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A wild quasi-étale double cover over a single smooth genus-two exceptional curve in characteristic two*

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Abstract

Over an algebraically closed field of characteristic two, the surface $b^2+(t^5+t^2a^2)b+a^5=0$ has one singular point, and its minimal resolution has one exceptional curve C: the smooth curve of genus two with affine equation $y^2+y=x^5$, with $C^2=-1$. We show that this surface has a connected Galois double cover, étale away from the singular point, whose ramification along C is fierce: ramification index one and a purely inseparable residue extension of degree two. A run of this project proved that no such wild cover exists when the formal neighbourhood of the exceptional curve is that of the zero section of a negative line bundle, as for graded surface cones. The example shows that this hypothesis cannot be dropped, even for a Gorenstein singularity with one smooth exceptional curve. The weighted homogeneous surface $b^2+t^5b+a^5=0$ has the same exceptional curve, with the same self-intersection, and every connected Galois cover of it that is étale away from the singular point is tame along that curve; so the two germs are not formally isomorphic. The example and its proof come from a run of GPT-6 Astra (ChatGPT Pro mode); they were checked, the comparison was added and the text was written by Claude Opus 5.5 at effort max.

1 Results

Throughout, k is an algebraically closed field. For a normal surface germ, a cover *étale on the punctured germ* is a finite cover by a normal germ that is étale away from the closed point. Let $R=\bigoplus_{d\geq 0}R_d$ be a graded ring with $R_0=k$ and let A be its completion at $R_+=\bigoplus_{d>0}R_d$; the *radial valuation* of $\mathrm{Frac}(A)$ is $v(f)=\min\{d : f_d\neq 0\}$ for $0\neq f=\sum_d f_d\in A$.

Theorem A ([SA12, Lemma 4.16 and Proposition 4.17, pp. 105–107])**.** *Let k have characteristic $p>0$, let $R=\bigoplus_{d\geq 0}R_d$ be a finitely generated normal graded k-domain of dimension two with $R_0=k$, and let A be its completion at R_+. Let B/A be the normal extension belonging to a connected finite Galois cover of the punctured spectrum of A. Assume the positive grading is effective, so that its occurring degrees have greatest common divisor one. Then every extension to* $\mathrm{Frac}(B)$ *of the radial divisorial valuation of* $\mathrm{Frac}(A)$ *is tame. Thus a normal surface germ whose completed resolution along its exceptional curve is the completion of the zero section in a negative line bundle over a smooth proper curve has no wild inertia at that exceptional valuation in any finite cover étale on the punctured germ.*

*AI-produced. The example and its proof are due to a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode); Corollary C (ii)–(iii), the proof of Lemma 2.3 given here and the text are due to Claude Opus 5.5 (Anthropic) at effort max, and Lemma 2.6 to a second agent of the same model and effort; see Section 3. No human has checked this text. To the extent that any rights exist in it, they are dedicated to the public domain under CC0 1.0.

From now on k has characteristic two. Put

$$F = b^2 + (t^5 + t^2a^2)\,b + a^5, \qquad F_0 = b^2 + t^5b + a^5,$$
$$H = d^2 + (c^4 + t^3 + t^4c^2)\,d + tc^5 + t^2c^7 + t^4c^6 + t^6c^5,$$

let S, S_0 and T be the affine surfaces $F = 0$, $F_0 = 0$ in $\mathbb{A}^3_{a,b,t}$ and $H = 0$ in $\mathbb{A}^3_{c,d,t}$, and let o be the origin. Let $\varphi\colon T \to S$ and $\sigma\colon T \to T$ be given by

$$\varphi^*a = c^2 + t^2c, \qquad \varphi^*b = t^2d + c^5, \qquad \varphi^*t = t,$$
$$\sigma^*c = c + t^2, \qquad \sigma^*d = d + c^4 + t^6c + t^8, \qquad \sigma^*t = t.$$

Theorem B ([SA12, Example 4.60, p. 143])**.** *Let k have characteristic two.*

(i) S is a normal Gorenstein surface whose only singular point is o. The weighted blow-up of $\mathbb{A}^3$ with weights $(2,5,1)$ on (a,b,t) restricts to a resolution $f\colon Y \to S$ which is an isomorphism over $S \setminus \{o\}$. Its exceptional fibre is one smooth curve C, the smooth projective model of $y^2 + y = x^5$, of genus two, with $C^2 = -1$. The discrepancy of C is -3.

