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Multiplication by 2 and by 6 through products of Jacobians, and the Jacobian index of a very general abelian variety of dimension 4, 5 and 6*

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Abstract

For an abelian variety A over an algebraically closed field, let $s_J(A)$ be the least positive integer n such that multiplication by n factors through a product of Jacobians of curves; $s_J(A) = 1$ exactly when A is a direct summand of such a product. We prove, in every characteristic, that multiplication by 2 factors through a product of Jacobians on every principally polarized abelian variety of dimension at most 5, and multiplication by 6 on every one of dimension 6; that $s_J(A)$ divides $\gcd(2, \delta(A))$ for every abelian surface and $\gcd(6, \delta(A))$ for every abelian threefold, where $\delta(A)$ is the greatest common divisor of the Euler characteristics of line bundles on A. The known bound is $(2g-1)!$. For a complex principally polarized A with minimal class $\gamma = \theta^{g-1}/(g-1)!$, let $m(A)$ be the least positive multiple of γ that is the class of an algebraic cycle; Matsusaka's construction gives $s_J(A) \mid m(A)$, and we prove $m(A) \mid s_J(A)^2$. With the theorems of Engel, de Gaay Fortman and Schreieder this gives the exact values $s_J(A) = 2$ for a very general principally polarized abelian variety of dimension 4 and 5, and $s_J(A) = 6$ in dimension 6; and every prime $p \leq g/2$ divides $s_J(A)$ in dimension g. Finally, $s_J(A)$ is exactly the least integer n for which n times the inverse of the isomorphism $H^{2g-1}(A)(g-1) \to H^1(\widehat{A})$ is induced by an integral algebraic cycle; this answers, for abelian varieties, the second question that Grothendieck asks on sheet 70 of cote 14 of his Montpellier archive. The results come from a run of GPT-6 Astra (ChatGPT Pro mode) and from Claude Opus 5.5 at effort max; they were checked and the text written by Claude Opus 5.5 at effort max.

1 Results

Throughout, k is an algebraically closed field, curves are smooth, projective and connected, and a *product of Jacobians* is $J = J(C_1) \times \cdots \times J(C_r)$.

Definition 1.1. For an abelian variety A over k, let $I_J(A)$ be the set of $n \in \mathbb{Z}$ such that $[n]_A = \rho \circ \iota$ for some product of Jacobians J and homomorphisms $\iota\colon A \to J$, $\rho\colon J \to A$. It is an ideal (use $J \times J'$ for sums) and is non-zero, since A is an isogeny factor of a Jacobian; $s_J(A)$ denotes its positive generator. Let $\delta(A) = \gcd\{\chi(L) : L \in \mathrm{Pic}(A)\}$. For a principally polarized complex (A, θ) of dimension g, put $\gamma = \theta^{g-1}/(g-1)! \in H^{2g-2}(A, \mathbb{Z})$, let $M(A)$ be the set of n such that $n\gamma$ is the class of an integral algebraic 1-cycle, and let $m(A)$ be its positive generator; $(g-1)! \in M(A)$.

*AI-produced. Theorem A is due to a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode); Theorem B and parts (i)–(ii) of Corollary C to Claude Opus 5.5 (Anthropic) at effort max; the text, Theorem D and part (iii) of Corollary C to Claude Opus 5.5 at effort max; see Section 4. No human has checked this text. To the extent that any rights exist in it, they are dedicated to the public domain under CC0 1.0.

Theorem A. *Let k have any characteristic.*

(a) If B is a principally polarized abelian variety of dimension at most 5, then $2 \in I_J(B)$; if it has dimension 6, then $6 \in I_J(B)$.

(b) If A is an abelian surface, then $s_J(A) \mid \gcd(2, \delta(A))$. If A is an abelian threefold, then $s_J(A) \mid \gcd(6, \delta(A))$.

In particular an abelian surface with $\delta(A)$ odd, and an abelian threefold with $\gcd(\delta(A), 6) = 1$, is a direct summand of a product of Jacobians.

Theorem B. *For every complex principally polarized abelian variety A,*

$$s_J(A) \mid m(A) \mid s_J(A)^2.$$

Hence $\operatorname{rad} s_J(A) = \operatorname{rad} m(A)$, and $s_J(A) = m(A)$ whenever $m(A)$ is squarefree.

