Two-field Hecke tests for vanishing of $K_{4k}(\mathbb{Z})$ through degree 200

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Abstract

From the pointwise strict Selmer estimate in Alpöge's Step 17 we deduce $K_{4k}(\mathbb{Z}) = 0$ for $1 \le k \le 50$. At each Galois element the estimate permits a different representation. For the two fields and weights used here, interpolation on inertia forces one admissible representation to have the prescribed residual traces everywhere. Classical Hecke operators and signed ternary unitary operators then test all possibilities. In the 24 degrees from 108 through 200, two integral compact determinants over $\mathbb{Q}(\sqrt{-3})$ and two determinant residues over $\mathbb{Q}(\sqrt{-7})$ suffice together with the classical calculations. The largest matrix has rank 20,825.

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1 Statement and arithmetic inputs

For a finite abelian group G, write $G[\ell^\infty]$ for its ℓ-primary part. Set $e = 2k$, so the degree under study is $2e = 4k$. Alpöge raised the question of consequences for higher K-groups in footnote 6 of the *Remark* [1]. The two-field reduction through degree 104 was developed in [4]. Mok and Zheng's proof through degree 120 [5] prompted us to ask whether the pointwise Selmer estimate alone suffices for our signed Hecke tests. We answer this using the interpolation argument from our degree-72 supplement [2]. The two calculations agree at a shared Hecke operator in degrees 108–120 (Table 2). The twenty multiples of four from 124 to 200 extend beyond their theorem.

Theorem 1. *Assume the pointwise statement* `OneCharU_s` *of [1] at level* 2*, trivial character, coefficients* $\mathbb{Z}_\ell$*, for* $E = \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7})$*, even* $2 \le e \le 100$ *and primes* $\ell \ge 2^{31}$*. Then*

$$K_{4k}(\mathbb{Z}) = 0 \qquad (1 \le k \le 50).$$

A nonzero K-group would give a prime ℓ and a nonzero rational strict Selmer group. For each element g of the auxiliary Galois group, the pointwise bound would give an admissible representation matching the target trace at g. We first prove that one of the available representations must match everywhere. We then use three sets of Hecke tests over $\mathbb{Q}(\sqrt{-3})$ to leave a finite set of possible primes, and three over $\mathbb{Q}(\sqrt{-7})$ to eliminate them. The underlying rational group is the same in both fields. All determinant and gcd calculations are independent of Step 17.

For odd ℓ, write $A_\ell = \mathrm{Cl}(\mathbb{Q}(\mu_\ell))[\ell^\infty]$ and let ω be the mod-ℓ cyclotomic character. The cyclotomic comparison [7, Thm. 5.1] identifies, after choosing a twist basis,

$$K_{2e}(\mathbb{Z})/\ell \simeq (A_\ell/\ell)^{(\omega^{-e})}.$$

The character is even. Published cyclotomic class-group computations exclude every odd $\ell < 2^{31}$ [8], and the known 2-primary calculation excludes 2 [9, Cor. VI.9.8]. All positive-degree groups under consideration are finite. It therefore suffices to work with primes $\ell \ge C = 2^{31}$.

For degrees through 104 we use the odd-discriminant, pointwise proof of Mok and Zheng. They identify the contributions of our earlier archive; for example their directed split-place tests in degrees 48–72 use the archived matrices and logs [5, §7]. The numerical comparison below uses their separately published records for degrees 108–120.

1.1 From a rational class to one representation over each field

Take $E_d = \mathbb{Q}(\sqrt{-d})$ for $d = 3, 7$, level 2, trivial character and coefficients $\mathbb{Z}_\ell$. A nonzero cyclotomic class as above gives a nonzero rational strict class with coefficients $D = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-e)$. Indeed its class-field extension over $\mathbb{Q}(\mu_\ell)$ is unramified and splits at the unique prime over ℓ, which is principal. Descent has degree $\ell - 1$ prime to ℓ. The descended class injects into $H^1(\mathbb{Q}, D)$ since $0 < e < \ell - 1$ gives $D^{G_\mathbb{Q}} = 0$. Its unramified local condition at 2 is strict because multiplication by the nonzero scalar $2^{-e} - 1$ on D is surjective. Thus the class is unramified on every finite inertia group and locally zero at both 2 and ℓ. These are the defining conditions on the rational strict Selmer group $S_\ell(e)$. In this specialization `vlen_one` identifies its length with Vlen, for either auxiliary field [4, §2]. Mok and Zheng give the equivalence with $K_{2e}(\mathbb{Z})/\ell \neq 0$ in Proposition 2 of [5]. We use the forward implication.

