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The Gvozdenović–Laurent neighborhood bound for the stability number

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Abstract

We prove the neighborhood bound conjectured by Gvozdenović and Laurent for the copositive sum-of-squares hierarchy for the stability number. One way to bound the size of a stable set is to fix r of its vertices in all possible ways and use the level-zero relaxation to bound how many more vertices can be added in each case. We prove that level r of the hierarchy on the original graph already certifies the upper bound obtained by combining all these cases. To combine the local certificates, we view the chosen stable sets as cliques in the complement and take a weighted sum over their vertex orderings. A vertex-deletion identity then lets us reduce the clique size one step at a time while preserving a sum-of-squares certificate. The resulting certificate belongs to the smaller cone of Peña, Vera, and Zuluaga. As a consequence, if a graph has stability number $\alpha \geq 2$ and no single edge deletion increases α, then level $\alpha - 2$ is exact, one level earlier than the general guarantee.

1 Introduction

A *stable set* (or *independent set*) in a finite simple graph G is a set of pairwise nonadjacent vertices. The maximum size of such a set is the *stability number* (or *independence number*) $\alpha(G)$. Let $\mathcal{S}_r(G)$ be the stable sets of size r. Once $S \in \mathcal{S}_r(G)$ is chosen, the vertices available to extend it induce $G - N_G[S]$, where the closed neighborhood $N_G[S]$ consists of S and its neighbors. Any stable set in this graph can be adjoined to S, and every maximum stable set of G arises this way for some S. Thus

$$\alpha(G) = r + \max_{S \in \mathcal{S}_r(G)} \alpha(G - N_G[S]) \qquad (1 \leq r \leq \alpha(G)).$$

If u_S is an upper bound on $\alpha(G - N_G[S])$ for each $S \in \mathcal{S}_r(G)$, then $r + \max_{S \in \mathcal{S}_r(G)} u_S$ is an upper bound on $\alpha(G)$.

To certify an upper bound t on $\alpha(H)$ for a nonempty graph H, give each vertex a nonnegative real variable x_v and set $s_H = \sum_{v \in V(H)} x_v$ and $Q_H = x^{\mathsf{T}}(I + A_H)x$, where A_H is the adjacency matrix. For a stable set T, take x to be its indicator vector, with $x_v = 1$ for $v \in T$ and $x_v = 0$ otherwise. Then $s_H = Q_H = |T|$, since no edge has both endpoints in T. Thus $tQ_H - s_H^2$ takes the value $|T|(t - |T|)$. If this quadratic is nonnegative for all $x \geq 0$, then $t \geq \alpha(H)$. The matrix of this quadratic is $t(I + A_H) - J$, where J is the all-ones matrix. Nonnegativity on $x \geq 0$ means that the matrix belongs to the *copositive cone*, the set of symmetric matrices M satisfying $x^{\mathsf{T}}Mx \geq 0$ for all $x \geq 0$.

To certify this nonnegativity, de Klerk and Pasechnik [2] introduced a hierarchy of sum-of-squares tests. Fix t and write $F_t = tQ_H - s_H^2$. Throughout, a polynomial in x is called *SOS* if substituting $x_v = z_v^2$ makes it a sum of squares of real polynomials in z. This substitution ranges over all $x \geq 0$.

Level zero tests F_t for an SOS decomposition, and level one tests $s_H F_t$. Since $s_H > 0$ for nonzero $x \geq 0$, either decomposition certifies $t \geq \alpha(H)$. A level-zero decomposition $F_t = \sum_i q_i^2$ gives a level-one decomposition $s_H F_t = \sum_{v,i} (z_v q_i)^2$ after substitution. At level one the polynomials being squared can be arbitrary cubic forms in z. This freedom can certify smaller values of t.

At each integer level $r \geq 0$ the multiplier is s_H^r. Searching for an SOS decomposition is a semidefinite feasibility problem, and minimizing over the values of t certified at this level gives the bound

$$\vartheta^{(r)}(H) = \min\{t \in \mathbb{R} : s_H^r(tQ_H - s_H^2) \text{ is SOS}\}. \tag{1}$$

Every feasible t is an upper bound on $\alpha(H)$, so taking the minimum in (1) gives $\vartheta^{(r)}(H) \geq \alpha(H)$. These upper bounds are nonincreasing in r, since multiplication by s_H carries any certificate to the next level. A level is *exact* when its bound equals $\alpha(H)$. For the graph $\varnothing$ with no vertices, we set $\vartheta^{(r)}(\varnothing) = 0$.

At level zero one recovers Schrijver's strengthening of the Lovász theta number [5]. In 2026, Baek and Vargas [1, Theorem 2.1] proved the de Klerk–Pasechnik conjecture that level $\alpha(G) - 1$ is exact for every nonempty G.