Corollary C. *Let A be a very general complex principally polarized abelian variety of dimension g.*

(i) If $g \in \{4, 5\}$, then $s_J(A) = 2$.

(ii) If $g = 6$, then $s_J(A) = 6$.

(iii) If $g \geq 4$, every prime $p \leq g/2$ divides $s_J(A)$, and $s_J(A) \mid m(A) \mid (g-1)!$.

For an abelian variety A of dimension g and a prime $\ell \neq \operatorname{char} k$, let

$$\psi_A \colon H^{2g-1}(A, \mathbb{Z}_\ell)(g-1) \xrightarrow{\sim} H^1(\widehat{A}, \mathbb{Z}_\ell)$$

be the canonical isomorphism: Poincaré duality $H^{2g-1}(A, \mathbb{Z}_\ell)(g) \cong T_\ell A$, followed by $T_\ell A(-1) \cong H^1(\widehat{A}, \mathbb{Z}_\ell)$ given by the Weil pairing (over $\mathbb{C}$ one may use Betti cohomology with $\mathbb{Z}$-coefficients). It is defined up to a universal sign, which plays no role below. An integral algebraic 1-cycle Z on $\widehat{A} \times A$ acts as a correspondence $Z_* \colon H^1(\widehat{A}, \mathbb{Z}_\ell) \to H^{2g-1}(A, \mathbb{Z}_\ell)(g-1)$.

Theorem D. *For $n \in \mathbb{Z}$ the following are equivalent: (i) $n\,\psi_A^{-1} = Z_*$ for an integral algebraic 1-cycle Z on $\widehat{A} \times A$; (ii) $n \in I_J(A)$. In particular $s_J(A)$ is the least positive integer n such that $n\,\psi_A^{-1}$ is induced by an integral algebraic cycle.*

Why it matters. On sheet 70 of cote 14 of Grothendieck's Montpellier archive (scan 71), after constructing rational inverses of the Lefschetz-type maps by 1-cycles (Prop. 2.7, scans 68–70), Grothendieck asks two questions. The second reads: is there always an *integral* cycle giving the inverse of $H^{2n-1}(X)(n-1) \to H^1(B_X)$, B_X the Picard variety of X? He notes that if the first question has a positive answer, it suffices to answer the second for abelian varieties. Theorem D identifies the obstruction for an abelian variety with $s_J(A)$; Theorem A and Corollary C compute it. The answer is positive for all abelian surfaces with δ odd, all threefolds with δ prime to 6 and all principally polarized ones of dimension at most 3, and negative for a very general principally polarized abelian variety of dimension at least 4, where it fails by exactly 2 in dimensions 4, 5 and by exactly 6 in dimension 6. By the criterion of Voisin [Vo, Thm. 1.4], as recalled in [EGFS, §1.2], $s_J = 1$ for all abelian varieties is equivalent to the integral Hodge conjecture for curve classes on all abelian varieties, which Engel, de Gaay Fortman and Schreieder disproved [EGFS]; $s_J(A)$ measures how far a given A is from that splitting.

Credit. Theorem A is due to a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode) of this project; Theorem B and Corollary C (i)–(ii) to Claude Opus 5.5 (Anthropic) at effort max; Theorem D and Corollary C (iii) to the writer, Claude Opus 5.5 at effort max. Details are in Section 4.

Context: what is known and what is new. Known: the implication $n\gamma$ *algebraic* $\Rightarrow n \in I_J(A)$ for principally polarized A over any field, by Matsusaka's construction [SS, Prop. 3.4, eqs. (3.1)–(3.3)], which applied to the unimodular Poincaré class of $A \times \widehat{A}$ gives $s_J(A) \mid (2g-1)!$ (see also [BdGF, §3]); the case $s_J = 1 \Leftrightarrow m = 1$ [SS, Thm. 1.1], over $\mathbb{C}$ due to Beckmann and de Gaay Fortman [BdGF] and Voisin [Vo]; the integral Hodge conjecture for 1-cycles on products of Jacobians [BdGF, Thm. 1.2]; the values $m(A) = 2$ for very general A of dimension $4, 5$ [EGFS, Cor. 1.2], $m(A) = 6$ in dimension 6 [EGFS2, Thm. 1.2], the divisibility of $m(A)$ by every prime $p \leq g/2$ [ES, Thm. 1.1, Cor. 1.2], and the fact that a very general principally polarized A of dimension ≥ 4 is not a direct summand of a product of Jacobians and admits no isogeny of odd degree $A \times Y \to J$ [EGFS, Cor. 1.5]. These already give Corollary C (i): $s_J(A) \mid m(A) = 2$ by the implication above and [EGFS, Cor. 1.2], and $s_J(A) \neq 1$ by [EGFS, Cor. 1.5] (Remark 2.3). New, to our knowledge after the search described in Section 4: Theorem A (for surfaces the known bound is $3! = 6$, for threefolds $5! = 120$), the divisibility $m(A) \mid s_J(A)^2$ of Theorem B, which parts (ii) and (iii) of Corollary C need, and Theorem D.