Both auxiliary fields have class number one by the Minkowski bound and are disjoint from $\mathbb{Q}(\mu_\ell)$, by their discriminants. The level and these discriminants are prime to ℓ, the coefficient ramification index is one, and $2e \leq 200 < \ell - 1$. These verify the field and weight

conditions for the trivial character. Fix the coefficient embedding ι. Write $B = \omega^e \oplus 1 \oplus \omega^{-e}$ on G_E. If at some g all admissible traces differed from the target modulo ℓ, their characteristic-zero differences would be nonzero units. The pointwise estimate would then give finite Selmer length bounded by zero. Thus $S_\ell(e) \neq 0$ implies

$$\text{for every } g \in G_E \text{ there exists an admissible } \rho_g \text{ such that } \operatorname{tr} \bar{\rho}_g(g) = \operatorname{tr} B(g). \tag{P}$$

All reductions use the fixed valuation on $\overline{\mathbb{Q}}_\ell$. This is the only Selmer implication used below. Its quantifier extends over every Galois element, beyond the finite set of Frobenius elements used for the numerical tests.

2 From pointwise traces to a single representation

Proposition 2. *For E, e, ℓ as in Theorem 1, condition* (P) *gives one admissible characteristic-zero representation ρ with $\bar{\rho}^{\mathrm{ss}} \simeq B$ on all of G_E.*

Proof. We give the finite-group argument from the degree-72 supplement [2]. Let s be the number of distinct residual trace functions, and choose one admissible representative and a stable lattice for each. The rank-three systems for fixed ι occur in the full compact space of §3, by [5, §4]. Distinct systems occupy distinct Hecke eigenspaces by strong multiplicity one and Chebotarev. There are at most two lattice classes and the coefficient representation of highest weight $(e - 1, 0, 1 - e)$ has dimension e^3 by the Weyl formula. The classical branch consists of twists of weight-$2e + 1$ newforms of level d, with at most $\varphi(d) \leq 6$ twists, by the conductor calculation in [5, §3]. The projective index of $\Gamma_1(d)$ is at most 24. The valence bound at infinity thus gives dimension at most $2(2e + 1) + 1 = 4e + 3$, counting all complex eigenforms including their coefficient conjugates. In particular

$$s \leq 2e^3 + 6(4e + 3), \qquad 2es \leq 400\,483\,600 < \ell - 1.$$

Put $F = \mathbb{Q}(\zeta_d)$. Form the finite joint image Γ of the chosen residual representations, ω, and $\mathrm{Gal}(F/E)$; let H be the image of G_F. Then Γ/H is cyclic. Both admissibility predicates have ordinary flags at each place above ℓ, with inertia characters $\omega^e, 1, \omega^{-e}$ after reduction. This follows by intersecting the flags with a stable lattice. The finite characters allowed in the pinned `LevelAt` and `LevelAway` conditions [1] factor through the action on μ_2 or μ_{2d}; they are trivial on G_F. In particular each system is unramified over F away from ℓ.

The inertia groups over ℓ normally generate H. Indeed the quotient by their normal closure would give an everywhere-unramified Galois extension of F. Minkowski's lower bound for the root discriminant of a totally imaginary field of degree n is $b_n = (\pi/4)(n^n/n!)^{2/n}$; it increases since the successive ratios of $n^n/n!$ increase. For $d = 3$, $b_4 > \sqrt{3}$. For $d = 7$, $b_{48} > 7^{5/6}$, so

the relative degree would be at most seven. The Minkowski ideal bound for $\mathbb{Q}(\zeta_7)$ is below five, and the residue degrees of 2 and 3 are three and six. Thus its class number is one. Every nontrivial group of order at most seven has a nontrivial abelian quotient, which class field theory excludes.

