THE DUAL SMALE MEAN VALUE CONJECTURE FOR DEGREE NINE: INVERSION GRACE–WALSH–SZEGO, RECIPROCAL POLAR BOXES, AND FORMAL LEAN 4 / OUTWARD-ROUNDED VERIFICATION

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ABSTRACT. Let $P(z) = z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C}[z]$ be a complex polynomial of degree $n \geq 2$ with $a_n \neq 0$. The Dual Smale Mean Value Conjecture (Dubinin–Sugawa and Ng, 2009) asserts that

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{n},$$

with equality if and only if $P(z) = \frac{1-(1-cz)^n}{nc}$ for some $c \in \mathbb{C} \setminus \{0\}$. Previously established for $n \leq 7$ (Hinkkanen–Kayumov–Khammatova, 2025), odd polynomials of all degrees (Tang, 2025), and unconditionally for $n = 8$ (Besevic, 2026), the conjecture remained open for all general degrees $n \geq 9$. Degree $n = 9$ introduces a new qualitative obstruction absent at $n = 8$: the single-variable Grace–Walsh–Szego (GWS) envelope $9|H_9(a, v)|$ drops below 1 at $a = 0$ (since $H_9(0, v) = G_8(v)$ has $\min_{v\in\overline{\mathbb{D}}} 9|G_8(v)| \approx 0.7794 < 1$), so $P(1)$ alone can no longer separate small inverse critical points w_j from the pole $w_j = 0$ of $P(\zeta_j)/\zeta_j$.

In this paper we prove the Dual Smale Mean Value Conjecture unconditionally for all complex polynomials of degree $n = 9$, together with strict uniqueness of the extremal family. Our proof introduces three key ingredients: (i) a *2-Block Inversion Grace–Walsh–Szego Theorem* with an exact single-variable factorization in $u = az_2$, eliminating $|w_j| \leq 0.160$ globally in 21 milliseconds without singularity at $w_j = 0$; (ii) an exact reciprocal polar-box formulation of the secondary critical-point quotients $280 \cdot 9\, P(\zeta_j)/\zeta_j = \sum_{r=0}^{6} e_r(w_{-j}) B_r(z_j)$ in $z_j = 1/w_j$ (eliminating $1/w_j^7$ denominator blowup) paired with exact dyadic simplex convex-combination pruning; and (iii) a two-layer audited proof consisting of a Lean 4.36.0 formalization (2,000 lines, 28 theorems, zero `sorry`, zero `native_decide`, standard axioms only) and a self-contained outward-rounded IEEE-754 Rust certifier (1,927 lines, exact integer coefficients $[1260, -420, 210, -126, 84, -60, 45, -35]/280$) that exhausts 100% of $\overline{\mathbb{D}}^7$ in 193.53 seconds on 10 CPU cores.

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1. INTRODUCTION

In 1981, Smale [8] posed the mean value conjecture asking whether every complex polynomial $P(z) = z + a_2 z^2 + \cdots + a_n z^n$ admits a critical point ζ with $|P(\zeta)/\zeta| \leq 1 - 1/n$ (or $\leq K$ for a universal constant). In 2026, the primal conjecture was disproved with sharp global supremum $K^* = 4$ [1, 4].

The *Dual Smale Mean Value Conjecture*, formulated independently by Dubinin–Sugawa [3] and Ng [7], asks for the sharp lower bound on the maximum over all critical points. Our main result resolves the first open general degree $n = 9$:

Theorem 1.1 (Main Theorem — Dual Smale Mean Value Conjecture for $n = 9$)**.** *Let $P(z) = z + a_2 z^2 + \cdots + a_9 z^9 \in \mathbb{C}[z]$ be any complex polynomial of degree 9 with $a_9 \neq 0$. Then*

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{9}. \tag{1}$$

Moreover, equality holds in (1) *if and only if $P(z) = \frac{1-(1-cz)^9}{9c}$ for some $c \in \mathbb{C} \setminus \{0\}$.*

Prior to this work, the dual conjecture was established for $n \leq 7$ by Hinkkanen, Kayumov, and Khammatova [5], for odd polynomials of arbitrary degree by Tang [9], unconditionally up

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to the general lower bound $(n-1/2)^{1/n}/n^2$ by Jatar and Ng [6], and unconditionally for all complex polynomials of degree $n=8$ by the author [2].

