d = 1 - z - D_1 = arb$ and $y = h - H = zr + ba$. In both rows $e < .00000402$, $d < 5 \cdot 10^{-9}$, $y < .001$ and $S < 19$. The first error is at most $[eh(h+1) + y(2h+1) + hd]/C + zar + ba^2$, the second at most $e(S-1) + br^2$, and the third at most $yS + he(S-1) + d + (z + hb)r^2$. These give the displayed bounds. If $t \le 4$, the first expression places the odd integer M in $(9, 11)$ or $(51, 53)$, respectively. Hence $t > 4$. The last expression now puts the even integer $N - hv - (h-1)$ in $(-2, 2)$. Its vanishing gives $7 < t < 15$ and $26 < t < 34$ in the two rows. Inserting these intervals in the first two expressions gives $9 < M < 13$, $4 < v < 7$ in the first row and $51 < M < 55$, $16 < v < 18$ in the second. Oddness proves the table. □

Lemma 13.21. *Let $K = \mathbb{Q}(\zeta_{16})^+$ in the two complementary four-codegree case, and let $p = 2/D$, $q = 2/E$ at an embedding maximizing E. For a generator g of $\mathrm{Gal}(K/\mathbb{Q})$, write $p_i = g^i(p)$, $q_i = g^i(q)$, $x_i = p_i/q_i$. Neither q_1 nor q_3 can be at most $4 \cdot 10^{-6}$. In particular, if $\mathrm{Tr}(E/D) > 10^6$, its largest conjugate is x_2.*

Proof. We may reverse the generator and assume $e = q_1 \le 4 \cdot 10^{-6}$. Put $r = q_0/e$, $a = x_0$, $Q = q_2$, $B = q_3$, $z = p_1$, $v = p_2$, $w = p_3$, $b = v/Q$, and $c = w/B$. We have $0 < a, r \le 1$ and $a \le b, c$. The faithful Fourier moduli are

$$P = \sqrt{(p_0 - v)^2 + (z - w)^2}, \qquad T = \sqrt{(q_0 - Q)^2 + (e - B)^2}.$$

The two mixed Fourier inequalities are

$$16P^2 + 8T \le 8, \qquad 8(2(p_0 + v) - 1)^2 + 2\sqrt{8}\,|2(q_0 + Q) - 1| \le 8.$$

Since $T \ge (1 - 4e)/\sqrt{2} > 0.707$, we have $v < 0.4$ and $v + |z - w| \le p_0 + \sqrt{2}P < 0.55$. Thus $z, w > 1/5$. If $\min(Q, B) \le 1/25$, then $T > 0.9599$ and $|2(q_0 + Q) - 1| > 0.9199$. It follows that $v < p_0 + 0.142$; the left side of the second inequality is greater than $8(0.715)^2 + 2\sqrt{8}(0.9199) > 8$. Consequently $1/25 < Q, B < 24/25$.

By Lemma 2.4, $H = L(gx)$ and $J = L(xg(x))$ are positive odd integers, where $L(v) = \mathrm{Tr}(qv)$. The expansion

$$J = rza + zb + Qbc + Bca$$

gives $1 \le J \le (2+c)b \le 27b$. In particular $v > 1/1000$. There is an additional quadratic unit

$$U = \frac{q_0 q_2}{q_1 q_3} \in \mathbb{Q}(\sqrt{2}).$$

Indeed the conjugate ratios of the d-number E are units, and translation by g^2 fixes this product while g inverts it. Thus $U = (3+2\sqrt{2})^n$ for $n \in \mathbb{Z}$. Since $U = rQ/B < 24$, one has $n \le 1$. For fixed U and Q, the identities

$$r = \frac{U(1-Q-e)}{Q+Ue}, \quad B = 1-Q-e(1+r), \quad H = rz + eb + \frac{Q(1-z-v)}{B} + a\left(B - \frac{Qer}{B}\right)$$

determine r, B and then a once the odd integer H is chosen. The coefficient of a is positive since $B > 1/25$.

The interval check starts from $(Q, z, v) \in [1/25, 24/25] \times [1/5, 4/5] \times [1/1000, 2/5]$ and $0 \le e \le 4 \cdot 10^{-6}$. For each admissible odd H recover r, B, a as above. Impose the two Fourier inequalities, $0 \le a \le \min(1, b, c)$, $w = 1-z-v-aer \ge 1/5$, and odd positive J. Multiplying Gram dominance by p_0 and retaining two terms on the right gives the last test, for $i = 1, 2, 3$,

$$p_i^2 + p_0 \sum_{k=1}^{3} p_{k+i}^2/p_k \ge a(q_i^2 + r q_{1+i}^2).$$

Bisect the widest input interval, treating $-5 \le n \le 1$ separately and $0 \le U \le (3+2\sqrt{2})^{-6}$ together. Outward operations on $10^{-9}\mathbb{Z}$ and integer square-root comparisons discard every box. Since $x_j > 250000$ implies $q_j < 4 \cdot 10^{-6}$, the trace assertion follows. □

Proposition 13.22. *In either common-four case assume* $\mathrm{Tr}(x) > 10^6$ *and let* j *maximize* $j(x)$. *It is the involution fixing* $\sqrt{2}$. *Put* $W = j(q)/q$. *Then*

$$(L(jx), L(xj(x)), L((jx)^2), L(W^2)) = (3, 1, 11, 5)$$

at conductor 24*; at conductor* 16 *the other possibility is* $(7, 3, 53, 17)$.

Proof. In the cyclic case Lemma 13.21 identifies j; in the biquadratic case every nonidentity element is an involution. In the notation of Lemma 13.20, put $e = 1/250000$, $C = q_l + q_m = 1 - b(1+r)$, $A = x_l$, $B = x_m$. Lemma 13.19 gives $b = q_j < e$. The first two moments h, k and the remaining ones in that lemma are positive odd integers by Lemma 2.4. Pairing indices gives

$$h = zr + ba + q_l B + q_m A, \qquad k = za(1+r) + CAB.$$

There are two further inequalities common to both fields. First put $S_r = r + r^{-1}$, $S_s = s + s^{-1}$. These are the two conjugates of $W + W^{-1}$ over the rationals, and $L(W^2) + 1 = (1-C)S_r + CS_s$. If $S_r \ne S_s$, the algebraic integer $L(W^2) + 1 - S_r$ has nonzero norm, giving

$$(1-C)C(S_r - S_s)^2 \ge 1.$$

Second, Gram dominance for the simultaneous columns, multiplied by p_0 and retaining two terms on the signed side, gives for $i \ne 0$

$$p_i^2 + e(1/z + 1/p_l + 1/p_m) \ge a(q_i^2 + r q_{i+j}^2).$$

For clarity, the only field-dependent inputs are the Fourier bounds. At conductor 24 these are $dP^2 + 2\sqrt{d}\,|Q| \le 8$, $dQ^2 \le 16$ at each character of conductor d. The character trivial on j has $|Q| > 1 - 4e$, so its conductor is 8 or 12; also $1/5 < z < 4/5$. The conductor-24 inequality gives $|q_l - q_m| \le \sqrt{2/3} + e$, hence $q_l, q_m > 1/12$. Using $a \le \min(A, B)$ in the expression for k gives $k \le 48 \min(p_l, p_m)$. The bounds $p_l p_m < (4/5)^2/4$ and $q_l q_m > (1 - 2e - 1/12)/12$ give $k < 5$; the formula for h gives $h < 12$.

At conductor 16 the quadratic and faithful mixed inequalities give

$$8(2(z+p_0)-1)^2 + 2\sqrt{8}(1 - 2(q_0+b)) \le 8, \qquad 16((z-p_0)^2 + (p_l - p_m)^2) + 8(q_l - q_m) \le 8.$$

Thus $.22 < z < .8$ and $q_l, q_m > 1/25$. The same moment formulas yield $k < 7$, $h < 25$, and $k \le 100\min(p_l, p_m)$. Here $S_r \ne S_s$: equality would give $[\mathbb{Q}(W) : \mathbb{Q}] \le 2$, so $jW = W$ and $W = W^{-1} = 1$. Then $jq = q$, contradicting (13.5). Since $S_s < 26$, the norm inequality already gives $r < .003$: otherwise $(1 - C)C(S_r - S_s)^2 < 2e \cdot 335^2 < 1$.