On each inertia group choose τ mapping by ω to a generator of $\mathbb{F}_\ell^\times$. Surjectivity follows since F is unramified at ℓ. The three inertia eigenvalues of τ are distinct in every block. In a diagonalizing basis, its powers are obtained by letting $x \in \mathbb{F}_\ell^\times$ vary in $\operatorname{diag}(x^e, 1, x^{-e})$, with cyclotomic coordinate x. An inertia element in the kernel of ω has the form $1 + N$ in each block, $N^3 = 0$. Its powers are $1 + yN + \binom{y}{2}N^2$, $y \in \mathbb{F}_\ell$. These families and their conjugates generate H; each contains the identity.

Choose t lifting a generator of Γ/H, with $t = 1$ for a trivial quotient. If each trace function failed somewhere on tH, choose th_i where the i-th fails. Write each h_i as a word in the preceding families. Concatenate the words with a separate variable for each position, and prefix t. Each trace difference is a nonzero Laurent polynomial because setting all other words to the identity recovers its value at th_i. Their product is nonzero. In each multiplicative variable the exponents lie between $-es$ and es; in each additive variable the degree is at most $2s$. Clearing the negative powers, the degree in each variable is strictly smaller than the size of its respective grid. Repeated univariate interpolation gives a grid point with nonzero product. This is a genuine group element contradicting (P). Hence one trace function agrees on all of tH.

Let V be its global semisimplification. Its restriction to H is semisimple: in an irreducible Γ-summand the sum of the simple H-submodules is nonzero and Γ-stable. The three characters of $B|_H$ are distinct on inertia. If one, b, were absent from $V|_H$, the image of $\overline{\mathbb{F}}_\ell[H]$ on $V|_H \oplus B|_H$ is a semisimple algebra, so there is a lift a of the central projector onto the b-part. Extending equality of traces on tH linearly to ta would give $0 = b(t)$. Thus all three occur once and their lines are Γ-stable since the characters extend to Γ. Each global character differs from its target by a character of Γ/H. Independence of the three characters on H in the equality on tH makes all three twists trivial at t, hence on the cyclic quotient. □

For degree 124 the bound is $2es \le 59\,292\,088$. It is the degree in *each variable* that matters; there is no bound on the length of the concatenated word. The proposition allows eigenvalues from two split places to be added on one system. Mok and Zheng instead take a product over all branches separately at each single place [5, eq. (1)].

Alternative using the full Step 17 construction. Our degree-104 paper [4] uses the Grothendieck–Montpellier joint-representation theorem to obtain, from a nonzero Selmer group, a deformation point whose representation has the prescribed residual traces everywhere. In a field where the three Bernoulli values are units, their Corollary ES.5 excludes the classical and essentially self-dual branches for that point, leaving the signed compact test [3, K17-S17-JOINT and ES.5]. For the pointwise hypothesis of Theorem 1, we instead test the three possibilities for the representation in Proposition 2.

3 Which representations the matrices detect

Put $\eta_d = \chi_{-d}$, $b_{j,d} = |\operatorname{num}(B_{j,\eta_d}/j)|$ in lowest terms and $A_d(e) = b_{e+1,d}b_{2e+1,d}$. A fixed witness is either a classical system $1 \oplus W|_{G_E}$, a genuinely self-dual cuspidal rank-three system, or a cuspidal system distinct from its dual. For the second possibility, cyclic descent and the adjoint lifting theorem identify it with an adjoint of a weight-$e+1$ newform. The conductor argument in [5, §6.3] gives level d and character η_d after a scalar twist. Indeed projective inertia at d has order at most two. Scalar inertia and the case where Frobenius exchanges its two lines both give even determinant at -1, contrary to odd weight. In the remaining principal series, twisting one line to become unramified leaves inertial ratio η_d. This fixes the

nebentypus since all other primes have been cleared. Scalar twists leave the adjoint unchanged. The direct classical branch may have any odd nebentypus at level d after a twist. The third possibility uses the signed compact space, including any system isomorphic to a twist of its dual but distinct from its dual itself.