1.1. **New Structural Obstructions at Degree $n=9$.** Normalize P by scaling $z\mapsto\zeta_1 z$ so that a critical point of minimal modulus lies at $\zeta_1=1$. Writing the remaining 7 critical points as $\zeta_k=1/w_k$ $(k=2,\dots,8)$, every w_k lies in the closed unit disk $\overline{\mathbb{D}}=\{w\in\mathbb{C}:|w|\le 1\}$, and

$$P'(z)=(1-z)\prod_{k=2}^{8}(1-w_k z).$$

Integrating from 0 to 1 yields the 7-variable symmetric multiaffine form

$$P(1)=\int_0^1(1-u)\prod_{k=2}^{8}(1-uw_k)\,du=\sum_{m=0}^{7}\frac{(-1)^m}{(m+1)(m+2)}\,e_m(w_2,\dots,w_8), \tag{2}$$

with diagonal polynomial $G_9(w)=P(1)\big|_{(w,\dots,w)}$ and single-variable GWS envelope $H_9(a,v)=P(1)\big|_{(a,v,\dots,v)}$. Passing from $n=8$ to $n=9$ creates two qualitative barriers that break the degree-8 reduction directly:

(1) **Failure of single-variable H_9 at the origin.** At degree $n=8$, $8|H_8(a,v)|>1$ holds for every $a\in\overline{\mathbb{D}}$ outside two narrow crescent regions $S_8^{\pm}\subset\{|w|\ge 0.67\}$; in particular $H_8(0,v)=G_7(v)$ satisfies $8|G_7(v)|\ge 8/7>1$, so $a=0$ and the entire inner disk $|w|\le 0.67$ were eliminated by $P(1)$ alone. At degree $n=9$, however,

$$H_9(0,v)=G_8(v),\qquad \min_{v\in\overline{\mathbb{D}}}9|H_9(0,v)|=\frac{9}{8}\min_{v\in\overline{\mathbb{D}}}8|G_8(v)|\approx 0.77942<1.$$

Hence $P(1)$ alone *cannot* exclude $w_j=0$ (nor $|w_j|$ arbitrarily small)! Because the secondary critical-point quotient $P(\zeta_j)/\zeta_j$ has a pole of order 7 as $w_j\to 0$, eliminating a uniform disk $|w_j|\le R_{\mathrm{inv}}$ around 0 requires a genuinely new global argument coupling $P(1)$ with $P(\zeta_j)/\zeta_j$.

(2) **Near-zero dip of G_9 and 14-dimensional scaling.** The polynomial $G_9(w)$ has a complex conjugate pair of roots just outside $\overline{\mathbb{D}}$ at $|w|\approx 1.00166$, $\arg(w)\approx\pm 79.10^\circ$, causing $9|G_9(w)|$ to dip down to ≈ 0.017305 (compared to 0.6928 at $n=8$) across a broad lune $|w|\in[0.839,1]$, $\arg(w)\in\pm[62.0^\circ,85.25^\circ]$. Moreover, the remaining domain lives in $\mathbb{C}^7\cong\mathbb{R}^{14}$ rather than $\mathbb{C}^6\cong\mathbb{R}^{12}$.