For each permissible conductor assignment and each odd h, k in these ranges, first enclose r using $0 < ba < e$ in the equation for h, then enclose a using the equation for k. The interval variables are $(z, q_l - q_m, p_l - p_m)$, initially in $[.2, .8] \times [0, .82] \times [-1, 1]$ at conductor 24 and in $[.2, .8] \times [0, 1] \times [-1, 1]$ at conductor 16. With $p_0, q_0, b \in [0, e]$, impose the displayed bounds and $0 < r \le 1$, $0 < a \le \min(1, A, B)$. Bisection of the longest edge, rounding outwards to $10^{-9}\mathbb{Z}$, leaves the following enclosures. The full tests are linked for conductors twenty-four and sixteen.

c	$K^{\langle j \rangle}$	h	k	z	s	a
24	$\mathbb{Q}(\sqrt{2})$	3	1	$[.228, .332]$	$[.138, .185]$	$(0, .232]$
16	$\mathbb{Q}(\sqrt{2})$	3	1	$[.228, .323]$	$[.138, .183]$	$(0, .270]$
16	$\mathbb{Q}(\sqrt{2})$	7	3	$[.228, .233]$	$[.054, .058]$	$(0, .409]$

In the first row equality $S_r = S_s$ would make W a unit of degree at most two inverted by j. Its smaller conjugate $s < 1$ would be a positive power of $2 - \sqrt{3}$ or $5 - 2\sqrt{6}$, neither possible in the displayed interval. Thus the same norm inequality gives $r < .003$ here as well. Lemma 13.20 now gives the asserted moments. □

Lemma 13.23. *Let $K = \mathbb{Q}(\zeta_c)^+$ for $c = 16$ or 24, and let j be the nontrivial automorphism over $\mathbb{Q}(\sqrt{2})$. Let z, x be totally positive integral d-numbers. Suppose $N(x)$ is odd and $q = z/\operatorname{Tr}(z)$ satisfies $\operatorname{Tr}(qx) = 1$. At $c = 16$ assume also $N(z) \in 2\mathbb{Q}^{\times 2}$. Write $L(v) = \operatorname{Tr}(qv)$ and $W = jz/z$. The vector $(L(j(x)), L(xj(x)), L(j(x)^2), L(W^2))$ cannot be $(3, 1, 11, 5)$ at either conductor, or $(7, 3, 53, 17)$ at conductor sixteen.*

Proof. The unit-orbit descriptions in Lemma 13.12 and Lemma 13.19 reduce the test to finite rings. Remove the rational integer factor of z, which cancels in q.

Conductor twenty-four. In the integral basis $1, \sqrt{2}, \sqrt{3}, (\sqrt{2} + \sqrt{6})/2$, write $[v]_0 = \operatorname{Tr}(v)/4$. For $(H, K_0, M, V) = (3, 1, 11, 5)$ the five moment identities say

$$[z(x, j(x), xj(x), j(x)^2, W^2)]_0 = [z]_0(1, H, K_0, M, V)$$

coordinatewise. Reduce these equations modulo 8 and 11 independently. For each of the eight seeds z_*, multiply by the residue subgroup generated by U_1, U_2, U_3 from Lemma 13.12. One obtains W even for a nonunit seed by starting with $j(z_*)/z_*$ and multiplying by $j(U_i)/U_i$ at each step. For x use the four seeds of odd norm, the same units, and all odd scalar multipliers. The numbers of residue classes of z admitting an x are, in seed order,

8	0	8	0	0	0	0	0	16
11	14	0	12	12	32	14	32	0.

The two residue calculations have disjoint seed supports. A four-modulus check also gives the exclusion for arbitrary integral x.

Conductor sixteen. Take $y = \sqrt{2 + \sqrt{2}}$, $w = \sqrt{2 - \sqrt{2}}$. Modulo 17 the four evaluations starting with $(\sqrt{2}, y, w) = (6, 5, 8)$ identify $\mathcal{O}_K/17$ with $\mathbb{F}_{17}^4$. The involution j shifts by two. Normalize unit vectors to first coordinate one; the three generators $3 + 2\sqrt{2}, (1 + y)^2, (1 + w)^2$ give a group G of order 128 (all generator orders divide eight). The norm classes allow only the seeds $2 - y$, $4 - 2\sqrt{2} - y + w$ for z, and 1 for x. For Z in either normalized seed translate, put $T = \sum Z_i$. Retain $\sum_i Z_{i+2}^2/Z_i = VT$. All surviving Z have $T \ne 0$; for $U \in G$ the only possible X proportional to U is then $TU/\sum_i Z_i U_i$, omitting zero denominators. The residue calculation gives 44 possible Z for $V = 5$ and 12 for $V = 17$. None admits an X satisfying the remaining three scalar equations. □

Proof of Theorem 13.1. If there were no spherical structure, Theorems 13.18, 13.17 and Lemma 13.19 would give the common four-rows and $\operatorname{Tr}(x) > 10^6$. Clear the denominators of q to write $q = z/\operatorname{Tr}(z)$ with z a totally positive integral d-number. Lemma 2.4 gives $N(x)$ odd. At conductor sixteen (13.5) gives $N(z) \in 2\mathbb{Q}^{\times 2}$. Proposition 13.22 now forces one of the moment vectors excluded by Lemma 13.23. □

14. The quartic field of conductor forty

Theorem 14.1. *Every complex rank-six fusion category with $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ admits a spherical structure.*

Write $F = \mathbb{Q}(\sqrt{5})$. The Fourier bounds leave a signed complement F and two rational ordinary characters. Induction then identifies certain square equations modulo four in the two orders. Their solution counts rule out the remaining codegrees.

Lemma 14.2. *The totally positive d-numbers z in the ring of integers of $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ are precisely the totally positive elements*

$$h\sqrt{d}\,(1+\sqrt{2})^{i}\phi^{j}(3+\sqrt{10})^{l}, \qquad h \in \mathbb{Z}_{>0}, \quad d \in \{1,2,5,10\}, \quad i,j,l \in \mathbb{Z}, \qquad \phi = (1+\sqrt{5})/2.$$

Here $N_K(z) = h^4 d^2$. If $\operatorname{Tr}_K(z) \le 1001$, the three exponents have absolute values at most $7, 12, 3$, respectively.

Proof. All ramification indices are at most two, so an invariant ideal has square norm, say m^2. The quadratic Pell equations give totally positive unit generators $(1+\sqrt{2})^2, \phi^2, (3+\sqrt{10})^2$. Equation (13.2) now writes z as $\sqrt{m}$ times the asserted product of units. Since $\sqrt{m} \in K$, we have $m = dh^2$ with $d \in \{1,2,5,10\}$. Conversely, every positive element of the displayed form is integral and has invariant ideal. Each real conjugate of a relative norm is at most $(\operatorname{Tr}_K(z)/2)^2$ by the arithmetic–geometric mean inequality. Applying this to each generator and its conjugate gives the stated exponent bounds. □

Proposition 14.3. *Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ be the Frobenius–Perron dimension field in rank six. Suppose one of the two character algebras has complementary codegrees f, f' generating $F = \mathbb{Q}(\sqrt{5})$. Write H for its dimension-orbit codegree, D in the ordinary sector and E in the signed sector. Then*

$$ff' = 5k^2, \qquad k = a^2 + b^2 \quad (a,b \in \mathbb{Z}), \qquad \operatorname{Tr}_{K/\mathbb{Q}}(z^2/H) \simeq_{\mathbb{Q}} \langle 1,1,1,1\rangle.$$

Choose $f > f'$ and put $\phi = (1+\sqrt{5})/2$. Then $f = k\sqrt{5}\,\phi^m$ with m positive and odd, and $\operatorname{Tr}_F(1/f) = F_m/k$, where F_m is the mth Fibonacci number. In particular $ff' = 45$ and 180 are impossible. At the prime over five, normalize $v(\sqrt{5}) = 1$. If $v(H)$ is even, $5^{-v(H)/2}H$ has nonsquare residue in $\mathbb{F}_{25}$.