Why the full lattice level suffices. Every admitted cuspidal rank-three system is conjugate self-dual and belongs to a stable packet of the definite ternary unitary group. Its coefficient highest weight is $(e-1, 0, 1-e)$. The local level-two characters are trivial. Select the spherical members away from d and the compact member at infinity, as in [4, §3]. At the ramified prime d, write K_d for the standard self-dual lattice stabilizer and K_d^+ for the kernel of its determinant sign on reduction. The ramified local calculation of [5, §4] constructs a member with a K_d^+-fixed vector by induction from a torus character trivial on the intersection of the Borel with K_d^+. The selected members occur together by the multiplicity formula for the stable packet ([10, Lem. 2.2], [5, §4]). The other coset in K_d/K_d^+ is represented by the rational central element $-I$. It acts trivially at infinity because the three weights sum to zero, and fixes the finite level vectors away from d. Its diagonal rational action on automorphic forms is trivial. It therefore fixes the local vector at d as well. This gives the full stabilizer used in the lattice enumeration.

Now suppose the system is distinct from its dual. Let J be conjugation together with exchange of the two coefficient variables. An eigenvector v belonging to this system and Jv have distinct Hecke systems in characteristic zero. Otherwise strong multiplicity one and conjugate self-duality would make the system self-dual. Hence $v - Jv \neq 0$. If a tested operator commutes with J, its eigenvalue on v is also its eigenvalue on $v - Jv$. If that eigenvalue reduces to zero, the integral characteristic polynomial implies divisibility of the determinant, including when the vector itself reduces to zero. In detail, J acts on the coefficient lattice and its negative part is cut out by the integral idempotent $(1-J)/2$ at our odd primes. The characteristic-zero eigenvalue is a root of the characteristic polynomial on this summand. Reducing that polynomial at λ gives the assertion. Mok and Zheng use this argument at inert primes in [5, §6.1]. At a split prime conjugation interchanges H_w and $H_{\bar{w}}$, so their sum also commutes with J.

4 From Frobenius traces to a finite set of primes

Write $u_Q = 1 + Q^e + Q^{2e}$. At a geometric Frobenius of residue norm Q, scaling the target trace by Q^e gives u_Q. At a split rational prime q, $Q = q$, and the neighbor operator H_w has eigenvalue q times the sum of the normalized Satake parameters. Hence $q^{e-1}H_w$ has target u_q, as does the opposite direction, by [5, eq. (3)].

At the inert rational prime 2 in E_3 the residue norm is 4. Of the 18 distance-two neighbors in the unitary tree, one lies toward a fixed end, one at the original height and sixteen away. Satake normalization gives coefficient $\sqrt{16} = 4$ at each extreme and leaves the middle coefficient one. Thus H_2 has eigenvalue $4z + 1 + 4z^{-1}$; the trace with base-change roots $z, 1, z^{-1}$ is $(H_2 + 3I)/4$ on the eigenline [5, §6.1]. The rational prime 7 splits in E_3. Our shifted operators on the negative eigenspace of J are

$$X = 4^{e-1}(H_2 + 3I) - u_4 I, \qquad Z = 7^{e-1}(H_w + H_{\bar{w}}) - 2u_7 I.$$

Put $G_e = \gcd(|\det X|, |\det Z|)$. A witness distinct from its dual over E_3 forces $\ell \mid G_e$, since its two directed eigenvalues over 7 belong to one system by Proposition 2.