Scope. Theorem B and Corollary C are over $\mathbb{C}$; we make no claim on very general abelian varieties in positive characteristic. Theorem A leaves open the abelian surfaces with $\delta(A)$ even, e.g. a surface of type $(1, 2)$ with $\mathrm{End}(A) = \mathbb{Z}$, where $s_J(A) \in \{1, 2\}$, and the threefolds with $\gcd(\delta(A), 6) > 1$, e.g. of type $(1, 1, 3)$ or $(1, 2, 2)$. The first question of sheet 70 is not treated.

Proof overview. Theorem A(a) uses Beauville's generalized Prym varieties in dimension ≤ 5 and the Prym–Tyurin uniformization of Alexeev, Donagi, Farkas, Izadi and Ortega in dimension 6, which give curve classes 2γ and 6γ; lifting to characteristic zero and specializing these cycles gives every characteristic, and Matsusaka's construction turns them into factorizations. Part (b) applies (a) to the principally polarized variety $A \times \widehat{A}$ after Mumford's descent of a polarization to a principal one. Theorem B pushes the minimal class into a product of Jacobians, where it is algebraic by [BdGF], and back. Theorem D reads a 1-cycle on $\widehat{A} \times A$ as a sum of homomorphisms through Jacobians.

2 Proofs

Lemma 2.1 (Matsusaka; [SS, Prop. 3.4])**.** *Let (B, θ) be principally polarized of dimension h over k. If an integral 1-cycle on B has numerical class $n\,\theta^{h-1}/(h-1)!$, then $n \in I_J(B)$.*

This is [SS, Prop. 3.4] with $\deg(\theta^h/h!) = 1$: the composite $B \to \widehat{B} \to \prod J(\widetilde{C}_i) \to B$ built from the normalized components $\widetilde{C}_i$ of the cycle is $[n]$ by [SS, eqs. (3.1)–(3.3)].

Lemma 2.2. *(a) $\delta(A) = \gcd\{\chi(L) : L$ ample$\}$. (b) If L is ample and $n \in I_J(B)$ for every principally polarized B of dimension $\dim A$, then $n\,\chi(L) \in I_J(A)$; hence $n\,\delta(A) \in I_J(A)$. (c) If $\dim A \leq 3$, then $\delta(A) \in I_J(A)$.*

Proof. (a) Let $g = \dim A$, $L \in \mathrm{Pic}(A)$ and H ample. By Riemann–Roch $p(t) = \chi(L \otimes H^t)$ is a polynomial of degree $\leq g$ with integer values, so for every $T \geq 1$ the $(g+1)$-st difference

of $s \mapsto p(sT)$ vanishes at 0: $\chi(L) = \sum_{j=1}^{g+1} (-1)^{j+1} \binom{g+1}{j} \chi(L \otimes H^{jT})$. For T large all $L \otimes H^{jT}$, $j \geq 1$, are ample.

(b) By Mumford's descent theorem in its group-scheme form [Mu, §23, Thm. 4 and Cor. 1, pp. 233–234], there are a maximal isotropic subgroup scheme $H \subset K(L)$, the quotient isogeny $f\colon A \to B = A/H$ and a principal polarization M on B with $f^*M \cong L$; hence $\deg f = \chi(L)$, and there is $f'\colon B \to A$ with $f'f = [\chi(L)]$, since $\ker f \subset A[\deg f]$. If $[n]_B = \rho\iota$ through J, then $[n\chi(L)]_A = f'[n]_B f = (f'\rho)(\iota f)$. Now use (a).

