An exponential lower bound on the exact stabilizer rank

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Abstract

For the single-qubit state $|T\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$, we prove that the minimum number of stabilizer states in an exact linear decomposition of $|T\rangle^{\otimes n}$ grows exponentially in n. The proof combines the tripartite stabilizer normal form with a spectral point obtained from OpenAI's preprint on Euclidean geometry in the asymptotic spectrum of tensors.

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AI disclosure. The contributors had previously considered using spectral points to lower-bound stabilizer rank. ChatGPT Astra 6.2 obtained the exponential lower bound in Theorem 1 by applying results from OpenAI's November 24, 2026 release [OAI26]. ChatGPT also assisted with proof simplification, literature review, and writing. The contributors substantially revised the manuscript, verified all mathematical arguments and citations, and take full responsibility for the paper.

1 Introduction

The n-qubit state space $(\mathbb{C}^2)^{\otimes n}$ has computational basis $\{|x\rangle : x \in \{0,1\}^n\}$. For $b \in \{0,1\}$, define the single-qubit Pauli operators by $X|b\rangle = |1-b\rangle$, $Z|b\rangle = (-1)^b|b\rangle$, $Y = iXZ$, and $I|b\rangle = |b\rangle$. The n-qubit Pauli group consists of n-fold tensor products of these operators multiplied by phases in $\{\pm 1, \pm i\}$. A *Clifford unitary* preserves this group under conjugation. The *stabilizer* of a pure n-qubit state $|\psi\rangle$ is the subgroup of Pauli operators P satisfying $P|\psi\rangle = |\psi\rangle$. The state $|\psi\rangle$ is a *stabilizer state* if its stabilizer has order 2^n, or equivalently if $|\psi\rangle = U|0\rangle^{\otimes n}$ for a Clifford unitary U [AG04, Theorem 1].

Clifford gates, stabilizer-state preparations, and measurements of Hermitian Pauli operators are called *stabilizer operations*. Circuits using these operations can be simulated efficiently on a classical computer [AG04]. A pure state on qubits that is not a stabilizer state is called a *magic state*. In the model of Bravyi and Kitaev [BK05], the standard magic state

$$|T\rangle = \frac{|0\rangle + e^{i\pi/4}|1\rangle}{\sqrt{2}}$$

is Bravyi and Kitaev's H-type state $|A_0\rangle$ [BK05, § VI] and implements $T = \mathrm{diag}(1, e^{i\pi/4})$ by magic-state injection [MT24, § 3.3]. For a nonzero vector $|\psi\rangle$ on n qubits, the *exact stabilizer rank* $\chi(|\psi\rangle)$ is the smallest r such that

$$|\psi\rangle = \sum_{i=1}^{r} c_i|s_i\rangle, \qquad c_i \in \mathbb{C}, \quad |s_i\rangle \text{ stabilizer states on } n \text{ qubits.}$$

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Low-rank decompositions of magic states are used in classical simulation algorithms for circuits with relatively few non-Clifford gates [BSS16, BG16, BBCC+19].

Bravyi, Smith, and Smolin [BSS16, App. C] proved that $\chi(|H\rangle^{\otimes n}) = \Omega(\sqrt{n})$, improved to $\Omega(n)$ by Peleg, Shpilka, and Volk [PSV22, Thm. 1.1]. Here $|H\rangle$ is the Hadamard $+1$ eigenstate, Clifford-equivalent to $|T\rangle$ [PSV22, App. A.2]. Lovitz and Steffan [LS22] proved exponential lower bounds for certain products of nonidentical states. Mehraban and Tahmasbi [MT24] proved a nearly quadratic lower bound for $|T\rangle^{\otimes n}$ at fixed Euclidean error $0 < \delta < 1$. Sinha and Lovitz [SL26] extended it to tensor powers of any non-stabilizer single-qubit state. Qassim, Pashayan, and Gosset [QPG21] bounded $\chi(|T\rangle^{\otimes n})$ by $O(2^{(\log_2 3)n/4})$. We use OpenAI's results [OAI26] to prove the exponential lower bound conjectured in [BSS16, § II].