The other two possibilities over E_3. Let T_q act on $S_m(\Gamma_0(3), \eta_3)$. For the direct classical branch $m = 2e + 1$; the necessary determinant factors at 2 and 7 are

$$C_2 = \det(T_2^2 - (2^{2e} - 1)^2 I), \qquad C_7 = \det(T_7 - (1 + 7^{2e})I).$$

For the self-dual branch $m = e + 1$. Put

$$J_2 = \det((T_2^2 + 2^{e+1}I)^2 - (1 + 2^{2e})^2 I), \qquad J_7 = \det(T_7^2 - (1 + 7^e)^2 I).$$

At 2 one takes the square of rational Frobenius, using $\eta_3(2) = -1$. For example the two allowed adjoint values of a_2^2 are $(2^e - 1)^2$ and $-(2^e + 1)^2$; their product of shifted factors is precisely the polynomial defining J_2. At split 7, $\eta_3(7) = 1$. The trace formulas in [5, §§3,6] give the displayed determinants up to signs. A fixed witness in these branches forces $\ell \mid \gcd(C_2, C_7)$ or $\ell \mid \gcd(J_2, J_7)$, respectively.

5 The second-field determinant

Testing a whole modulus at once. Over E_7 the prime 2 splits. The sum of the two neighbor operators, scaled by 2^{e-1}, has block matrix $\left(\begin{smallmatrix} A & B \\ D & 0 \end{smallmatrix}\right)$ on the signed parts of the two lattice classes. For either direction a standard lattice has six standard neighbors and one nonstandard neighbor. A nonstandard lattice has seven standard neighbors; this gives the zero lower-right block [11, §3], [5, §5]. Its eigenvalue s at a putative witness satisfies $s \equiv t = 2u_2 \pmod{\lambda}$. At every candidate prime t is a unit, hence $s \neq 0$. Writing a characteristic-zero eigenvector as (x, y) gives $Ax + By = sx$ and $Dx = sy$. If $x = 0$, then $y = 0$, so the standard-class component x is nonzero and satisfies

$$(s^2 I - sA - BD)x = 0.$$

The polynomial $F(T) = \det(T^2 I - TA - BD)$ is integral at ℓ by the local coefficient lattice in [4, §4]. It follows that $F(t) \equiv 0 \pmod{\lambda}$. The stored matrix W evaluates the images of the signed basis polynomials at integral pairs satisfying $x \cdot y = 0$. In characteristic zero, $\det W = \det(V)F(t)$, where V evaluates the basis at the same pairs. Since $\det V$ is integral, $\ell \mid \det W$ is necessary. If $\ell \mid F(t)$, multiplication by the integral determinant $\det V$ preserves that divisibility. The test thus only needs the recorded numerator $\det W$. Checking $\gcd(\det W, M) = 1$ excludes every prime divisor of M at once, including when M is composite.

Classical tests at the split prime 2 over E_7. The remaining two necessary determinants are

$$L^{(7)} = \det(T_2^3 - (1 + 2^{2e})^3 I \mid S_{2e+1}(\Gamma_1(7))), \qquad D^{(7)} = \det(T_2^2 - (1 + 2^e)^2 I \mid S_{e+1}(\Gamma_1(7))).$$

The full first space includes every odd character. The twist that puts a classical form at level 7 has value at 2 of order dividing three, so cubing removes it. For the adjoint descent the exact character is η_7, with $\eta_7(2) = 1$. Its trace is $a_2^2/2^e - 1$, so the target requires $a_2^2 = (1 + 2^e)^2$. The full space for $D^{(7)}$ contains that character space. A unit value of the full determinant excludes the required subsystem.

Why more classical Hecke primes need not help. There is an obstruction valid in every odd weight $m \geq 3$. Write $\eta_3 = \chi_{-3}$. If $\ell \nmid 6$ divides $b_{m,3}$, the full level-three cusp space of weight m and character η_3 contains a nonzero residual eigenvector with eigenvalues $1 + \eta_3(q)q^{m-1}$ for every $q \nmid 3\ell$. Indeed the Eisenstein series E_m^{1,η_3}, normalized to have first coefficient one, has constant $c = -B_{m,\eta_3}/(2m)$ at infinity and zero at the other cusp, by Fricke interchange [12]. Let $\theta = \sum_{a,b \in \mathbb{Z}} \exp(2\pi i z(a^2 + ab + b^2))$, the weight-one hexagonal theta series, and let $\eta(z)$ denote Dedekind's eta function. The level-three eta quotient $\eta(z)^9/\eta(3z)^3$ has weight three, character η_3 and order one at cusp 0. Its product expansion has constant one at infinity. Hence $U = \theta^{m-3}\eta(z)^9/\eta(3z)^3$ has the required weight and character, integral Fourier coefficients, and cusp constants one and zero. The cusp form $E_m^{1,\eta_3} - cU$ reduces to the nonzero Eisenstein eigensystem. On restriction to G_{E_3}, the character η_3 disappears. For $m = e + 1$ the

