The Strong Factorial Conjecture for Univariate Binomials

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Abstract

We prove that two consecutive positive factorial moments of a nonzero univariate binomial cannot both vanish over a field of characteristic zero. Equivalently, the two corresponding rational moment polynomials are coprime for every support and every consecutive pair of orders. A Padé resultant bounds the orders at which a common factor can occur. At the remaining orders, prime divisors of short factorial products and the widths of shared Newton polygon edges exclude possible factor degrees. An exact finite certificate completes this range.

1 The problem and the proof

Let K be a field of characteristic zero. The factorial functional on $K[X]$ is the K-linear map $\mathcal{L}$ defined by $\mathcal{L}(X^j) = j!$. Edo and van den Essen's Strong Factorial Conjecture asserts that a polynomial with r nonzero terms has a nonzero factorial moment in every run of r consecutive positive orders [3]. We prove the case of two terms in one variable.

Theorem 1. *For all integers $a \geq 0$, $d \geq 1$, $N \geq 2$ and every nonzero $f = X^a(A + BX^d) \in K[X]$, at least one of $\mathcal{L}(f^{N-1})$ and $\mathcal{L}(f^N)$ is nonzero.*

For a single monomial this follows from the definition. If $AB \neq 0$, put $t = B/A$ and define

$$P_m(t) = \sum_{k=0}^{m} \binom{m}{k} (am+1)_{dk}\, t^k, \qquad R_m(x) = x^m P_m(1/x), \qquad \mathcal{L}(f^m) = A^m (am)!\, P_m(B/A). \tag{1}$$

Here $(y)_j = y(y+1)\cdots(y+j-1)$ and $(y)_0 = 1$. The integral polynomials R_m are monic, with nonzero constant term. It suffices to prove $\gcd(R_{N-1}, R_N) = 1$ over $\mathbb{Q}$, since a rational Bézout identity holds over every K. Conversely, a nonconstant common factor gives a common root over $\mathbb{C}$ and hence two consecutive zero moments.

Rocks studied consecutive coprimality for two monomials and proved several cases, using the same coefficient polynomials [8, Section 3.4]. For the constant-plus-power support $\{0, d\}$, Filaseta and Rocks proved that, for each fixed d, the orders at which P_m is irreducible have lower density at least $\log 2$ [4, Theorem 1.1]. Our conclusion concerns every pair of consecutive orders and allows both exponents to vary.

Our proof uses prime divisibility at small orders and combines it with a complex estimate at large orders. A Padé truncation of $e^z R_{N-1}(-z^d)$, after normalization, has integral coefficients and a long gap in its Taylor series. A Wronskian gives a nonzero resultant with a factor lying above the roots of a putative common factor. Primes dividing the factor's constant term give a lower bound for the resultant by divisibility. The complex estimate, averaged over rotated roots, gives an incompatible upper bound. The remaining orders are finite. There the prime sieve and Newton polygons leave only a short list of full polynomial gcd computations.

For an example of the prime sieve, take $a = 8$, $d = 3$, $N = 25$ and $p = 101$. In both factorial intervals $(8m, 11m]$, $m = 24, 25$, the only multiple of p is 202. All binomial coefficients are p-adic units. The coefficient valuations of P_{24} jump from zero to one at $k = 4$, and those of P_{25} at $k = 1$. The lower Newton polygons give positive reciprocal root valuations $1/21$ and $1/25$, respectively. Every root of R_{25} has positive valuation, since 202 appears in the first product of three factorial arguments. Thus the two orders have no common root. In general, primes in the first few factorial arguments constrain the constant term of a common factor. Newton polygons then restrict the factor degrees allowed at those primes.

2 Reduction of the exponent range

Write $P_m = P_{m,a,d}$ for the normalized moment polynomial and $R_m(x) = x^m P_m(1/x)$ for its monic reciprocal. Put $b = a + d$ and $s = d^2/b$. For $x \in \mathbb{R}$ and even m, the Gamma integral gives

$$R_m(x) = \frac{1}{(am)!} \int_0^\infty u^{am}(x + u^d)^m e^{-u}\, du > 0.$$

This observation will also be used in the arithmetic argument.

Lemma 2 (Root annuli and separation of the shapes)**.** *For $a \geq 1$, every root x of R_m satisfies*

$$am < |x|^{1/d} \leq (a + d)m.$$

For any $a \geq 0$, the pair P_{N-1}, P_N is coprime if $a \geq d(N - 2)$. If $N \geq 25$, it is also coprime if $6a \geq (5N - 11)d$.

The proof is in Appendix A. It uses the root annuli of the unit-step Gamma polynomials and the fact that these annuli multiply under composition in the binomial basis. The strengthened inner radius matters because the factorial arguments in the two moments grow at different rates.

Lemma 3 (Constant plus a power)**.** *For $a = 0$ and every $d \geq 1$, the consecutive polynomials $P_{N-1,0,d}, P_{N,0,d}$ are coprime over $\mathbb{Q}$.*

Proof. In this case $R'_N = NR_{N-1}$, by coefficient comparison. The squarefreeness proof in Appendix B therefore gives the result. □

Lemma 4 (Endpoint normalization)**.** *For $a \geq 1$ define*

$$G_m = \left(\frac{((a + d)m)!}{(am)!} \right)^{1/m}, \qquad P_m(w/G_m) = 1 + w^m + \sum_{k=1}^{m-1} c_{m,k} w^k.$$

For $1 \leq k < m$ one has

$$0 < c_{m,k} \leq \binom{m}{k} \exp\left(-\frac{s\, k(m - k)}{2m} \right). \tag{2}$$

For $N \geq 2$ the centers satisfy

$$G_N/G_{N-1} > (N/(N - 1))^{d/2}.$$

Proof. The consecutive differences of $\log((am+dk)!)$ are the logarithms of rising products of length d. For $1 \le k < m$ the second difference is

$$\sum_{j=1}^{d} \log\left(1 + \frac{d}{am + d(k-1) + j}\right) \;\ge\; \frac{d^2}{(a+d)m}.$$

Indeed, $\log(1+v) \ge v/(1+v)$ for $v \ge 0$. Subtracting the quadratic with this second difference and comparing with the chord through the endpoints proves (2).

For the second claim, let $f(u) = \log\Gamma(u+1)$ and $H_0(x) = (f((a+d)x) - f(ax))/x$. The trigamma series gives $f''(u) > 1/(u+1)$, by comparison with the integral. Since $ax \ge 1$ for $x \ge 1$,

$$H_0'(x) = \frac{1}{x^2}\int_{ax}^{(a+d)x} u f''(u)\, du > \frac{d}{2x}.$$

Integrate from $N-1$ to N. □

Lemma 5 (Ratio bound)**.** *For $a \ge 1, d \ge 1, N \ge 25$, if $s \ge \sigma_N$, where*

$$\sigma_N = \begin{cases} 2\log N + 3 & (25 \le N < 200), \\ 2\log N + 2 & (N \ge 200), \end{cases}$$

then P_{N-1}, P_N are coprime.

Proof. Suppose $s \ge \sigma_N$. The coefficient bound in (2) is symmetric in $k, m-k$. Write $\sigma_N = 2\log N + C$ and take $m = N-1$ or N. The following rational constants bound the short and long parts of the coefficient sum.

N	A	E	q	η
$25 \le N < 200$	.224	1.218	.52	7/8
$N \ge 200$	.368	1.033	.24	19/20

For $1 \le k \le 4$, use $\binom{m}{k} \le N^k/k!$ in (2). The factor N^k cancels $\exp(-k\log N)$. The resulting bound is $(AE^k)^k/k!$, since

$$A \ge e^{-C/2}, \qquad E \ge \exp\left(\frac{\log N + C/2}{N-1}\right).$$

For $5 \le k \le m/2$, put $u = k/m$ and use $\binom{m}{k} \le (e/u)^k$. The logarithm of the resulting bound, divided by k, is $1 - \log u - \sigma_N(1-u)/2$. Its second derivative in u is $1/u^2 > 0$, so it is bounded above by the larger value at $u = 5/m$ and $u = 1/2$. The corresponding exponentials are bounded by $eAE^5/5$ and $2e\, e^{-\sigma_N/4}$. On a row with fixed C, the quotient $(\log N + C/2)/(N-1)$ decreases because its derivative has numerator $1 - 1/N - \log N - C/2 < 0$. The second endpoint exponential decreases because σ_N increases. Thus it suffices to check the bounds on E and q at $N = 25$ and $N = 200$, respectively.

The two rows satisfy

$$2\sum_{k=1}^{4} \frac{(AE^k)^k}{k!} + \frac{2q^5}{1-q} < \eta.$$

The supplement checks this rational inequality and encloses the exponentials at the two endpoints. By symmetry the displayed sum bounds $\sum_{k=1}^{m-1} c_{m,k}$, including the middle coefficient when m is

even. At a root w of $P_m(w/G_m)$ with $|w| \le 1$, $1 \le |w|^m + \sum_{k=1}^{m-1} c_{m,k} < |w|^m + \eta$. Apply the same argument to $1/w$ in the reversed polynomial when $|w| \ge 1$. Every root in the normalized variable then satisfies

$$(1-\eta)^{1/m} < |w| < (1-\eta)^{-1/m}.$$

At a common root the two values of $|w|$ have ratio G_N/G_{N-1}. By Lemma 4 this is too large for the two annuli, since

$$\frac{d}{2}\log\left(\frac{N}{N-1}\right)\frac{N(N-1)}{2N-1} \;\ge\; \frac{d(N-1)}{2(2N-1)} \;>\; -\log(1-\eta).$$

For the last inequality, $d > s \ge \sigma_N$ gives $d \ge 10$ in the first row and $d \ge 13$ in the second. The minimum of the rational lower bound in each row occurs at the smallest N and the smallest d. The resulting bounds are $120/49$ and $2587/798$, which exceed $\log 8$ and $\log 20$, respectively. The supplement encloses both logarithms and checks the lower bounds on d. This contradicts a common root. □

Lemma 6 (Support reduction)**.** *If $N \ge 25$ and $P_{N-1,a,d}, P_{N,a,d}$ have a common root, then*

$$a \ge 5, \qquad 2 \le d < 3a, \qquad a < d(N-2), \qquad 6a < (5N-11)d, \qquad s < \sigma_N,$$

where $s = d^2/(a+d)$.

Proof. Lemma 3 disposes of $a = 0$. Lemma 20 covers $a = 1, d \ge 3$ and $a \ge 2, d \ge 3a$. The remaining small positive lower exponents and the adjacent exponents are dealt with in Proposition 19. These exclusions give $a \ge 5$ and $2 \le d < 3a$. Lemma 2 gives the two bounds on the shape. Lemma 5 gives the bound on s. □

2.1 Terminals through twenty-four

Proposition 7. *Theorem 1 holds for $2 \le N \le 24$.*

Proof. Lemma 3 covers $a = 0$. At $N = 2$, R_1 has a real root and R_2 is positive on $\mathbb{R}$ by its gamma integral. Take $a \ge 1$ and $3 \le N \le 24$. The weak annulus in Lemma 2 gives $a < d(N-2)$ if the two polynomials share a root.

Write $s = d^2/(a+d)$ and use the endpoint normalization of Lemma 4. If $s \ge 20$, the mixed coefficients at either order have sum less than $1/2$. Indeed their upper bound in (2) is symmetric, and for $k \le m/2$ the exponential term is at most e^{-5k}. Thus the sum is bounded by

$$2\big((1+e^{-5})^{24} - 1\big) < 2\big(e^{6/35} - 1\big) < \frac{12}{29} < \frac{1}{2}.$$

Here $e^{-5} < 1/140$, and $e^x < 1/(1-x)$ for $0 < x < 1$. In the normalized variable the roots have moduli between $2^{-1/m}$ and $2^{1/m}$, by comparison of $1 + w^m$ with the mixed terms and by reversal. Since $d > s \ge 20$,

$$\frac{d}{2}\log\left(\frac{N}{N-1}\right)\frac{N(N-1)}{2N-1} \ge \frac{d(N-1)}{2(2N-1)} \ge 4 > \log 2.$$

The center ratio in Lemma 4 therefore separates the annuli. A common root requires $s < 20$.

We next show that $d < 43$. If $d \ge 43$, set $r = \lfloor d/43 \rfloor$. Then $43 \le d/r < 86$, and the interval $(aN/r, (aN+2d)/r]$ has length at least 86. Its upper endpoint is below $86[N(N-2)+2] \le 45580$. The bound $s < 20$ gives $a > d^2/20 - d \ge d^2/43 \ge rd$, so the left endpoint exceeds dN. The first

record prime gap larger than 86 starts at 360653 [7, Table 8]. It follows that the interval contains a prime $p > dN$, and that rp belongs to the first $2d$ factorial terms at order N.

Since $a < d(N-2)$, we have $p^2 > (a+d)N$. Each factorial interval $(am, (a+d)m]$ contains at most one multiple u_m of p, and all binomial coefficients are p-adic units. A positive valuation of a reciprocal root has the form $1/w_m$, where $w_m = 1 + \lfloor ((a+d)m - u_m)/d \rfloor$. Equal positive valuations at both orders would imply $u_N - u_{N-1} \in [a+1, a+2d-1] \subset (0, p)$, which is impossible. Thus all roots of a common factor would be p-adic units. But the first $2d$ terms at order N contain rp, so $R_N \bmod p$ has at most one root away from zero, counted with multiplicity. A monic common divisor reduces with its degree unchanged, and would have nonzero constant term and even degree at least two. This contradiction proves $d < 43$.

It remains to test the integer triples

$$3 \le N \le 24, \qquad 1 \le d \le 42, \qquad 1 \le a < d(N-2), \qquad d^2 < 20(a+d).$$

The supplementary verifier exhausts this set and finds 217 952 triples. At each triple it forms the two monic reciprocals modulo 1 000 000 007, a prime certified by trial division, and computes their gcd by Euclidean division. Every gcd is one. A monic integral common divisor would remain a nonconstant common divisor on reduction. This proves the proposition. □

3 Prime bounds and the finite range

Take $a, d \ge 1$, $N \ge 25$, $d < 3a$, $6a < (5N-11)d$, and $s = d^2/(a+d) < \sigma_N$, as in Lemma 6. In this section m runs through $N-1, N$, $b = a+d$, $M = bN$, and F is a nonconstant monic rational common divisor of $R_m(x) = x^m P_m(1/x)$. By Gauss's lemma $F \in \mathbb{Z}[x]$, and $F(0) \neq 0$. Write $h = \deg F$ and $H = dh$. The reciprocal of even order is positive on $\mathbb{R}$, as is seen by integrating $(x+u^d)^m$ against the gamma density of shape $am+1$. Hence h is even.

In fact $h < N-1$. To see this, put $n = N-1$ and $u = an+1$. If $P_n \mid P_{n+1}$, the quotient is $1+ct$. Its linear coefficient would have to be both

$$c_1 = (n+1)(u+a)_d - n(u)_d, \qquad c_2 = \frac{(u+dn)_b}{(u)_a}.$$

Set $A = (u)_b$, $B = (u)_a(u)_d$, and $f(z) = (u+z)_b$. Then $(u)_a(c_2 - c_1) = f(dn) - (n+1)A + nB$. Convexity gives $f(dn) \ge nf(d) - (n-1)A$, and $f(d)/A = (u+b)_d/(u)_d > A/B$. Thus $f(d) - 2A + B > (A-B)^2/B \ge 0$, proving $c_2 > c_1$.

A sufficiently large prime can occur in a common root valuation only through a multiple close to the end of each factorial product. The locations of those multiples must agree to within one coefficient step after translation by b. We first use this observation to restrict primes in the two initial factorial blocks.

For a prime p put $y_m(k) = v_p([t^k]P_m)$ and

$$U_m(j) = y_m(m) - y_m(m-j) = v_p\left(\prod_{i=0}^{dj-1} (bm-i)\right) - v_p\binom{m}{j}.$$

A lower Newton slope λ of P_m is a valuation of roots of R_m. Its width is the number of roots at that valuation, counted with multiplicity. Equivalently the extremes of the maximizing indices of $U_m(j) - \lambda j$ have this difference.

Lemma 8 (Degree allocations at a prime)**.** *For a common slope with reduced denominator v, let w be the smaller of its two widths. The common factor can use only a number of roots from $\{0, v, 2v, \dots, w\}$ at that slope. Consequently h belongs to the sum of these sets over the common slopes, at every prime. In particular each positive valuation of a root of F is at least $1/h$.*

Proof. At each slope the roots of a common factor occur with at most the smaller multiplicity. Its Newton polygon has integral vertices, so each width in that polygon is divisible by the reduced denominator of the slope. The last assertion follows since such a width is at most h. □

Lemma 9 (Prime support of short blocks)**.** *Set*

$$C_N = \max\{a + 2d - 1, \sqrt{M}\}, \qquad B_h = \min\{dN, \max(2H, C_N)\}.$$

Every prime divisor of $F(0)$ or of either block $\prod_{i=1}^{H}(am + i)$ is at most B_h.