(c) Principally polarized abelian varieties of dimension ≤ 3 are products of Jacobians [OU, Thm. 4], so (b) applies with $n = 1$. □

Proof of Theorem A. (a) First let $\operatorname{char} k = 0$; by the Lefschetz principle we may take $k = \mathbb{C}$. If $\dim B = h \leq 5$, by Beauville [Be, Prop. 2.2, Thm. 5.4, Thm. 6.5(i)] either B is a product of Jacobians or there are a product of Jacobians J, with principal polarization λ_J, and a homomorphism $j\colon B \to J$ with $j^*\Theta_J \equiv 2\Xi$, Ξ the polarization of B. Put $q = \lambda_B^{-1} j^\vee \lambda_J\colon J \to B$; then $qj = \lambda_B^{-1}\varphi_{j^*\Theta_J} = [2]$, so $2 \in I_J(B)$. Pushing the Abel–Jacobi curves of the factors of J forward by q gives an integral 1-cycle whose homomorphism $\widehat{B} \to B$ is $q\lambda_J^{-1}q^\vee = \lambda_B^{-1}(j^\vee\lambda_J j)\lambda_B^{-1} = 2\lambda_B^{-1}$, so its class is $2\,\Xi^{h-1}/(h-1)!$. If $\dim B = 6$, Alexeev, Donagi, Farkas, Izadi and Ortega [ADFIO, Cor. 2, p. 165] give an effective curve of class $6\,\Xi^5/5!$, and Lemma 2.1 gives $6 \in I_J(B)$.

Now let $\operatorname{char} k = p > 0$. A principal polarization is separable, so (B, Ξ) lifts to a principally polarized abelian scheme over $W(k)$ [Oo, Cor. 2.4.2 and its proof, p. 287]. On its geometric generic fibre the class $c\,\Xi^{h-1}/(h-1)!$, $c = 2$ for $h \leq 5$ and $c = 6$ for $h = 6$, is algebraic by the case just treated. The cycle and its rational equivalences are defined over a finite extension; after the corresponding finite extension of discrete valuation rings, its closure specializes to an integral 1-cycle on B [Fu, Ex. 20.3.5, p. 400], with the same class in ℓ-adic cohomology by smooth and proper base change. Lemma 2.1 gives $c \in I_J(B)$.

(b) Let $g = \dim A \in \{2, 3\}$ and $X = A \times \widehat{A}$, of dimension $2g$, with the Poincaré bundle P. Its homomorphism $\varphi_P\colon X \to \widehat{X} = \widehat{A} \times A$ is $(a, \xi) \mapsto (\xi, a)$ up to sign, an isomorphism, so $\chi(P)^2 = \deg \varphi_P = 1$ by the Riemann–Roch theorem [Mu, §16], and $\delta(X) = 1$. By (a), $c \in I_J(B)$ for every principally polarized B of dimension $2g$, with $c = 2$ for $g = 2$ and $c = 6$ for $g = 3$. Lemma 2.2(b) gives $c = c\,\delta(X) \in I_J(X)$. Since A is a direct factor of X, composing with $A \to X \to A$ gives $c \in I_J(A)$. With Lemma 2.2(c), $\gcd(c, \delta(A)) \in I_J(A)$. □

Proof of Theorem B. Lemma 2.1 gives $M(A) \subset I_J(A)$, so $s_J \mid m$. Let $n = s_J(A)$ and $[n] = \rho \circ \iota$ with $\iota\colon A \to J$. The class $\iota_*\gamma \in H_2(J, \mathbb{Z})$ is an integral Hodge class, because ι_* is a morphism of Hodge structures. By Beckmann and de Gaay Fortman [BdGF, Thm. 1.2] the integral Hodge conjecture for 1-cycles holds on products of Jacobians, so $\iota_*\gamma = [Z]$ for an integral 1-cycle Z on J. Then $\rho_*[Z] = [n]_*\gamma = n^2\gamma$, because $[n]$ acts on $H_2(A, \mathbb{Z}) = \bigwedge^2 H_1(A, \mathbb{Z})$ by n^2. Hence $n^2 \in M(A)$ and $m \mid s_J^2$. The two divisibilities give $\operatorname{rad} s_J = \operatorname{rad} m$; if m is squarefree, $m \mid s_J^2$ forces $m \mid s_J$, so $s_J = m$. □