trace-zero adjoint of the residual pair has the three target characters. For $m = 2e + 1$ twisting the pair by ω^{-e} and adjoining the trivial character gives the same triple. Thus these primes can pass every good classical Hecke trace test in this field. An ambient eigenvector alone does not produce a Selmer class. The second field lets us test a hypothetical class without this ambiguity.

The Bernoulli numbers in the records. For each new degree $D = 2e = 4k$ we compute the four rational numbers $B_{m,\eta_d}/m$, for $d = 3, 7$ and $m = e + 1, 2e + 1$, using the formula

$$B_{m,\eta_d} = d^{m-1} \sum_{a=1}^{d-1} \eta_d(a) B_m(a/d).$$

The least common multiple of their four reduced denominators is at most 3969. The largest prime dividing any $\gcd(A_3, A_7)$ is 193. These quantities give a convenient common description of the candidate moduli across the range. The first-field branch gcds below show directly why these candidate moduli suffice.

6 The twenty-four new degrees

Let M_1, M_2 be the positive cofactors obtained by removing the recorded small factors from $b_{e+1,3}, b_{2e+1,3}$, respectively. Every removed factor is smaller than 2^{31}. The cofactors may be composite and can retain small factors. Write $G = \gcd(|\det X|, |\det Z|)$ for the two shifted determinants over the first field, and W for the integral evaluation matrix of the second-field test. The records give the following exact identities in each of the 24 degrees.

$$M_1 \mid G, \qquad \gcd(G/M_1, M_1) = \gcd(G, M_2) = 1,$$
$$\gcd(M_1 M_2,\, 42 u_2 A_7) = 1, \qquad \gcd(\det W, M_i) = 1 \quad (i = 1, 2).$$

Every prime factor of G/M_1 is smaller than 2^{31}. This last check needs no full factorization. Repeatedly divide G/M_1 by all integers from 2 to 10,000. The remainder is one in all degrees except 108, where it is 60 754 717. Thus the prime factors of G above the cutoff coincide, with multiplicity, with those of $b_{e+1,3}$ throughout this range.

The records also give two gcds $D_{\mathrm{cl}} = \gcd(|C_2|, |C_7|)$, $D_{\mathrm{ad}} = \gcd(|J_2|, |J_7|)$ per degree over E_3. Every prime divisor of D_{cl} or D_{ad} above the cutoff divides $A_3(e)$. To verify this, strip successive gcds with $A_3(e)$ until coprime and check the remaining factors are small. Over E_7 the stored values satisfy $\gcd(L^{(7)}, M_1 M_2) = \gcd(D^{(7)}, M_1 M_2) = 1$. For example, in degree 124 the first-field classical and adjoint spaces have dimensions 40 and 20, and the two second-field spaces have dimensions 245 and 121. Reconstructing all four records from Fourier expansions reproduces the archived gcds and residues exactly. This calculation is independent of the rank-550 and rank-4,960 compact calculations.

Suppose $\ell \geq 2^{31}$ divides the K-group. Over E_3 Proposition 2 places a fixed witness in one of the three cases, so ℓ divides at least one of $D_{\mathrm{cl}}, D_{\mathrm{ad}}, G$. By the support checks, $\ell \mid M_1 M_2$. Apply the proposition again over E_7, allowing a different representation. The determinants $L^{(7)}, D^{(7)}$ exclude the classical and self-dual cases. In the remaining case $2u_2$ is a unit and the signed determinant $\det W$ would be divisible by ℓ, contradicting its coprimality with the applicable M_i. This excludes every large prime.