Proof. If $p > dN$, then $p^2 > M$ because $a < d(N - 2)$, and all binomial coefficients are p-adic units. Each full factorial interval $(am, bm]$ has at most one multiple u_m of p. If present, it yields only one positive slope, equal to $1/w_m$ with $w_m = 1 + \lfloor (bm - u_m)/d \rfloor$. A common positive slope would require the same w_m in both orders. It would give

$$u_N - u_{N-1} \in [b - d + 1, b + d - 1] = [a + 1, a + 2d - 1]. \tag{3}$$

This interval is positive and below p, so it contains no multiple of p.

Now let $p > \max(2H, C_N)$ and suppose that F has a positive slope λ at p. By Lemma 8, $\lambda \geq 1/h$. Since $p^2 > M$, the count of terminal multiples gives $U_m(j) < 1 + dj/p$ for $j > 0$. The maximizing value of $U_m(j) - \lambda j$ is at least zero, so every maximizing index is less than $2h$. Throughout this range $j < p$ and $dj < p$. The binomial valuation cannot decrease because $j!$ is a unit. The factorial valuation can rise only once and by one. The weighted ordinate therefore decreases strictly between jumps. A positive slope must tie zero to the index of the single factorial jump, giving slope $1/w_m$ with $w_m = 1 + \lfloor (bm - u_m)/d \rfloor$. As before, a shared slope implies (3), which is impossible since $p > C_N$.

These two cases show that $p > B_h$ cannot divide $F(0)$. If it divides one of the first H factorial terms, all the coefficients at $k \geq h$ in P_m vanish modulo p. Then $R_m \bmod p$ is a power of x times a polynomial of degree at most $h - 1$. Its monic divisor $F \bmod p$ has degree h and nonzero constant term, a contradiction. □

3.1 The degree bound

Lemma 10 (Degree bound from the first factorial block)**.** *If $aN + 1 > \max\{H + 13, 279H/262\}$, then $2H < C_N$. The starting-point inequality holds whenever $a \geq 2d$, or whenever $N \geq 275$ and $aN \geq 1375$. In particular the second condition covers $a \geq 5$ at these terminals and every $a \geq 1$ when $N \geq 3150$.*

Proof. Theorem 1(a) of Laishram and Shorey [5] gives a prime divisor greater than $2H$ of the first H consecutive integers beginning at $aN + 1$, under the stated starting-point inequality. Lemma 9 then gives $2H < C_N$. If $a \geq 2d$, the inequality follows from $h \leq N - 1$ and $N \geq 25$.

It remains to establish the starting-point inequality for $N \geq 275$, $aN \geq 1375$, and $a < 2d$. Put $X = aN$, $w = X/40$, and consider three intervals of length w whose left endpoints are

$$X, \qquad X + 2w + 4d, \qquad X + 6w + 12d.$$

Take each interval open at its left endpoint. Their union is contained in $(X, X+E]$ with $E = 7w + 12d < X/3$, since $d < 3a$. Each interval contains a prime. Indeed $X \geq 1375$. For $X \leq 7160$, $7160 < X \leq 11750$, and $11750 < X < 89693$, respectively, the upper endpoint $4X/3$ lies below $9551, 15683, 155921$, where the record prime gaps first exceed $34, 36, 72$, respectively [7, Table 8]. In each range w exceeds the corresponding gap. For $X \geq 89693$, Dusart [2, Proposition 5.4] gives a prime within $x/\log^3 x < w$ of each left endpoint $x \in [X, 4X/3)$.

Discard any interval meeting

$$I_j = [(M-b)/j, (M+2d)/j] \qquad (1 \leq j \leq 4).$$

The envelope misses I_1, since $M - b > (3/2)(1-1/N)X > 4X/3$. If I_j, $2 \leq j \leq 4$, meets the envelope, then $M + 2d > jX$. For $j' < j$ this implies

$$(M-b)/j' > X + (X - a - 3d)/3 > X + E,$$

where $E/X < 7/40 + 36/275 < (1 - 10/275)/3$. Thus at most one I_j meets the envelope. Its width is at most $(a+3d)/2 < 5d/2$, smaller than the separation of the three intervals. We obtain two primes $p_1 < p_2$ in distinct retained intervals.

For either retained prime p, we have $p > X \geq m$ and $p^2 > M$ because $b < 4a$. Therefore $y_m(k) = \lfloor (am + dk)/p \rfloor$. The avoidance of the I_j gives the same $q = \lfloor bm/p \rfloor \in \{1, 2, 3\}$ at both orders and $(q+1)p - bm > 2d$. The right endpoints of the constant-ordinate intervals before the jumps are

$$k_j(m) = \left\lfloor \frac{jp - am - 1}{d} \right\rfloor \qquad (1 \leq j \leq q).$$

Every positive edge before the last of these endpoints joins two such endpoints and has slope at most $1/\lfloor p/d \rfloor$. This follows by subtracting their floor expressions, since an edge of height J then has width at least $J\lfloor p/d \rfloor$. The edge from $k_q(m)$ to m has height one and width

$$w_m = \lceil (bm - qp + 1)/d \rceil \leq \lceil p/d \rceil - 2 < \lfloor p/d \rfloor.$$

All remaining points lie on or above this final segment. Its slope is larger than all earlier slopes, and $w_N > w_{N-1}$ since $b > d$. Neither final slope is shared.

For an interior positive edge of height J_e in P_N, its width is $w_e = J_e p/d + \epsilon_e$, with $|\epsilon_e| < 1$. If the factor uses r_e of its roots, the quantity $r_e J_e / w_e$ is an integer by Lemma 8. Summing over these edges yields

$$h = h_0 + Jp/d + \epsilon, \qquad 0 \leq h_0 < (p - X)/d, \qquad J \in \{0, 1, 2\}.$$

Here h_0 counts slope-zero roots and J is at most the total interior height $q - 1$. The error satisfies $|\epsilon| < 2$ when $J > 0$, and is zero when $J = 0$. For p_1, p_2, the intervals of possible h with equal positive J are disjoint because $p_2 - p_1 > p_1 - X + 4d$, by the placement of the intervals. Intervals for unequal J are disjoint as well, since $X > 2E + 4d$. Thus $J = 0$ at both primes. The smaller prime comes from the first or second interval, giving

$$H < p_1 - X \leq 3w + 4d < .12X.$$

As $X \geq 1375$, this implies the required starting-point inequality. □

3.2 Bounds from records of prime gaps

For an even record gap G, write X_G for the lower prime endpoint of the first larger record gap in [7, Table 8]. Every real interval $(x, y]$ with $x \geq 2$, $y < X_G$ and $y - x \geq G$ contains a prime. Otherwise the consecutive primes straddling x have gap greater than G and lower endpoint below X_G.

Lemma 11 (Uniform gap and degree bounds)**.** *Let $G \geq 52$ be an even record gap such that*

$$G \geq 4\sigma_N, \qquad G\left(\frac{N(5N-11)}{6}+2\right) < X_G. \tag{4}$$

Then $d < G/2$. If Lemma 10 applies, then also $H < G$.

Proof. Put $g = G/2$. If $d \geq g$, take $r = \lfloor d/g \rfloor$. Then $g \leq d/r < 2g$, so $(aN/r, (aN + 2d)/r]$ has length at least G. Its upper endpoint is below X_G by $6a < (5N - 11)d$ and (4). Its left endpoint exceeds dN, since $s < \sigma_N$ and $d \geq g \geq 2\sigma_N$ give $a > d^2/\sigma_N - d \geq d^2/g \geq rd$. A prime p in the interval divides rp among the first $2d$ terms of the upper factorial block, contrary to Lemma 9.

Suppose now that $2H < C_N$ and $H \geq G$. Take $r = \lfloor H/G \rfloor$. The interval $(aN/r, (aN + H)/r]$ has length at least G, and its upper endpoint is less than

$$2G\left(\frac{N(5N-11)}{6h}+1\right) \leq G\left(\frac{N(5N-11)}{6}+2\right) < X_G.$$

The left endpoint exceeds two since $r \leq H/G < dN/G$ and $aG/d > G/3$. If $C_N = \sqrt{M}$, then $aN/r \geq aGN/H > 2aG\sqrt{M}/b > \sqrt{M}$. If $C_N = a + 2d - 1$, then $2H < C_N$ and $h \geq 2$ give $a > 2d$. Since $d < G/2$, we have $r < N/2$ and $aN/r > 2a > C_N$. Thus in both cases the interval supplies a prime $p > C_N$ dividing one of the first H terms. This contradicts Lemma 9. □

Table 1 gives choices of G for Lemma 11. Use the first applicable row. The inequalities in (4) follow by checking the right endpoint of each row, since $2\log N + 3$ is an increasing upper bound for σ_N.

Upper terminal N	G	X_G
50	72	155 921
100	114	1 349 533
300	154	17 051 707
1 000	222	189 695 659
2 000	282	1 294 268 491
6 000	354	10 726 904 659
10 000	456	42 652 618 343
19 999	468	182 226 896 239
32 767	516	461 690 510 011
65 535	602	2 614 941 710 599
3 269 017	1132	43 841 547 845 541 059

Table 1: Prime-gap bounds used to control d and dh. The record endpoints are from [7, Table 8].

Lemma 12 (Refinement using the actual factorial position)**.** *Assume $2H < C_N$ and $H \leq H_0$. Let $G \geq 52$ be an even record gap. If $H_0/G < N/2$ and $aN + H_0 < X_G$, then $H < G$ and $d < G/2$.*

Proof. If $H \geq G$, the interval with $r = \lfloor H/G \rfloor$ in the second part of Lemma 11 has upper endpoint below X_G by assumption, as well as length at least G. The two comparisons with C_N in that proof still hold, using $r \leq H_0/G < N/2$ in the second case. The resulting prime gives the same contradiction. □

In particular $H \leq 281$ and $d \leq 140$ for $1001 \leq N \leq 3295$ whenever Lemma 10 applies. Starting with $G = 354$ from Table 1, apply Lemma 12 successively with

G	upper terminal N	X_G
320	5000	3 842 610 773
292	4150	2 300 942 549
288	3467	1 453 168 141
282	3295	1 294 268 491

At each step $aN + H_0 < (5/6)D_0N^2 + H_0 < X_G$, where $d \leq D_0$ and $H \leq H_0$ are the preceding bounds.

3.3 Exact computation in the finite range

Proposition 13. *Under the support reduction of Lemma 6, the consecutive polynomials are coprime for $a \geq 5$, $d \geq 2$, and $25 \leq N < 3150$.*

Proof. Use $d^2 < 19b$, which follows from $\sigma_N < 19$ in this range, and use the gap caps

N	25–50	51–100	101–300	301–1000	1001–3149
$d \leq$	35	56	76	110	140.

These come from Lemmas 11 and 12. For $2200 \leq N < 3150$, the analytic exclusions in Appendix D also require $b > N$ and $97d < 3a$.

For each N, a, the inequalities $6a < (5N - 11)d$, $2 \leq d < 3a$, the displayed cap, and the two additional inequalities when $N \geq 2200$ give an integer interval $[d_0, d_1]$ for d. When $N \geq 275$ or $a \geq 2d$, Lemma 10 reduces the bound in Lemma 9 to $\min(dN, C_N)$, independent of h. Otherwise use dN. The resulting bound is nondecreasing in d, including where it switches to dN. So a prime exceeding the bound at d_1 in either first $2d_0$-term block eliminates the whole interval. The remaining gaps are tested individually with their first $2d$ terms and the conditions $a < d(N - 2)$, $d^2 < 19b$.

For each remaining support, try the possible even degrees $2 \leq h < N-1$. Test the starting-point inequality directly when $N < 275$. Impose the degree bound when it applies, and test the full first H terms with Lemma 9 in both orders.

The remaining degrees are tested at odd primes $aN < p \leq M$. Here $p > m, d$, $p^2 > M$, and $am < p$. The ordinate is $\lfloor (am + dk)/p \rfloor$ and $q_m = \lfloor bm/p \rfloor \leq 3$. Thus the lower hull uses at most five points

$$(0, 0), \qquad \left(\left\lfloor \frac{jp - am - 1}{d} \right\rfloor, j - 1 \right) \quad (1 \leq j \leq q_m), \qquad (m, q_m).$$

Every omitted point is to the left of a retained point of the same ordinate and lies on or above the retained lower hull, which is nondecreasing. Compute the allowed degree sum from Lemma 8 at each prime and intersect it with the surviving degrees. For example $N = 25, a = 5, d = 8, p = 127$ gives hull slopes $1/16, 1/8$ and $1/16, 1/9$ at orders 24 and 25, respectively. Only degrees zero and sixteen are allowed at this prime.

The supplementary verifier exhausts these integer intervals. The largest factorial argument is 1 156 520 634. A segmented sieve stores smallest odd prime factors through that limit. Trial divisions from this table and cross multiplication of slopes perform the tests above. In prime caps the square-root threshold is replaced by its integer part, checked by squaring. The initial prime-cap test leaves 21 583 groups (N, a). The individual gap test leaves 74 362 supports. The degree and prime-block tests exclude 62 593, and the Newton allocations exclude 11 705. For the remaining 64 supports, the verifier forms both monic reciprocals modulo the prime 1 000 000 007 and checks by Euclidean division that their gcd is one. The modulus is checked by trial division. A monic integral common divisor would retain its degree on reduction, so these computations prove rational coprimality. □

4 An exponential Padé estimate

Let F be a monic irreducible common factor of the reciprocal moment polynomials R_{N-1}, R_N. Write $h = \deg F$, $H = dh$, $b = a + d$, $M = bN$, $m = N - 1$, $s = d^2/b$, and $\alpha = d/b$. We use the root bounds $ai < |x|^{1/d} \le bi$ for $R_i(x) = 0$ with $i = m, N$, from Lemma 2. In particular, $H \log(aN) < \log|F(0)| < H \log M$. The Padé construction works on $z^d = -x$, where the root bound is linear in the moment order. Its key feature is that an integral polynomial approximates the same polynomial to high order after multiplication by each of the d exponentials $e^{-\omega z}$, $\omega^d = 1$. The resulting Wronskian makes at least one of the approximants nonzero above every root of F. The comparison of an integer resultant with its local estimates adapts the Laguerre construction in Nguyen-Dang's preprint [6, Proposition 3.9]. The following proof derives the rotated truncations, their Wronskian and both local estimates for the binomial moments.

Lemma 14 (Padé resultant inequality)**.** *Assume $N \ge 200$, $a, d \ge 1$ and $d < 3a$. Set $r = (b-1)m$, and put*

$$K_\pm = \tfrac{1}{2} \pm \tfrac{\alpha}{14}, \qquad \rho_0 = b(1 - \alpha K_+) = a + dK_-,$$
$$g = \max\{b \log(b/\rho_0) - b + \rho_0,\ \rho_0 - a - a \log(\rho_0/a)\}.$$

There is a monic irreducible factor G of $F(-z^d)$ of degree $D = hk$, where $1 \le k \le d$, and $K = |G(0)|$ satisfies $K^d = |F(0)|^k$. Write $V = \log(M+1)$. At each complex root z of G one may take the gain

$$L(z) = \max\left\{(m! e^{-mg-1})^{1/(m+1)}, \frac{am - \Re z}{d}\right\}.$$

With $\Lambda = (\prod_{G(z)=0} L(z))^{1/D}$, one has

$$\log \Lambda < \sum_{p \mid K} \min\left\{\frac{\mathrm{v_p}(K)}{D}, \frac{1}{p-1}\right\} \log p + \frac{\omega(K) V}{N}. \tag{5}$$

Here $\omega(K)$ is the number of distinct prime divisors. In addition, the first entry in $L(z)$ can use $g = 0$ when $\Re z \le am$. For $N \ge 200$ the first entry is greater than $me^{-1.01-g}$. For $d = 1$ the additional gain $m\sqrt{a}\, e^{-1/2}/(1 + a^{-1/2})$ is available.

Proof. Let $B(z) = (am)! R_m(-z^d)/m!$ and let $Q(z)$ be the Taylor polynomial of $e^z B(z)$ through degree r. For $0 \le j \le bm$ and $v = bm - j$, direct expansion gives

$$[z^j] e^z B(z) = \frac{v!}{m!} [w^v] (1+w)^{am} ((1+w)^d - 1)^m. \tag{6}$$

Indeed the term indexed by i in R_m contributes $(-1)^{m-i}\binom{m}{i}(am+di)!/(m!(am+di-v)!)$, taken as zero if $am+di<v$. Thus Q is integral, its top coefficient is d^m, and the coefficients of degrees $r+1$ through bm vanish.