We overcome both obstructions cleanly:

- **2-Block Inversion Grace–Walsh–Szegő (Stage 1d).** Whenever $9|P(1)|\le 1$, Stage 1b (two-block GWS on $P(1)$) forces the six largest moduli to satisfy $|w_{(2)}|\ge R_2=0.350$ and $|w_{(3)}|,\dots,|w_{(7)}|\ge R_3=0.590$. Inverting the six other variables $z_k=1/w_k$ transforms $a^7(\prod z_k)\cdot 9P(\zeta_a)/\zeta_a$ into a symmetric multiaffine form on the closed disk $|z_k|\le 1/R_3$. Applying GWS collapses the 5-tuple to a single z_2 and factors the entire expression in the single variable $u=az_2$ (Theorem 3.2), excluding $|w_k|\le 0.160$ globally in 21 ms.
- **Exact reciprocal polar-box Q_j evaluation.** Writing $z_j=1/w_j$, the map $w_j\mapsto 1/w_j$ sends any polar sector $[r_0,r_1]\times[\theta_0,\theta_1]$ $(r_0>0)$ to the exact polar sector $[1/r_1,1/r_0]\times[-\theta_1,-\theta_0]$ with zero geometric overapproximation, turning $280\cdot 9P(\zeta_j)/\zeta_j$ into a pure polynomial in (w_{-j},z_j) with no w_j^7 denominator blowup.

1.2. **Two-Layer Audited Architecture.** Every algebraic identity, coincidence theorem, local-ball estimate, and pruning/splitting soundness theorem is formally verified in Lean 4.36.0 (`lean/DualSmale/`, 2,000 lines, 28 theorems, zero `sorry`, zero `native_decide`, axioms strictly `[propext, Classical.choice, Quot.sound]`). Every finite domain exhaustion is certified by a standalone outward-rounded IEEE-754 Rust certifier (`verifier/src/`, 1,927 lines, zero compiler warnings, zero quadrature floats) executing in 193.53 seconds on 10 CPU cores.

2. Exact Integer Identities at Degree Nine

Since $\operatorname{lcm}(2,6,12,20,30,42,56,72) = 2520 = 280 \cdot 9$, multiplying by 2520 clears all denominators into small exact integers representable without rounding error in IEEE-754 binary64:

Proposition 2.1 (Exact Integer-Coefficient Identities at Degree $n = 9$)**.** *For all $w = (w_2, \dots, w_8) \in \mathbb{C}^7$, $a, v \in \mathbb{C}$, and any $j \in \{2, \dots, 8\}$ with $w_j \neq 0$, letting $z_j = 1/w_j$ and letting $w_{-j} \in \mathbb{C}^6$ denote the remaining six coordinates:*

$$\begin{aligned} 2520\,P(1) &= 1260 - 420e_1(w) + 210e_2(w) - 126e_3(w) + 84e_4(w) \\ &\quad - 60e_5(w) + 45e_6(w) - 35e_7(w), \end{aligned} \tag{3}$$

$$\begin{aligned} 2520\,G_9(v) &= 1260 - 2940v + 4410v^2 - 4410v^3 + 2940v^4 \\ &\quad - 1260v^5 + 315v^6 - 35v^7, \end{aligned} \tag{4}$$

$$\begin{aligned} 2520\,H_9(a,v) &:= 2520\,P(1)\big|_{(v,v,v,v,v,v,a)} \\ &= (1260 - 2520v + 3150v^2 - 2520v^3 + 1260v^4 - 360v^5 + 45v^6) \\ &\quad + a(-420 + 1260v - 1890v^2 + 1680v^3 - 900v^4 + 270v^5 - 35v^6), \end{aligned} \tag{5}$$

$$\begin{aligned} 2520\,\frac{P(\zeta_j)}{\zeta_j} &= (1260 - 420z_j) + e_1(w_{-j})(-420z_j + 210z_j^2) + e_2(w_{-j})(210z_j^2 - 126z_j^3) \\ &\quad + e_3(w_{-j})(-126z_j^3 + 84z_j^4) + e_4(w_{-j})(84z_j^4 - 60z_j^5) \\ &\quad + e_5(w_{-j})(-60z_j^5 + 45z_j^6) + e_6(w_{-j})(45z_j^6 - 35z_j^7). \end{aligned} \tag{6}$$

Proof. All four identities are verified algebraically in Lean 4 (`exact_p1_deg9_integer_identity`, `exact_G9_integer_identity`, `exact_H9_integer_identity`, and `exact_qj_deg9_zpoly_identity` in `DualSmale/PolynomialReduction.lean`). □

Thus $9|P(1)| > 1 \iff |2520\,P(1)| > 280$, and $9|P(\zeta_j)/\zeta_j| > 1 \iff |2520\,P(\zeta_j)/\zeta_j| > 280$.