In Table 1, r_3 is the rank of each of the two first-field matrices, r_7 is the rank after eliminating the nonstandard lattice class over the second field, and $\delta = \gcd(A_3, A_7)/4$.

Degree 200. At $e = 100$ the four rational denominators have least common multiple 63 and $\gcd(A_3, A_7) = 4$. The moduli are

$$M_1 = \frac{b_{101,3}}{2 \cdot 257 \cdot 317}, \qquad M_2 = \frac{b_{201,3}}{2 \cdot 53 \cdot 337 \cdot 7109 \cdot 7253}.$$

D	r_3	r_7	bits (M_1, M_2)	δ	D	r_3	r_7	bits (M_1, M_2)	δ
108	360	3276	128,423	$37 \cdot 109$	156	1092	9880	290,723	79^2
112	405	3654	191,480	1	160	1183	10660	302,755	1
116	450	4060	193,513	47	164	1274	11480	316,802	1
120	495	4495	174,512	1	168	1365	12341	312,798	43^2
124	550	4960	221,547	1	172	1470	13244	343,838	1
128	605	5456	228,581	1	176	1575	14190	344,878	1
132	660	5984	229,582	67^2	180	1680	15180	341,857	37
136	726	6545	250,618	1	184	1800	16215	369,909	1
140	792	7140	258,654	1	188	1920	17296	376,959	1
144	858	7770	234,650	37^2	192	2040	18424	374,947	193
148	936	8436	280,682	1	196	2176	19600	390,1000	1
152	1014	9139	277,719	1	200	2312	20825	402,994	1

Table 1: Signed matrix ranks, modulus bit lengths and common Bernoulli divisors.

Both are composite. The signed matrices over the first field have rank 2,312. The matrices $W \bmod M_i$ have size 20,825 × 20,825. Both determinants are units in the respective rings $\mathbb{Z}/M_i\mathbb{Z}$. This excludes all their prime factors simultaneously.

Combining these exclusions with the small-prime results and the pointwise odd-discriminant calculations through 104 in [5, §§5–7] proves Theorem 1.

7 Arithmetic cross-checks

7.1 Comparison with Mok and Zheng

For $e = 54, 56, 58, 60$, Mok and Zheng use two inert rational primes, 2 and 5, over E_3. They cover the branches separately at each prime. For the surviving candidates they use the inert prime 3 in $\mathbb{Q}(i)$, including the classical level-eight and adjoint level-sixteen tests. Since the Gaussian discriminant and the level share a factor of two, they give a separate argument transferring the pointwise inequality to this case [5, §6.5]. Our second probe over E_3 is split 7, and our final probe is split 2 over E_7. These choices give a common integer to compare and two distinct final tests.

In the convention of [5, §6.1], their shift at the inert prime 2 on the same negative space is

$$(2^{4e} + 2^{2e-2} + 1)I - 2^{2e-2}H_2 = -X.$$

The predicted sign of their determinant relative to ours is therefore $(-1)^{r_3}$. We compared the full signed integers in the archived records and obtained this identity in all four degrees. Table 2 also compares their Gaussian test modulus M_{G} with our M_1M_2. After cancellation every discrepancy is smaller than 2^{31}. Thus the two lists of large candidate primes agree without factoring either modulus.

D	r_3	$\det_{\mathrm{MZ}} / \det X$	$M_{\mathrm{G}}/(M_1M_2)$	Gaussian rank	E_7 rank
108	360	+1	60 754 717	819	3276
112	405	−1	1/507 907	910	3654
116	450	+1	1/147 671	1015	4060
120	495	−1	1/788 267	1120	4495

Table 2: Comparison with the fixed Mok–Zheng records [6].

For each Gaussian record we also checked that the evaluation determinant and the shifted determinant were units modulo M_{G}, and that their product equalled the stored numerator modulo M_{G}. The companion script compare_mok_zheng.py replays these equalities and

Table 2 from the archived integers. Degrees 124–200 use our classical and compact records in the supplement.