(ii) T is a normal surface, φ is finite of degree two and étale over $S \setminus \{o\}$, $\varphi^{-1}(o)$ is one point o', and σ generates $\mathrm{Gal}(k(T)/k(S)) \cong \mathbb{Z}/2$. The surface T is singular at o', and o' is not a rational singularity; T is smooth elsewhere.

(iii) The valuation v_C of C has a unique extension w to $k(T)$. Its ramification index is one, its residue extension is purely inseparable of degree two, its different exponent is one, and σ acts trivially on its residue field. So the inertia group at w is all of $\mathbb{Z}/2$: the cover φ is wildly ramified along C.

In the coordinates $x = a/t^2$ on Y and $z = c/t$ on T the cover is $z^2 + tz = x$.

Corollary C. *In the notation above:*

(i) The completion of Y along C is not isomorphic to the completion of the total space of a negative line bundle on a smooth proper curve along its zero section.

(ii) S_0 is a normal surface whose only singular point is o; the same weighted blow-up gives a resolution of S_0 with one exceptional curve $C_0 \cong C$, with $C_0^2 = -1$; and every connected finite Galois cover of the punctured spectrum of $\widehat{\mathcal{O}}_{S_0,o}$ is tame along C_0.

(iii) The complete local rings $\widehat{\mathcal{O}}_{S,o}$ and $\widehat{\mathcal{O}}_{S_0,o}$ are not isomorphic.

Why it matters. Starting from Grothendieck's notes on the fundamental group in cote 29 of his Montpellier archive, the run [SA12] studied the topological finite generation of $\pi_1(X \setminus Y)$, and of its quotient by the wild inertia at the codimension-one components of Y, for X normal and proper and $Y \subset X$ closed [SA12, §4.1]. Where Y has isolated points at which X is singular, one resolves X there (at a regular point, a cover étale off the point extends, by purity), and a cover étale over $X \setminus Y$ may then be wildly ramified along the new exceptional curves, which a tame fundamental group of the resolution does not see. Theorem A excludes this when the exceptional curve has a linear formal neighbourhood. Theorem B shows that the hypothesis is needed: one smooth exceptional curve of genus two, on a Gorenstein surface, already carries a fierce cover. Corollary C shows that the dual graph and the self-intersection do not decide the matter: S and S_0 have the same exceptional curve C with $C^2 = -1$, and only S has a wild cover.

Credit. Theorem A and Theorem B, with their proofs, are due to a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode) of this project [SA12]. Corollary C (i) makes precise the run's remark that the hypothesis of Theorem A cannot be discarded. Corollary C (ii)–(iii) and the proof of Lemma 2.3 given here are due to the writer, Claude Opus 5.5 at effort max; Lemma 2.6 is due to the note's independent checker, Claude Opus 5.5 at effort max. Details are in Section 3.

Context: what is known and what is new. Known: wild covers of surface singularities, étale off the closed point, exist in positive characteristic: Artin's wild $\mathbb{Z}/2$-actions on $k[[u,v]]$ [Ar75] and coverings of rational double points [Ar77], the wild $\mathbb{Z}/p$-quotients of $k[[u,v]]$ of Lorenzini–Schröer [LS20, LS23], the wild quotients of Ito–Schröer [IS] by special p-groups acting diagonally on products of Hermitian curves, and Matsumoto's $\mathbb{Z}/p$-coverings of rational double points by regular germs [Ma, §6.2 and Lemma 6.6]. In all of them the source of the cover is regular or its base is a rational double point, and then every exceptional curve over the base is rational (Lemma 2.6; for quotients of regular germs see also [IS, p. 5]). Here C has genus two, and T is not a rational singularity. Cutkosky and Srinivasan [CS, Thm. 1(b)] show that if the prime-to-p completion of the local fundamental group is finite, the exceptional curves of a good resolution are rational and form a tree; here C has genus two, so that completion is infinite. For strongly F-regular singularities the local fundamental group is finite of order prime to p [CST, Thm. A], so their covers are tame; S lies far outside that class, since its discrepancy -3 is below -1. New, to our knowledge after the search described in Section 3: the example of Theorem B and Corollary C.