3. 2-Block Inversion Grace–Walsh–Szego Theorem

To eliminate the neighborhood of $w_j = 0$ without losing uniformity at the pole $z_j = 1/w_j \to \infty$, we establish an inversion form of the Grace–Walsh–Szego coincidence theorem.

Lemma 3.1 (Reciprocal Elementary Symmetric Symmetry)**.** *For any $m \geq 0$, any $u_1, \dots, u_m \in \mathbb{C} \setminus \{0\}$, and any $0 \leq r \leq m$,*

$$e_r(u_1, \dots, u_m) \prod_{k=1}^{m} u_k^{-1} = e_{m-r}(u_1^{-1}, \dots, u_m^{-1}).$$

Proof. Formalized by induction on m in `DualSmale/InversionGWS.lean` (`elemSym_mul_prod_inv`). □

Define the three exact integer degree-5 polynomials

$$\begin{aligned} U_0(u) &:= -60 + 420u - 1260u^2 + 2100u^3 - 2100u^4 + 1260u^5, \\ U_1(u) &:= 45 - 300u + 840u^2 - 1260u^3 + 1050u^4 - 420u^5, \\ U_2(u) &:= -35 + 225u - 600u^2 + 840u^3 - 630u^4 + 210u^5. \end{aligned}$$

Theorem 3.2 (2-Block Inversion Grace–Walsh–Szego and $u = az_2$ Factorization)**.** *Let $r_{\mathrm{inv}} \geq 0$, $R_2 > 0$, $R_3 > 0$. Suppose $a \in \mathbb{C} \setminus \{0\}$ satisfies $|a| \leq r_{\mathrm{inv}}$, $w_1 \in \mathbb{C}$ satisfies $|w_1| \geq R_2$, and $w_{\mathrm{rest}} = (w_2, \dots, w_6) \in \mathbb{C}^5$ satisfies $|w_k| \geq R_3$ for all $k \in \{2, \dots, 6\}$. Setting $\zeta_a = 1/a$, there exists a single complex number u in the closed disk $|u| \leq r_{\mathrm{inv}}/R_3$ such that*

$$|a|^7 \cdot \left|2520\,\frac{P(\zeta_a)}{\zeta_a}\right| \geq \left(|w_1| \prod_{k=2}^{6} |w_k|\right) \left(|U_2(u)| - r_{\mathrm{inv}}\left(1 + \frac{1}{R_2}\right)|U_1(u)| - \frac{r_{\mathrm{inv}}^2}{R_2}|U_0(u)|\right). \tag{7}$$

Proof outline (formalized in `DualSmale/InversionGWS.lean`). Multiplying (6) (with $w_j = a$ and $w_{-j} = (w_1, w_2, \ldots, w_6)$) by $a^7 w_1^{-1} \prod_{k=2}^{6} w_k^{-1}$ and applying Lemma 3.1 produces a 5-variable symmetric multiaffine polynomial $F_{\mathrm{inv}}(a, z_1; z_2, \ldots, z_6)$ in the reciprocals $z_k = w_k^{-1}$ (`inv_gws_multiaffine_repr`), where $|z_1| \le 1/R_2$ and $|z_k| \le 1/R_3$ for $k = 2, \ldots, 6$. By the scaled Grace–Walsh–Szego coincidence theorem on the closed disk $|z| \le 1/R_3$ (`grace_walsh_szego_disk`), there exists a single z_* with $|z_*| \le 1/R_3$ such that $F_{\mathrm{inv}}(a, z_1; z_2, \ldots, z_6) = F_{\mathrm{inv}}(a, z_1; z_*, \ldots, z_*)$. Setting $u := az_*$ (so $|u| \le r_{\mathrm{inv}}/R_3$), direct algebraic expansion (`exact_inv_gws_u_factorization`) gives the exact identity

$$F_{\mathrm{inv}}(a, z_1; z_*, \ldots, z_*) = \big(U_2(u) + aU_1(u)\big) + z_1 a\big(U_1(u) + aU_0(u)\big).$$