Theorem 1. *Under the Euclidean prices, exact extension, and geodesic extension results of [OAI26, Theorems 1.1(1),(2) and 3.1(2), Corollary 3.4], there exists an absolute constant $c > 0$ such that for every integer $n \geq 1$,*

$$\chi(|T\rangle^{\otimes n}) \geq 2^{cn}.$$

2 Proof of Theorem 1

We use Strassen's asymptotic spectrum of tensors [Str88]. Let S be a finite set of at least two parties. A tensor on S is a vector in $\bigotimes_{a\in S} V_a$, where the V_a are finite-dimensional complex vector spaces. Given tensors $U \in \bigotimes_a V_a$ and $V \in \bigotimes_a W_a$, their direct sum $U \oplus V$ is obtained by embedding the two tensors in disjoint blocks of $\bigotimes_a (V_a \oplus W_a)$ and adding them. Their tensor product is grouped as an element of $\bigotimes_a (V_a \otimes W_a)$. We call U a *restriction* of V if $U = (\bigotimes_a L_a)V$ for linear maps $L_a : W_a \to V_a$. In the following, we use plain letters U, V for general tensors and ket notation $|\cdot\rangle$ for quantum states (viewed as tensors).

We next define spectral points. Intuitively, each spectral point is a potential function on tensors.

Definition (Spectral point)**.** A spectral point on S is a nonnegative real-valued function ϕ on tensors on S, monotone under restrictions and satisfying

$$\phi(0) = 0, \qquad \phi(1) = 1, \qquad \phi(U \oplus V) = \phi(U) + \phi(V), \qquad \phi(U \otimes V) = \phi(U)\phi(V),$$

where 1 is the scalar tensor with one-dimensional spaces at all parties.

In particular, ϕ is invariant under nonzero scaling and invertible local linear maps, and has value 1 on every nonzero product of local vectors. Since an ordinary sum is a restriction of a direct sum, $\phi(U + V) \leq \phi(U) + \phi(V)$ for tensors in the same space.

We now show that proving the exponential lower bound on $\chi(|T\rangle^{\otimes n})$ reduces to constructing a spectral point ϕ satisfying certain pairwise constraints, together with a witness state $|\Psi\rangle$ having a sufficiently large value of $\phi(|\Psi\rangle)$. Intuitively, this reduction relies on the tripartite stabilizer normal form of Bravyi, Fattal, and Gottesman [BFG06].

The stabilizer rank bound. Fix four parties $S = \{A, B, C, M\}$, where M is an auxiliary party that always carries a one-dimensional space $\mathbb{C}^1$. For each positive integer j, put $V_j = (\mathbb{C}^{2^j})^{\otimes 3} \otimes \mathbb{C}^1$, carrying $\mathbb{C}^{2^j}$ at each of A, B, C and $\mathbb{C}^1$ at M. We identify V_j with the $3j$-qubit space $(\mathbb{C}^{2^j})^{\otimes 3}$ by dropping the one-dimensional factor at M. For distinct parties $a, b \in S$, write $|E_{ab}\rangle = |00\rangle + |11\rangle$ for the Einstein–Podolsky–Rosen (EPR) state on a, b (with one-dimensional factors at the remaining parties), and write $|G\rangle = |000\rangle + |111\rangle \in V_1$ for the three-party Greenberger–Horne–Zeilinger (GHZ) state on A, B, C.

Suppose ϕ is a spectral point on S satisfying

$$\phi(|E_{AB}\rangle),\ \phi(|E_{BC}\rangle),\ \phi(|E_{CA}\rangle) \le 2^{2/3}. \tag{1}$$

Since $|G\rangle$ is a sum of two nonzero product vectors, subadditivity gives $\phi(|G\rangle) \le 2$. By the tripartite stabilizer normal form [BFG06, Section VI, Theorem 5], every stabilizer state $|s\rangle \in V_j$ is equivalent, under local Clifford unitaries on A, B, C, to a tensor product of e pairwise EPR states among $\{A, B, C\}$, g copies of $|G\rangle$, and local stabilizer states, with $2e + 3g \le 3j$. By (1) and multiplicativity,