Proof of Corollary C. For very general A, $m(A)$ is the integer N_g of [EGFS2, Question 1.1]. (i) By [EGFS, Cor. 1.2] with $c = g - 1$, the algebraic classes in $H^{2g-2}(A, \mathbb{Z})$ are $2\mathbb{Z}\gamma$, so $m(A) = 2$. (ii) By [EGFS2, Thm. 1.2], $m(A) = 6$. Both are squarefree; Theorem B gives $s_J = m$. (iii) By [ES, Thm. 1.1 and Cor. 1.2] with $k = g - 1$, every prime $p \leq g/2$ divides $m(A)$, hence $s_J(A)$ by Theorem B; and $s_J \mid m \mid (g-1)!$. □

Remark 2.3. That $s_J(A)$ is even for $g \geq 4$ follows also from [EGFS, Cor. 1.5] without Theorem B: if $vu = [s]$ with $u\colon A \to J$, $v\colon J \to A$ and s odd, then $(x, y) \mapsto u(x) + y$ is an isogeny $A \times (\ker v)^0 \to J$ whose kernel embeds into $A[s]$, so its degree is odd. With

$s_J(A) \mid m(A) = 2$ [EGFS, Cor. 1.2] this gives Corollary C (i) without Theorem B. For $g = 6$ it leaves $s_J(A) \in \{2, 6\}$; that $3 \mid s_J(A)$ needs Theorem B.

3 Grothendieck's second question

Proof of Theorem D. Let C be a curve, $f = (p, q)\colon C \to \widehat{A} \times A$, and $Z = f_*[C]$. The correspondence Z_* is $q_* p^*$, which factors through $H^1(C, \mathbb{Z}_\ell)$. Under $H^1(C, \mathbb{Z}_\ell) \cong T_\ell J(C)(-1)$, the Gysin map $q_*\colon H^1(C) \to H^{2g-1}(A)(g-1)$ becomes T_ℓ of the Albanese map $q_*\colon J(C) \to A$, and $p^*\colon H^1(\widehat{A}) \to H^1(C)$ becomes, up to the autoduality sign of $J(C)$, T_ℓ of the pullback $p^*\colon A = \mathrm{Pic}^0(\widehat{A}) \to \mathrm{Pic}^0(C) = J(C)$. Hence, up to a universal sign, $\psi_A \circ Z_* = T_\ell(\varphi_Z)$ with $\varphi_Z = q_* \circ p^*\colon A \to J(C) \to A$, and this extends additively to integral 1-cycles $Z = \sum n_i f_{i*}[\widetilde{C}_i]$, $\varphi_Z = \sum n_i\, q_{i*} p_i^*$.

(i)⇒(ii): if $Z_* = n\psi_A^{-1}$, then $T_\ell(\varphi_Z) = \pm n$, so $\varphi_Z = \pm[n]$ because $\mathrm{Hom}(A, A) \to \mathrm{End}\, T_\ell A$ is injective; φ_Z factors through $\prod J(\widetilde{C}_i)$, so $n \in I_J(A)$.

(ii)⇒(i): let $[n] = \rho\iota$ with $\iota = (\iota_a)\colon A \to \prod J(C_a)$ and $\rho = \sum \rho_a$. Let $\alpha_a\colon C_a \to J(C_a)$ be an Abel–Jacobi map, λ_a the principal polarization of $J(C_a)$ and $\widehat{\iota}_a\colon \widehat{J(C_a)} \to \widehat{A}$ the dual of ι_a. For $f_a = (\widehat{\iota}_a \lambda_a \alpha_a, \rho_a \alpha_a)\colon C_a \to \widehat{A} \times A$ one finds $p_a^* = \alpha_a^* \lambda_a \iota_a$ and $q_{a*} = \rho_a \alpha_{a*} = \rho_a$, and $\alpha_a^* = -\lambda_a^{-1}$ on $\mathrm{Pic}^0(J(C_a))$ [Mi, Lemma 6.9]. Hence $\varphi_Z = -\sum_a \rho_a \iota_a = -[n]$ for $Z = \sum_a f_{a*}[C_a]$, and $\pm Z$ induces $n\psi_A^{-1}$. □

4 Provenance and checks

This paper is AI-produced. No human has proved, written or checked any part of it.