Regrouping as $U_2(u) + a(1 + z_1)U_1(u) + z_1 a^2 U_0(u)$ and applying the reverse triangle inequality with $|a| \le r_{\mathrm{inv}}$ and $|z_1| \le 1/R_2$ yields (7) (`inversion_gws_2block_deg9`). □

Remark 3.3. Because $U_2(0) = -35 \neq 0$, the right-hand side of (7) stays strictly bounded away from zero throughout $|u| \le r_{\mathrm{inv}}/R_3$, whereas the required threshold $280\, r_{\mathrm{inv}}^7 / \prod_{k=2}^{7} R_k$ is of order 10^{-2} at $r_{\mathrm{inv}} = 0.160$. Even a tiny two-dimensional branch-and-bound over u certifies (7) with a factor of over $60\times$ safety margin in 21 milliseconds.

4. Stage 0 Quantitative Local Ball and Strict Uniqueness

Around the extremal point $(a, v) = (1, 1)$, we prove an explicit quantitative lower bound for $H_9(a, v)$ in Lean 4, complementing the universal all-n second-order quadratic form theorem (`universal_local_quadratic_lower_bound`).

Theorem 4.1 (Stage 0 Quantitative Local Ball for H_9)**.** *For all $(a, v) \in \overline{\mathbb{D}}^2 \setminus \{(1, 1)\}$ satisfying $|a - 1| \le 1/100$ and $|v - 1| \le 1/100$, we have strictly*

$$9\,|H_9(a, v)| > 1.$$

Proof outline (formalized in `DualSmale/LocalMinimum.lean`). Writing $a = 1 + d_1$, $v = 1 + d_2$ with $d_k = x_k + iy_k$ and $2x_k + x_k^2 + y_k^2 \le 0$ (since $|1 + d_k| \le 1$), exact expansion of (5) gives $9H_9(1 + d_1, 1 + d_2) = Q + R$, where

$$Q = 1 - \frac{1}{8}d_1 - \frac{3}{4}d_2 + \frac{3}{14}d_1 d_2 + \frac{15}{28}d_2^2,$$

and every term of R carries a factor of d_2^2 times at least one additional factor of d_1 or d_2, with sum of absolute coefficients equal to $15/8 < 2$. Hence $|R| \le \frac{2}{100}|d_2|^2$. Meanwhile, separating $\mathrm{Re}(Q) = 1 + A + B_{\mathrm{re}}$ (with $B_{\mathrm{re}} \ge 0$ since $x_1, x_2 \le 0$) and $\mathrm{Im}(Q) = C_{\mathrm{im}} + B_{\mathrm{im}}$ yields the exact positive-definite quadratic lower bound

$$2A + C_{\mathrm{im}}^2 \ge \frac{1}{8}x_1^2 + \frac{3}{4}x_2^2 + \frac{9}{448}y_1^2 + \frac{27}{224}y_2^2 + \frac{27}{224}(y_1 - y_2)^2,$$

whose coefficients strictly dominate both the cross term $|2C_{\mathrm{im}}B_{\mathrm{im}}| \le \frac{1}{500}y_1^2 + \frac{1}{25}y_2^2$ and the remainder margin $(1 + \frac{2}{100}|d_2|^2)^2 - 1$. Thus $|Q| > 1 + \frac{2}{100}|d_2|^2 \ge 1 + |R|$, giving $|Q + R| > 1$ (`stage0_H9_local_ball`). □

5. Reciprocal Polar Boxes and Soundness of Outward-Rounded Certification

The Rust certifier (`verifier/src/`) uses strict IEEE-754 outward rounding (`f64::next_down`/`f64::next_up`) on every elementary arithmetic operation and conservative 4-ULP outward bounds on sin/cos. Each coordinate $w_k = re^{i\theta}$ in a polar sector $[r_0, r_1] \times [\theta_0, \theta_1]$ is enclosed inside a rotated rectangle (`PBox`) with exact `f64` center c_k, exact rotation vector rot_k, and rigorous half-widths $h_{u,k}, h_{v,k}$ and radius rad_k.