Scope. We treat one example in characteristic two. We do not claim that every nonlinear formal neighbourhood carries a wild cover. The point o' of T is a singular, nonrational double point (Lemma 2.6). The run also resolves T (one curve, the Frobenius twist of C, with self-intersection -2) and computes the maximal pro-2 quotients of the local fundamental groups of T and S [SA12, Prop. 4.61, Cor. 4.62]; they are not used here. The example bears on tameness only, not on the finite generation of any fundamental group.

Proof overview. S is a hypersurface with an isolated singularity. The weighted blow-up has two smooth charts covering the exceptional fibre, $y^2 + (1 + tx^2)y + x^5 = 0$ and $\xi^2(1+\xi+\rho\tau^2)+\tau^5 = 0$; the first shows C and its genus, and the divisor of t on the second gives $C^2 = -1$. The function $x = a/t^2$ is a unit along C whose residue is not a square in $k(C)$, so $z^2 + tz = x$ defines a separable quadratic extension in which t stays a uniformizer and the residue field becomes $k(C)(\sqrt{x})$. The substitution $c = tz$, $d = (b + c^5)/t^2$ turns F into t^4H, which presents the normalization T; the cover is étale where $t \neq 0$ by the derivative t^2, and where $t = 0$ by a Jacobian computation. Corollary C follows from Theorem A, applied to S and to the graded ring of S_0.

2 Proofs

2.1 The surfaces and their resolution

Lemma 2.1. *F and F_0 are irreducible, and S and S_0 are normal Gorenstein surfaces whose only singular point is o.*

Proof. Both polynomials are monic of degree two in b over the factorial ring $k[a,t]$; a factorization would give a root in $k(t)(a)$. Put $h = t^5 + t^2a^2$ (resp. $h = t^5$) and $w = b/h$;

a root gives $a^5/h^2 = w^2 + w$ with $w \in k(t)(a)$. For the valuation $-\deg_a$ at $a = \infty$ over $k(t)$, the left side has value $-5 + 2\deg_a h$, which is -1 (resp. -5), an odd negative number, while $w^2 + w$ has value $2v(w)$ if $v(w) < 0$ and a value ≥ 0 otherwise. So F and F_0 are irreducible. In characteristic two,

$$F_a = a^4, \quad F_b = t^2(t^3 + a^2), \quad F_t = t^4 b; \qquad (F_0)_a = a^4, \quad (F_0)_b = t^5, \quad (F_0)_t = t^4 b.$$

On either surface, $a = 0$ forces $t = 0$ through the second derivative and then $b^2 = 0$; so the singular locus is $\{o\}$. A hypersurface is Gorenstein and satisfies Serre's condition (S_2), and (R_1) holds; so both are normal. □

Lemma 2.2. *The weighted blow-up of Theorem B (i) restricts to resolutions $f\colon Y \to S$ and $f_0\colon Y_0 \to S_0$ which are isomorphisms over the complement of o. The exceptional fibre of each is a single smooth curve isomorphic to the smooth projective model C of $y^2 + y = x^5$, of genus two and self-intersection -1. The discrepancy of C in Y is -3.*

Proof. Let $I_m \subset k[a, b, t]$ be spanned by the monomials $a^i b^j t^l$ of weight $2i+5j+l \geq m$, and $X = \operatorname{Proj} \bigoplus_{m\geq 0} I_m s^m$. The Rees algebra is generated by $as, as^2, bs, \ldots, bs^5, ts$, so $X \to \mathbb{A}^3$ is projective; it is an isomorphism over $\mathbb{A}^3 \setminus \{o\}$, since there one coordinate is a unit and its powers lie in every I_m. Its exceptional divisor is $E = \operatorname{Proj} \bigoplus_m I_m/I_{m+1} = \mathbb{P}(2, 5, 1)$, and X is covered by $D_+(ts)$, $D_+(as^2)$ and $D_+(bs^5)$, since $D_+(as) \subset D_+(as^2)$ and $D_+(bs^j) \subset D_+(bs^5)$.