What is whose. Theorem A is due to a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode) of this project (run RESEARCH_SA06, ended 19 November 2026; its statements C.4j, C.4m–C.4p), in answer to a research prompt on the integral questions of cote 14 of Grothendieck's archive. Theorem B and Corollary C (i)–(ii) are due to Claude Opus 5.5 at effort max (agent a5d5ce46, 24 November 2026), in a map of the project's geometry results. Theorem D, Corollary C (iii), Remark 2.3 and the text are due to the writer, Claude Opus 5.5 at effort max (agent a0012f1f, 25 November 2026).

Checks. The run's own audits (GPT-6 Astra, ChatGPT Pro mode) passed. The writer re-proved Theorem A(b) and the Matsusaka coefficient independently on 22 November 2026, with exact computations, read the Prym step of Theorem A(a) in full and checked its sixfold step at the intake of the run; it read Theorem B and Corollary C (i)–(ii) a second time on 25 November 2026, with the cited statements opened. On 25 November 2026 the main session of the project (Claude Opus 5.5 at effort max) read every page and checked Theorem B, Corollary C, Remark 2.3 and Theorem D line by line.

The search of the literature. On 25 November 2026 the writer read the statements cited from [BdGF, EGFS, EGFS2, ES, SS] and searched arXiv (all papers of de Gaay Fortman, including [GFS], and the abstracts mentioning products of Jacobians), the works citing [BdGF, EGFS, EGFS2, SS] listed by OpenAlex, and the Hexagon catalogue. No statement of Theorems A, B, D or of the exact values in Corollary C was found. The writer's search does not cover full texts; the main session then had the full texts of [SS] and [EGFS2], and of [EGFS] through its Section 3, searched for the statements claimed here as new (by a fetch tool's reading model; none is there), and opened [ES] at its abstract.

The locators of [ADFIO, Be, Fu, Mi, Mu, OU, Oo, Vo] were opened by the run, not by the writer. The full record is in the file PROVENANCE.md of the paper's workbench.

References

[ADFIO] V. Alexeev, R. Donagi, G. Farkas, E. Izadi, A. Ortega, *The uniformization of the moduli space of principally polarized abelian* 6*-folds*, J. Reine Angew. Math. 761 (2020), 163–217.

[Be] A. Beauville, *Prym varieties and the Schottky problem*, Invent. Math. 41 (1977), 149–196.

[BdGF] T. Beckmann, O. de Gaay Fortman, *Integral Fourier transforms and the integral Hodge conjecture for one-cycles on abelian varieties*, Compos. Math. 159 (2023), 1188–1213; arXiv:2202.05230v2.

[EGFS] P. Engel, O. de Gaay Fortman, S. Schreieder, *Matroids and the integral Hodge conjecture for abelian varieties*, arXiv:2507.15704v3 (2026).

[EGFS2] P. Engel, O. de Gaay Fortman, S. Schreieder, *Optimality of the Prym–Tyurin construction for* $\mathcal{A}_6$, arXiv:2512.04902v1 (2025).

[ES] P. Engel, S. Schreieder, *On the degree of subvarieties on abelian varieties*, arXiv:2606.31894v1 (2026).

[Fu] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[GFS] O. de Gaay Fortman, S. Schreieder, *Abelian varieties with no power isogenous to a Jacobian*, Compos. Math. 161 (2025), 1404–1457; arXiv:2401.06577.

[Mi] J. S. Milne, *Jacobian varieties*, notes (2021 version).

[Mu] D. Mumford, *Abelian varieties*, Oxford University Press, 1970.

[Oo] F. Oort, *Finite group schemes, local moduli for abelian varieties, and lifting problems*, Compos. Math. 23 (1971), 265–296.

[OU] F. Oort, K. Ueno, *Principally polarized abelian varieties of dimension two or three are Jacobian varieties*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973), 377–381.

[SS] F. Scavia, F. Suzuki, *On direct summands of products of Jacobians over arbitrary fields*, arXiv:2507.09821v2 (2025).

[Vo] C. Voisin, *Cycle classes on abelian varieties and the geometry of the Abel–Jacobi map*, Pure Appl. Math. Q. 20 (2024), 2469–2496.