For $\omega^d=1$ put $E_\omega(z)=e^{-\omega z}Q(\omega z)$. Let $T=e^{-z}Q-B$, which vanishes to order at least $bm+1$. Since $B(\omega z)=B(z)$, we have $E_\omega-B=T(\omega z)$. Form the Wronskian of B and the d functions E_ω with derivative orders $0,\dots,d$. Factor $e^{-\omega z}$ out of each E_ω column. The resulting polynomial determinant has degree at most $dm+dr=dbm$. Its top coefficient comes from row zero of the B column and the determinant $((-\omega)^i)_{1\le i\le d,\,\omega^d=1}$, which is a Vandermonde determinant times $\prod_{\omega^d=1}(-\omega)$. The top coefficient is nonzero.

To calculate the vanishing order, subtract the B column from the original E_ω columns. The invertible change of columns taking the sums $d^{-1}\sum_{\omega^d=1}\omega^{-(bm+j)}T(\omega z)$, for $1\le j\le d$, retains one residue class of exponents in each column. Their orders are at least $bm+j$. In each determinant term the derivatives subtract at most $\sum_{i=0}^{d} i$ from the total order, so the Wronskian vanishes to order at least dbm. Removing the exponential factors preserves this order. Hence the polynomial determinant is a nonzero multiple of z^{dbm}. At any nonzero root of B the zeroth row consequently has $Q(\omega z)\neq 0$ for at least one ω. Choose G to be the minimal polynomial of such a rotation above a root of F. Then $\mathrm{Res}(G,Q)$ is a nonzero integer, and every root of F has exactly $k=D/h$ lifts in G. The norm identity follows by multiplying $z^d=-x$ over them.

For $\ell\ge 1$, a beta integral gives the tail coefficient

$$[z^{bm+\ell}]e^zB(z)=\frac{(-1)^m}{m!(\ell-1)!}\int_0^1 t^{am}(1-t^d)^m(1-t)^{\ell-1}\,dt.$$

With $c_v=[t^v](1+t+\cdots+t^{d-1})^m$, the same coefficient is

$$(-1)^m\binom{m+\ell-1}{m}\sum_v c_v\frac{(am+v)!}{(am+v+m+\ell)!}. \tag{7}$$

Let p be prime and $e=v_p(z)>0$ in an embedding into $\overline{\mathbb{Q}}_p$, normalized by $v_p(p)=1$, and set $\mu=1/(p-1)$. The base-p digit sum $s_p(n)$ satisfies $s_p(n)/(p-1)\le\log_p(n+1)$, since $p^{u/(p-1)}\le u+1$ for each digit u, and the product of the $u+1$ counts integers whose digits do not exceed those of n. If $e\le\mu$, the coefficient of z^{r-j} in (6) contains the integer factor $(m+j)!/m!$. By Legendre's formula its valuation is at least $j\mu-\log_p(M+1)$, since $m+j\le bm$. The valuation after multiplication by z^{r-j} is at least $re-\log_p(M+1)$. If $e>\mu$, the p-adic exponential converges and $Q(z)$ is the negative tail at a root of G. In (7) put $n=am+v\le(b-1)m$. Legendre's formula gives

$$v_p\left(\frac{n!}{(n+m+\ell)!}\right)\ge-(m+\ell)\mu-\mu s_p(n)\ge-(m+\ell)\mu-\log_p(M+1).$$

After multiplication by $z^{bm+\ell}$, the bound increases with ℓ and has its minimum at $\ell=1$. In both cases,

$$v_p(Q(z))\ \ge\ (bm+1)e-N\min(e,\mu)-\log_p(M+1). \tag{8}$$

For $e=0$, integrality gives $v_p(Q(z))\ge 0$.

Now let z be a complex root and set $R=|z|$. Integrate each $v!$ in (6) and use the nonnegative coefficients of the polynomial in w. Also sum the beta tail before taking its absolute value. These two operations give

$$\frac{|Q(z)|}{R^{bm+1}}\le\frac{1}{m!}\min\left\{\begin{array}{l}\displaystyle\int_0^\infty e^{-Ru}(1+u)^{am}((1+u)^d-1)^m\,du,\\[2mm]\displaystyle\int_0^1 e^{u\Re z}(1-u)^{am}(1-(1-u)^d)^m\,du\end{array}\right\}. \tag{9}$$

In the second integral $1-(1-u)^d \le du$. When $am > \Re z$, integrating $u^m e^{-(am-\Re z)u}$ on the positive ray gives the second gain in the lemma. When $\Re z \le am$, the integral is at most one. If $R \le m\rho_0$, maximizing the exponent $u\rho_0 + a\log(1-u)$ gives the upper bound $e^{m(\rho_0 - a - a\log(\rho_0/a))}$. If $R \ge m\rho_0$, bound the first integrand using $((1+u)^d-1)^m \le (1+u)^{dm}$ and reserve e^{-u} for integration. The maximum of the remaining exponent divided by m is $g_+(\rho_0 - 1/m)$, where $g_+(v) = b\log(b/v) - b + v$. Since $\rho_0 - b/2 = a/2 - s/14 > 19a/56 > 1/m$, the interval $[\rho_0 - 1/m, \rho_0]$ lies in $(b/2, b)$, where $g_+'(v) = 1 - b/v > -1$. Therefore $g_+(\rho_0 - 1/m) < g_+(\rho_0) + 1/m$ and $|Q(z)| < R^{bm+1}/L(z)^N$. Stirling's lower bound gives the simpler gain once $0.01(m+1) + \frac{1}{2}\log(2\pi/m) > 0$. This expression increases for $m \ge 199$, since its derivative is $0.01 - 1/(2m)$. At 199 it is positive by $2\pi > 6$ and $e^4 > 199/6$, using the first five terms of the exponential series. When $d = 1$, use the first integral directly, with $((1+u)^d - 1)^m = u^m$. The tangent inequality

$$\log(1+u) \le \log(1+v) - \frac{v}{1+v} + \frac{u}{1+v}, \qquad v = a^{-1/2},$$

bounds the constant exponent by $m/2$, since $\log(1+v) - v/(1+v) \le v^2/2$. The coefficient of $-u$ is $R - am/(1+v) > m\sqrt{a}/(1+v)$. Integrating u^m against that exponential proves the additional gain.

Take the product of the complex bounds over the D roots and compare with the prime factorization of $\mathrm{Res}(G, Q)$ using (8). At a prime dividing K the sum of the positive e's is $v_p(K)$, and the sum of $\min(e, \mu)$ is at most $\min(v_p(K), D\mu)$. Cancellation of the common term $(bm+1)\log K$ gives (5). □

For later estimates, the two losses defining g, divided by s, are the functions

$$K_+^2 \int_0^1 \frac{u\,du}{1 - \alpha K_+ u}, \qquad K_-^2 \int_0^1 \frac{u\,du}{1 - \alpha + \alpha K_- u}. \tag{10}$$

The first integrand increases since K_+ increases and $1 - \alpha K_+ u$ decreases. For the second, put $D_- = 1 - \alpha + \alpha K_- u$. Its derivative with respect to α is

$$\frac{uK_-}{D_-^2}\left(K_-(1 - u/2) - \frac{1-\alpha}{7}\right),$$

which is nonnegative for $0 \le u \le 1$, since $K_-/2 \ge 25/112 > 1/7$ on $0 \le \alpha \le 3/4$. At $\alpha = .03, .1, 1/3, .5, .75$, upper bounds for both functions are respectively $.128, .134, .160, .176, 7/32$.

We also need the following refinement of the local estimate. Partial fractions in the tail formula yield

$$[z^{bm+\ell}]e^z B(z) = \frac{(-1)^m d^m}{(\ell-1)!} \sum_{j=1}^{\ell} \frac{(-1)^{j-1}\binom{\ell-1}{j-1}}{\prod_{i=0}^{m}(am + di + j)}.$$

Indeed the coefficient of $1/(x+j)$ in $1/(x+1)_\ell$ is

$$\left(\prod_{\substack{1 \le u \le \ell \\ u \ne j}} (u - j)\right)^{-1} = \frac{(-1)^{j-1}}{(\ell-1)!}\binom{\ell-1}{j-1}.$$

Apply this to $x = am + di$ in the factorial expansion of the tail. The remaining sum over i is $\sum_i (-1)^i \binom{m}{i}/(x + di) = d^m m!/\prod_{i=0}^{m}(x + di)$. The coefficient of $1/(x + di)$ on the right is

$m!/\prod_{t\neq i}(t-i) = (-1)^i\binom{m}{i}$. In both identities, matching the simple poles leaves a polynomial that tends to zero at infinity.

For $e > \mu$, the tail term indexed by ℓ, j has valuation at least

$$(bm+1)e + mv_p(d) - v_p\left(\prod_{i=0}^{m}(am+di+j)\right) + (\ell-1)(e-\mu),$$

using $v_p((\ell-1)!) \le (\ell-1)\mu$. At fixed j the weakest bound is at $\ell = j$. Taking the infimum gives

$$v_p(Q(z)) \ge (bm+1)e + mv_p(d) - \sup_{j\ge 1}\left\{v_p\left(\prod_{i=0}^{m}(am+di+j)\right) - (j-1)(e-\mu)\right\}. \tag{11}$$

4.1 Rotations and the prime support

Write $U_i(j)$ for the reverse Newton ordinates

$$U_i(j) = v_p((bi-dj+1)_{dj}) - v_p\binom{i}{j}, \qquad i = N-1, N.$$

For $L_p = \lfloor \log_p M \rfloor$, the dj factorial arguments contain at most $dj/p^\nu + 1$ multiples of p^ν for $1 \le \nu \le L_p$, and none for $\nu > L_p$. Summing and dropping the binomial valuation gives $U_i(j) \le dj/(p-1) + L_p$. The concave hull has the same affine upper bound, so its value at h bounds the sum of the largest h valuations of roots of R_i. In particular

$$v_p(F(0)) \le H/(p-1) + L_p, \qquad p > 2H \implies v_p(F(0)) \le L_p. \tag{12}$$

Also every prime of either initial block $(ai+1)_H$ divides $F(0)$. Otherwise $F \bmod p$, of degree h and nonzero constant term, would divide a power of x times a polynomial of degree less than h.

Lemma 15 (Cyclotomic restriction)**.** *Above any one root of F, the roots of G in Lemma 14 are a union of regular rotation cosets of common size $k_0 \mid d$. If $q = d/k_0 \ge 2$, and the starting-point inequality of Lemma 10 holds, then*

$$2H < P = M^{1/(q+1)}, \qquad p \mid F(0) \implies p \le P. \tag{13}$$

If $k_0 = d$, then $k = d$.

Proof. Let x be a root of F and adjoin μ_d to $\mathbb{Q}(x)$. Over this field the extension generated by $z^d = -x$ is Galois, and $\sigma \mapsto \sigma(z)/z$ identifies the group with a subgroup of μ_d, of order k_0. Any other lift differs from z by a root of unity already adjoined. This proves the coset assertion and shows that z^{k_0} is a q-th root of $-x$ in $\mathbb{Q}(x, \mu_d)$. At $p > d$ the cyclotomic extension is unramified, by lifting roots of unity of order prime to p from the residue field. Thus every positive integer-normalized local valuation of x in $\mathbb{Q}(x)$ is a multiple of q. With normalization $v_p(p) = 1$ it is at least q/h, and $v_p(F(0))$ is also a multiple of q.

Suppose $p > H + d$, $p^{q+1} > M$, and $p \mid F(0)$. For a selected slope $\lambda \ge q/h$, the affine upper bound puts every maximizing index of $U_i(j) - \lambda j$ below $qh/(q-1) \le 2h$. In this range $j!$ is a unit. The factorial jumps are spaced by at least $h+1$ indices, with heights at most q, and the binomial valuation cannot decrease. Thus the maximizing indices can only be zero and the first factorial jump. By (12), the norm exponent is exactly q. The edge at each order must have height q, since

it accounts for all selected positive slopes. The equal widths then place the difference of the two reached multiples of p^q in $[a+1, a+2d-1]$. Consequently every norm prime is at most

$$\max\{H+d,\ M^{1/(q+1)},\ (a+2d-1)^{1/q}\}.$$

The starting-point inequality supplies an initial-block prime above $2H$ by Lemma 10. If $(a+2d-1)^{1/q} > 2H$, then $s < (1+\alpha)/(4h^2)$, and $b < dN$ gives

$$\frac{M}{(a+2d-1)^{3/2}} > \frac{1}{\sqrt{s}(1+\alpha)^{3/2}} > \frac{2h}{(1+\alpha)^2} > 1.$$

It follows that $M^{1/(q+1)} > (a+2d-1)^{1/q}$ in this case. If the last entry in the maximum is at most $2H$, the initial-block prime directly gives $M^{1/(q+1)} > 2H$. Both cases prove (13). □

A coset of size $n \geq 2$ places at least $\lfloor n/2 \rfloor$ points in the closed left half-plane, where the gain is at least $m(1-\alpha)/\alpha$. We will use a stronger average when α is small.

Lemma 16 (Angular average)**.** *Assume $N \geq 200$ and $\alpha < .03$, and put $g_0 = 1.01 + .128s$ and $C' = (1-\alpha)/(2\alpha)$. In a coset of size $n \geq 2$, order the arguments θ_j by absolute value, and put $u_j = |1 - e^{i\theta_j}|^2$. Let r be the number of $j > 0$ with $u_j < 4\alpha$. The logarithm of the geometric mean of the gains is at least*

$$\log m + (1-1/n)\log C' + \frac{2\log(2n/\pi) - g_0}{n} - \frac{r}{n}(g_0 + \log(2(1-\alpha))) - J, \tag{14}$$

where

$$\frac{r}{n} \leq \frac{2}{\pi}\arcsin\sqrt{\alpha} < .111, \qquad J = \frac{2\log 2}{\pi}\big(\arcsin\sqrt{\alpha} + \sqrt{\alpha(1-\alpha)}\big) < .153.$$

For $n \leq 9$ one has $r = 0$ and may use $J = 2(\log 2)\alpha(n^2-1)/(3n)$. A weaker bound, used below, takes $C = C'/2$ in (14), replaces $\log(2(1-\alpha))$ by $\log(1-\alpha)$, and omits J.

Proof. Where $u_j \geq 4\alpha$, the gain is at least $mC'u_j(1 - 2\alpha/u_j)$. For nonnegative cosine use $|z| \leq bm$ in $am - \Re z$. For negative cosine use $|z| > am$. Use the baseline me^{-g_0} at the nearest argument and at the other r arguments. The cyclotomic product gives $\prod_{j>0} u_j \geq (2n/\pi)^2$, because $|\theta_0| \leq \pi/n$ and $|\sin(n\theta_0/2)/\sin(\theta_0/2)| \geq 2n/\pi$. The removed factors $C'u_j$ are less than $2(1-\alpha)$. The angle spacing gives $|\theta_j| \geq \pi j/n$ and the bound on r/n. For the retained factors, apply $\log(1-y) \geq -2(\log 2)y$ for $0 \leq y \leq 1/2$. The mean correction is bounded by the integral of the decreasing function $2(\log 2)\min\{1/2, \alpha/(2\sin^2(\pi x/2))\}$ over $0 \leq x \leq 1$, which equals J. If $n \leq 9$, then $\sin^2(\pi/(2n)) > .03$ and the discrete sum gives the claimed improvement using $\sum_{j=1}^{n-1}\csc^2(\pi j/(2n)) = 2(n^2-1)/3$. Finally the difference between the strong and weak bounds is $(1-(r+1)/n)\log 2 - J > 0$. □

4.2 Large primes in the norm

Lemma 17 (Two possible edge heights)**.** *Assume $d \geq 2$ and let $p > 2H + d$, $p \mid F(0)$, and $p^4 > M$. Suppose neither interval $[bi - 6H, bi]$, $i = N-1, N$, contains a positive multiple of p^3. In particular this holds if $p^3 > M$. The selected Newton edge at order i joins zero to the first jump, of width w_i and height $e_i \in \{1, 2\}$. Write $u_i = bi - dw_i + q_i$, $1 \leq q_i \leq d$, for the reached multiple of p. The heights and widths satisfy*

$$\frac{e_N}{w_N} = \frac{e_{N-1}}{w_{N-1}}, \qquad v_p(F(0)) \leq \min(e_N, e_{N-1}).$$

If the heights are equal, $u_N - u_{N-1} \in [a+1, a+2d-1]$. Otherwise $v_p(F(0)) = 1$, the widths are in ratio two, and p divides an integer in one of

$$b(N-2) + [2-d, 2d-1], \qquad b(N+1) + [2-d, 2d-1]. \tag{15}$$

In this case p^2 has a positive multiple in one of $[bi - 2H, bi]$. In both cases $u_N \neq u_{N-1}$ and

$$u_N - u_{N-1} \in [b - H - d + 1, b + H + d - 1], \qquad p^{v_p(F(0))} \mid (u_N - u_{N-1}).$$

Proof. The slope λ is at least $1/h$. The affine count with error three puts all maximizing indices below $6h$, and the absence of terminal cubes improves this to $4h < p$. The binomial valuation is nondecreasing there. A factorial jump raises the ordinate by at most two, and successive jumps are more than $2h$ apart. Thus the weighted ordinates $U_i(j) - \lambda j$ strictly decrease between jumps and from one jump index to the next. Only zero and the first jump can maximize. This proves the first assertions.