Proof. Both character factors are totally real, so all simples are self-dual. Their orthonormal basis for the unit-coefficient pairing gives $A \perp B \simeq 6\langle 1\rangle$, with

$$A(z) = \operatorname{Tr}_K(z^2/H), \qquad B(y) = \operatorname{Tr}_F(y^2/f).$$

The determinant of A is a square, since both $\operatorname{disc}(K)$ and $N(H)$ are squares (Lemma 14.2). Thus $\det B = 5/N(f)$ modulo squares gives $N(f) = 5k^2$.

Put $w = (5+\sqrt{5})/2 = \sqrt{5}\,\phi$. The invariant ideals (f) and (kw) coincide by equality of their norms. Their quotient is therefore a totally positive unit, an even power of ϕ. This gives the asserted f and its reciprocal trace by Binet's formula. In the basis $1, \phi$ the form $\operatorname{Tr}_F(y^2/w)$ has identity Gram matrix, so $B \simeq k\langle 1,1\rangle$.

At each odd $p \neq 5$ the algebra $K \otimes \mathbb{Q}_p$ is unramified. Remove the common rational p-power of H. Such scaling preserves the Hasse invariant of a rank-four form of square determinant: the factor is $(d,d)_p^6 (d, \det A)_p^3 = 1$. The resulting form is unimodular, hence has Hasse invariant one.

Since $A \perp B$ is standard, $(k,k)_p = (k,-1)_p = 1$. Thus each $p \equiv 3 \pmod 4$ divides k to even order, so $k = a^2 + b^2$. Multiplication by $a + bi$ identifies B with $\langle 1, 1\rangle$, and Witt cancellation gives the assertion about A [24, Chapters III–IV].

Finally put $L = \mathbb{Q}_5(\sqrt{2})$, the unramified quadratic extension of $\mathbb{Q}_5$, so $K \otimes \mathbb{Q}_5 = L(\sqrt{5})$. Rational scaling does not change the isometry class of A at five, by its square determinant and the scaling calculation above. If $v(H)$ is even, take H to be a unit. Choose a unit $h \in L$ with the same residue. Hensel's lemma makes H/h a square. Thus A has the form $B_0 \perp 5B_0$, where $B_0(y) = \mathrm{Tr}_{L/\mathbb{Q}_5}(2y^2/h)$ is unimodular of rank two. Its determinant modulo squares is $2/N_{L/\mathbb{Q}_5}(h)$. The Hasse invariant at five of $B_0 \perp 5B_0$ is the quadratic character of this determinant: in a diagonal unit basis use $(5,5)_5 = (5,-1)_5 = 1$, while a Hilbert symbol of two units is one. It must equal one. Since two is a nonsquare modulo five, the residue norm of h is a nonsquare in $\mathbb{F}_5^\times$, equivalently the residue of h is a nonsquare in $\mathbb{F}_{25}^\times$. □

Theorem 14.4. *Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ be the dimension field of a rank-six category. Unless it has a pseudo-unitary Galois conjugate, the signed complementary codegrees are distinct irrational conjugates in $\mathbb{Q}(\sqrt{5})$ and*

$$6837/10000 < \mathrm{Tr}_K(1/E) < 143/200, \qquad \mathrm{Tr}_K(E/D) > 94$$

after transport maximizing E.

Proof. Theorem 11.1 permits us to assume $x = E/D$ primitive. Write $p = 1/D$, $q = 1/E$, $s = \mathrm{Tr}(p)$, $t = \mathrm{Tr}(q)$ and let j fix $\sqrt{2}$. Only the pair q_0, q_j can be simultaneously small: the other two pair bounds are $q_i \geq (t - 1/\sqrt{5})/4$ in unnormalized coordinates.

Suppose first the signed complementary codegrees are rational. Lemma 14.2 gives square norm. Thus the order-discriminant identity makes gh a square for two rational axes of codegrees g, h. A quadratic factor of common rational codegree must instead be $\mathbb{Q}(i)$ with codegree $4n^2$. For $t < 18/25$ the smaller denominator is at most seven. The local congruences at three and five from Theorem 12.21 eliminate $(5,5), (6,6)$; Lemma 12.14 excludes three in either sector. The square condition leaves

$$(4,4), \quad (4,9), \quad (4,16), \quad (4,25), \quad (7,7).$$

At half mass one has $s = t = 1/2$. At $(4,h)$ with $h = 9, 16, 25$, the signed evaluations at V_0 must have the form $(1,b)$, $1 \leq b \leq h/4$. Indeed $s \geq 1/2$ by Lemmas 12.15 and 12.5, so the first entry is one. Thus $s = 3/4 - b/h$ and $t = 3/4 - 1/h$.

The ordinary complementary masses $u = 17/36, 7/16, 41/100$ are impossible. None is a sum of two reciprocals of integers at least three. For a quadratic pair of reduced mass $u = a/b$, Lemma 5.4 gives squared gap at least $(a^2 - 4)/b^2 > 1/8$ in these three cases, while (2.3) bounds it by $1/10, 1/8, 1/40$ at conductors $5, 8, 40$. At signed half mass the first shift lies in $1/2 + \mathbb{Z}_{\geq 0}$ and the mixed moment is at least $1/2$. First dispose of an ordinary short pair in $\mathbb{Q}(\sqrt{2})$. Its squared reciprocal gap is $1/8$, so at conductor eight $8O_\eta \geq 2$. The mixed bound in (2.3) forces $T_\eta = 0$, hence $Q_8 = 0$ and $q_0 + q_j = t/2 = 1/4$. In this case $q_j \geq 1/8$ already. Otherwise Lemma 12.5 leaves rational ordinary codegrees or $5 \pm \sqrt{5}$. At every shift the former give zero Gram correction. The golden pair contributes at most $(5-3)/2 = 1$ to the ordinary squared length, the signed contribution is at least $1/2$, and hence the correction is at most $1/2$. The two-variable bound of Theorem 12.21, with this correction, gives $q_j > 1/76$ in 19473 boxes, hence $A \leq 38$. Of the primitive quartic polynomials only two satisfy parity at this cap: their (A, B, C, H) are $(16, 62, 64, 1)$ and $(32, 254, 368, 1)$. The positive odd-moment vectors in their outer boxes number 15 and 197; none has least weight at the least root (moment calculation).

If $t = 23/36$ or $11/16$, omit the Gram correction entirely and use only the two linear column bounds on $a = p_0/q_0, b' = p_0/q_j$. Use $h_j \in s + \mathbb{Z}_{\geq 0}$ as before and $K_j \geq 1 + \lfloor 1/4 + b^2/h \rfloor - (1/4 + b^2/h)$, since the complementary contribution is $1/4 + b^2/h$. At each allowed s the same box argument

gives $q_j > 1/300$, hence $A \le 207$. At $t = 71/100$ we will use $A \le 550$. If $q_j \ge 1/400$ this is immediate. Otherwise $|Q_8| > 7/10$ gives $|P_8| < 51/1000$, so $p_j < (s + 51/1000)/2 \le 381/1000$ while $q_j \ge (t - 1/\sqrt{2})/4$. The two other nonidentity coordinates are at least $(t - 1/\sqrt{5})/4$. Substitution gives $A < 550$. For $t \ge 5/7$ the separate estimates give $A < 1 + 4t/(t - 1/\sqrt{2}) < 399$.

At trace cap 550, impose the minima, separate and mixed Fourier bounds, $2/5 \le s \le t$, $t \ge 23/36$, $s + t \ge 1$, and the three first-shift inequalities. Of 594 polynomials only $(64, 62, 16, 1), (166, 167, 34, 1)$ survive. For each impose $t \ge 18/25$, the $(7, 7)$ masses $t = 5/7$, $s = 4/7, 5/7$, or the displayed $(4, h)$ masses and mixed moments. The separation test eliminates both. Suppose instead the signed codegrees form a quadratic pair of discriminant c. Again $s \ge 1/2$: an ordinary codegree two would force a rational signed codegree two, and three is excluded by Lemma 12.14. Apply Lemmas 12.15 and 12.5. For $c = 5$ the short-pair lemma leaves $5 \pm \sqrt{5}$ or $t \ge 2/3$. For the longer pairs add the first-shift inequality fixing the complementary field; at $c = 8$ also impose $t - |Q_8| \ge 2/94$. The separating weights exclude all ratios in the following ranges:

c	t at least	A through	polynomials
8	1/2	94	75
40	3/4	94	75
5	2/3	94	75
5	143/200	510	553

At $c = 8$ the cap 94 is Proposition 12.7; at $c = 40$ the initial row bound of Theorem 12.8 gives $t \ge 3/4$ and the same cap. In the last row $A < 1 + 4/(143/200 - 1/\sqrt{2}) < 511$.