The chart $D_+(ts)$ is $\operatorname{Spec} k[x, y, t]$ with $x = a/t^2$, $y = b/t^5$, and $E = \{t = 0\}$ on it. Since $F(t^2x, t^5y, t) = t^{10}G_1$ with $G_1 = y^2+(1+tx^2)y+x^5$, and $F_0(t^2x, t^5y, t) = t^{10}(y^2+y+x^5)$, the strict transforms there are $G_1 = 0$ and $y^2 + y + x^5 = 0$. They are smooth: $\partial_y G_1 = 1 + tx^2$ and $\partial_x G_1 = x^4$ have no common zero, and likewise for the second. They meet E in $C^{\mathrm{aff}} = \{y^2 + y + x^5 = 0\} \subset \mathbb{A}^2$.

The chart $D_+(as^2) \cap D_+(bs^5)$. Put $\rho = b/a^2$, $\xi = a^5/b^2$, $\tau = ta^2/b$, so that $a = \rho^2\xi$, $b = \rho^5\xi^2$, $t = \rho\tau$. The monomial $\rho^\alpha \xi^\beta \tau^\gamma$ equals $a^{-2\alpha+5\beta+2\gamma} b^{\alpha-2\beta-\gamma} t^\gamma$, of weight α; the exponent matrix has determinant -1. The ring of this chart is spanned by the Laurent monomials in a, b times powers of t of non-negative weight, so it is $k[\rho, \tau, \xi, \xi^{-1}]$, with $E = \{\rho = 0\}$. Since $F(\rho^2\xi, \rho^5\xi^2, \rho\tau) = \rho^{10}\xi^2 G_2$ with $G_2 = \xi^2(1 + \xi + \rho\tau^2) + \tau^5$, and the same with $\xi^2(1 + \xi) + \tau^5$ for F_0, the strict transforms are $G_2 = 0$ and $\xi^2(1 + \xi) + \tau^5 = 0$. They are smooth, as ∂_ξ of either equation is ξ^2, a unit.

The exceptional fibre. The strict transform Y lies in the zero locus of Fs^{10}, whose intersection with E is the zero locus of the initial form $b^2 + t^5 b + a^5$ in $\mathbb{P}(2, 5, 1)$; the same holds for Y_0. At a point of that locus with $t = 0$ we have $b^2 = a^5$, so $a \neq 0 \neq b$, and the point lies in the second chart, with $\rho = \tau = 0$ and $\xi^2(1 + \xi) = 0$, hence $\xi = 1$. So the exceptional fibre of Y, and that of Y_0, is $C^{\mathrm{aff}} \cup \{P_\infty\}$ with $P_\infty = (\rho, \tau, \xi) = (0, 0, 1)$; it is a proper curve, smooth by the two charts (at P_∞ the curve is $\xi^2(1+\xi)+\tau^5 = 0$ with $\partial_\xi = 1$), and irreducible since $y^2 + y + x^5$ is (the argument of Lemma 2.1 with $h = 1$). Hence it is the smooth projective model C of $y^2 + y = x^5$, and Y, Y_0 are smooth along it.

Genus. Let $\pi\colon C \to \mathbb{P}^1_x$ be the double cover. Over $x \neq \infty$, $\pi_*\mathcal{O}_C$ is free on $1, y$. At infinity put $u = 1/x$ and $v = y/x^3$: then $v^2 + u^3 v + u = 0$, which is smooth at $(0, 0)$ since its u-derivative is $1 + u^2 v$, and $\pi_*\mathcal{O}_C$ is free on $1, v$. As $y = u^{-3}v$, we get $\pi_*\mathcal{O}_C \cong \mathcal{O} \oplus \mathcal{O}(-3)$, so $\chi(\mathcal{O}_C) = 1 + (1 - 3) = -1$ and $g(C) = 2$ (this is also the Artin–Schreier formula [Sti, Prop. 3.7.8]).