The statements for equal heights follow by subtracting the u_i. For unequal heights the smaller width is the reduced denominator of the slope and is at most h. Twice the reached multiple with the shorter width minus the other belongs to (15). The wider edge reaches a multiple of p^2 within $2H$ of the terminal. In either case, $|w_N - w_{N-1}| \leq 1/\lambda \leq h$ gives the stated interval for the difference. If the two multiples were equal then $w_N > w_{N-1}$, since $b > d$. The upper binomial numerator interval would contain the lower one, with the same factorial valuation at the first jump. This gives $e_N \leq e_{N-1}$, contradicting the common positive slope. The norm bound on $v_p(F(0))$ gives the last divisibility. □

Lemma 18 (Prime sums used below)**.** *Let $S(x) = \sum_{p \leq x} \log p/(p-1)$. Then for $x > 1$,*

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right), \qquad S(x) < \log x + .2.$$

For $x \geq 350$ the latter bound improves to $\log x - .49$, and for $\log x \geq 8$ to $\log x - .54$.

Proof. The prime count is Dusart's Theorem 6.9 [1]. Theorem 6.11 gives an upper bound for $\sum_{p \leq x} (\log p)/p$ of $\log x + E + .2/\log x + .2/\log^2 x$ when $x > 2974$. Add the prime-power correction using $E = -\gamma_{\mathrm{E}} - \sum_p \log p/(p(p-1))$ and $\gamma_{\mathrm{E}} > .57$. This proves the claimed bound at $\log x \geq 8$. For $1 < x < 3000$, the excess $S(x) - \log x$ decreases between primes. Enumerate primes by trial division and bound the excess at each prime using the expansion $\log r = 2\sum_{j=0}^{9} z^{2j+1}/(2j+1) + \epsilon$, where $r \in [1, 2]$, $z = (r-1)/(r+1)$, and $0 \leq \epsilon \leq 2z^{21}/(21(1-z^2))$. Normalize arguments by powers of two. This gives an upper bound below .144 on the excess, with its maximum at $p = 3$. Starting at $x = 350$ and checking that endpoint as well gives an upper bound below $-.499$ through 3000. □

5 Completion of the proof

The support and initial-order reductions leave the range of Lemma 6. Proposition 13 covers $25 \leq N < 3150$ there. Proposition 21 covers $3150 \leq N < e^{15}$, and Proposition 23 covers $N \geq e^{15}$. Together with Proposition 7, these give rational coprimality in every case of (1) and hence Theorem 1.

Electronic supplement. The electronic supplement accompanying this article contains two parts. Its exact verification programs cover the initial and central orders, the finite boundary cases, and the bounded comparisons in the Padé estimates. The submitted proof appendix *Positive-exponent Boundary* gives the three contour arguments used in Appendix C, with their exact constants and verification programs.

AI assistance and review. ChatGPT (OpenAI) assisted with mathematical arguments, literature checks, exposition, proof review and verification code.

A Proofs of the root bounds and additional boundary cases

A.1 Bounds for a Gamma polynomial

For $\gamma > 0$ put

$$G_{m,\gamma}(z) = \sum_{k=0}^{m} \binom{m}{k} (\gamma)_k z^{m-k}, \qquad h_0 = m - 1.$$

We first show that every zero z of $G_{m,\gamma}$ satisfies

$$|z| \le \gamma + h_0, \qquad \Re z \le -\gamma + h_0.$$

Let $T = \gamma + m$. The polynomial is $\det(zI + J)$, where $J = D + iK$ is complex symmetric, with real diagonal $D_{jj} = T + 1 - 2j$ for $1 \le j \le m$ and real symmetric off-diagonal entries $K_{j,j+1} = \sqrt{j(T-j)}$. To check the representation, the characteristic polynomial of the leading $j \times j$ block is

$$\sum_{k=0}^{j} (-1)^{j-k} \frac{j!}{k!} \binom{T-1-k}{j-k} z^k.$$

This expression has the initial terms $1, z - (T-1)$ and satisfies the determinant recurrence, whose two coefficients are $z - (T + 1 - 2j)$ and $(j-1)(T+1-j)$. At $j = m$, replacing z by $-z$ and multiplying by $(-1)^m$ gives $G_{m,\gamma}$.

The lower bound on the eigenvalue real parts of J follows from its real diagonal D. To bound its numerical radius, take $0 < \theta < \pi$ and set $q = \tan(\theta/2)$. For the Hermitian part $H_\theta = (\cos\theta)D + (\sin\theta)K$ of $e^{-i\theta}J$, the positive vector with entries

$$v_j = \sqrt{\binom{T-1}{j-1}}\, q^{j-1}$$

satisfies $(H_\theta v)_j = (T-1)v_j$ for $j < m$, and $(H_\theta v)_m = [T - 1 - \gamma(1 - \cos\theta)]v_m$. These identities follow by substituting the adjacent ratios in the off-diagonal terms. The top eigenvector of H_θ can be taken positive by maximizing its Rayleigh quotient with nonnegative coordinates and using the connected off-diagonal chain. Its scalar product with the displayed inequality gives $\lambda_{\max}(H_\theta) < T - 1$. An alternating diagonal conjugates K to $-K$, and continuity handles the endpoint angles. Hence $|z| \le T - 1$ for each eigenvalue z of $-J$. For $m \ge 2$, equality could only occur in an endpoint direction. An eigenvector would then be supported in a single extreme diagonal coordinate, which the adjacent entry prevents. Thus the modulus bound is strict for $m \ge 2$.

Coefficient comparison also gives the equation

$$zG'' + (1 - \gamma - m - z)G' + mG = 0.$$

The constant term is positive. At a repeated nonzero zero the term zG'' would have strictly smaller order than every other term, so the zeros are simple. Let z_i minimize the modulus and write $r = |z_i|$. The real part of the differential equation at z_i, using $G''(z_i)/G'(z_i) = 2\sum_{j\neq i}(z_i - z_j)^{-1}$, gives

$$r - \gamma \;\geq\; -\Re z_i - \gamma = \sum_{j\neq i} \frac{|z_j|^2 - r^2}{|z_j - z_i|^2} \;\geq\; 0. \tag{16}$$

Equality $r = \gamma$ forces $z_i = -\gamma$ and all moduli γ. Since their sum is $-m\gamma$, all zeros would then equal $-\gamma$. Thus $r > \gamma$ for $m \geq 2$.

When $m \geq 2$ and $\gamma \geq 15h_0$, we show the stronger $r > \gamma + h_0/6$. Vieta's formulas give $\sum z_j = -m\gamma$ and $\sum z_j^2 = m\gamma^2 - mh_0\gamma$. In particular,

$$\sum_j |z_j|^2 \geq m\gamma^2 + mh_0\gamma.$$

Set $u_j = z_j + \gamma$ and $s_0 = h_0/\gamma$. The two bounds on z_j above give $|\Re u_j| \leq h_0$. Since $\sum u_j = 0$ and $\sum u_j^2 = -mh_0\gamma$, it follows that $\sum |u_j|^2 \leq mh_0\gamma(1 + 2s_0)$. Suppose $r \leq \gamma + h_0/6$, and set $\eta = (r^2 - \gamma^2)/(\gamma h_0)$. Then $0 \leq \eta \leq 181/540$ and $|u_i|^2 \leq r^2 - \gamma^2$, since $\Re z_i \leq -\gamma$. Write $q_j = |z_j|^2 - r^2$. We have

$$\sum_j q_j \geq m\gamma h_0(1 - \eta), \qquad 0 \leq q_j \leq \gamma h_0(2 + s_0 - \eta), \qquad \sum_j |z_j - z_i|^2 \leq m\gamma h_0(1 + 2s_0 + \eta).$$

The last inequality follows from $\sum u_j = 0$. Weighted Cauchy–Schwarz in (16) now gives

$$r - \gamma \geq m\frac{(1 - \eta)^2}{(2 + s_0 - \eta)(1 + 2s_0 + \eta)} \geq m\,\frac{128881}{741455} > \frac{h_0}{6}.$$

The rational expression decreases in s_0 and η on the stated ranges, so its value at $s_0 = 1/15, \eta = 181/540$ gives the middle bound. This contradiction proves the stronger inner radius.

A.2 Products of Gamma annuli

For monic polynomials of degree m in the binomial basis, write

$$\left(\sum_{k=0}^{m} \binom{m}{k} a_k z^{m-k}\right) \diamond \left(\sum_{k=0}^{m} \binom{m}{k} b_k z^{m-k}\right) = \sum_{k=0}^{m} \binom{m}{k} a_k b_k z^{m-k}.$$

The upper and lower root radii multiply under this operation. For the upper bound, if the first polynomial is $\prod_{j=1}^{m}(z + \delta_j)$ and Q^{pol} is the polarization of the second, the composed value at z equals $\prod_j \delta_j\, Q^{\mathrm{pol}}(z/\delta_1, \ldots, z/\delta_m)$. If $|z|$ exceeds the product of the two upper radii, all the arguments lie in the open exterior of the disk for the second polynomial. The Grace–Walsh–Szegő theorem [10, Theorem 4.1] gives nonvanishing there. The lower bound follows by reversal, which commutes with $\diamond$. With strict radii one can first tighten the bounds on the finitely many roots of each factor.

Use $\gamma = am + 1$ and the identity

$$(\gamma)_{dk} = d^{dk} \prod_{j=0}^{d-1} ((\gamma + j)/d)_k.$$

It expresses R_m, after scaling the variable by d^d, as the composition of the d Gamma polynomials of shapes $(\gamma + j)/d$. Consequently

$$(\gamma)_d \le |x| \le (\gamma + d(m-1))_d \qquad (R_m(x) = 0),$$

strictly for $m \ge 2$. This gives the first assertion of Lemma 2. At the two orders $\gamma = am + 1$, the inner radius for $m = N$ is at least the outer radius for $m = N - 1$ when $a \ge d(N-2)$. The strict inner bound gives separation even at equality and at $N = 2$. If $N \ge 25$ and $6a \ge (5N - 11)d$, then $a \ge 15d$. Each of the d unit-step shapes for R_N is at least $15(N-1)$. The stronger bound gives the lower radius $(aN + 1 + d(N-1)/6)_d$, and

$$aN + 1 + d(N-1)/6 \ge a(N-1) + 1 + d(N-2).$$

This proves the remaining separation in Lemma 2.

A.3 Small positive lower exponents and adjacent exponents

Proposition 19. *For $N \ge 25$, the pair P_{N-1}, P_N is coprime on each of the following supports:*

$$1 \le a \le 4, \quad 1 \le d < 3a, \qquad \text{or} \qquad a \ge 1, \quad d = 1.$$

Proof. Suppose first $N \ge 3150$. We may assume $6a < (5N - 11)d$, by Lemma 2. In both domains $d < 3a$, and Lemma 10 applies because $aN \ge 1375$. Thus the primes of the common-factor norm in Lemma 14 are supported up to C_N as in Lemma 9.

If $1 \le a \le 4$ and $d < 3a$, then $b \le 15$, $a + 2d - 1 \le 25$, and $s \le 121/15$. Use the prime threshold $X = \sqrt{15N}$, which is at least C_N and exceeds 150. By Lemmas 18 and 22, the right side of (5) is less than

$$\tfrac{1}{2}(\log N + \log 15) - .45 + 2.48\sqrt{15/N}.$$

Here $\log(M+1)/\log X < 2.001$, and $2.001(1 + 1.2762/\log X) < 2.48$. The loss bound $g \le 7s/32$ in (10) gives $\log \Lambda > \log N - 2.776$. The difference exceeds 0.17 at $N = 3150$ and increases with N.

For $d = 1$ and $a \ge 2$ take $X = \sqrt{(a+1)N} = C_N$, since $a + 1 < (5/6)(N-1)$. The prime-sum and prime-count bounds give instead

$$\tfrac{1}{2}(\log N + \log(a+1)) + .2 + 2.60\sqrt{(a+1)/N}.$$

Use the $d = 1$ gain from Lemma 14. The surplus is greater than

$$\tfrac{1}{2}\log N - .701 - \tfrac{1}{2}\log(3/2) - \log(1 + 1/\sqrt{2}) - 2.60\sqrt{5/6} > .21 \qquad (N \ge 3150).$$

The case $a = d = 1$ is in the preceding domain.

For $25 \le N < 3150$, the supplemental checker `verify_small_support.cpp` treats the two stated support families together. It retains the inequalities $a < d(N-2)$, $6a < (5N - 11)d$ and $d^2 < 19(a + d)$, using Lemmas 2 and 5, and uses the gap caps in Lemmas 11 and 12. It groups the first factorial block test by (N, a) and then applies the individual block and Newton slope tests of

Section 3. The resulting counts are 12 520 groups and 18 904 supports. Of these, 4 357 are excluded by the degree-dependent prime bounds and 4 853 by comparing Newton polygons. For the other 9 694, the full monic reciprocals are coprime modulo 1 000 000 007. The checker verifies primality by trial division and computes the modular gcd by exact Euclidean division. A monic common factor over $\mathbb{Q}$ would be integral and would retain its degree modulo this prime. Thus these tests establish coprimality throughout the finite range. □

B The constant-plus-power boundary

Write $F_{n,d}(t) = \sum_{k=0}^{n} \binom{n}{k}(dk)!t^k$ and $R_n(z) = z^n F_{n,d}(1/z)$. We prove that R_n is squarefree for all positive integers n, d. The proof uses a coefficient bound for large d, Newton residuals for the small prime powers, and a uniform contour estimate for the remaining large orders.

B.1 The coefficient bound

For integers $1 \le u \le v$ one has

$$\frac{u!}{(v!)^{u/v}} \le (u/v)^{u/2}.$$

Indeed $(k!)^{1/k}/\sqrt{k}$ is nondecreasing. The comparison at $k, k+1$, after raising to positive powers, is $(k!)^2(k+1)^{k(k-1)} \le k^{k(k+1)}$. This follows from $k! \le ((k+1)/2)^k$ and $(1+1/k)^{k+1} \le 4$. For the latter bound, $(x+1)\log(1+1/x)$ decreases for $x > 0$. Also $k\log(n/k) \ge (\log 2)\min(k, n-k)$, by concavity of $-x\log x$ on $[1/2, 1]$ and by $n/k \ge 2$ for $k \le n/2$.

After scaling t by $((dn)!)^{-1/n}$, the polynomial is $1 + t^n + \sum_{k=1}^{n-1} b_k t^k$, with

$$0 < b_k \le C_k := \binom{n}{k} q^{\min(k,n-k)}, \qquad q = 2^{-d/2}.$$

The bound follows from the factorial inequality at dk, dn. If $2^d \ge 4n^2$, then $q \le 1/(2n)$. By symmetry of C_k and $q^{\min(k,n-k)} \le q^k + q^{n-k}$, both weighted sums $\sum(k/n)b_k$ and $\sum(1-k/n)b_k$ are bounded by $(1+q)^n - 1 < 1$. At a repeated root of modulus at most one, subtract t/n times the derivative from the polynomial to obtain a contradiction to the second bound. At a repeated root of modulus at least one, divide t/n times the derivative by t^n and use the first. We have proved that

$$2^d \ge 4n^2 \implies F_{n,d} \text{ is squarefree.} \tag{17}$$

B.2 Small exponents

For $d = 1$, $R_n(z) = n!\sum_{j=0}^{n} z^j/j!$. Its derivative and itself cannot share a root, since the two truncated exponentials differ by $z^n/n!$. To cover $2 \le d \le 5$, we give the prime-power argument. The factorial units and the digit structure of the Newton polygon used here occur in Rocks's work on factorial moment polynomials [8].