For $f, f' = 5 \pm \sqrt{5}$ one has $s = t = 1/2$ and the short evaluation is $\beta = 1$ or ϕ^2. In either case $K_j \ge 1/2$ for the shift exchanging $\sqrt{5}$ (Theorem 12.24). For $\beta = 1$ the first shift is half-integral and the preceding trace-38 calculation applies. For $\beta = \phi^2$ it is integral. Again $q_j \ge 1/8$ for an ordinary pair in $\mathbb{Q}(\sqrt{2})$. In all remaining cases the Gram correction at every shift is at most $1/2$, and $|Q_5| \le \sqrt{5}/10$ by Lemma 12.6. The two-variable bound now gives $q_j > 1/48$ in 46225 boxes. Since $|Q_{40}| \le 1/\sqrt{10}$, the two remaining nonidentity weights also exceed $1/48$. Thus $A \le 24$. The shifted complementary polynomial $(X^2 - 3X + 1)^2$ makes A, C even and B, H odd; $2\,\mathrm{Tr}(\beta^r/f) = 2, 5$ makes the normalized moments of orders two and three even and odd, respectively.

Only $(A, B, C, H) = (14, 47, 26, 1), (18, 43, 18, 1)$ remain. The same moment calculation gives 15 and 23 positive weight vectors in their outer boxes. Only one has least weight at the least root, at moments $(6, 81)$ in the second polynomial. Its largest weight exceeds 0.85, whereas Fourier inversion gives $\max_i 2q_i \le (2t + \sqrt{5}/10 + 1/\sqrt{10})/2 < 0.78$.

It remains to exclude $2/3 \le t \le 6837/10000$. By Proposition 14.3, $u = 1 - t = F_m/k$, with m positive and odd and k a sum of two squares. The signed conductor-five bound $u^2 \le 1/10 + 4/(5k^2)$ gives

$$10F_m^2 - k^2 \le 8, \qquad 3163k/10000 \le F_m \le k/3.$$

Together these imply $k \le 132$ and $F_m \le 44$. The odd-indexed Fibonacci numbers in this range are $1, 2, 5, 13, 34$. For the last three the first inequality requires $k \ge 16, 42, 108$, respectively, while the second gives $k \le 15, 41, 107$. The first two force $k = 3, 6$, neither a sum of two squares. □

Theorem 14.5. *In the remaining case of Theorem 14.4,* $\mathrm{Tr}_K(E/D) > 1001$ *and the ordinary complement consists of two rational characters with codegrees* $(4, 4n^2)$, $n \ge 2$, *or* $(8, 8)$.

Proof. Put $A = \mathrm{Tr}_K(E/D)$ and let j fix $\sqrt{2}$. First suppose $A > 1001$. Since $p_0/q_0 \le 1$ and $s < 1$, the pair bounds $q_i \ge (t - 1/\sqrt{5})/4$ for $i \ne 0, j$ give $q_0 \le q_j < 1/1000$, and $p_0 < 1/1000$. Then $|Q_8| = t - 2(q_0 + q_j) > 679/1000$. Consequently the mixed projection and the separate conductor-forty estimate yield

$$|P_8| \le \sqrt{(2 - \sqrt{8}\,|Q_8|)/8} < 1/10, \qquad |P_{40}| \le 1/\sqrt{20} < 224/1000.$$

Since $P_8 = 2(p_0 + p_j) - s$ and $P_5 + P_{40} = 2(p_0 - p_j)$, we obtain $|P_5| > s - 328/1000$. The separate bound $|P_5| \le 1/\sqrt{5}$ gives $s < 776/1000$.

Suppose there is a distinct irrational ordinary pair g, g', and put $w = 1/g - 1/g'$ and $N_o = gg'$. Its quadratic field has conductor $c = 5, 8$ or 40. Write $U > 1$ for the least totally positive unit. Since g/g' is a nontrivial power of U up to conjugation, $w^2 \ge [(U-1)/(U+1)]^2(1-s)^2$. For $c = 8$ the multiplier is $1/2$, whereas the twist-fixed projection gives $w^2 \le 1/8 - Q_8^2/4 < 1/8 - (679/1000)^2/4$, strictly below $(1/2)(224/1000)^2$. At $c = 40$ the multiplying factor is $9/10$ from $U = 19 + 6\sqrt{10}$. Then $w^2 \le 1/40$ again contradicts the unit-ratio bound.

At $c = 5$, (2.3) gives $P_5^2 + 2w^2 \le 1/5$. Since $s \ge 1/2$ and $w^2 = (1-s)^2 - 4/N_o$, this implies

$$8/N_o \;\ge\; \tfrac{2}{3}(1 - 328/1000)^2 - 1/5 > 1/10.$$

Here we minimized $(s-328/1000)^2 + 2(1-s)^2$ over the real line. Thus $N_o < 80$. In Proposition 14.3 the only possible k at $5k^2 < 80$ are one and two, as three is excluded. The row trace $L_o = g + g'$ is a multiple of five, at least ten and at most $N_o/2$. Thus only $L_o = 10, N_o = 20$ remains, with $g, g' = 5 \pm \sqrt{5}$.

Let e be the idempotent of this ordinary quadratic factor. Its different is $(\sqrt{5})$ and $(g) = (2\sqrt{5})$, so the conductor formula of Lemma 2.6 gives $2e \in R$. The involution $1 - 2e$ has squared length one and unit coefficient $1 - 2(1/g + 1/g') = 0$, hence is a signed nontrivial basis element. Its Frobenius–Perron value is one, so it is an actual invertible. Its positive lift is one on the four dimension characters and has a constant sign on the conjugate pair. Its zero unit coefficient would give $t = 1/2$, outside the permitted interval.

An ordinary quadratic pair of common rational codegree is likewise impossible. All simples are self-dual since the signed algebra is totally real. The ordinary field discriminant would be a positive quadratic field discriminant times a square, whereas $N(D)g^2$ is a square. Thus both ordinary characters are rational and their codegrees $g \le h$ are even: in the g-row the four signed dimension induction evaluations equal one and the two remaining signed evaluations are equal integers, giving $g = 4 + 2a$ by the unit-coefficient trace and symmetry. The product gh is a square by the discriminant calculation. Since $1/g + 1/h > 224/1000$ and $g \ge 4$, one gets $(4, 4n^2), (6, 6)$ or $(8, 8)$. For $(6, 6)$ the two rows must pair at three by Lemma 2.7: all quartic residue branches contain the unramified quadratic field from $\sqrt{2}$. Their squared row difference twelve would then be divisible by nine. Lemma 13.2 excludes $(4, 4)$, since it forces $t = 1/2$.

It remains to exclude $94 < A \le 1001$. Write $N = ff' = 5k^2$ and $u = 1 - t = F_m/k$ by Proposition 14.3. If $k < 16$, then $285/1000 < F_m/k < 3163/10000$ gives $F_m \le 4$. Since m is odd, only $F_m = 1, 2$ remain, and the interval forces $F_m = 2$, $k = 7$, impossible for a sum of two squares. Consequently $N \ge 1280$ and the amplified signed projection gives

$$|Q_5|^2 \le 1/5 - 2(u^2 - 4/N) < (21/100)^2.$$

The unit bounds in Lemma 14.2 give 1065 primitive ratios through trace 1001, including 75 through trace 94. For the others use the separate and mixed projections, $s \ge 1/2$, $t > 6837/10000$, the last conductor-five bound, and $\mathrm{Tr}(q\sigma x) \ge s$ for σ fixing F. Only the following two tuples lack a strict rational separating certificate, with $4x = A + u_8\sqrt{8} + u_5\sqrt{5} + u_{40}\sqrt{40}$ up to conjugacy:

$$(A, u_8, u_5, u_{40}) = (166, 58, 74, 26), \qquad (564, 199, 252, 89).$$

Apply (13.4) with σ fixing F. Starting with the four conjugates of $\sigma x - x$, adjoin their products by gx until the Hermite basis stabilizes. In the basis $1, \sqrt{2}, \phi, \sqrt{2}\phi$ this gives the two diagonal basis matrices $\mathrm{diag}(16, 4, 16, 4)$ and $\mathrm{diag}(4, 2, 4, 2)$, respectively. The second tuple forces $4t \in \mathbb{Z}$, incompatible with the permitted interval for t. The first gives $16t \in \mathbb{Z}$ and therefore $t = 11/16$, together with $4\sqrt{2}\,Q_8 \in \mathbb{Z}$. At this fixed mass the same linear system supplies two strict certificates giving $-7/10 < Q_8 < -54/100$. This interval has no integral multiple of $1/(4\sqrt{2})$.