$$\phi(|s\rangle) \le (2^{2/3})^e \phi(|G\rangle)^g \le 2^{2e/3+g} \le 2^j.$$

Now let $0 \neq |\psi\rangle \in V_j$ and write a minimal stabilizer decomposition $|\psi\rangle = \sum_{i=1}^{r} c_i |s_i\rangle$ with $r = \chi(|\psi\rangle)$, $c_i \neq 0$, and stabilizer states $|s_i\rangle \in V_j$. By subadditivity and invariance under nonzero scaling,

$$\phi(|\psi\rangle) \le \sum_{i=1}^{r} \phi(c_i|s_i\rangle) = \sum_{i=1}^{r} \phi(|s_i\rangle) \le r\, 2^j \quad \Longrightarrow \quad \chi(|\psi\rangle) \ge 2^{-j}\phi(|\psi\rangle). \tag{2}$$

From a witness to T states. By (2) and multiplicativity of ϕ, any state $|\Psi\rangle \in V_j$ with $2^{-j}\phi(|\Psi\rangle) > 1$ already satisfies $\chi(|\Psi\rangle^{\otimes n}) \ge (2^{-j}\phi(|\Psi\rangle))^n$, so its stabilizer rank grows exponentially under tensor powers. Universality of Clifford and T gates lets us transfer this lower bound to tensor powers of $|T\rangle$. Theorem 1 therefore reduces to the following claim. For completeness, we include its derivation from Claim 1.

Claim 1. *There exist a spectral point ϕ on $S = \{A, B, C, M\}$ satisfying (1), an integer $j \ge 1$, and a state $|\Psi\rangle \in V_j$ such that $2^{-j}\phi(|\Psi\rangle) > 1$.*

Proof of Theorem 1 assuming Claim 1. Let ϕ, j, and $|\Psi\rangle \in V_j$ be as in Claim 1. For any spectral point, the sublevel sets $\{U \in V_j : \phi(U) \le b\}$ are closed in the Euclidean topology [CHNVZ25, arXiv v2, Corollary 2.4 and Section 3], so the superlevel set $\mathcal{O} = \{U \in V_j : \phi(U) > 2^j\}$ is open. By nonzero-scale invariance of ϕ, the open set $\mathcal{O}$ contains the normalized state with its global-phase orbit $\{e^{i\theta}|\Psi\rangle/\||\Psi\rangle\|_2 : \theta \in \mathbb{R}\}$. Since states prepared from $|0\rangle^{\otimes 3j}$ by finite Clifford+T circuits are dense in the unit sphere of V_j up to global phase [BMR+00, Section 3], there exists a finite Clifford+T circuit using t T gates that prepares a state $|\Phi\rangle \in V_j$ lying in $\mathcal{O}$, i.e., with $\phi(|\Phi\rangle) > 2^j$. Because every stabilizer state $|s\rangle \in V_j$ satisfies $\phi(|s\rangle) \le 2^j$, we necessarily have $t \ge 1$.

By standard magic-state injection [MT24, Section 3.3], this circuit can be implemented on $|0\rangle^{\otimes 3j} \otimes |T\rangle^{\otimes t}$ using postselected stabilizer operations, producing a nonzero scalar multiple of $|\Phi\rangle$. Since Clifford unitaries, stabilizer-state preparations, and fixed-outcome Pauli measurements map each stabilizer state to a scalar multiple of a stabilizer state or to zero [AG04], they cannot increase exact stabilizer rank on a nonzero output. Moreover, tensoring minimal decompositions shows that χ is submultiplicative under tensor products. Repeating the preparation n times in parallel and applying (2) to $|\Phi\rangle^{\otimes n} \in V_{jn}$ together with multiplicativity of ϕ, we obtain

$$\chi(|T\rangle^{\otimes n})^t \ge \chi(|T\rangle^{\otimes tn}) \ge \chi(|\Phi\rangle^{\otimes n}) \ge 2^{-jn}\phi(|\Phi\rangle^{\otimes n}) = \left(2^{-j}\phi(|\Phi\rangle)\right)^n.$$

Taking t-th roots proves Theorem 1 with $c = t^{-1}\log_2(2^{-j}\phi(|\Phi\rangle)) > 0$. □

It therefore remains to prove Claim 1.