Let $d = p^\nu$ with p prime and $\nu \ge 1$, and put $c = (d-1)/(p-1)$. For an integer $u \ge 0$, write $U_p(u!) = p^{-v_p(u!)}u!$. If $u = \sum_i u_i p^i$, Wilson's theorem, applied after taking out multiples of p one level at a time, gives

$$U_p(u!) \equiv (-1)^{v_p(u!)} \prod_i u_i! \pmod{p}.$$

The digit shift from k to dk and Legendre's formula therefore give

$$F_{n,d}(T/p^c) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} w_k T^k, \qquad w_k = \frac{(dk)!}{p^{ck}k!} \in \mathbb{Z}_p^\times, \qquad w_k \equiv (-1)^{ck} \pmod{p}.$$

Write $n = \sum_i a_i p^i$ and $s_i = \sum_{j \le i} a_j p^j$, with $s_{-1} = 0$. The lower Newton polygon has an edge from s_{i-1} to s_i for each $a_i > 0$, of slope h_i/e_i with $e_i = p^i$ and $h_i = (p^i - 1)/(p-1)$. Indeed the coefficient ordinate is $v_p(n!) - v_p((n-k)!)$. For $k = s_i - \delta$ in that segment the difference between the ordinate and the line through the two endpoints is

$$\frac{\mathrm{digitsum}_p(\delta) - \delta/p^i}{p-1} \ge 0.$$

This is Legendre's formula, since $n - s_i$ and δ have disjoint digit positions. Equality occurs exactly at the indices spaced by e_i, the slopes strictly increase, and h_i/e_i is in lowest terms.

Divide each coefficient on the edge by its power of p and take ratios to the leftmost coefficient. Wilson's unit formula then gives the edge polynomial

$$\sum_{j=0}^{a_i} \frac{a_i!}{(a_i - j)!} (\eta_i Y)^j, \qquad \eta_i = (-1)^{h_i + ce_i}.$$

Here only digit a_i changes in the factorial ratio, and the change in valuation is jh_i. The displayed polynomial is a scaled reversal of $E_{a_i}(Z) = \sum_{j=0}^{a_i} Z^j/j!$ over $\mathbb{F}_p$. It is squarefree because $E'_{a_i} = E_{a_i - 1}$ and $E_{a_i} - E_{a_i - 1} = Z^{a_i}/a_i!$.

For completeness, edge polynomials multiply at any fixed slope u/e in lowest terms. Substitute $T = \pi^{-u} Z$ in a valued extension with $\pi^e = p$, divide each polynomial by the power giving minimum coefficient valuation zero, and reduce. The reductions multiply. The surviving indices in a polynomial over $\mathbb{Q}_p$ are all congruent modulo e, so after removing the lowest monomial the same identity holds in the edge variable $Y = Z^e$. A square divisor would thus give a repeated factor in the edge polynomial at any one of its slopes. The squarefreeness above rules this out.

B.3 A uniform contour estimate

We now assume $n \ge 10000$ and $6 \le d \le \sqrt{n}$. Set $A = (d/e)^d$ and $v = z/(n^d A)$. The exponential integral for R_n, with its contour split at $d/2$, is

$$R_n(n^d A v) = n^{dn+1} I(v), \qquad I(v) = \int_0^\infty (Av + s^d)^n e^{-ns}\, ds = K(v) + J(v),$$

where K and J integrate over $[0, d/2]$ and $[d/2, \infty)$, respectively. Both are polynomials in v. We bound J by a Gaussian at its saddle and K by an endpoint approximation. Their logarithmic derivatives are separated where a zero of I can occur.

The outer bound. For $\theta = k/n$, the entropy bound on $\binom{n}{k}$ and $(dk)! \le e\, dk (dk/e)^{dk}$ give

$$\frac{[\binom{n}{k}(dk)!]^{1/k}}{n^d A} \le q_d(\theta)(e\, dk)^{1/k}, \qquad q_d(\theta) = \theta^{d-1}(1-\theta)^{-(1-\theta)/\theta}.$$

The endpoint values use continuous limits. We have $q_d \le q_6$. The sign of the derivative of $\log q_6$ is the sign of $6\theta + \log(1-\theta)$. Its positive-to-negative change occurs at $\theta = 1 - u$ with $0 < u < 1/300$,

since $6(299/300) > \log 300$. There $\log q_6 = 5\log(1-u) + 6u < u$, so $q_6 < 300/299$. If $k \ge n/10$, then $d \le \sqrt{n}$ gives

$$k^{-1}\log(e\,dk) \le 10(1 + \tfrac{3}{2}\log n)/n < 2/125.$$

Thus the normalized coefficient root is less than $(300/299)(125/123) < 21/20$. If $k < n/10$, the bounds $\binom{n}{k}^{1/k} \le en/k$ and $(e\,dk)^{1/k} \le ed$ give instead $e^2 d\theta^{d-1} < 21/20$. The latter unit bound uses $k \le 2^{k-1} \le (ed)^{k-1}$. Summing the resulting finite geometric series excludes $|v| \ge R = 21/10$.

The large saddle. Put $B = 11/5$ and $\delta = 1/25$. For $|v| \le B$ the equation $Av = s^{d-1}(d-s)$ has a unique solution $s_0 = \alpha_0 + i\beta_0$ satisfying $|s_0 - d| < \delta$. On $|s-d| = \delta$ the modulus on the right divided by A is at least $\delta(1-\delta)e^d/d > B$. Bernoulli's inequality gives the lower estimate, and $e > 27/10$ verifies the strict comparison at $d = 6$, where the bound is smallest. Rouché's theorem gives the solution, which is simple and analytic in v on a neighborhood of the disk. Use the phase

$$\Phi(s,v) = d\log s + \log(1 + Av/s^d) - s, \qquad S = \Re\Phi(s_0, v), \qquad a_0 = 1 - (d-1)/s_0.$$

At s_0, $\Phi_s = 0$, $\Phi_{ss} = -a_0$, and

$$\Re a_0 > 9/(10d), \qquad |a_0| < 11/(10d), \qquad S > \log A - 1/100. \tag{18}$$

For the first two bounds use $|a_0 - 1/d| = (d-1)|s_0 - d|/(d|s_0|) < \delta/d$. For the height, use $\Phi(s_0, v) = (d-1)\log s_0 + \log d - s_0$. The loss is at most $\delta/d + \delta^2/[2d(1-\delta/6)] < 1/100$ by the logarithm series.

Deform the contour of J to its horizontal translate by $i\beta_0$, with the vertical connector at $d/2$. The integrand is entire and the connector at positive infinity tends to zero. On $s = s_0 + t$, $|t| \le d/8$ we have $|s| > 6d/7$. With $q_t = B(35/81)^6$, differentiating the convergent logarithm series gives

$$d^2|\Phi_{sss}| \le (7/6)^3\left[2 + \frac{36q_t(1+q_t)}{(1-q_t)^3} + \frac{18q_t}{(1-q_t)^2} + \frac{2q_t}{1-q_t}\right] < 8.$$

For this inequality use $|Av/s^d| \le B(35/81)^d$. In the differentiated series the corresponding terms have factors $d^2, d, 1$, each giving its largest bound at $d = 6$. The cubic remainder is at most $4|t|^3/(3d^2)$. By (18) its contribution to the local integral is at most

$$\frac{4ne^{nS}}{3d^2}\int_{\mathbb{R}} |t|^3 e^{-17nt^2/(60d)}\,dt < 17e^{nS}/n.$$

Completing the Gaussian costs less than $18e^{nS - 9dn/1280}/n$, by bounding the Gaussian tail by the integral with the extra factor $8|t|/d$. The complete value $U_0 = e^{n\Phi(s_0,v)}\sqrt{2\pi/(na_0)}$ has modulus greater than $2\sqrt{d/n}\,e^{nS}$.

We give the bounds on the remaining pieces of this contour. For $2d/3 \le x \le 4d/3$, let $q_c = B(5/9)^6$. On $s = x + i\beta_0$, the logarithm series and $|\beta_0|/x < 1/100$ give

$$\frac{d^2}{dx^2}\Re\Phi(x + i\beta_0, v) \le -\frac{d}{x^2}\left[\frac{1 - 10^{-4}}{(1+10^{-4})^2} - \frac{6q_c}{(1-q_c)^2} - \frac{q_c}{1-q_c}\right] < -\frac{3}{16d}.$$

Here $|Av/s^d| \le B(5/9)^d$, and $d(5/9)^d$ decreases for $d \ge 6$. Integrating from $x = \alpha_0$ bounds the height by $S - 3(x-\alpha_0)^2/(32d)$. Outside the local interval this costs at least $d/1000$ in height, over length less than d. At $4d/3$ the loss is greater than $d/100$. For $x \ge 4d/3$, the real derivative is at most $(3/4)/(1 - B(5/18)^6) - 1 < -1/5$, so the far ray costs at most $5e^{nS - dn/100}/n$.

On $[d/2, 2d/3]$ the upper envelope $[BA + (x+\delta)^d]e^{-x}$ increases. For $x + \delta < d - 1$, the function $(x+\delta)^{d-1}(d-x-\delta)$ increases, and its value at $x = d/2$ exceeds BA since $(e/2)^d > B$. The envelope divided by A is at most $Be^{-2d/3} + r_\ell^d$ there, with $r_\ell = (101/150)(141/100)$ from $e^{1/3} < 141/100$. The corresponding bound on the vertical connector is $Be^{-d/2} + r_v^d$ with $r_v = 38/45$, from $e^{1/2} < 5/3$. Both bounds are below $e^{-d/50}$. After multiplication by $e^{d/50}$ their two exponential terms decrease in d, and at 6 the comparisons follow from

$$B(10/27)^3 + r_\ell^6/(22/25) < 1, \qquad B(10/27)^2 + r_v^6/(22/25) < 1.$$

By (18) the height of these pieces is below $S - d/100$. Their lengths are $d/6$ and at most δ. The phase charts used on the horizontal contour are valid since $|Av/s^d| \le B(2/e)^d < 1$ for $x \ge d/2$.

Summing the contour bounds gives an error at most e^{nS} times

$$40/n + de^{-dn/1000} + (d+\delta)e^{-dn/100} < 44/n.$$

The last comparison uses $e^x \ge x^3/6$ in the middle term and $e^x \ge x^2/2$ in the last. Consequently $J = U_0(1+\epsilon)$ with $|\epsilon| < 22/\sqrt{dn} < 1/10$. The error is analytic on a neighborhood of the disk. Cauchy's estimate at radius $1/10$ gives $|\epsilon'| < 1$ on $|v| \le R$. Also

$$U_0'/U_0 = n\frac{d - s_0}{dv} - \frac{a_0'}{2a_0}, \qquad s_0' = \frac{A}{ds_0^{d-2}(d - 1 - s_0)}.$$

Bernoulli's inequality gives $|s_0'| \le de^{-d}/(1-\delta)^2 < 1$, and hence $|a_0'/(2a_0)| < 1$ by (18). We obtain

$$\left|J'/J - n(d - s_0)/(dv)\right| < 3 \qquad (|v| \le R). \tag{19}$$

The expression at zero is understood by continuity.

The endpoint. For $0 \le x \le d/2$,

$$\log(1 + x^d/((2/5)A)) \le 19x/20.$$

At $x = d/2$ this is $1 + (5/2)(e/2)^d < e^{19d/40}$. After multiplication by $e^{-19d/40}$, both terms decrease with d, since $e^{21/40} \le (5/3)(40/39) < 2$. At $d = 6$ use $e < 49/18$ and the rational comparison

$$1 + (5/2)(49/36)^6 < \sum_{j=0}^{12} (57/20)^j/j!.$$

To pass to smaller x, the logarithm divided by x has derivative of the sign of $du/(1+u) - \log(1+u)$, where $u = x^d/((2/5)A)$. This function is nonnegative up to $u = d - 1$ by differentiation. Thereafter its first term is at least $d - 1 > 19d/40$, while the logarithm is bounded by its endpoint value.

For $|v| \ge 2/5$, write $K = (Av)^n(1 + D(v))/n$. Subtract the full exponential integral and use $|(1+z)^n - 1| \le n|z|(1+|z|)^{n-1}$. For D' differentiate $(1 + x^d/(Av))^n$ inside the integral. The same endpoint envelope gives

$$|D(v)| \le \frac{d!20^{d+1}}{(2/5)An^{d-1}} + e^{-dn/2}, \qquad |D'(v)| \le \frac{d!20^{d+1}}{(2/5)^2An^{d-1}}.$$

Both are below $1/1000$, since $d! \le edA$, with the largest majorants at $d = 6, n = 10000$. In particular

$$K(v) \ne 0, \qquad |K'/K - n/v| < 1 \qquad (|v| \ge 2/5). \tag{20}$$

For $|v| \le 1/2$, the upper envelope $(A/2 + x^d)e^{-x}$ attains its maximum on $[0, d/2]$ at an endpoint, by the sign of its derivative. At the right endpoint it is below $A/2$ because $e^{-d/2} + 2(e^{1/2}/2)^d \le (5/8)^6 + 2(5/6)^6 < 1$. Thus $|K| \le (d/2)(A/2)^n$. The lower estimate for J gives $|J| > (3/2)\sqrt{d/n}(Ae^{-1/100})^n$. It follows that $|K/J| < \sqrt{dn}(50/99)^n/3 < 1$, using $\sqrt{dn} \le n$ and $\sum_{j\ge1} j(50/99)^j < 3$. Therefore every zero of I lies in $1/2 < |v| < R$. At such a zero, $I' = J(J'/J - K'/K)$, and (19) and (20) give

$$|J'/J - K'/K| > n|s_0|/(d|v|) - 4 > n/3 - 4 > 0.$$

This proves simplicity in the entire stated contour range.

B.4 Completion and finite verification

If $n \ge 10000$ and (17) fails, then $d < 2\log_2(2n) < \sqrt{n}$. The second inequality follows since $u - 2 - 4\log_2 u$ is positive at $u = 100$ and increasing for $u \ge 100$. Hence the contour estimate and the small-exponent argument cover every such order. For $n < 10000$, the remaining pairs have $6 \le d \le 29$ and $2^d < 4n^2$. The case $n = 1$ is linear.

Here is a small-degree test for that finite range. Let $n = p + r$ with p prime and $p > d\max(1, r)$. Then $r < p$ and $p^2 - dn = p(p - d) - dr > 0$. The Newton ordinate of $F_{n,d}$ is

$$y(k) = \lfloor dk/p \rfloor + \mathbf{1}_{\{k \bmod p > r\}}.$$

It is zero up to r. For $r < k < p$ it exceeds dk/p, and for $p \le k \le n$ it equals d. Thus the polygon has a horizontal part of width r and a slope d/p of width p. Every repeated rational irreducible divisor has degree at most $n/2 < p$, so by the slope denominator it can use only the horizontal part. It is monic integral in the reciprocal and reduces with nonzero constant term. Lucas's congruence gives $R_n \bmod p = z^p(R_r \bmod p)$. Indeed for $k \le r$ the binomial coefficient reduces to $\binom{r}{k}$, and the other coefficients of $F_{n,d}$ vanish modulo p. Thus a repeated divisor would reduce to a repeated divisor of $R_r \bmod p$.

Every repeated rational irreducible divisor has degree at least two. For even n, the exponential integral $F_{n,d}(t) = \int_0^\infty (1 + tu^d)^n e^{-u}\, du$ is positive on the real axis. For odd n, its derivative is positive there. In either case it has no repeated real root. It follows that a gcd of degree at most one between the modular R_r and its derivative excludes every repeated rational factor of R_n. The same argument applies to a direct modular test of R_n at any prime greater than dn.

The supplementary verifier exhausts 202 021 pairs in the remaining range. It uses a shifted prime in 201 927 cases, where the tested degree is at most 35. In the other 94 cases it tests the full polynomial at 1 000 000 007, with degree at most 148. The primes are obtained by an Eratosthenes sieve in the shifted tests and by trial division for the full tests. Every modular gcd has degree at most one. This completes the squarefreeness proof and thus Lemma 3.

C The large relative gaps

We separate the prime-power gaps from the three contour estimates. The arithmetic proof for prime-power gaps is given below. The electronic appendix *Positive-exponent Boundary* gives the quantitative contour bounds, their derivations and the exact constant checks.

Lemma 20. *The pair $P_{N-1,a,d}, P_{N,a,d}$ is coprime for every $N \ge 2$ on $a = 1, d \ge 3$, and on $a \ge 2, d \ge 3a$.*

Proof. Proposition 7 covers $N \le 24$. Thus we may impose the ratio bound of Lemma 5 throughout this proof.