The file `field_checks/forty/verify.py` regenerates the census, checks the two Hermite bases and verifies all separating weights. □

Corollary 14.6. *Both dyadic orders are local with residue field $\mathbb{F}_2$, and $4 \mid k$ in $ff' = 5k^2$. Write*

$$H = h_H \sqrt{d_H}\, \xi^{i_H} \phi^{j_H} \theta^{l_H} \quad (H = D, E), \qquad f = k\sqrt{5}\, \phi^m, \qquad \xi = 1 + \sqrt{2}, \quad \theta = 3 + \sqrt{10}.$$

Then $3 \mid j_D, j_E, m$ and $1 - t = F_m/k$ with $m \equiv 3 \pmod 6$, on choosing $f > f'$.

Proof. Neither ordinary rational axis splits off at two, since its idempotent has nonintegral trace. Their sum cannot split off either, for the same reason, and the quartic completion is a field. Thus the ordinary order is local with residue field $\mathbb{F}_2$; its reduction modulo two gives the same conclusion for the signed order. The signed F-row ψ agrees with an ordinary short row r modulo $2\mathcal{O}_F$, as F is unramified. Hence $f = \sum \psi_i^2 \equiv \sum r_i^2 \equiv 0 \pmod{4\mathcal{O}_F}$.

Let j negate $\sqrt{5}$ and fix $\sqrt{2}$. The simultaneous column for j is a unit (its coordinates are ratios of conjugate d-numbers), hence reduces to one. In the quartic coordinates H/jH the radical contributes a sign, j fixes ξ, and θ reduces to one. In the complementary coordinate $f/jf = \phi^{2m}$. Since $j\phi = -\phi^{-1}$ and $\bar{\phi}$ has order three in $\mathbb{F}_4^\times$, we get $3 \mid j_D, j_E, m$. The remaining assertions are Proposition 14.3. □

14.1. **A congruence from the second indicator.**

Lemma 14.7. *In the remaining conductor-forty case choose the lift of positive second Frobenius–Schur indicator for each simple. Let $\Phi : R \to T$ be the isometry of underlying lattices obtained by matching these basis elements. For every induction constituent C one has*

$$\Phi([FC]_R) = [FC]_T \pmod{4T}.$$

For every $a, b \in R$ one also has

$$\Phi(a^2 b) = \Phi(a)^2 \Phi(b) \pmod{4T}.$$

Proof. The two character algebras are commutative and totally real. Thus all simple lifts are self-dual: every character takes the same value on a lift and on its dual. The pivotal maps on the two lifts differ by a sign, so the pivotal transpositions and the indicators have opposite signs. In a symmetric self-dual object the multiplicity space of an indicator--1 simple has a nondegenerate alternating form and has even dimension. Its ordinary and signed multiplicities therefore agree modulo four in the stated basis.

The pivotal fusion category $\mathcal{S}$ is unimodular, so $I(\mathbf{1})$ is symmetric Frobenius in $Z(\mathcal{S})$ [23, Lemma 7.1]. Each constituent A_χ is self-dual: the rows $S_{A_\chi^*, Y} = S_{A_\chi, Y^*}$ and $S_{A_\chi, Y}$ agree by the character formula and $[FY^*] = [FY]$. Its multiplicity is one, so the symmetric pairing restricts nondegenerately to it. The pivotal forgetful functor preserves this self-duality.

For the product assertion, let j be the pivotal map and $f : Y \to Y^*$ a symmetric self-duality. Then $j_X \otimes f \otimes 1_{X^*}$ is a symmetric self-duality on XYX^*: pivotal transposition fixes it by $f^* j_Y = f$ and $j_X^* j_{X^*} = 1$. Commutativity and self-duality identify its class with $[X]^2[Y]$. This proves the assertion on basis elements. Extend linearly in b and expand the square in a: the mixed terms have coefficient two, and products agree modulo two. □

Let $C = FA$ for an ordinary rational character label A of codegree h.

Lemma 14.8. *The signed coordinates of C are $(E/h, (h/2-2)f/h)$. In particular every $(z, w) \in T$ satisfies*

$$\operatorname{Tr}_K z + (h/2 - 2) \operatorname{Tr}_F w \in h\mathbb{Z}. \tag{14.1}$$

On the ordinary dimension orbit and the two complementary labels, with A listed first among the latter, the row $2hS_{A,-}$ takes integral values γ, m, b, respectively. They satisfy

$$\gamma > 0, \qquad 4\gamma + m + b = h, \qquad 4\gamma^2 + m^2 + b^2 \le h(h+m)/2. \tag{14.2}$$

Proof. On the signed dimension orbit all four entries of $2hS_{A,-}$ are one. The two remaining entries agree by Galois invariance. Since the six entries sum to h, their common value is $h/2-2$. Symmetry gives the signed coordinates of C, and the integral unit coefficient of $C(z,w)$ gives (14.1).

On the ordinary side, rationality and Galois invariance again give integral entries constant on each orbit. Symmetry identifies γ with $h\,\mathrm{FPdim}(C)/D$, and the unit coefficient gives the sum in (14.2). For the inequality take the A,A entry of $S\Theta S$. It equals $S_{A,A}$ since A has twist one and the center has trivial anomaly. The row of this self-dual label is real. Its squared norm is equally divided between the two sign sectors by orthogonality with the JA row. The signed twist terms cancel in J-pairs. Of the ordinary mass $1/2$, at least $M=(4\gamma^2+m^2+b^2)/(2h^2)$ has twist one, counting the J-translates. Hence $2M-1/2\le S_{A,A}=m/(2h)$. □

Proposition 14.9. *The forgetful image of an induction label of smallest ordinary complementary codegree has the following coordinates:*

ordinary codegrees	$K\times\mathbb{Q}^2$	$K\times F$	*parameters*
$(4,4n^2)$	$(D/4,m,-n^2m)$	$(E/4,0)$	$m=0,2$
$(8,8)$	$(\gamma D/8,m,m)$	$(E/8,f/4)$	$(\gamma,m)=(1,2),(2,0)$

In the first row all signed quartic traces, and all ordinary quartic traces when $m=0$, are divisible by four. In the second row both induction labels have isomorphic forgetful images, their ordinary $16S$-submatrix has every entry m, and $8\mid k$.

Proof. The signed columns are given by Lemma 14.8. For $h=4$, (14.2) gives $\gamma=1$, $b=-m$, $-1\le m\le 2$. The signed complementary value is zero and the residue character is common, so m is even. Symmetry gives the other evaluations; pairing the column with an arbitrary integral element gives the trace assertions.

For $h=8$ let m_i,c be the diagonal and off-diagonal entries of the symmetric $16S$-submatrix. The row bound becomes

$$4\gamma_i+m_i+c=8,\qquad (m_i-2)^2+c^2\le 36-4\gamma_i^2.$$

Thus $\gamma_i=1,2$: the value three would require $m_i=2,c=0$, contrary to the first equation. The ordinary rows coincide modulo two on the FP induction column, so the γ_i agree, as do the $m_i=m$. The two rational character rows r_1,r_2 each have four odd entries ±1, one ±2 and a zero. If they agreed modulo four their nonzero difference would be supported at the single position with entries ±2, contradicting orthogonality to the Perron row. Thus one coordinate of $(r_1-r_2)/2$ is odd. The signed induction columns agree, so by Lemma 14.7 their ordinary difference belongs to $4R$. Character inversion gives its coefficient vector $(m-c)(r_1-r_2)/8$, forcing $16\mid m-c$. The two displayed bounds now give exactly the second row of the table. The images agree in both quotients and hence in the sphericalization. Finally $f/4\equiv m\equiv 0 \pmod{2\mathcal{O}_F}$ by the common residue character, giving $8\mid k$. □

14.2. **The valuations of the two global dimensions.** Write v_2 for the rational 2-adic valuation. On K use the integral valuation at the prime over two, normalized by $v(\sqrt{2})=1$. Thus v restricts to $2v_2$ on $\mathbb{Q}$. The induction columns just computed supply both the scale and the parity information in the following comparison.