A direct single-place test in degree 124. One could also extend Mok and Zheng's method [5, eq. (1), §§6–7] by multiplying the branch determinants at each place before taking the gcd across places. Over E_3 the saved inert-2 determinant supplies the first compact test. The inert-5 test would require a new determinant of rank 550. Over $\mathbb{Q}(i)$ the three classical tests at 3 give units modulo our M_1M_2. Completing this route would require identifying the large primes left by the first-field gcd, covering them in the Gaussian tests including the rank-1,240 compact determinant, and extending the Gaussian specialization of [5, §6.5] to $e = 62$.

Two prime exclusions independent of Step 17. For prime $\ell > e + 3$, let $j = \ell - 1 - e$ and let $A^{(i)}$ be the i-th Teichmüller component of $\mathrm{Cl}(\mathbb{Q}(\zeta_\ell))/\ell$. The cyclotomic comparison of [7, Thm. 5.1] identifies the mod-ℓ rank of $K_{2e}(\mathbb{Z})$ with $\dim A^{(j)}$. Reflection gives $\dim A^{(j)} \leq \dim A^{(1-j)}$, and Herbrand's theorem [13] gives $A^{(1-j)} = 0$ when the ordinary Bernoulli numerator B_j is nonzero modulo ℓ. The following exact residues eliminate two actual divisors of the test moduli, each occurring to the first power.

D	weight over E_3	ℓ	j	$B_j \bmod \ell$
176	89	50 177 369 101	50 177 369 012	36 559 167 625
196	197	63 231 194 447	63 231 194 348	54 382 770 099

In particular the entire corresponding ℓ-primary subgroup vanishes in each case. The determinants reported above used the full moduli, so these exclusions provide independent checks.

Reproducing the arithmetic. The file K-theory-108-200-supplement.zip (SHA-256 prefix `815b94885f0ca838`) contains both exact first-field determinants and both second-field determinant residues in each degree. The first-field ranks are $r_3 = \sum_{j=1}^{\lfloor (k-1)/3 \rfloor} j(k-3j)$, from the signed invariant basis. For the second field the standard-class rank is $r_7 = \binom{k+1}{3}$ by the character calculation in [4, §4]. The first-field generators recover the Hecke matrices over the rationals from integral polynomial evaluations and check the resulting matrix identities. The second-field generator works directly at the two moduli. Each generation checks the first and last rows against a direct sum over the oriented neighbor paths. At degree 124 the two evaluations of the determinants, run separately and in parallel on the same matrices, agree.

The short verifier recomputes the Bernoulli inputs and gcds, and checks the recorded residues, including the two classical-branch support checks over E_3 and the two unit tests over E_7 in each degree. For the 46 calculations using block elimination, it checks that each recorded pivot determinant is a unit and that their product is the reported residue. Two prime-field calculations use direct determinants. Checking saved residues and pivot products does not derive them anew from the matrix entries. The supplement includes the algorithms and inputs for an independent reconstruction of those entries and a fresh determinant computation. From an extracted copy, `python verify.py` checks the records. We also regenerated the 48 first-field classical and adjoint gcds in degrees 108–200 from Fourier expansions. With the extracted directory as its argument, the companion script check_classical_branches.py reconstructs these gcds and both degree-124 level-seven residues from Fourier series. The independent ordinary Bernoulli calculation can be repeated with `python reflection.py -degrees 176 196`.

8 A consequence in higher degrees

Fix an odd prime ℓ and put $h = (\ell - 1)/2$. Under the hypothesis of Theorem 1, at every positive integer k with

$$k \equiv i \pmod{h} \quad \text{for some } 1 \le i \le 50, \qquad K_{4k}(\mathbb{Z})[\ell^\infty] = 0.$$

Indeed $\omega^{-2k} = \omega^{-2i}$, so the two groups have the same dimension modulo ℓ by [7, Thm. 5.1]. Finiteness gives the assertion about the whole primary part. This is useful for removing a fixed prime in higher degrees. For instance, in degree 204 a repetition of one of these 50 indices requires $\ell - 1 \mid 2(51 - i)$, hence $\ell \le 101$. All such primes already lie in the published small-prime range. The corollary by itself gives no further large-prime exclusion at degree 204.

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