For $a = 1$, write

$$Q_{m,\gamma,d}(x) = \sum_{k=0}^{m} \binom{m}{k} (\gamma)_{dk} x^{m-k}.$$

At terminal N the actual shapes are N and $N+1$. Coefficient comparison gives, with $\gamma = N$,

$$Q'_{N,\gamma,d} = NQ_{N-1,\gamma,d}, \qquad \gamma Q_{N,\gamma+1,d} = (\gamma + dN)Q_{N,\gamma,d} - dxQ'_{N,\gamma,d}.$$

In fact the lower reciprocal is $Q_{N-1,N,d}$, so at a common root the derivative of $Q_{N,N,d}$ vanishes. The second identity then makes the upper reciprocal a nonzero scalar multiple of $Q_{N,N,d}$ there. Conversely, a multiple root of $Q_{N,N,d}$ makes both reciprocals vanish. The prime-power argument in the next subsection gives squarefreeness at $d = 3, 4, 5, 7, 8, 9$. The separate gap-six contour proof gives it at $d = 6$ for $N \ge 10000$.

For $d \ge 10$, the lower-one contour proof gives squarefreeness of $Q_{N,N,d}$ whenever $N \ge \max(5000, 128(d+1))$. Its two contour contributions have separated logarithmic derivatives at a zero, so the derivative of their sum is nonzero. For $a \ge 2, d \ge 3a$, the weighted contour proof in the electronic appendix gives coprimality for $N \ge \max(4000, 128(a+d))$. Here one must compare two integrals with the actual moment orders. With $w = A/(BN^d)$ and $g(x) = x^a(w + x^d)$, their integrands after the common dilation are $g(x)^N e^{-Nx}$ and $g(x)^{N-1}e^{-Nx}$. Write I_U, I_L and J_U, J_L for their two contour contributions. At a zero of the first integral, $I_U = -I_L \ne 0$. The verified packet estimates at the two selected saddles x_U, x_L give

$$\left|\frac{J_U}{I_U}\right| < \frac{24\sqrt{N/(N-1)}}{|g(x_U)|}, \qquad \left|\frac{J_L}{I_L}\right| > \frac{\sqrt{N/(N-1)}}{5|g(x_L)|}, \qquad \left|\frac{g(x_U)}{g(x_L)}\right| > 150.$$

Thus $|J_U| < |J_L|$ and the second integral is nonzero. The proof estimates the extra amplitude e^{-x} in the order-$N-1$ contour integral when the phase is written as $\log g(x) - x$.

If $N < \max(4000, 128b)$, where $b = a + d \le 4d/3$, a potential failure has $d \le 24$. Indeed $s \ge 3d/4$. For $N \ge 200$ and $d \ge 25$, one has

$$N < 512d/3, \qquad \frac{3d}{4} > 2\log(512d/3) + 2.$$

At $d = 25$ the margin exceeds $3/100$, and its derivative $3/4 - 2/d$ is positive. For $N < 200$, use $\sigma_N \le 2\log 199 + 3 < 14 < 3d/4$.

For $a = 1, d \ge 10$ below the contour threshold, a potential failure has $d \le 19$. For $20 \le d \le 38$ the threshold is 5000, and

$$\frac{d^2}{d+1} \ge \frac{400}{21} > 2\log(5000) + 2.$$

For $d \ge 39$ the threshold is $128(d+1)$. The difference $d^2/(d+1) - 2\log(128(d+1)) - 2$ exceeds 18 at 39, and its derivative $1 - 1/(d+1)^2 - 2/(d+1)$ is positive. For $N < 200$ and $d \ge 20$, the upper bound $\sigma_N < 14 < 400/21$ suffices. The electronic comparison script certifies the displayed logarithmic inequalities by rational bounds.

The supplemental finite checker therefore needs only the families

a	d	terminals N
1	6	$25 \le N < 10000$
1	$10 \le d \le 19$	$25 \le N < 5000$
$2 \le a \le \lfloor d/3 \rfloor$	$6 \le d \le 24$	$25 \le N < \max(4000, 128(a+d))$.

Before any further exclusions this is 338 071 cases. Every case satisfies $a < d(N-2)$, and a possible common factor has $s < 21$ by Lemma 5. The block-prime support and Newton allocation arguments in Lemmas 9 and 8 apply here with these hypotheses. The stronger degree bound is used only when its starting-point condition is verified for that degree. In the subsequent Newton check, each odd prime $p > aN$ satisfies $p > N, p > d$ and $p^2 > (a+d)N$ throughout these finite families. Thus the sparse floor ordinates used in Section 3 are exact also in this range.

The supplemental program `verify_positive_boundary.cpp` retains 326 146 cases after imposing the uniform bound $d^2 < 21(a+d)$. It tries the first available sparse Newton test, applies the prime-block bounds at the remaining degrees, and then uses additional primes as needed. The exact exclusions are

Family	Supports	Prime blocks	Newton polygons
$a \geq 2,\ d \geq 3a$	266 421	12 901	253 520
$a = 1,\ 10 \leq d \leq 19$	49 750	966	48 784
$a = 1,\ d = 6$	9 975	535	9 440

Every possible even degree is excluded, and no modular fallback is needed. □

C.1 The prime-power gaps

We prove the squarefreeness needed above for $q = p^r$, where p is prime and $r \geq 1$. Use $n = N$, $\gamma = n$ and

$$T(t) = t^n Q_{n,\gamma,q}(1/t), \qquad R(z) = z^{qn} T((-1)^{q+1} z^{-q}), \qquad M = \gamma + qn.$$

Both changes of variable are unramified away from zero, and both $T(0)$ and $R(0)$ are nonzero. Write $D = d/dz$, $\theta = zD$ and define

$$\mathcal{A} = z - zD + \gamma + q - 1, \qquad \Pi_{n,q}(t) = \prod_{j=0}^{n-1} (t + qj), \qquad B_n = \Pi_{n,q}(\mathcal{A})1.$$

The rank condition. The polynomial R solves $LR = 0$ for the monic operator

$$L = z^{-q-1} \left[\theta \prod_{\ell=0}^{q-1} (\theta - M - \ell) - z^q (\theta - qn) \right].$$

Indeed consecutive coefficients $r_k = [z^{q(n-k)}]R$ satisfy $(k+1)r_{k+1} = (-1)^{q+1}(n-k)(\gamma + qk)_q r_k$, which is the equation at $z^{q(n-k)}$ after multiplying by z^{q+1}. Put $U = 1 - D$. We will use the adjoint solutions

$$f_\omega(z) = e^{-\omega z} z^{1-M} B_n(\omega z), \qquad \omega^q = 1.$$

To derive them, set

$$\begin{aligned} C(x) &= (1-x)^{-\gamma} \left(\frac{1 - (1-x)^q}{qx} \right)^{-n-1}, \\ A_0(x) &= (1-x)((1-x)^q - 1), \\ b_0(x) &= (M-1)((1-x)^q - 1) + qn. \end{aligned}$$

The residue $n![x^n](C(x)e^{zx})$ equals B_n. In fact the substitution $v = 1 - (1-x)^q$ makes it

$$q^n n! [v^n] (1-v)^{-(\gamma+q-1)/q} e^{z(1-(1-v)^{1/q})}.$$

With $v = qt$ the generating function in n is $(1-qt)^{-(\gamma+q-1)/q} e^{z(1-(1-qt)^{1/q})}$. Writing it as $F(t,z)$, differentiation gives $(1-qt)F_t = \mathcal{A}F$, $F(0,z) = 1$. This proves the claimed residue formula. Logarithmic differentiation gives $A_0 C' = ((n+1)A_0/x - A_0' + b_0)C$. Taking the residue of $(e^{zx} A_0 C x^{-n-1})'$ therefore yields

$$[zA_0(D) + b_0(D)]B_n = z(U^{q+1} - U)B_n + (M-1)(U^q - 1)B_n + qnB_n = 0.$$

The divergence form $L = z^{-q} D z^{M+q} D^q z^{-M} - D + qn/z$ gives $L^* = (-1)^{q+1} z^{-M} D^q z^{M+q} D z^{-q} + D + qn/z$. For $f_1 = e^{-z} z^{1-M} B_n$ the first term becomes $e^{-z} z^{-M} [zU^{q+1} + (M-1)U^q] B_n$, using $U^q z = zU^q - qU^{q-1}$. The last two terms give $e^{-z} z^{-M} [-zU + qn + 1 - M] B_n$. This proves $L^* f_1 = 0$, and substitution of ωz gives the remaining solutions.

Write $L = \sum_{k=0}^{q+1} a_k D^k$. The bilinear expression

$$\sum_{k=1}^{q+1} \sum_{i=0}^{k-1} (-1)^i (a_k f_\omega)^{(i)} R^{(k-1-i)}$$

has derivative $f_\omega LR - RL^* f_\omega = 0$, by cancellation of adjacent summands. It is zero because it is a rational function of z times $e^{-\omega z}$, which cannot be a nonzero constant. At a multiple root $z_0 \neq 0$ of R, the first two coordinates of its jet vanish, and its coordinates $R'', \dots, R^{(q)}$ are not all zero by uniqueness for the ordinary equation $LR = 0$. They give a common nonzero nullvector of the restricted rows of these expressions. In the column multiplying $R^{(j)}$, $2 \leq j \leq q$, the term of highest derivative of f_ω is $(-1)^{q-j} f_\omega^{(q-j)}$, since $a_{q+1} = 1$. The restricted rows are therefore triangular transforms of the jets $f_\omega, \dots, f_\omega^{(q-2)}$. Removing the common power gauge and the nonzero exponential row factors gives a column dependence among $\omega^j (U^j B_n)(\omega z_0)$, $0 \leq j \leq q-2$. The relation $Uz = zU - 1$ shows that $(zU)^j$ differs from $z^j U^j$ by a combination of $z^i U^i$, $i < j$. Using $\mathcal{A} = zU + \gamma + q - 1$ and $z_0 \neq 0$, the dependence becomes

$$[\Pi_{n,q}(\mathcal{A}) P(\mathcal{A}) 1](\omega z_0) = 0 \quad \text{for every } \omega^q = 1$$

for some nonzero P of degree at most $q-2$. The dependent evaluation matrix has algebraic entries, so this condition transfers to $\overline{\mathbb{Q}}_p$.

Excluding the dependence. Normalize $v(p) = 1$, work in a complete algebraically closed extension of $\mathbb{Q}_p$, and let $\mathcal{O}$ be its valuation ring. Set $b_r = 1 + p + \cdots + p^{r-1}$. Every interval of qk consecutive integers contains qk/p^ℓ multiples of p^ℓ for $1 \leq \ell \leq r$. Thus $v((\gamma)_{qk}) \geq b_r k$, and $v(z_0) \geq b_r/q$. Otherwise the leading term of $R(z_0)$ would have uniquely least valuation. In particular $v(u) \geq b_r$ for $u = z_0^q$.

Here is the invariant under descent from $q = p^r$ to p^{r-1}. Allow an operator and seed

$$A = c - zD + z + \sum_{i \geq 1} a_i z^{p^i}, \quad c \in \mathbb{Z}_p, \quad a_i \in p\mathbb{Z}_p, \qquad h \in \mathcal{O}[[z]], \quad \bar{h} = \overline{h(0)} \neq 0.$$

For $u \neq 0$, $v(u) \geq b_r$, and primitive $P \in \mathcal{O}[t]$ with $\deg \bar{P} \leq q-2$, the cyclic coefficient vector of $\Pi_{n,q}(A) P(A) h$ modulo $z^q - u$ is nonzero. Here the component j of the vector of a series $\sum f_k z^k$ is $\sum_{k \geq 0} f_{qk+j} u^k$. These sums converge since $v(u) > 0$, and their simultaneous vanishing is equivalent to vanishing at the q distinct roots of $z^q = u$.

At $r = 1$ reduce the vector modulo the maximal ideal. The reduced ring is $k[z]/(z^p)$, where k is the residue field, h reduces to a constant unit, and $\Pi_{n,p}(\bar{A}) = \bar{A}^n$. The matrix of $\bar{A} = z - zD + \bar{c}$

is triangular with distinct diagonal entries $\bar{c}-j$, $0\le j<p$. They are all the elements of $\mathbb{F}_p$. Its vector 1 is cyclic because $\bar{A}^j 1$ is monic of degree j for $j<p$, so 1 has a nonzero projection to each eigenspace. Vanishing would force $\bar{P}$ to vanish at all $p-1$ nonzero eigenvalues, impossible by its degree.

At $r\ge 2$ use the integral unit series

$$E(z)=\exp\left(\sum_{i\ge 0} z^{p^i}/p^i\right)=\prod_{p\nmid m}(1-z^m)^{-\mu(m)/m},$$

where μ is the Möbius function. The product identity follows by taking formal logarithms and summing μ over divisors. Each factor and its inverse are integral because their exponents are in $\mathbb{Z}_p$ and the binomial coefficients are integral there by continuity from nonnegative integers. The conjugation $E^{-1}AE$ subtracts $zD\log E=\sum_{i\ge 0} z^{p^i}$ from A. It preserves vanishing after changing the seed to $E^{-1}h$, since the series and their inverses converge at every point of the fiber. In residue $j \bmod p$, set $y=z^p$, $b=a_1-1$, and $w=by/p$. Write $e_j\in\{0,\dots,p-1\}$ for the representative of $c-j \bmod p$. The new block operator is e_j+pL_j, with

$$L_j=\frac{c-j-e_j}{p}-w\frac{d}{dw}+w+\sum_{i\ge 1}\frac{(a_{i+1}-1)p^{p^i-1}}{b^{p^i}}w^{p^i}.$$

This is in the same class. If $E^{-1}h=\sum_{j=0}^{p-1} z^j\widetilde{h}_j(y)$, the new seed $h_j(w)=\widetilde{h}_j(pw/b)$ reduces to the constant $\overline{h(0)}(-1)^j/j!\neq 0$, using the first p coefficients of e^{-z}. Put $q'=q/p$. The new fiber parameter is $u_*=(b/p)^{q'}u$, with $v(u_*)\ge b_r-q'=b_{r-1}$. If the original cyclic vector vanishes, the block vectors

$$\Pi_{n,q}(e_j+pL_j)P(e_j+pL_j)h_j \pmod{w^{q'}-u_*}$$

all vanish. Indeed for each $y^{q'}=u$, the evaluations on the p roots of $z^p=y$ separate these p residue blocks by a Vandermonde matrix.

If $e_j\neq 0$, absorb the first factor into the seed $\widehat{h}_j$. This still reduces to a constant unit. The cyclic vectors of $L_j^i\widehat{h}_j$, $0\le i<q'$, form a unit basis since their reductions are polynomials of successively increasing degree with unit leading coefficient in $k[w]/(w^{q'})$. The higher vectors are integral. Expanding $P(e_j+pL_j)$ in divided Taylor coefficients $P^{[i]}(e_j)$ and inverting the unit basis gives $v(P^{[i]}(e_j))\ge q'-i>0$ for $i<q'$: all terms with $i\ge q'$ are divisible by $p^{q'}$. Hence

$$\bar{P}(t)=(t^{p-1}-1)^{q'}H(t),\qquad \deg H\le q'-2.$$

The first unit coefficient of P occurs at an index at most $q'-2$. Choose a coefficient κ of $P(pt)$ of least valuation, so $v(\kappa)\le q'-2$. Then $Q(t)=P(pt)/\kappa$ is primitive and $\deg\bar{Q}\le q'-2$. In the unique block $e_j=0$, $\Pi_{n,q}(pL_j)=p^n\Pi_{n,q'}(L_j)$, and division by $p^n\kappa$ contradicts the induction hypothesis at q'. This proves the invariant.

Finally the original $\mathcal{A},1$ belong to this class. Scaling the dependent polynomial P to be primitive, the rank condition contradicts the invariant. This proves squarefreeness at every prime-power gap needed in this appendix.

D Intermediate terminal orders

Proposition 21. *Assume the conditions of Lemma 6. For $2200\le N<e^{15}$, a common factor requires $\alpha<.03$ and $b>N$. For $3150\le N<e^{15}$ there is no common factor.*

Write $t = \log(1/\alpha)$, $V = \log(M+1)$ and $\sigma = 2U + 2$, where $U = \log N$. Throughout this appendix, $\log m = U + \log(1 - 1/N) > U - 1/(N-1) > U - .001$. From Lemma 11 we may use $d \le 233$ below 20,000 and $d \le 565$ throughout the appendix. If X bounds the prime divisors of the lifted norm K, the right side of (5) is at most

$$S(X) + \pi(X)V/N, \qquad S(X) = \sum_{p \le X} \frac{\log p}{p-1}. \tag{21}$$

In the applications below $M \le X^j$, $M \ge 7N$ and $X \ge 9$ give

$$\frac{V}{\log X} \le j + \frac{\log(1 + 1/M)}{\log X} < j + \frac{1}{7N \log 9} < j + .001.$$

The baseline from Lemma 14 is $\log \Lambda > U - 1.011 - g$. The weaker constant 1.016 also suffices for the estimates of proper cosets.