Proposition 14.10. *In the setting of Corollary 14.6, write $H=h_H\sqrt{d_H}\,\xi^{i_H}\phi^{3r_H}\theta^{l_H}$ for $H=D,E$. Then $i_D\equiv i_E$ and $l_D\equiv l_E$ modulo two. In the case $\gamma=2$ of Proposition 14.9 all four exponents are even. In particular E/D is an integer times a unit times either 1 or $\sqrt{5}$ in the parametrization of Lemma 14.2.*

Proof. For the simultaneous column W_σ, pair $W_\sigma-1$ with the column of the smallest ordinary codegree $h=4,8$. The ordinary complementary evaluations of W_σ are one. In the signed

complement they are $f/\sigma f$. Thus Lemma 14.7 and Proposition 14.9 give a single congruence

$$\gamma(\mathrm{Tr}_K(D/\sigma D)-4)\equiv \mathrm{Tr}_K(E/\sigma E)-4+(h/2-2)(\mathrm{Tr}_F(f/\sigma f)-2)\pmod{4h}.$$

The last term vanishes modulo $4h$. Only $h=8$ and a nontrivial action on F require checking. There the trace difference is $(\phi^m-\phi^{-m})^2$, divisible by 16 because the rational part of $(2+\sqrt{5})^{m/3}$ is even.

Negating $\sqrt{5}$ alone gives $\mathrm{Tr}_K(H/\sigma H)/4\equiv 3^{l_H}\pmod 8$: it is the product of the rational parts of ϕ^{6r_H} and θ^{2l_H}, with $\phi^6=9+4\sqrt{5}$, $\theta^2=19+6\sqrt{10}$. Negating both radicals gives 3^{i_H} instead, since $\xi^2=3+2\sqrt{2}$. Thus for $a=i,l$ the two congruences are $\gamma(3^{a_D}-1)\equiv 3^{a_E}-1\pmod h$. These give matching parities at $\gamma=1$ and even exponents at $\gamma=2$, $h=8$. The sign conditions on E/D now exclude a $\sqrt{2}$ component in its radical factor. □

Proposition 14.11. *In the residual conductor-forty case write $H=h_H\sqrt{d_H}\xi^i\phi^{3J}\theta^l$ for $H=D,E$ as in Corollary 14.6. Set $s=\mathrm{Tr}(1/D)$, $b=v_2(k)$, and let $\delta_H=1$ if $d_H=5$ and l_H is odd, and zero otherwise. Then $d_H\in\{2,5,10\}$ and*

$$v_2(h_D)=2+\delta_D-v_2(s),\qquad v_2(h_E)=1+\delta_E+b.$$

Whenever $\gamma=1$ the two dimensions have the same dyadic valuation, $\delta_D=\delta_E$, and

$$b=1-v_2(s).$$

For $(4,4n^2)$ with n even, the ordinary complementary evaluations of the codegree-four induction column both vanish.

Proof. For $H=E$ its quartic trace form is a sum of four rational squares by Proposition 14.3. For $H=D$ the ordinary complement is a sum of two squares in either case: in the eight case each weight is $1/8$, and $\langle 2,2\rangle\simeq\langle 1,1\rangle$. Witt cancellation gives the same quartic form. At five the argument in Proposition 14.3 therefore applies to both. If $d_H=1$ the unit remaining at five after removing the scalar has square residue in $\mathbb{F}_{25}$: ξ has norm -1, a square modulo five, and ϕ,θ have rational residue. This excludes $d_H=1$.

Write $\xi^{-i}=a+c\sqrt{2}$, $(2+\sqrt{5})^{-J}=u+w\sqrt{5}$, $\theta^{-l}=v+y\sqrt{10}$. Here a,v are odd and $(c,y,u,w)\equiv(i,l,1-J,J)\pmod 2$; also $4\mid w$ for even J since $(2+\sqrt{5})^2=9+4\sqrt{5}$. Positivity gives $i+l\equiv[2\mid d_H]$ and $J+l\equiv[5\mid d_H]\pmod 2$. Extract the rational coefficient of h_H/H:

d_H	2	5	10
$h_H\,\mathrm{Tr}(1/H)/4$	$cuv+5awy$	$awv+2cuy$	$auy+cwv.$

In each column its valuation is δ_H. Reversing the exponents also gives $v_2(\mathrm{Tr}\,H)=2+v_2(h_H)+v_2(d_H)+\delta_H$. For $H=D$ the first scale formula follows immediately. For $H=E$ use $t=1-F_m/k$ with $4\mid k$, $m\equiv 3\pmod 6$, and $F_m\equiv 2\pmod 4$ by the Fibonacci recurrence.

The signed FP induction column has quartic value E/D and residue γ. At $\gamma=1$ this is a unit, so d_D,d_E have the same parity and $v_2(h_D)=v_2(h_E)$. The only odd seed is 5 and $l_D\equiv l_E\pmod 2$ by Proposition 14.10. Thus $\delta_D=\delta_E$ and $b=1-v_2(s)$.

For the codegrees $(4,4n^2)$ write $m_C\in\{0,2\}$ for the ordinary evaluation of the four-column. Comparing the two squared lengths modulo eight by Lemma 14.7 gives

$$\mathrm{Tr}\,E-\mathrm{Tr}\,D\equiv 4m_C^2(1+n^2)\pmod{128}.\tag{14.3}$$

For even n, $v_2(s)=-2-2v_2(n)$, so both traces are divisible by 128. Thus $m_C=0$. □

Proposition 14.12. *In the ordinary $(8,8)$ case of Proposition 14.9 one has $\gamma=1$ and $v_2(k)=3$. The dyadic valuations of D and E agree (normalized by $v_{\mathfrak{p}}(\sqrt{2})=1$) and belong to $\{9,10\}$.*

Proof. Put $b=v_2(k)\ge 3$ and $P_H=\mathrm{Tr}_K H\ \mathrm{Tr}_K(1/H)/16$ for $H=D,E$. When $d_H=5$, write

$$\xi^i=a+b'\sqrt{2},\quad (2+\sqrt{5})^r=u+v\sqrt{5},\quad \theta^l=c+d'\sqrt{10}.$$

Multiplying these expressions and their inverses gives $P_H = 5(-1)^l((avc)^2 - (2b'ud')^2)$. In particular $P_H \equiv 5 \pmod 8$ when l is even: then i is even, r is odd and a, v, c are odd.

At $\gamma = 1$, Proposition 14.11 gives $b = 3$ and a common valuation 8 ($d = 5$, l even), 9 ($d = 2, 10$) or 10 ($d = 5$, l odd). At $\gamma = 2$, Proposition 14.10 makes i, l even, whence $d_D = d_E = 5$. The signed evaluation E/D of the ordinary Frobenius–Perron induction class reduces to γ, hence has positive valuation. The trace formulas in the proof of Proposition 14.11 give $v_2(h_D) = 4$ and $v_2(h_E) = 1 + b$, so $b \geq 4$.

By Lemma 14.7 the two squared lengths of $C = FA_i$ agree modulo eight. Write $k = 2^b K_0$ and $F_m = 2A$ with K_0, A odd. The coordinates in Proposition 14.9 give the signed squared length

$$\frac{\operatorname{Tr}_K E}{64} + \frac{\operatorname{Tr}_F f}{16} = 2^{b-3} K_0 \left(\frac{P_E}{2^{b-1} K_0 - A} + 5A \right).$$

At $\gamma = 2$ the bracket vanishes modulo four, since $P_E \equiv A^2 \equiv 1 \pmod 4$. The ordinary length squared is $4P_D/3 \equiv 4 \pmod 8$, a contradiction. At $\gamma = 1$ with valuation eight the ordinary length squared is $P_D/3 + 1 \equiv 0 \pmod 8$. The signed length squared is $K_0(5(4K_0 - A) + 5A) \equiv 4 \pmod 8$, since odd residues modulo eight are self-inverse. This leaves only valuations nine and ten. □

14.3. **Counting square roots in the two local orders.** By Lemma 14.7 the lattice isometry identifies the solutions in $R/2R$ and $T/2T$ of each marked equation $bx^2 = y \pmod 4$. Here C is a forgotten induction column of smallest ordinary complementary codegree, $b = 1, C$, and y lies in the integral span of the unit and the columns C, W_j, W_l, W_{jl}.