5.1. **Reciprocal Polar-Box Evaluation of Q_j.** For any polar box $B_j = [r_0, r_1] \times [\theta_0, \theta_1]$ with $r_0 > 0$, the image of B_j under $w_j \mapsto z_j = 1/w_j$ is contained in the outward-rounded reciprocal polar box

$$B_j^{-1} := \big[\mathrm{down}(1/r_1),\, \mathrm{up}(1/r_0)\big] \times [-\theta_1, -\theta_0],$$

verified by `polar_reciprocal_norm_bounds`. Expanding (6) as a pure polynomial in (w_{-j}, z_j) around (c_{-j}, z_c) eliminates the w_j^7 denominator entirely, yielding an ultra-tight standalone lower bound in (w_{-j}, z_j) coordinates. For convex combinations with $P(1)$ and other Q_ℓ in the shared coordinates $w_k - c_k$, we substitute the exact algebraic identity (`reciprocal_linearization_identity`):

$$\frac{1}{w_j} - z_c = \left(\frac{1}{c_j} - z_c\right) - \frac{w_j - c_j}{c_j^2} + \frac{(w_j - c_j)^2}{c_j^2 w_j},$$

whose rigorous real-part lower bound is formalized in `reciprocal_linearization_lower_bound`.

5.2. **Rigorous Complex Ball Folding and Simplex Convex-Combination Pruning.** To accelerate the 2^7-mask polynomial center fold of $P(1)$ and the $2^6 \times 8$ fold of Q_j without losing soundness, `bounds.rs` uses rigorous midpoint-radius complex balls `CBall` $\{z \in \mathbb{C} : |z - (\mathrm{re} + i\,\mathrm{im})| \le \mathrm{err}\}$ with strict outward error accumulation ($\varepsilon_{\mathrm{rad}} = 4.5 \times 10^{-16} > 4 \cdot 2^{-53}$, $\delta_{\mathrm{tiny}} = 10^{-300} \gg 6 \cdot 2^{-1074}$).

When neither $P(1)$ nor a single Q_j exceeds 280 uniformly across a box, `optimal_combo_lb_rigorous` finds heuristic simplex weights via 16 steps of mirror ascent, quantizes them into *exact* dyadic weights $\omega_i = k_i/256$ ($k_i \in \mathbb{Z}_{\ge 0}$, $\sum k_i = 256$, exact in IEEE-754 binary64), and rigorously evaluates the combined linear bound. By `weighted_combo_re_le_max`,

$$\sum_i \omega_i \,\mathrm{Re}(d_i F_i(w)) \le \max_i |F_i(w)|$$

whenever $|d_i| \le 1$, so exceeding 280 rigorously prunes the box. Every binary split of a repeated polar box of multiplicity m partitions the group into $n_{\mathrm{left}} \in \{0, \dots, m\}$ copies on the left and $m - n_{\mathrm{left}}$ on the right; completeness up to permutation is certified by `sym_multiset_binary_split_cover`.

6. Complete Six-Stage Certification and Proof of Theorem 1.1

Table 1. Complete outward-rounded certification summary for degree $n = 9$ (10-core Apple Silicon run, exact integer coefficients /280, zero warnings).

Stage	Certified region / bound	Leaves / nodes	Time
Stage 0	$\max(\lvert a-1\rvert, \lvert v-1\rvert) \le 0.01 \Rightarrow 9\lvert H_9(a,v)\rvert > 1$	Lean 4 proof	—
Stage 1a	12 GWS safe disks for $9\lvert G_9(w)\rvert > 1$ (2D)	1,230	2.8 ms
Stage 1b	6 two-block GWS thresholds $R_2, \dots, R_7$ (4D)	1,773,354	3.28 s
Stage 1c	Wedge $[0.805, 1] \times [-12°, 12°]$ via H_9 (4D)	466,960	0.41 s
Stage 1d	Inversion GWS $\lvert w_k\rvert \le 0.160$ (2D in u)	27,377	21.0 ms
Stage 2	43 mixed half-plane seeds ($n_L \in \{1,2,3\}$, 14D)	59,927,355	64.99 s
Stage 3	6 same-half-plane seeds ($n_L = 0$, 14D)	45,060,411	124.83 s
Total	**100% of $\overline{\mathbb{D}}^7$ (strict uniqueness)**	**107,256,687**	**193.53 s**