Return to the stable sets S of a fixed size $1 \leq r \leq \alpha(G)$. Taking $u_S = \vartheta^{(0)}(G - N_G[S])$ for each $S \in \mathcal{S}_r(G)$ gives the upper bound

$$\alpha(G) = r + \max_{S \in \mathcal{S}_r(G)} \alpha(G - N_G[S]) \leq \underbrace{r + \max_{S \in \mathcal{S}_r(G)} \vartheta^{(0)}(G - N_G[S])}_{t}.$$

Which level j on the full graph G certifies this upper bound t, in the sense that $\vartheta^{(j)}(G) \leq t$?

Gvozdenović and Laurent [3, Conjecture 2] conjectured that level $j = r$ suffices for every such stable-set size r. They proved $\vartheta^{(r)}(G) \leq t$ for $1 \leq r \leq \min\{6, \alpha(G) - 1\}$ and also for $r = 7$ when $\alpha(G) = 8$ [3, Theorem 2]. The remaining intermediate levels are $7 \leq r \leq \alpha(G) - 2$ [1, Remark 2.6].

Both the low-order and terminal proofs give certificates in the cone $\mathcal{Q}_n^{(r)}$ of Peña, Vera, and Zuluaga [7]. Here $M \in \mathcal{Q}_n^{(r)}$ means that $(\sum_i x_i)^r x^\mathsf{T} M x$ is a sum of degree-r monomials times squares of linear forms in x, plus a homogeneous polynomial of degree $r + 2$ with nonnegative coefficients. A related hierarchy of Vargas, Vera, and Dickinson [8] allows the multiplier to vary and converges finitely to $\alpha(G)$.

Theorem 1 (Gvozdenović–Laurent neighborhood bound)**.** *For every graph G and every integer $1 \leq r \leq \alpha(G)$,*

$$\vartheta^{(r)}(G) \leq r + \max_{S \in \mathcal{S}_r(G)} \vartheta^{(0)}(G - N_G[S]). \tag{2}$$

For t equal to the right-hand side, $t(I + A_G) - J$ belongs to $\mathcal{Q}_{|V(G)|}^{(r)}$.

For example, deleting a vertex of the five-cycle C_5 and its two neighbors leaves a single edge H. Here $Q_H = s_H^2$, so level zero certifies the upper bound 1 on $\alpha(H)$. With $r = 1$ the theorem combines these five local certificates into a level-one certificate for $\alpha(C_5) \leq 2$.

The recursion of Baek and Vargas closes at a maximum stable set because no further vertex can be added. At an intermediate level r, the maximum in (2) admits a level-zero certificate on each $G - N_G[S]$. We weight each certificate by $\prod_{v \in S} x_v$ and sum over all orderings of all such S. This weighted sum supplies the base case at level r. We then apply the recursion of Baek and Vargas [1, Lemma 2.4] with parameter t to obtain the certificate on G.

In particular, if $\alpha(G) \geq 2$ and deleting any single edge leaves $\alpha(G)$ unchanged, then level $\alpha(G) - 2$ is already exact (Corollary 3).

2 Proof of the neighborhood bound

Every monomial in x becomes a square under $x_v = z_v^2$, so polynomials with nonnegative coefficients in x, including s_H and Q_H, are SOS. Closure under sums and products implies that feasibility in (1) persists when t or r increases. For nonempty H, every feasible t satisfies $t \geq \alpha(H)$ by evaluation at the indicator of a maximum stable set. The value $t = n = |V(H)|$ is feasible, since $n\sum_v x_v^2 - s_H^2 = \sum_{\{u,v\}\in\binom{V(H)}{2}}(x_u - x_v)^2$ and $Q_H - \sum_v x_v^2$ has nonnegative coefficients. The cone of SOS polynomials in z of degree at most $2r+4$ is closed [4, Section 3.8], so the minimum is attained. Thus

$$\alpha(H) \leq \vartheta^{(r+1)}(H) \leq \vartheta^{(r)}(H) \leq |V(H)|. \tag{3}$$

Proof of Theorem 1. First suppose $1 \leq r < \alpha(G)$. Set

$$a = \max_{S\in\mathcal{S}_r(G)} \vartheta^{(0)}(G - N_G[S]), \qquad t = r + a.$$

Choosing S inside a maximum stable set gives $a \geq \alpha(G) - r > 0$. We will construct a certificate for t at level r.