Self-intersection. On $D_+(ts)$ the function t is a local equation of C. On the second chart $t = \rho\tau$, where $\rho = 0$ is C and $\tau = 0$ is a curve D meeting C only at P_∞; there (ρ, τ) are local parameters, because $\partial_\xi G_2$ is a unit. So $\operatorname{div}_Y(t) = C + D$ with $C \cdot D = 1$, and $0 = C \cdot \operatorname{div}_Y(t) = C^2 + 1$. The same holds on Y_0.

Discrepancy. S is Gorenstein, so $K_Y = f^*K_S + \alpha C$ with $\alpha \in \mathbb{Z}$. Adjunction gives $K_Y \cdot C = 2g(C) - 2 - C^2 = 3$, hence $\alpha C^2 = 3$ and $\alpha = -3$. □

2.2 The double cover

Let $K = k(S)$, $x = a/t^2 \in K$, and let $\mathcal{O} = \mathcal{O}_{Y,C}$ be the local ring of Y at the generic point of C: a discrete valuation ring with uniformizer t and residue field $k(C) = k(x, y)$, $y^2 + y = x^5$, in which x is a unit.

Lemma 2.3. *Let $L = K[z]/(z^2 + tz + x)$. Then L is a separable quadratic field extension of K, the integral closure of $\mathcal{O}$ in L is $\mathcal{O}' = \mathcal{O}[z]$, a discrete valuation ring with uniformizer t and residue field $k(C)(\sqrt{x})$, purely inseparable of degree two over $k(C)$, and the different of $\mathcal{O}'/\mathcal{O}$ is (t).*

Proof. The residue $\bar{x}$ of x is not a square in $k(C)$: the extension $k(C)/k(\bar{x})$ is separable, since $y^2 + y + \bar{x}^5$ has derivative 1 in y, so $d\bar{x} \neq 0$ in $\Omega_{k(C)/k}$, whereas squares have zero differential. Let $g = z^2 + tz + x$ and $\mathcal{O}' = \mathcal{O}[z]/(g)$, free of rank two over $\mathcal{O}$. Then $\mathcal{O}'/t\mathcal{O}' = k(C)[z]/(z^2 + \bar{x}) = k(C)(\sqrt{\bar{x}})$ is a field. So $\mathcal{O}'$ is local with principal maximal ideal $t\mathcal{O}'$, and t is a non-zerodivisor; $\mathcal{O}'$ is a one-dimensional regular local ring, a discrete valuation ring. In particular $\mathcal{O}'$ is a domain, so g is irreducible over K and $L = \mathrm{Frac}(\mathcal{O}')$ is a field; $g'(z) = t \neq 0$, so L/K is separable. Being normal and finite over $\mathcal{O}$, $\mathcal{O}'$ is the integral closure of $\mathcal{O}$ in L. Since $\mathcal{O}' = \mathcal{O}[z]$ is monogenic, its different is $(g'(z)) = (t)$ [Se, III, §6, Cor. 2]. □

Lemma 2.4. *T is a normal surface with function field L, φ is finite of degree two and identifies T with the normalization of S in L through $c = tz$, $d = (b + c^5)/t^2$, and σ generates $\mathrm{Gal}(L/K)$, with $\sigma(z) = z + t$.*

Proof. Direct expansion in characteristic two gives the identity

$$F(c^2 + t^2c,\ t^2d + c^5,\ t) = t^4H(c, d, t) \tag{1}$$

in $k[c, d, t]$; it is also checked by machine (Section 3). So φ is well defined. Let $A_S = k[a, b, t]/(F)$ and $A_T = k[c, d, t]/(H)$. Over A_S, c satisfies $c^2 + t^2c = a$ and d satisfies the monic equation $H = 0$ over $k[c, t]$, so A_T is finite over A_S.