Proposition 14.13. *For $(4, 4n^2)$ with n odd and $v_2(k) = 2$, the common valuation $m = v(D) = v(E)$ is neither six, seven, nor eight. For $(8, 8)$ with $v_2(k) = 3$ and $\gamma = 1$, it cannot be nine.*

Proof. Let $F_2 = \mathbb{Q}_2(\sqrt{5})$, $\pi = \sqrt{2}$, and write A for either completed order. Its normalization is $O = \mathcal{O}_{F_2}[\pi] \times B$, with $B = \mathbb{Z}_2^2$ in the ordinary sector and $B = \mathcal{O}_{F_2}$ in the signed sector. Let $c = 2$ for $(4, 4n^2)$ and $c = 3$ for $(8, 8)$. Since all simples are self-dual, A is self-dual for the unit-coefficient trace. The quartic weight is $1/H$ ($H = D, E$), and the complementary weights have denominator 2^c up to units. The different of $F_2(\pi)$ is (2π), since F_2 is unramified. Lemma 2.6 thus gives

$$I = O^{\#} = \pi^{m-3}\mathcal{O}_{F_2}[\pi] \times 2^c B \subseteq A \subseteq O, \qquad [O : A] = [A : I] = 2^{m-3+c}.$$

All square tests below take place in $O/4I$.

The first valuation can be counted by hand. When $m = 6$, put $\overline{O} = \mathbb{F}_4[\pi]/(\pi^2) \times S$, where $S = \mathbb{F}_2^2$ or $\mathbb{F}_4$, and $U = (A + 2O)/2O$, $V = \{\bar{z} : 2z \in A\}$. Here $4O \subset A$. The reduction of 4τ on $(X + Y\pi, s)$ is $\ell = \operatorname{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(X) + \operatorname{Tr}_{S/\mathbb{F}_2}(s)$, with the sum trace in the split case. Indeed $8/H$ is one modulo π by Corollary 14.6 and Proposition 14.11, and the complementary weights of 4τ are one modulo two. Thus $V \subset \ker \ell$ and the index gives $\dim U + \dim V = 7$. Locality gives $U = \mathbb{F}_2\mathbf{1} + (\pi W, 0)$ with $W \subset \mathbb{F}_4$. Squaring a lift of $(\pi e, 0)$ shows $(e^2, 0) \in V$, hence $\operatorname{Tr}(e^2) = 0$. Consequently $W = \mathbb{F}_2$ and $V = \ker \ell$.

A solution of $x^2 = 1 \pmod{4A}$ has the form $x = 1 + 2y$ with $y \in O$. Indeed both factors of $(x - 1)(x + 1)$ would have valuation one in the quartic component if $v(x - 1) = 1$, and the other components are unramified. Membership is then equivalent to $\bar{y} \in V$ and $y^2 + y \in A$. Every $b \in V$ with $b^2 + b \in U$ has such a lift: changing y by $2z$ changes $y^2 + y$ by $2z$ modulo $4O$. Two successful lifts of the same b differ by an element of A. The number of solutions is therefore $|\{b \in V : b^2 + b \in U\}|/|U|$. Write $b = (x_0 + x_1\pi, s)$. One has $x_1 \in \mathbb{F}_2$ in both sectors. In the ordinary sector $x_0 \in \mathbb{F}_2$ and $s = (t, t)$. In the signed sector $x_0, s \in \mathbb{F}_4$ have the same trace. There are thus 8 and 16 admissible b, respectively. Division by $|U| = 4$ gives two ordinary and four signed square roots.

Valuation eight. Here n is odd, $d_H = 5$, $v_2(h_H) = 4$ and i_H, l_H are odd by Proposition 14.11. Thus $h = H/16$ is one modulo $2\mathcal{O}_{K,2}$: the two units ξ, θ each reduce to $1 + \pi$, and $\sqrt{5}, \phi^{3r}$ reduce to one.

Also $128 \mid \operatorname{Tr} H$ in both sectors. Equation (14.3) and $v_2(1+n^2)=1$ give $m_C=0$. Consequently $C=(4h,0)$ in either normalization.

We compare $x^2 = 1 + C \pmod{4A}$. Any solution has $x = 1 + 2y$ in O, as at valuation six. Write $[a+b\pi]_0 = a$ for $a, b \in \mathcal{O}_{F_2}$, and tr for the trace from F_2 to $\mathbb{Q}_2$ or its reduction from $\mathbb{F}_4$ to $\mathbb{F}_2$. In the ordinary order $2y \in A$ forces $\operatorname{tr}(\bar{y}_K) = 0$, where the bar denotes reduction modulo π: the quartic contribution to $4\tau(2y)$ is $\operatorname{tr}([y_K/h]_0)$ and both rational contributions are even. The residues of $y^2 + y - C/4$ are therefore one on the quartic branch and zero on the rational branches. By locality this element is outside A, contradicting the square equation.

In the signed order put $u = 4/f \equiv 1 \pmod 2$. Hensel's lemma gives $y_K = 2r$ with $y_K^2 + y_K = 1 + h$ and y_F with $y_F^2 + y_F = 1$; the latter has residue of trace one. We claim $\operatorname{tr}(\bar{r}) = 1$. Write $h = 1 + 2e$. The first equation gives $\bar{r} = 1 + \bar{e}$. Normalization gives

$$8 = 8\tau(1) = \operatorname{tr}([h^{-1}]_0) + 2\operatorname{tr}(u).$$

Since $h^{-1} \equiv 1 - 2e \pmod 4$ and $\operatorname{tr}(u)$ is even, $\operatorname{tr}(\bar{e}) = 1$, proving the claim. For $(w, z) \in A$, locality gives $\bar{w} = \bar{z} \in \mathbb{F}_2$. Therefore

$$2\tau\big((4r, 2y_F)(w, z)\big) = \operatorname{tr}([rw/h]_0) + \operatorname{tr}(uy_F z) \equiv \bar{w}\big(\operatorname{tr}(\bar{r}) + \operatorname{tr}(\bar{y}_F)\big) = 0 \pmod 2.$$

Self-duality gives $2y = (4r, 2y_F) \in A$. Now $x = 1 + 2y$ satisfies $x^2 = 1 + C + 4\mathbf{1}$, a signed solution to the same equation.

Valuations seven and nine. Let j negate $\sqrt{5}$ and l negate $\sqrt{2}$, each fixing the other radical. We count solutions of the following equations in $A/2A$, with all right sides read modulo $4A$:

m	equations		
7	$x^2 = 2C - W_j - W_l - W_{jl}$	$x^2 = 2 - W_l$	$Cx^2 = 2C$
9	$x^2 = C - W_l$	$x^2 = -1 - C$	$Cx^2 = 2W_j + 2W_{jl} - C.$

In each row the sets of possible counts are disjoint:

m	R	T
7	$(0,0,16), (0,2,0), (2,0,0), (2,2,0)$	$(0,0,0), (0,2,16)$
9	$(0,0,8), (1,0,8), (1,0,16), (1,1,0)$	$(0,0,0), (0,0,16), (0,1,8)$

Use coordinates $1, \phi, \pi, \pi\phi$ and the two coordinates of B, with $\phi^2 = 1 + \phi$. Work modulo 16 at $m = 7$ and 32 at $m = 9$; these moduli kill $O/4I$. Write $v = \pi^m/H$. Modulo the indicated coordinate modulus its possible values have the form

$$a(\sqrt{5})^e \xi^i \phi^h \theta^u, \qquad a \text{ odd}, \quad e = 0, 1, \quad i + u \equiv 1, \quad h + u + e \equiv 0 \pmod 2, \quad 3 \mid h.$$

These are the signs and the residue condition of Lemma 14.2 and Corollary 14.6 applied to $1/H$. At every valuation the signed weight $2^c/f$ ranges through odd multiples of $\sqrt{5}\phi^h$ with $h \equiv 3 \pmod 6$. On the ordinary side run over odd n when $c = 2$. In each loop all powers can be taken through their periods in the finite unit group. The four columns are obtained from the ratios of Galois conjugates and from Proposition 14.9. Retain the normalization $\tau(1) = 1$ to the available precision.