A spectral point with a strict gap. For a spectral point ϕ on any finite set of parties, the *spectral distance* $d_\phi(a,b) = \log_2 \phi(|E_{ab}\rangle)$ for $a \neq b$, with $d_\phi(a,a) = 0$, is a pseudometric (distinct parties may have distance zero). Indeed, it is nonnegative by restriction to a product vector, symmetric, and satisfies the triangle inequality because $|E_{ac}\rangle$ is a restriction of $|E_{ab}\rangle \otimes |E_{bc}\rangle$ via the functional $|i\rangle|j\rangle \mapsto \mathbf{1}_{i=j}$ at b. Recently, OpenAI [OAI26] proved a converse for Euclidean distances. We summarize the three results from [OAI26] used to prove Claim 1.

Euclidean prices (planar case) [OAI26, Thm. 1.1(1),(2)]
For a finite noncollinear set $J \subset \mathbb{R}^2$, let $W(J)$ be half the perimeter of its convex hull. There exists a spectral point ϕ on J such that

$$W(J)\, d_\phi(a,b) = \|a-b\|_2 \qquad (a,b \in J).$$

Exact extension [OAI26, Thm. 3.1(2)]
Fix a spectral point ϕ on J and give each site $i \in J$ a nonnegative integer weight w_i. One can add a site p and extend ϕ to a spectral point ϕ' that keeps the value of every old tensor U when p carries a one-dimensional factor, with

$$\sum_{i\in J} w_i d_{\phi'}(p,i) = \sup_{\substack{h\in\mathbb{Z}_{\geq 1} \\ 0\neq U\in\bigotimes_{i\in J}\mathbb{C}^{2^{hw_i}}}} \frac{\log_2 \phi(U)}{h}.$$

Midpoint uniqueness [OAI26, Cor. 3.4]
If two sites x and y are each at spectral distance $\frac{1}{2}d_\phi(a,b)$ from both a and b, then $d_\phi(x,y) = 0$.

Proof of Claim 1. Place A, B, C at the vertices of a unit equilateral triangle in $\mathbb{R}^2$ and M at the midpoint of AB. The convex hull of $S = \{A, B, C, M\}$ has half-perimeter $W(S) = 3/2$, so the Euclidean prices theorem gives a spectral point ϕ on S with

$$d_\phi(a,b) = \tfrac{2}{3}\|a-b\|_2 \qquad (a,b \in S).$$

In particular, $d_\phi(A,B) = d_\phi(B,C) = d_\phi(C,A) = 2/3$, so (1) holds.

Suppose for contradiction that $\phi(U) \leq 2^j$ for all $j \geq 1$ and $U \in V_j$. Applying the exact extension theorem to ϕ on $J = S$ with weights $w_A = w_B = w_C = 1$ and $w_M = 0$ yields an extension ϕ' whose spectral distance $d = d_{\phi'}$ satisfies

$$d(p,A) + d(p,B) + d(p,C) = \sup_{j\geq 1,\, 0\neq U\in V_j} \frac{\log_2 \phi(U)}{j} \leq 1.$$

The old distances are preserved, so each pair of distances from p to A, B, C sums to at least $2/3$. Their total is at most 1, so all three inequalities are tight and $d(p,A) = d(p,B) = d(p,C) = 1/3$. Both p and M are at distance $1/3 = \frac{1}{2}d(A,B)$ from A and B, so midpoint uniqueness [OAI26, Cor. 3.4] gives $d(p,M) = 0$. By the triangle inequality, $\frac{1}{3} = d(p,C) = d(M,C) = \frac{1}{\sqrt{3}}$, a contradiction (Figure 1). Thus some $j \geq 1$ and $|\Psi\rangle \in V_j$ satisfy $\phi(|\Psi\rangle) > 2^j$. □

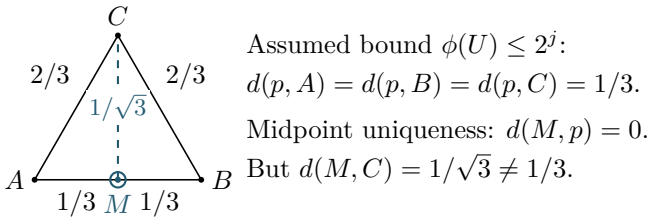


Figure 1: The extension forces p to be a midpoint of AB, contradicting $d(M,C) = 1/\sqrt{3}$.

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