Lemma 22. *For each of the pairs*

$$(x_0, c) = (21, .22), (34, .30), (43, .33), (100, .39), (117, .425), (137, .449), (150, .45), (343, .49)$$

one has $S(x) < \log x - c$ for $x \ge x_0$. For the prime count one has

$$\frac{\pi(x) \log x}{x} < \begin{cases} 1.256 & (x \ge 40), \\ 1.224 & (x \ge 117), \end{cases}$$

with the smaller bound on the overlap.

Proof. For $S(x) - \log x$ below 3000, check the listed left endpoints and the primes, since the function decreases between primes. For the prime count, check 40 and 117 and the primes up to 300, since between primes the multiplier $\log x/x$ decreases for $x > e$. For $x \ge 300$, Lemma 18 gives $1 + 1.2762/\log x \le 1 + 1.2762/\log 300 < 1.224$. The supplement encloses the logarithms at all these checkpoints. □

D.1 Proper rotation cosets

Let k_0 be the size of each rotation coset in Lemma 15. Suppose $q = d/k_0 \ge 2$, and put $P = M^{1/(q+1)}$. Then all norm primes are at most P and $4d \le 2H < P$. In particular,

$$P^q < \min\{N/(4\alpha), N^2/4\}. \tag{22}$$

The first inequality uses $M = dN/\alpha < PN/(4\alpha)$, and the second uses $M < dN^2$.

If $k_0 = 1$ and $d \ge 3$, then $P^{d+1} = M < dN^2$. Together with $4d < P$ this gives $4(4d)^d < N^2$. In particular $d \log(4d) < 2U$, so $d < U$ because $\log(4d) \ge \log 12 > 2$. Use $X = (UN^2)^{1/4} > P$ and $s < 3U/4$ in the baseline with $g \le 7s/32$. The resulting surplus over (21) is at least

$$U - 1.016 - \frac{21}{128}U - \log X + .22 - 4.001 \cdot 1.256\, X/N.$$

Here $\log X = U/2 + (\log U)/4$. The derivative of the terms linear or logarithmic in U is $1 - 21/128 - 1/2 - 1/(4U) > 0$, and X/N decreases because $-1/2 + 1/(4U) < 0$. The surplus therefore exceeds the certified value 1.10 at $N = 2200$. If $k_0 = 1, d = 2$, take $X = (2N^2)^{1/3}$ and $s < 1.5$.

Now $\log X = (2U + \log 2)/3$, so the surplus again increases. With the prime-sum correction .33 in place of .22, it exceeds .95 at $N = 2200$.

Suppose $k_0 \geq 2$ and $\alpha \geq .03$. For $\alpha \in [\ell, u]$ put $X = \sqrt{N/(4\ell)}$. By (22), $X > P$ and $M < X^3$. At least a third of each coset lies in the left half-plane. The one-sided gain from Lemma 14 gives the lower bound

$$U - .006 - \tfrac{2}{3}(1.01 + \kappa\sigma) + \tfrac{1}{3}\log((1-u)/u)$$

on $\log \Lambda$, using $g \leq \kappa s$. In every row $X > 40$, so (21) is bounded above by $\log X + c + 3.001 \cdot 1.256\, X/N$. The lower bound on the gain has slope $1 - 4\kappa/3 > 1/2$ in U. The logarithmic part $\log X$ of the upper bound has slope $1/2$, and X/N decreases. It remains to check the following margins at $N = 2200$.

$[\ell, u]$	κ	c	surplus exceeds
$[.03, .1]$	.134	$-.30$	1.35
$[.1, 1/3]$	.160	.20	.75
$[1/3, .75]$	.218	.20	.14

For $\alpha < .03$ and $q \geq 3$, take $X = N^{2/3}/4^{1/3} > P$. From $4d < P$ we obtain $M < dN^2 < XN^2/4 = X^4$. If D bounds d, the baseline exceeds $U - 1.016 - .00384D$. Subtracting the upper bound $\log X - .22 + 4.001(X/N)(1 + 1.2762/\log X)$ gives a surplus that increases for fixed D, since $\log X$ has slope $2/3$ in U and $(X/N)(1 + 1.2762/\log X)$ decreases. At $N = 2200, D = 233$ and $N = 20{,}000, D = 565$ it exceeds 1.08 and .68, respectively.

For $\alpha < .03$ and $q = 2$, use $X = N/2$, $M < X^3$ and (21) to bound the norm charge by $U + .862$. When $d = 4$, half of the size-two coset has gain at least $m(1-\alpha)/\alpha$, and the mean log gain exceeds $U + 1.21$. For $6 \leq d \leq 18$, the small-coset form of Lemma 16, with $k_0 = d/2$, gives a gain greater than $U + 1.89$. For $d \geq 20$, use $g_0 \leq 3.18$, $g_0/k_0 \leq .10868$, $r/k_0 < .111$ and $J < .153$ in (14), together with $\log C' > t - .7237$ and $1 - 1/k_0 \geq .9$. Dropping the positive $2\log(2k_0/\pi)/k_0$ leaves a gain exceeding $U + 1.80$. This proves that the rotation coset is full.

D.2 Full cosets at moderate and large ratios

First take $2200 \leq N < 20{,}000$. Here $a \leq (97/3)d$, $d \leq 233$ and $H \leq 467$. Thus $aN + H < 189{,}695{,}659 = X_{222}$. Lemma 12 gives $H < 222$ and $d \leq 110$. For $N \leq 3500$ these caps give $aN + H < X_{154} = 17{,}051{,}707$, so $d \leq 76$.

Let $\ell \leq \alpha \leq u$ and order the d rotations by increasing absolute argument, starting at index zero. An arc centered at angle zero and containing $j + 1$ arguments has length at least $2\pi j/d$, by the rotation spacing. Thus $c_j = \cos(\pi j/d)$ bounds the cosine in position j.

For a nonnegative upper bound c on the cosine, set $W = (1 - u - c)/u$. In the second integrand in (9), write $v = 1 - e^{-x/d}$. Use $v \leq x/d$ and $\Re z/m \leq bc$ to bound the logarithm of the original integrand, divided by m, by

$$-\frac{1-\alpha-c}{\alpha}x + \log(1 - e^{-x}) \leq -Wx + \log(1 - e^{-x}).$$

For $W > 0$ the derivative on the right vanishes only at $e^{-x} = W/(1+W)$, and the second derivative is negative. The maximum is $-J(W)$, where $J(W) = (1+W)\log(1+W) - W\log W$. The original integral over $0 \leq v \leq 1$ is at most $e^{-mJ(W)}$. Since $J(W) < 1 + \log(1+W) \leq 1 + \log(100/3) < 5$, Stirling's inequality gives $(m!e^{mJ(W)})^{1/(m+1)} > me^{-1.01+J(W)}$. Indeed $.01(m+1) + \frac{1}{2}\log(2\pi/m) - 5$ is positive at $m = 2199$ and has derivative $.01 - 1/(2m) > 0$. The linear gain in Lemma 14 is at least mW. Together the two gains give $\Phi_u(c) = \max\{-1.01 + J(W), \log W\}$ after subtracting

$\log m$. For $c < 0$ use $W = (1-u)/u$ in the integrand and replace $\log W$ in this maximum by $\log(W(1-c))$. For $c \geq 0, W \leq 0$ use the baseline $-1.01 - \kappa(u)du$, where $\kappa(u) = g/s$ at $\alpha = u$ in (10). In the smaller terminal band we also use the lower bound

$$-1.01 - \frac{d}{u}\left(c - (1-u) - (1-u)\log\frac{c}{1-u}\right)$$

when this improves the baseline. It follows by discarding the beta tail factor and maximizing the remaining exponent. The loss at α is zero for $c \leq 1-\alpha$ and otherwise the same formula with α in place of u. For $c > 1-\alpha$ the derivative of the loss is

$$\frac{d}{\alpha^2}\left(\log\frac{c}{1-\alpha} - c + 1 - \alpha\right) \geq 0.$$

For the inequality, $\log(c/(1-\alpha))$ is the integral of $1/x \geq 1$ from $1-\alpha$ to $c \leq 1$. This permits evaluating the loss at the upper endpoint u.

By Lemma 10, a prime bound is $\max(a + 2d - 1, \sqrt{M})$. It lies between X_1 and X_0, where

$$X_0 = \max\{d(1 + 1/\ell), \sqrt{dN/\ell}\}, \qquad X_1 = \sqrt{\max(7, d/u)N}.$$

The upper bound for the right side of the norm inequality is therefore

$$\log X_0 - c_0 + 2.001(1 + 1.2762/\log X_1)X_0/N. \tag{23}$$

Use $c_0 = .39$ in the first band, where $X_1 > 100$, and $c_0 = .45$ in the second, where $X_1 > 150$. In U, $\log X_0$ is the maximum of a constant and an affine function of slope $1/2$. The quotient X_0/N is the maximum of two decreasing functions, and $1/\log X_1$ also decreases. Thus the norm bound grows at most at rate $1/2$. The angular lower bound $\log(N-1) + d^{-1}\sum_j \Phi_u(c_j)$ grows at rate $N/(N-1) > 1$. Since $d\ell \leq s < 2U + 2$, the first eligible order is bounded below by $\max\{N_{\min}, \exp((d\ell - 2)/2)\}$. For each d and each interval of the following subdivision, the interval checker evaluates the surplus at that lower bound. Intervals with $d(1/\ell - 1) < 5$ contain no eligible support.

$N_{\min}$	$d_{\max}$	interval widths	surplus exceeds
2200	76	.005 to .05, then .01	.070
3500	110	.02	.089

Now take $20{,}000 \leq N < e^{15}$. The gap table gives the successive caps $D = 257{,}300{,}565$ at starting orders $20{,}000, 32{,}768, 65{,}536$. In each range $D/N < .015$. Since $(1+\alpha)^2/\alpha$ decreases on $[.03, .75]$,

$$\frac{(1+\alpha)^2 b}{N} \leq \frac{D}{N}\frac{(1+\alpha)^2}{\alpha} < .015\frac{1.03^2}{.03} < 1.$$

Hence $\sqrt{M}$ bounds the norm primes. Using $S(\sqrt{M}) < \log\sqrt{M} - .49$, the right side of the norm inequality is below

$$\tfrac{1}{2}(U + \log b) - .49 + 2.46\sqrt{b/N}. \tag{24}$$

At angles with $\cos\theta \leq 1-\alpha$ the beta integral has gain at least $me^{-1.01}$. The exceptional fraction is at most $\arccos(1-\alpha)/\pi + 1/d$. For $\ell \leq \alpha \leq u$, use $\eta \geq \arccos(1-u)/\pi$. At $d \geq 10$, at least a fraction .45 lies in the left half-plane, where one may use the greater gain $m(1-\alpha)/\alpha$. Thus, with $S_0 = \min(2U+2, Du)$ and $b_0 = \min((2U+2)/\ell^2, D/\ell)$, the surplus exceeds

$$U - 1.011 - \kappa(\eta S_0 + u) + .45\max\{0, 1.01 + \log((1-u)/u)\} - \left(\tfrac{1}{2}(U + \log b_0) - .49 + 2.46\sqrt{b_0/N}\right).$$

Where S_0 is differentiable in U, its derivative is zero or two. Thus the lower bound on the gain grows at rate at least $1 - 2\kappa\eta$. The logarithmic term in the upper bound grows at most at rate $1/2 + 1/(2U + 2)$, since $(\log b_0)' \le 2/(2U + 2) < 1$. The latter inequality also makes $\sqrt{b_0/N}$ decrease. The checker verifies $1 - 2\kappa\eta > 1/2 + 1/(2U + 2)$ at each starting order, where the comparison is tightest. It also checks $\cos(\pi\eta) < 1 - u$ and the upper bound κ on g/s at u. The table records the least certified surplus across the three starting orders for each interval. For $d \le 9$ the same argument before adding the left half-plane gain uses $\kappa = .218, \eta = .42,\ s \le 6.75,\ b \le 300$, and gives a positive surplus at $N = 20{,}000$ increasing thereafter.

$[\ell, u]$	κ	η	surplus exceeds
$[.03, .036]$	.128	.086	.12
$[.036, .045]$	.129	.096	.20
$[.045, .07]$	.131	.120	.11
$[.07, .11]$	.134	.151	.21
$[.11, .17]$	.140	.189	.51
$[.17, .28]$	.150	.245	.67
$[.28, .45]$	.170	.315	.62
$[.45, .60]$	.190	.370	.53
$[.60, .75]$	.218	.420	.08

D.3 Small ratios with the upper exponent at most the order

Take $X = 1.03\sqrt{M}$, which is a prime bound by Lemma 10. Indeed $a + 2d - 1 < (1 + \alpha)b \le 1.03\sqrt{M}$. For $d \ge 20$, the angular estimate (14) gives

$$\log \Lambda - U > .95\big(\log(100/3) - .7237\big) - .002 - .05434 - .111\big(1.01 + .00384D + \log 2\big) - .153$$

with $d \le D$. In (21), use $X/N \le 1.03$ and $X > 1.03\sqrt{20N/.03}$. At $N = 2200, D = 233$ the surplus exceeds .17, and at $N = 20{,}000, D = 565$ it exceeds .08. For fixed D, the gain bound minus U is constant. After using $X/N \le 1.03$ in the norm bound, its remaining N-dependence is through the decreasing reciprocal of $\log(1.03\sqrt{20N/.03})$. For $10 \le d \le 19$, inserting each d in (14) gives $\log \Lambda > U + 2.41$, against a right side of the norm inequality below $U + 2.052$ by $X > 343$. For $4 \le d \le 9$ use the small-coset angular bound. The gain exceeds $U + 2.24$, against a right side of the norm inequality below $U + 2.177$ by $X > 137$.

For an antipodal coset with $d = 2$, write $c = |\cos\theta|$, $A_0 = (1 - \alpha)/\alpha$ and $B_0 = e^{-g_0}$. The two gains have product at least $m^2 A_0(1 + c)\max\{B_0, A_0 - c/\alpha\}$. On $0 \le c \le 1 - \alpha(1 + B_0)$ this is a concave quadratic in c and is linear increasing on the rest of its domain. The two endpoint values in the quadratic range, divided by $m^2 A_0$, are A_0 and $B_0(2 - \alpha(1 + B_0))$. The second is smaller because $A_0 > 32$ and $B_0 < 1$. This gives the lower bound $m^2 A_0 B_0(2 - \alpha(1 + B_0))$. In particular $\log \Lambda > U + t/2 - .2$. For $d = 3$ the small-coset bound gives $U + 2t/3 - .435 > U + t/2 - .2$. Put $\mu = \frac{1}{2}\log(b/N)$. Since $X > 117$, the right side of the norm inequality is at most

$$U + \mu + \log(1.03) - .425 + 2.001 \cdot 1.03 \cdot 1.224\, e^{\mu}.$$

Substitute $t = U - \log d + 2\mu$ in the gain $U + t/2 - .2$. The linear terms in μ cancel in the surplus, while the remaining negative multiple of e^{μ} is smallest at $\mu = 0$. The resulting bound increases with U and exceeds .96 at $N = 2200$. This proves the first conclusion of Proposition 21.

D.4 Small ratios with the upper exponent exceeding the order

It remains to exclude this case for $N \ge 3150$. The bounds $H < G,\ d \le D = G/2 - 1$ are shown in the following table. In addition to the main gap table, apply Lemma 12 successively using $G = 320, 292, 288, 282$ up to $N = 5000, 4150, 3467, 3295$, respectively. At every step the preceding caps give $aN + H < (5/6)D_{\text{old}}N_{\max}^2 + H_{\text{old}} < X_G$.

N from	N through	G	A_v	A_c	surplus exceeds
3150	3295	282	2.504	2.885	.023
3296	3467	288	2.506	2.886	.146
3468	4150	292	2.514	2.889	.31
4151	5000	320	2.522	2.892	.44
5001	6000	354	2.531	2.896	.86
6001	10000	456	2.58	3.01	.59
10001	19999	468	2.58	2.92	2.37
20000	32767	516	3	3	3.88
32768	65535	602	3	3	4.66
65536	3269017	1132	3	3	4.87

Let

$$w = \log(b/N), \quad v = \log M = 2U + w, \quad P = 4M^{1/3},$$
$$\zeta = \log P, \qquad t_D = U + w - \log D, \qquad \xi = t_D + .337.$$

We have $0 < w < \log(5D/6)$ and $P > \max(830, 5G/2)$ in every row. At $p > P$ we have $p > 2H + d$ and $p^3 > M$. Lemma 17 separates the prime divisors of the norm there according to whether the two Newton edge heights agree.