Let $F = \overline{G}$, $s = s_G$, and let $\mathcal{C}_k$ be the set of ordered k-tuples of distinct vertices forming a clique in F, for $k \geq 1$. For $C \in \mathcal{C}_k$, let U_C be its common neighborhood in F, and put

$$x_C = \prod_{v\in C} x_v, \qquad h_C = \sum_{v\in U_C} x_v, \qquad p_k = \sum_{C\in\mathcal{C}_k} x_C.$$

In particular, $p_1 = s$ and $p_2 = s^2 - Q_G$. Appending one vertex from U_C gives each ordered $(k+1)$-clique exactly once, so

$$\sum_{C\in\mathcal{C}_k} x_C h_C = p_{k+1}. \tag{4}$$

For $C \in \mathcal{C}_r$ with vertex set S, we have $U_C = V(G) \setminus N_G[S]$. Thus $H_C := G[U_C] = G - N_G[S]$ and $h_C = s_{H_C}$. Since $a \geq \vartheta^{(0)}(H_C)$, the polynomial $aQ_{H_C} - h_C^2$ is SOS. It is zero when $U_C = \varnothing$.

To sum these polynomials, write $b_k = \sum_{C\in\mathcal{C}_k} x_C h_C^2$. The square h_C^2 counts all ordered pairs in U_C with their monomial weights, allowing repetitions. The difference $h_C^2 - Q_{H_C}$ counts precisely the ordered edges of $F[U_C]$. Appending such an edge to C gives each ordered $(r+2)$-clique exactly once. Since each weight x_C/a is SOS, so is the weighted sum

$$L_r = \frac{1}{a}\sum_{C\in\mathcal{C}_r} x_C(aQ_{H_C} - h_C^2) = \left(1 - \frac{1}{a}\right) b_r - p_{r+2}. \tag{5}$$

The local certificates give L_r at size r. At size one, the target quadratic satisfies $(tQ_G - s^2)/t = (1 - 1/t)sp_1 - p_2$ since $p_1 = s$ and $p_2 = s^2 - Q_G$. Set $P_k = c_k s p_k - p_{k+1}$ with $c_k = 1 - k/t > 0$ for $1 \leq k \leq r$. We descend from r to 1 to show that $s^{r-k+1}P_k$ is SOS, which at $k = 1$ gives the required certificate. We use the local sum L_r to establish the base case $k = r$. Each subsequent step uses only identities for ordered cliques.

To start the induction, we eliminate p_{r+2} from (5) by counting one-vertex deletions of cliques, then eliminate b_r by completing a square. For the first step we sum the vertex-deletion identity of Baek and Vargas [1, equation (7) in the proof of Lemma 2.2] over ordered cliques in the complement, with weights x_C. For $k \geq 1$ and $C \in \mathcal{C}_{k+1}$, write C_{-j} for the tuple obtained by deleting its j-th entry. Then

$$D_k := sp_{k+1} + kp_{k+2} - (k+1)b_k = \sum_{C\in\mathcal{C}_{k+1}} x_C \left(s + kh_C - \sum_{j=1}^{k+1} h_{C_{-j}}\right). \tag{6}$$

Indeed, for fixed j, grouping by $B = C_{-j}$ gives $\sum_C x_C h_{C_{-j}} = \sum_B x_B h_B^2 = b_k$, and the other terms follow from (4). A vertex v adjacent to every entry of C has coefficient $1 + k - (k+1) = 0$ in parentheses. If v is adjacent to all but one entry, only deleting that entry contributes x_v to the last sum, again giving zero. This includes $v \in C$, when the deleted entry is v itself. All other vertices have coefficient one. Thus D_k has nonnegative coefficients. For any $c \in \mathbb{R}$, expanding the squares and using (4) gives

$$E_k(c) := b_k - 2csp_{k+1} + c^2 s^2 p_k = \sum_{C \in \mathcal{C}_k} x_C (h_C - cs)^2. \tag{7}$$

Both D_k and $E_k(c)$ are therefore SOS.

At $k = r$, equations (5) and (6) give $D_r + rL_r = sp_{r+1} - (1 + r/a)b_r$. Multiplying by $c_r = a/t$ and then adding $E_r(c_r)$ from (7) cancels the b_r terms, giving

$$c_r sP_r = c_r(D_r + rL_r) + E_r(c_r).$$

Since $c_r > 0$, this proves that sP_r is SOS.

For $1 \le k < r$, the relation $1 + kc_{k+1} = (k+1)c_k$ gives $(D_k + kP_{k+1})/(k+1) = c_k sp_{k+1} - b_k$. Adding $E_k(c_k)$ gives the one-step identity

$$c_k sP_k = \frac{D_k + kP_{k+1}}{k+1} + E_k(c_k).$$

Multiply by s^{r-k}. The induction hypothesis gives an SOS decomposition of $s^{r-k}P_{k+1}$, and the other terms on the right are SOS. Dividing by $c_k > 0$ proves that $s^{r-k+1}P_k$ is SOS. At $k = 1$, $ts^r P_1 = s^r(tQ_G - s^2)$, as required.