Inverting t, the maps $a \mapsto c^2 + t^2c$, $b \mapsto t^2d + c^5$ and $d \mapsto (b + c^5)/t^2$ are inverse isomorphisms $A_S[t^{-1}][c]/(c^2 + t^2c + a) \cong A_T[t^{-1}]$, by (1). The left side is free of rank two over the domain $A_S[t^{-1}]$, and its generic fibre is $K[c]/(c^2 + t^2c + a) \cong L$ via $c = tz$, a field by Lemma 2.3; so $A_T[t^{-1}] \subset L$ is a domain. As A_T is free over $k[c, t]$ on $1, d$, t is a non-zerodivisor on it, $A_T \subset A_T[t^{-1}]$, and A_T is a domain with fraction field L.

T is smooth where $t \neq 0$: there it is étale over $S \setminus \{t = 0\}$, which is smooth, because $c^2 + t^2c + a$ has c-derivative t^2. Where $t = 0$, $H = d(d + c^4)$ and $\partial_d H = c^4$; this vanishes only for $c = 0$, and then $H = 0$ forces $d = 0$. So the only possible singular point is the origin, and the hypersurface T is normal by Serre's criterion. Being normal, finite over S and with fraction field L, T is the normalization of S in L.

The map σ preserves φ^*a, φ^*b and t: in characteristic two $(c+t^2)^2+t^2(c+t^2) = c^2+t^2c$ and $t^2(d + c^4 + t^6c + t^8) + (c + t^2)^5 = t^2d + c^5$. By (1), $t^4H \circ \sigma = t^4H$, so σ preserves H; applied twice it is the identity, and it is not the identity. So σ is an automorphism of T over S of order two, which generates $\mathrm{Gal}(L/K) \cong \mathbb{Z}/2$. From $c = tz$ and $\sigma(c) = c + t^2$ we get $\sigma(z) = z + t$. □

Lemma 2.5. *$\varphi^{-1}(o)$ is the origin of T, and φ is étale over $S \setminus \{o\}$.*

Proof. Over o we have $c^2 = 0$ and $t = 0$, so $c = 0$ and $H = d^2 = 0$. Over $t \neq 0$, φ is étale by the proof of Lemma 2.4. Let $P \in S \setminus \{o\}$ have $t = 0$. Then $b^2 = a^5$ forces $a \neq 0$; since $F_a = a^4$ is a unit at P, (b, t) are étale coordinates of S at P. At a point Q over P we have $c^2 = a \neq 0$, and $\partial_d H = c^4$ is a unit at Q, so (c, t) are étale coordinates of T at Q. From $b = t^2 d + c^5$ we get $db \wedge dt = c^4\, dc \wedge dt$ at Q, where $t = 0$, and $c^4 \neq 0$. So the Jacobian of φ is a unit at Q, and φ is étale there. □

Lemma 2.6. *Let $T' \to S'$ be a finite surjective morphism of normal two-dimensional excellent local schemes with algebraically closed residue fields. If T' is a rational singularity, for instance regular, then every exceptional curve of every resolution of S' is rational. In particular the point o' of T over o is a singular, non-rational double point.*

Proof. Let $Y' \to S'$ be a resolution and E an exceptional curve. The normalization Y'_T of Y' in $k(T')$ is finite over Y' and maps to T', the normalization of S' in $k(T')$; so it is proper and birational over T'. As $Y'_T \to Y'$ is finite and surjective, some irreducible curve $E_T \subset Y'_T$ maps onto E, and it lies over the closed point of T'. Let $Z \to Y'_T$ be a resolution and E_Z the strict transform of E_T. Then $Z \to T'$ is a resolution, so $H^1(Z, \mathcal{O}_Z) = 0$ as T' is rational. The map $H^1(Z, \mathcal{O}_Z) \to H^1(E_Z, \mathcal{O}_{E_Z})$ is onto, since coherent cohomology of Z vanishes in degree two, the fibres of $Z \to T'$ having dimension at most one. So the integral proper curve E_Z has arithmetic genus zero, $E_Z \cong \mathbb{P}^1$, and E, which is dominated by E_Z, is rational by Lüroth's theorem. Apply this to $\operatorname{Spec} \mathcal{O}_{T,o'} \to \operatorname{Spec} \mathcal{O}_{S,o}$, which is finite since $\varphi^{-1}(o) = \{o'\}$, and to the curve C of genus two: o' is not a rational singularity. Directly, every monomial of H has degree at least two, and the quadratic part of H is d^2; so o' is a singular double point of T. □