Proof of Theorem 1.1. Normalize P so that a critical point of minimal modulus lies at $\zeta_1 = 1$, with remaining inverse critical points $w = (w_2, \dots, w_8) \in \overline{\mathbb{D}}^7$. Assume for contradiction (or equality characterization) that

$$\max\left(9|P(1)|,\ \max_{2 \le j \le 8,\ w_j \ne 0} 9\left|\frac{P(\zeta_j)}{\zeta_j}\right|\right) \le 1. \tag{8}$$

Step 1 (Modulus thresholds, Stage 1b). For each $k \in \{2,3,4,5,6,7\}$, Stage 1b certifies by 4D outward-rounded polar branch-and-bound on the two-block GWS form $F_k(v_1, v_2) =$

$2520\int_0^1(1-u)(1-uv_1)^k(1-uv_2)^{7-k}\,du$ that $|v_1|\le R_k$ and $|v_2|\le 1$ imply $|F_k(v_1,v_2)|>280$, where

$$(R_2,R_3,R_4,R_5,R_6,R_7)=(0.350,\ 0.590,\ 0.700,\ 0.764,\ 0.805,\ 0.834).$$

Sorting the moduli as $|w_{(1)}|\le|w_{(2)}|\le\cdots\le|w_{(7)}|$, we must therefore have $|w_{(k)}|>R_k$ for all $k\in\{2,\dots,7\}$. In particular, at most five coordinates satisfy $|w_k|\le 0.805$, and the six largest moduli satisfy $|w_{(2)}|>R_2=0.350$, $|w_{(3)}|,\dots,|w_{(7)}|>R_3=0.590$, and $\prod_{k=2}^7|w_{(k)}|>\prod_{k=2}^7 R_k$.

Step 2 (Global elimination of $|w_k|\le 0.160$, Stage 1d). Suppose $|w_{(1)}|\le 0.160$. If $w_{(1)}=0$, then P has degree ≤ 8, contradicting $a_9\neq 0$. Hence $0<|w_{(1)}|\le 0.160$. Applying Theorem 3.2 with $a=w_{(1)}$, $r_{\mathrm{inv}}=0.160$, $R_2=0.350$, $R_3=0.590$, and the 2D outward-rounded certification of Stage 1d (27,377 leaves in 21.0 ms) yields $|a|^7\cdot|2520\,P(\zeta_a)/\zeta_a|>280\,r_{\mathrm{inv}}^7\ge 280\,|a|^7$, so $9|P(\zeta_a)/\zeta_a|>1$, contradicting (8). Therefore every coordinate satisfies $|w_k|>0.160$.

Step 3 (Near-1 right wedge and strict uniqueness, Stage 0 + Stage 1c). Let $W_{\mathrm{near1}}:=[0.805,1]\times[-12^\circ,+12^\circ]$. Suppose some coordinate $w_k\in W_{\mathrm{near1}}$. Applying Grace–Walsh–Szegő (`grace_walsh_szego_unitDisk`) to the remaining six coordinates $w_{-k}\in\overline{\mathbb{D}}^6$ produces $v\in\overline{\mathbb{D}}$ such that $P(1)=H_9(w_k,v)$. By Theorem 4.1 (Stage 0) together with the 4D outward-rounded certification of Stage 1c (466,960 nodes in 0.41 s), we have $9|H_9(a,v)|>1$ for every $(a,v)\in(W_{\mathrm{near1}}\times\overline{\mathbb{D}})\setminus\{(1,1)\}$. Thus (8) forces $w_k=1$, $v=1$, and $P(1)=H_9(1,1)=1/9$.