The remaining search is intrinsic to O/I. Enumerate its unital subrings of order 2^{m-3+c} containing the four columns, with residue field $\mathbb{F}_2$ and $\tau(A) \subset \mathbb{Z}_2$. Here $I = 2^c O$. Use the filtration $U_d = (2^{-d}(A \cap 2^d O) + 2O)/2O$. The ambient residue algebra $O/2O$ has the perfect pairing $\lambda(xy)$, where $\lambda(X + Y\pi, s) = \operatorname{Tr}(X + Y) + \operatorname{Tr}(s)$, since $\pi^m/H \equiv 1 + \pi$ and $2^c/f \equiv 1$ modulo two. Self-duality gives $U_{c-1-d} = U_d^{\perp}$, and locality and orthogonality to 1 give $U_0 = \langle 1 \rangle$ or $\langle 1, (\pi, 0) \rangle$. For $c = 3$ the intermediate U_1 is a totally isotropic three-space containing U_0 and the characteristic vector $(\pi, 1_S)$. Lift a basis of U_d/U_{d-1} at each level d, taking higher digits at level a modulo U_a. This gives every lattice; retain those closed under products. For $m = 9$ the signed side must also satisfy $\tau(C^2) \equiv 3 \pmod 4$ by Lemma 14.7.

Each retained lattice has just 2^6 classes in $A/2A$. Substitution of these classes in the displayed equations gives the table, as verified for valuation seven and valuation nine. □

Corollary 14.14. *In the ordinary complementary pair $(4, 4n^2)$ the integer n is even. In the pair $(8, 8)$ both global dimensions have valuation ten at the prime above two, normalized by $v_{\mathfrak{p}}(\sqrt{2}) = 1$.*

Proof. If n is odd, Proposition 14.11 gives $v_2(k) = 2$ and the same valuation of D and E. It is six or eight for the seed $d_D = 5$, and seven for $d_D = 2, 10$. Proposition 14.13 excludes all three valuations. For $(8, 8)$, Proposition 14.12 leaves nine or ten, and Proposition 14.13 excludes nine. □

14.4. **The last two ordinary pairs.**

Theorem 14.15. *The ordinary complementary codegrees $4, 4n^2$ with even n do not occur in the residual conductor-forty case.*

Proof. Write $H = h_H\sqrt{d_H}\xi^{i_H}\phi^{3r_H}\theta^{l_H}$ for $H = D, E$. Put $e = v_2(n) \geq 1$, $s = \mathrm{Tr}_K(1/D)$ and $t = \mathrm{Tr}_K(1/E)$. The ordinary complementary terms give $s = (3n^2 - 1)/(4n^2)$, and hence $v_2(s) = -2 - 2e$. Let j negate $\sqrt{5}$ and fix $\sqrt{2}$, and let u negate $\sqrt{2}$ and fix $\sqrt{5}$. Apply (14.1) with $h = 4n^2$ to the signed simultaneous units for j and ju. Their complementary value ϕ^{2m} is the odd $(m/3)$th power of $9 + 4\sqrt{5}$, so its trace is two modulo sixteen. Therefore the quartic traces are four modulo sixteen. The trace calculations in Proposition 14.10 give

$$\tfrac{1}{4}\,\mathrm{Tr}_K(E/jE) \equiv 3^{l_E}, \qquad \tfrac{1}{4}\,\mathrm{Tr}_K(E/juE) \equiv 3^{i_E} \pmod 4.$$

Both exponents are even, and the same holds for D by that proposition. The exclusion of $d = 1$ in Proposition 14.11 and total positivity now force $d_D = d_E = 5$, r_D, r_E odd. Furthermore the same proposition gives $b = v_2(k) = 3 + 2e$, $v_2(t) = -2 - 2e$ and $v_2(h_D) = v_2(h_E) = 4 + 2e$.

Let W_j be the simultaneous induction unit for j and put

$$P = W_j + W_j^{-1} - 2, \qquad q = 2^{2e-2}.$$

We compare the unit coefficients of qP in the two orders. They must agree modulo two because the based reductions and the images of W_j agree there.

The ordinary complementary values of P vanish. Its signed complementary value $(\phi^m - \phi^{-m})^2$ equals 16 times an odd square: an odd power of $2 + \sqrt{5}$ has rational part two modulo four and has norm -1. Thus it is 16 (mod 128). On a quartic factor set $A = \phi^{3r_H}$, $B = \theta^{l_H}$ and $\Lambda = \mathbb{Z}_2[\sqrt{2}, \sqrt{5}]$. The value P_H of P is $(AB - A^{-1}B^{-1})^2$. As r_H is odd, $(A - A^{-1})/4$ is an odd integer and $A^{-1} \equiv \sqrt{5} \pmod{2\Lambda}$. As l_H is even and $\theta^2 = 19 + 6\sqrt{10}$, we have $B \equiv 1 \pmod{2\Lambda}$ and $(B - B^{-1})/4 \in \sqrt{10}\mathbb{Z}_2$. Consequently

$$\frac{AB - A^{-1}B^{-1}}{4} = \frac{A - A^{-1}}{4}B + A^{-1}\frac{B - B^{-1}}{4} \equiv 1 + c\sqrt{2} \pmod{2\Lambda}$$

for some $c \in \mathbb{F}_2$. Squaring gives $P_H \in 16(1 + 2\Lambda)$.

Since $h_H/H \in \Lambda$ and $\mathrm{Tr}_K \Lambda \subset 4\mathbb{Z}_2$, the error $q\,\mathrm{Tr}_K((P_H - 16)/H)$ has valuation at least $(2e-2)+5+2-(4+2e) = 1$. The signed complementary error is also even, since $v_2(1-t) = -2-2e$. Thus the ordinary unit coefficient of qP is $16qs$ modulo two, a dyadic unit. The signed coefficient is $16qt + 16q(1 - t) = 16q$ modulo two. This contradiction holds for every $e \geq 1$. □

Proposition 14.16. *The codegree pair $(8, 8)$ cannot occur.*

Proof. By Propositions 14.12 and 14.13 only the common ramified valuation ten remains. Then $v = 32/D$ is an integral dyadic unit, and $v \equiv 1 \pmod{2\mathcal{O}_{K,2}}$. Indeed in the notation of Proposition 14.12 one has $d = 5$, both i, l odd, and the units ξ, θ are $1 + \epsilon$ modulo two, where $\epsilon = \sqrt{2}$ and $\epsilon^2 = 0$ modulo two. The factors $\sqrt{5}, \phi^{3r}$ both reduce to one.

Write $v = v_0 + \epsilon v_1$ with $v_i \in \mathcal{O}_{F,2}$ and $F = \mathbb{Q}(\sqrt{5})$. For an element $x = (a + b\phi + \epsilon y, r, s)$ of the ordinary completed order we have $r \equiv s \equiv a \pmod 2$ by locality. Write tr for the trace on F. Normalization of the unit coefficient gives $\mathrm{tr}(v_0) = 12$, since $\tau(1) = \mathrm{tr}(v_0)/16 + 1/4$. It follows that

$$16\tau(x) = a\,\mathrm{tr}(v_0) + b\,\mathrm{tr}(v_0\phi) + 2\,\mathrm{tr}(v_1 y) + 2(r+s) \equiv b\,\mathrm{tr}(v_0\phi) \pmod 4.$$

The displayed trace factor is odd because $v_0 \equiv 1$ and $\mathrm{tr}(\phi) = 1$, so $4 \mid b$. Now the codegree-eight induction class is $C = (D/8, 2, 2)$, whence $4\tau(Cx) = 2a + b + r + s$. Thus $2a + r + s \equiv 0 \pmod 4$, and as $r \equiv a \pmod 2$ this gives $s - r \equiv 0 \pmod 4$. Both rational coordinates would agree modulo four on every basis element, contradicting the primitivity of $(r_1 - r_2)/2$ in the proof of Proposition 14.9. □

Proof of Theorem 14.1. In the absence of a spherical structure Theorem 14.5 gives two possibilities for the complementary ordinary codegrees. Corollary 14.14 and Theorem 14.15 exclude $(4, 4n^2)$ for all n. Proposition 14.16 excludes $(8, 8)$. □

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