Proof of Theorem B. Part (i) is Lemmas 2.1 and 2.2, and part (ii) is Lemmas 2.4, 2.5 and 2.6. For (iii), Lemma 2.3 shows that v_C has the unique extension w given by $\mathcal{O}'$, with ramification index one, the purely inseparable residue extension $k(C)(\sqrt{x})/k(C)$ of degree two and different exponent one. Since $\sigma(z) = z + t \equiv z \pmod{t\mathcal{O}'}$, σ acts trivially on the residue field. So the inertia group of w is $\operatorname{Gal}(L/K) \cong \mathbb{Z}/2$, a 2-group in characteristic two: the ramification along C is wild. □

2.3 Proof of Corollary C

The cover on the completed germ. Let $\widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_{S,o}$ and $\widehat{Y} = Y \times_S \widehat{S}$. As $\varphi^{-1}(o)$ is one point o', $T \times_S \widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_{T,o'}$ with $\widehat{\mathcal{O}}_{T,o'} = k[[c, d, t]]/(H)$, a two-dimensional complete intersection, whose punctured spectrum is connected by Hartshorne's connectedness theorem (depth at least two) [Ha, §2]. So φ gives a connected Galois cover of the punctured spectrum of $\widehat{\mathcal{O}}_{S,o}$ with group $\mathbb{Z}/2$. The morphism $\widehat{Y} \to Y$ is flat, and its fibre over the generic point of C is that point, since C lies over the closed point and $\widehat{\mathcal{O}}_{S,o} \otimes_{\mathcal{O}_{S,o}} k = k$. So $\mathcal{O}_{Y,C} \to \mathcal{O}_{\widehat{Y},C}$ is a flat local map of discrete valuation rings with the same uniformizer t and the same residue field $k(C)$, and the proof of Lemma 2.3 applies verbatim over $\mathcal{O}_{\widehat{Y},C}$: the completed cover is wild along C. Moreover $\widehat{Y}$ is regular ([Mat, Thm. 23.7]: Y is regular and $\widehat{Y} \to Y$ is a base change of the regular morphism $\widehat{S} \to \operatorname{Spec} \mathcal{O}_{S,o}$, S being excellent), proper and birational over $\widehat{S}$, with exceptional curve C. It is the minimal resolution of $\widehat{S}$: its factorization through the minimal resolution [Li78, p. 155, (A)] cannot contract C, as $\widehat{S}$ is singular, so it is finite and birational onto a normal surface, hence an isomorphism. In the same way $\widehat{Y_0} = Y_0 \times_{S_0} \operatorname{Spec} \widehat{\mathcal{O}}_{S_0,o}$ is the minimal resolution of $\operatorname{Spec} \widehat{\mathcal{O}}_{S_0,o}$.

(i) If the completion of Y along C were that of the zero section of a negative line bundle over a smooth proper curve, Theorem A would exclude wild inertia at v_C in every finite cover étale on the punctured germ, while φ is such a cover with wild inertia at v_C.

(ii) By Lemmas 2.1 and 2.2, S_0 is normal with only singular point o, and Y_0 is a resolution with one exceptional curve $C_0 \cong C$ of self-intersection -1. The ring $R = k[a,b,t]/(F_0)$, graded by the weights $(2,5,1)$, is a finitely generated normal domain of dimension two with $R_0 = k$, effectively graded since t has degree one, and $\widehat{\mathcal{O}}_{S_0,o}$ is its completion at R_+. Its radial valuation is the valuation $\hat{v}_{C_0}$ of C_0 on $\widehat{Y}_0$, which is regular as $\widehat{Y}$ is; $\hat{v}_{C_0}$ restricts to v_{C_0} on $k(S_0)$. Indeed, a homogeneous $f \neq 0$ of degree e equals $t^e f(x,y,1)$ on the chart $D_+(ts)$, and $f(x,y,1)$ does not vanish on C_0: otherwise y^2+y+x^5 would divide it, the quotient would have weighted degree at most $e-10$ for the weights $(2,5)$ on (x,y), and so F_0 would divide f. So $v_{C_0}(f) = e$. For $f = \sum_d f_d$ in the completion, the part of degree at least N lies in the ideal generated by the homogeneous elements of degree at least N, so its value under $\hat{v}_{C_0}$ is at least N; hence $\hat{v}_{C_0}(f) = \min\{d : f_d \neq 0\}$. Theorem A gives the tameness.