To establish strict uniqueness, consider the symmetric multiaffine form $F_6(w_{-k}):=2520\,P(1)\big|_{w_k=1}-280$ on $\overline{\mathbb{D}}^6$, whose diagonal polynomial is $Q(z):=2520\,H_9(1,z)-280$. By Stage 0 and Stage 1c, $Q(z)$ has a unique root $z=1$ in $\overline{\mathbb{D}}$, and $Q'(1)=280\cdot(-3/4)=-210\neq 0$, so $Q(z)=-35(z-1)\prod_{i=1}^5(z-a_i)$ with every $|a_i|>1$. Suppose some $w_\ell\in\overline{\mathbb{D}}\setminus\{1\}$ ($\ell\neq k$). For the polar derivative $D_{w_\ell}Q(z)=6Q(z)+(w_\ell-z)Q'(z)$, we have $D_{w_\ell}Q(1)=(w_\ell-1)Q'(1)\neq 0$, and for every $z\in\overline{\mathbb{D}}\setminus\{1\}$ (and $z\neq w_\ell$, since $D_{w_\ell}Q(w_\ell)=6Q(w_\ell)\neq 0$), the Laguerre identity

$$2\,\mathrm{Re}\left((1-z\overline{w_\ell})\frac{a-w_\ell}{a-z}\right)=(1-|z|^2)\left|\frac{a-w_\ell}{a-z}\right|^2+(1-|w_\ell|^2)+(|a|^2-1)\frac{|z-w_\ell|^2}{|a-z|^2}$$

is ≥ 0 for the simple boundary root $a=1$ and strictly >0 for each exterior root $a_1,\dots,a_5$. Hence $D_{w_\ell}Q$ is zero-free on all of $\overline{\mathbb{D}}$. Applying Grace–Walsh–Szegő to the remaining five coordinates implies $F_6(w_{-k})\neq 0$, contradicting $P(1)=1/9$. Thus $w_2=\cdots=w_8=1$, giving $P(z)=\frac{1-(1-z)^9}{9}$ (and $\frac{1-(1-cz)^9}{9c}$ upon unscaling).

Step 4 (Exhaustion of the remaining domain, Stages 2 and 3). If no coordinate lies in W_{near1} and all satisfy $|w_k|>0.160$, then every w_k lies in the exact four-sector partition $U_{\mathrm{in}}\cup U_{\mathrm{out}}\cup L_{\mathrm{in}}\cup L_{\mathrm{out}}$, where $U_{\mathrm{in}}=[0.160,0.805]\times[0,\pi]$, $U_{\mathrm{out}}=[0.805,1]\times[12^\circ,\pi]$, and $L_{\mathrm{in}}=\overline{U_{\mathrm{in}}}$, $L_{\mathrm{out}}=\overline{U_{\mathrm{out}}}$. By complex conjugation symmetry we may assume at most 3 coordinates lie in the lower half-plane ($n_L\le 3$), and by Step 1 at most 5 lie in $U_{\mathrm{in}}\cup L_{\mathrm{in}}$. This leaves exactly 43 mixed seeds ($n_L\in\{1,2,3\}$, Stage 2) and 6 same-half-plane seeds ($n_L=0$, Stage 3). Both stages certify strictly $\max(9|P(1)|,\max_j 9|P(\zeta_j)/\zeta_j|)>1$ across all 104,987,766 leaves (Table 1), completing the proof. □

7. Reproducibility and Software Artifacts

All Lean 4 proofs (28 theorems, 2,000 lines) and the outward-rounded Rust certifier (1,927 lines) are publicly available at `https://github.com/ivan-fr/dual-smale-n9` and verified automatically by GitHub Actions CI:

```
# 1. Verify all 28 Lean 4.36.0 theorems and audit axioms
cd lean && lake exe cache get
lake build DualSmale && lake env lean check_axioms.lean

# 2. Run Rust unit tests and full outward-rounded degree-9 certification
cd verifier && cargo test --release && cargo run --release
```

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