At level zero, each $aQ_{H_C} - h_C^2$ is a sum of squares of linear forms in x plus a quadratic with nonnegative coefficients [3, Section 6.1]. The multipliers of the squares in L_r and $s^{r-k}E_k(c_k)$ are x_C/a and $s^{r-k}x_C$, respectively. Both expand into degree-r monomials with positive coefficients. The terms $s^{r-k}D_k$ also have nonnegative coefficients, giving the required $\mathcal{Q}^{(r)}_{|V(G)|}$ certificate.

At $r = \alpha(G)$, all remaining graphs are empty, so $a = 0$ and (5) would divide by zero. The right-hand side of (2) is $\alpha(G)$, so by monotonicity it suffices to prove exactness one level earlier. If $\alpha(G) \ge 2$ and $S \in \mathcal{S}_{\alpha(G)-1}(G)$, two nonadjacent vertices in $H = G - N_G[S]$ would extend S to size $\alpha(G) + 1$. Thus each nonempty H is complete, with $Q_H = s_H^2$ and $\vartheta^{(0)}(H) = 1$. At least one H is nonempty since S can be chosen inside a maximum stable set. The bound just proved and (3) give exactness at level $\alpha(G) - 1$. If $\alpha(G) = 1$, G is complete and level zero is exact. In either case, multiplying the level-$\alpha(G) - 1$ certificate by s gives the required $\mathcal{Q}^{(\alpha(G))}_{|V(G)|}$ certificate. □

3 Conditions for earlier exactness

As at level $\alpha(G) - 1$, exactness at any $1 \le r < \alpha(G)$ follows if level zero is exact on each $G - N_G[S]$ for $S \in \mathcal{S}_r(G)$. Indeed, (2) then gives $\vartheta^{(r)}(G) \le r + \max_{S \in \mathcal{S}_r(G)} \alpha(G - N_G[S]) = \alpha(G)$. Clique partitions can certify these local equalities.

Corollary 2. *Let G be nonempty and $0 \le r < \alpha(G)$, with $\mathcal{S}_0(G) = \{\varnothing\}$. Suppose that for each $S \in \mathcal{S}_r(G)$, the vertex set of $H = G - N_G[S]$ can be partitioned into $\alpha(H)$ cliques. Then $\vartheta^{(r)}(G) = \alpha(G)$. In particular, this holds if all such H are perfect.*

Proof. Let $C_1, \dots, C_q$ partition $V(H)$ into cliques, with $q \geq 1$, and set $y_i = \sum_{v \in C_i} x_v$. Then

$$qQ_H - s_H^2 = q\left(Q_H - \sum_{i=1}^{q} y_i^2\right) + \sum_{1 \leq i < j \leq q} (y_i - y_j)^2.$$

The polynomial in parentheses sums $2x_u x_v$ over edges $\{u, v\}$ between cliques, so q is feasible at level zero. For $q = \alpha(H)$, (3) gives $\vartheta^{(0)}(H) = \alpha(H)$, also true for the empty graph by convention. The claim now follows from (2) for $r \geq 1$ and directly from $H = G$ for $r = 0$.

If H is perfect, so is $\overline{H}$ by Lovász's perfect graph theorem [6]. Thus $\chi(\overline{H}) = \omega(\overline{H}) = \alpha(H)$, and the color classes of an optimal coloring of $\overline{H}$ give the required cliques of H. □

An edge is *critical* if its deletion increases $\alpha(G)$, and G is *acritical* if it has no critical edge. For acritical G, Laurent and Vargas [5, Proposition 5] bounded $\vartheta^{(0)}(G - N_G[S])$ by two for every stable set S of size $\alpha(G) - 2$. The neighborhood bound then known gave exactness at this level for $2 \leq \alpha(G) \leq 8$. Theorem 1 removes the upper limit, with a clique partition supplying the local bound.

Corollary 3. *If G is acritical and $\alpha(G) \geq 2$, then $\vartheta^{(\alpha(G)-2)}(G) = \alpha(G)$.*

Proof. Fix $S \in \mathcal{S}_{\alpha(G)-2}(G)$ and let $H = G - N_G[S]$. Adjoining a maximum stable set of H to S gives $\alpha(H) \leq 2$. If $\alpha(H) \leq 1$, then H is empty or complete and has the required partition.

If $\alpha(H) = 2$, then $\overline{H}$ has an edge and no triangle. Two neighbors u, w of a vertex v in $\overline{H}$ would thus form an edge of H. Deleting this edge from G would make $S \cup \{u, v, w\}$ a stable set of size $\alpha(G) + 1$, so the edge would be critical. Thus $\overline{H}$ consists of disjoint edges and isolated vertices, and a two-coloring gives the two cliques of H required by Corollary 2. □

Use of generative AI

We used OpenAI's GPT-6.2 Astra to help develop the proof. We also used it in drafting and revising the manuscript. The authors take responsibility for its contents.

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