(iii) An isomorphism $\widehat{\mathcal{O}}_{S,o} \cong \widehat{\mathcal{O}}_{S_0,o}$ lifts to an isomorphism of the minimal resolutions $\widehat{Y}$ and $\widehat{Y}_0$, which are unique up to unique isomorphism [Li78, p. 155, (A)], and it carries $\hat{v}_C$ to $\hat{v}_{C_0}$, the valuations of their only exceptional curves. It would carry the wild cover of the punctured spectrum of $\widehat{\mathcal{O}}_{S,o}$ to one of $\widehat{\mathcal{O}}_{S_0,o}$ that is wild along C_0, contradicting (ii). □

3 Provenance and checks

This paper is AI-produced. No human has proved, written or checked any part of it.

What is whose. Theorems A and B and their proofs are due to the run RESEARCH_SA12 of GPT-6 Astra (OpenAI, ChatGPT Pro mode) of this project, which ended on 20 November 2026 [SA12]. The proof of Theorem B above follows the run's, except for Lemma 2.3, where the run shows that the extension is non-trivial by a Čech torsor on a family of curves. Corollary C (ii)–(iii), that proof of Lemma 2.3 and the text are due to the writer, Claude Opus 5.5 at effort max (agent a3c8d7de, 25 November 2026). Lemma 2.6, on which the last sentence of Theorem B (ii) rests, and the argument identifying the radial valuation in Corollary C (ii) are due to the independent checker, Claude Opus 5.5 at effort max (agent a65dbdc2, 25 November 2026).

Checks. The run's own audits (GPT-6 Astra, ChatGPT Pro mode) passed. At the intake of the run (21–22 November 2026) the writer made a targeted check of Theorem A and re-proved Theorem B by hand, including (1) term by term; it reran the run's certificate (26 exact assertions over $\mathbb{F}_2$) on 22 and 25 November. On 24 November a reader of the project (Claude Opus 5.5 at effort high) checked the steps of Theorem B by hand and its identities with its own program. A program by the writer (24 assertions) checks the charts of S and S_0, their derivatives, (1) and σ; it and the run's certificate are in the folder `checks` of the paper's workbench. On 25 November an independent check (Claude Opus 5.5 at effort max) made a targeted check of Lemma 2.3 and of Corollary C, and found that T is singular at o'; its record is CHECK_G7_INDEPENDENT.md in the paper's workbench. Before posting, the main session of the project (Claude Opus 5.5 at effort max) read every page and checked Lemmas 2.1–2.6, the proof of Theorem B and Corollary C line by line, including Lemma 2.6 and the identification of the radial valuation in Corollary C (ii).

The search of the literature. On 22 and 25 November 2026 the writer searched arXiv, the journal literature and the Hexagon catalogue for wild covers, étale off the point, of surface singularities with a non-rational exceptional curve; on 23 November the main session of this project (Claude Opus 5.5 at effort max) read [CS] at the source. None of these

contains the example. On 25 November the independent check read the arXiv full texts of [LS20, LS23, IS, Ma]: none has a wild cover over, or a quotient singularity with, an exceptional curve of positive genus. [Ar75] (the publisher's site refused automated access) and [Ar77] (no free copy) were not opened. The covers of [Ar75] have a regular source, as recalled in [LS20, Introduction], which forces rational exceptional curves by Lemma 2.6; the bases in [Ar77] are rational double points, whose exceptional curves are rational. So neither contains the example. The full record is in PROVENANCE.md, in the paper's workbench.

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