The primes with equal heights divide $\binom{a+2d-1}{2d-1}$, since they divide a number in $[a+1, a+2d-1]$ and exceed $2d-1$. Using $\binom{x}{j} \le (ex/j)^j$, their total logarithmic weight is at most

$$2d\big(1 + \log(a/d + 2) - \log 2\big) < 2d(t + .337) \le 2D\xi.$$

For the last step the function $x(\log(b/x) + .337)$ increases for $2 \le x \le D$, since $b > N$ and $D < N/4$. Their right side of the norm inequality is at most $2D\xi(1/(P-1) + V/(N\zeta))$.

For unequal heights, one of the two windows $[bm - 2H, bm]$, $m = N-1, N$, contains qp^2 with $1 \le q \le Q = M/P^2 = P/64$. At fixed q each window contains at most one such $p > P$, since consecutive integer squares multiplied by q are spaced by more than $2P > 2H$ there. Thus the digit charge is at most $2QV/N$. Put $Z = b(N-1) - 2H$. Since $Z > .9995M$ and $Q > 12$, the first sum in (5) on these primes is at most

$$v\,\frac{\sum_{q \le Q}\sqrt{q}}{\sqrt{Z} - \sqrt{Q}} \le \frac{1.002v}{64}\left(\frac{2}{3} + \frac{1}{2Q}\right) < .71v/64.$$

Here $p \ge \sqrt{Z/q}$ and $p \le \sqrt{M}$. For the middle bound use the trapezoid inequality for the concave square root and $\sqrt{Z} - \sqrt{Q} \ge \sqrt{M}(.9995 - 1/P) > \sqrt{M}/1.002$. The lower bound on Z/M follows from $1 - 1/N - 2H/N^2 > .9995$, with $H \le 467$ below 20,000 and $H \le 1131$ above.

The primes $p \le P$ contribute at most $S(P) + \pi(P)V/N$. We have $V < v + .001$ and

$$\frac{V}{\zeta} < 3 - \frac{3\log 4 - .001}{\zeta}.$$

Put $k = 3\log 4 - .001$ and $c_1 = 1.2762$. Both $3 - k/\zeta$ and

$$(3 - k/\zeta)(1 + c_1/\zeta) = 3 - \frac{k - 3c_1}{\zeta} - \frac{kc_1}{\zeta^2}$$

increase in ζ, since $k > 3c_1$. This also bounds both by three. In each row the upper corner has its largest N and $b/N = 5D/6$, so the checker evaluates there to obtain A_v, A_c. Combining the three ranges of primes, using $S(P) < \log P - .49$ and $P/(P-1) < 1.002$, gives the upper bound

$$R = \zeta - .49 + \frac{P}{N}\left(A_c + \frac{2(v + .001)}{64}\right) + \frac{.71v}{64} + \frac{2.004D\xi}{P} + \frac{2A_vD\xi}{N}. \tag{25}$$

For the lower bound let $E = D/b$. The errors in (14) satisfy $J < .883\sqrt{\alpha}$ and $r/d < .641\sqrt{\alpha}$. Indeed the derivative of $\arcsin x + x\sqrt{1-x^2}$ is $2\sqrt{1-x^2} \le 2$. For the second estimate, convexity bounds $(1-y)^{-1/2}$ on $0 \le y \le .03$ by the chord from 1 to $1/\sqrt{.97} < 1.0156$. Integrating the resulting upper bound $1 + .52x^2$ gives $\arcsin x \le x + .52x^3/3 < 1.006x$ for $0 < x^2 \le .03$. For $d \ge 20$, use $\log C' > t - .6932 - 1.031E$, $g_0 \le 1.01 + .128DE$, and $g_0/d \le 1.01/d + .128E$. At fixed b, the derivatives of $(1 - 1/x)(\log(b/x) - .6932)$ and $[2\log(2x/\pi) - 1.01]/x$, multiplied by x^2, are

$$\log(b/x) - .6932 - x + 1, \qquad 3.01 - 2\log(2x/\pi),$$

respectively. Both are negative for $d \le x \le D$, since $x \ge 20$ and $\log(b/x) \le t < U < 15$. Substituting D for d in these two decreasing terms and bounding the errors with E gives

$$\begin{aligned} L = U - .001 + (1 - 1/D)(t_D - .6932) + \frac{2\log(2D/\pi) - 1.01}{D} \\ - 1.159E - \big[.641(1.7032 + .128DE) + .883\big]\sqrt{E}. \end{aligned} \tag{26}$$

For $d = 2, 3,\ 4 \le d \le 9,\ 10 \le d \le 19$, use $D = 2, 3, 9, 19$ respectively in (25) and t_D, keeping the row's A_v, A_c. The same angular estimates used above give $L = U + ft_D - c$ with $(f, c) = (1/2, .2), (2/3, .435), (.75, .68), (.9, .925)$, respectively. For $10 \le d \le 19$ retain the positive term $2\log(2d/\pi)/d \ge 2\log(38/\pi)/19$ in (14).

For fixed U, the lower bound L is affine in w minus a nonnegative linear combination of e^{-w}, $e^{-w/2}$ and $e^{-3w/2}$. It is therefore concave. For R only the P/N and ξ/P terms have nonzero second derivatives in w. Their sum gives

$$R_{ww} = \frac{P}{9N}\left(A_c + \frac{2(v + .001 + 6)}{64}\right) + \frac{2.004D}{9P}(\xi - 6).$$

To prove $R_{ww} \ge 0$, multiply by $9P$ and compare the positive term with $2.004D(6 - \xi)$. The positive term increases in U, w because $P^2/N = 16e^{U/3 + 2w/3}$ and $v = 2U + w$. The term $D(6 - \xi)$ decreases in U, w. Where it is positive, it increases with D because its derivative in D is $7 - \xi > 1$. It suffices to check $w = 0$ at the smallest order in each row with its largest D.

For fixed w, L grows with U at rate at least 1.5, since its negative exponential terms increase towards zero. The U-derivative of $(P/N)(A_c + (v + .001)/32)$ equals

$$\frac{P}{N}\left(\frac{1}{16} - \frac{A_c + (v + .001)/32}{3}\right) < 0,$$

since $A_c > 2$. The two equal-height terms decrease because their derivatives have the signs of $1 - 2\xi/3$ and $1 - \xi$, and $\xi > 3$. Thus R grows at rate at most $2/3 + 2 \cdot .71/64$. The supplement checks the convexity inequality and the surplus at the two endpoints of w at the smallest N in each row. For $d \ge 20$ the certified margins are given in the table. For all four smaller-gap cases the margin exceeds 3.8. It follows that $L > R$ throughout each row, which contradicts (5) and completes the proposition.

E The range $N \geq e^{15}$

Proposition 23. *Suppose $U = \log N \geq 15$, $a, d \geq 1$, $d < 3a$, and*

$$s = \frac{d^2}{a+d} < 2U + 2, \qquad 6a < (5N - 11)d.$$

Then R_{N-1} and R_N are coprime.

Proof. Suppose there is a monic irreducible common factor F. The notation $m, b, M, h, H, \alpha, V, \Lambda$ is as in Lemma 14. Set $\sigma = 2U + 2$, $t = \log(1/\alpha)$, and $B = H^4/M$. The starting-point inequality of Lemma 10 is available throughout this range, and $\log m > U - .001$. The shape inequality gives $b < (5/6)dN$, hence $t < U$ and $d < \sigma N$. Thus $M < \sigma e^{3U}$. The comparison $e^{.24U} > \sigma + e^{-3U}$ at $U = 15$ extends to larger U because $.24e^{.24U} > 2$ there. It follows that $V < 3.24U$. We first force the full set of rotations in the Padé norm. The angular gain then rules out small H. For the remaining factors, counts of square multiples in short intervals bound the large-prime contribution to the norm.

Rotations and the range of the parameters

We record how the weak bound in Lemma 16 will be used. For a coset of size n and $\alpha < .03$, put $c_\alpha = \log(4/(1-\alpha))$. Discarding the positive $2\log(2n/\pi)/n$ term and the nonnegative term $-(r/n)\log(1-\alpha)$ gives the lower bound

$$U - .001 + (1 - 1/n)(t - c_\alpha) - g_0/n - (r/n)g_0. \tag{27}$$

Here $c_\alpha < 1.417 < t$ and $g_0 < 1.266 + .256U$. For $d \geq 2$ and $s < .27$, a full coset also gives $\log \Lambda > U + t/3 - .71$: at least a third of the rotations have log gain at least $\log m + t + \log(1-\alpha)$, and the baseline gain at every rotation exceeds $\log m - 1.01 - .128s$. Use $s < .27$ to bound the resulting loss.

Suppose first that $q = d/k_0 \geq 2$ in Lemma 15, and set $P = M^{1/(q+1)}$. The right side of (5) is bounded by $S(X) + \pi(X)V/N$ for any $X \geq P$. If $\alpha \in [\ell, u] \subseteq [.03, .75]$, the inequality $M = dN/\alpha < PN/(4\alpha)$ gives $P^q < N/(4\alpha)$. Take $X = (N/(4\ell))^{1/2}$, so $X > P$, $M < X^3$, and $V/\log X < 3.001$. Here $X \geq \sqrt{3N/4}$ and $\log(M+1) \leq \log M + 1/M$. Thus $\pi(X)V/N < 3.6X/N$, since $3.001(1 + 1.2762/\log\sqrt{3N/4}) < 3.6$ at $U = 15$. Using $S(X) < \log X - .49$, the surplus of the baseline $\log m - 1.01 - g$ over the upper bound is greater than

$$\tfrac{1}{2}(U + \log(4\ell)) - 1.011 - \kappa(2U+2) + .49 - \frac{3.6}{\sqrt{4\ell N}},$$

where $g \leq \kappa s$. Use $\kappa = .134, .160, 7/32$, respectively, for $[.03, .1]$, $[.1, 1/3]$, and $[1/3, .75]$, by (10). The smallest endpoint surplus at $U = 15$ is greater than .12. The slopes of the affine terms are $\frac{1}{2} - 2\kappa \geq 1/16 > 0$, and the last (subtracted) term decreases.

For $\alpha < .03$ and $q \geq 3$, the inequalities $M < dN^2 < PN^2/4$ give $P < X = N^{2/3}/4^{1/3}$ and $M < X^4$. The baseline exceeds $.744U - 1.267$. Here $V/\log X < 4.001$ and $4.001(1 + 1.2762/\log X) < 4.6$ at $U = 15$, so the norm bound is $\log X - .54 + 4.6X/N$. The surplus exceeds .85 at $U = 15$. The baseline grows at rate $.744 > 2/3$ and X/N decreases. If $\alpha < .03$ and $q = 2$, we have $P/N < 1/2$ and $V/\log P < 3.001$. The digit loss $\pi(P)V/N$ is less than $4.848P/N < 2.43$, since $P > 2H \geq 8$. For $d < 50$ this improves to .121 using $P/N < (50/N)^{1/3}$. In that case the gain exceeds $U - 1.22$ and the prime sum is below $\max\{8, (2U + \log 50)/3\} - .54$. For $d \geq 50$, the weaker bound in

Lemma 16, with $n = d/2$, gives gain greater than $.9715U + .96t - 1.608$ in logarithm. In (27), use $1/n \le .04$, $g_0/n \le 1.01/25 + 2(.128)(.03) < .049$, and $r/n < .111$. The prime sum and digit loss are less than $\max\{8, (U + 2t + \log\sigma)/3\} + 1.89$. The surplus is positive at $U = 15, t = 3.5$. Its two branches increase with U and t, since the derivatives of $(U + 2t + \log\sigma)/3$ in these variables are at most $17/48$ and $2/3$, respectively. We conclude that $k_0 = k = d$.

For this full coset, suppose $\alpha \ge .03$. If $b = 2$, then $a = d = 1$, and the norm primes are at most $\sqrt{2N}$. The prime-count bound and $S(\sqrt{2N}) < \log\sqrt{2N} - .49$ already contradict the baseline $U - 1.011 - .176/2$. Otherwise $b \ge 3$, so $\log\sqrt{M} > 8$. Since $b \le \sigma/(.03)^2 < N/4$, the degree and prime bounds give prime support at most $\sqrt{M}$. Here $V/\log\sqrt{M} < 2.001$ and $2.001(1 + 1.2762/(U/2)) < 2.35$ at $U = 15$. The right side of (5) is then less than

$$\tfrac{1}{2}(U + \log b) - .54 + 2.35\sqrt{b/N}. \tag{28}$$

For $d \ge 10$, at least .45 of the rotations lie in the left half-plane. On the first and last intervals $[\ell, u]$ in the table below, the log gain therefore exceeds $U - .001 - .55(1.01 + \kappa\sigma) + .45\log((1-u)/u)$. On the middle interval at least $17/30$ of the rotations have $|\arg z| \ge \pi/3$, hence $\Re z \le am$. There the gain is at least $me^{-1.01}$, so the logarithmic mean exceeds $U - 1.011 - (13/30)\kappa\sigma$. Using $b \le \sigma/\ell^2$, the limiting comparisons are

$[\ell, u]$	κ	gain lower bound at $U = 15$	upper bound at $U = 15$
$[.03, .1]$	.134	13.07	12.45
$[.1, .5]$	.176	11.54	11.08
$[.5, .75]$	7/32	10.09	9.41

The affine lower bounds have slope at least .75, whereas the logarithmic part of the upper bounds has slope at most $1/2 + 1/\sigma$ and the square-root term decreases. For $d \le 9$, use $s < 6.75$, $b \le 300$ and the baseline $U - 1.011 - 7s/32$ against (28). Hence $\alpha < .03$ is necessary.

If $b \le N$, the norm primes are at most $X = 1.03\sqrt{M}$. Use $\log X > 8$, $V/\log X < 2.001$, and $\pi(X)V/N < 2.42\sqrt{b/N}$. The right side of (5) is less than $\frac{1}{2}(U + \log b) + 1.91$, since $\log(1.03) - .54 + 2.42 < 1.91$. When $d \ge 10$, use $n = d$ in (27) with $1/n + r/n < .211$. This gives $\log\Lambda > .945U + .9t - 1.70$. Writing $.9t - \frac{1}{2}\log b = -.05\log b - .45\log s$ and using $\log b \le U$, the surplus is greater than $.395U - .45\log\sigma - 3.61 > 0$. This last expression increases, since $2/\sigma \le 1/16$. If $2 \le d \le 9$, then $s = d\alpha < .27$, so the gain $U + t/3 - .71$ gives a surplus greater than $U/3 - (\log 9)/2 - 2.62 > 0$, using $\log b = t + \log d$ and $t < U$. For $d = 1$, the last gain in Lemma 14 exceeds $U + \frac{1}{2}\log a - 1.195$ in logarithm. Against $\frac{1}{2}(U + \log b) + 1.91$ this leaves a positive margin, using $b = a + 1 \le 2a$. The shape inequality precludes $b > N$ when $d = 1$, so now $b > N$ and $d \ge 2$.

For any $x \ge \max(e^8, \log K)$, the prime sum in (5) is below $\log x + .461$. Below x use Lemma 18, and above x bound it by $\log K/(x-1) \le x/(x-1) < 1.001$. Suppose $B < 2048$. Use $B = h^4 s^2 b/N$ for the bound on s. For the bound on V, write $M = (B/h^4)^{1/3} e^{(4t+4U)/3}$ and use $h \ge 2$ and $t < U$. These give

$$s < 8\sqrt{2}, \qquad V < \tfrac{8}{3}U + 1.62, \qquad H < 10.08e^{(U+t)/3}.$$

The last bound follows from $H = (B/h)^{1/3}e^{(U+t)/3}$. Split the count of distinct norm primes at $N/16$ and use $\log K \le HV$. The digit loss is less than $.236 + 9.46UH/N$. Both coefficients follow from the prime count at $U = 15$. To see the decrease, divide the bound $\frac{8}{3}U + 1.62$ for V by $U - \log 16$ in the first coefficient, and use $(\frac{8}{3} + 1.62/U)^2/(1 - \log 16/U)$ in the second. For $d \ge 50$ the bound (27) gives $\log\Lambda > U + .98t - 1.72$, using $g_0 < 2.459$, $g_0/d < .025$, and $r/d < .111$. Splitting the prime sum at $\max(e^8, HV)$, the surplus is greater than

$$\tfrac{2}{3}U + (.98 - \tfrac{1}{3})t - \log(10.08(\tfrac{8}{3}U + 1.62)) - 2.417 - 95.3568\,Ue^{(t-2U)/3}.$$

This is concave in t. At $t = 3.5, U$ its values are greater than $3.59, 1.60$, respectively, at $U = 15$. At either endpoint the logarithmic term grows at rate less than $1/U$, while $Ue^{-jU/3}$ decreases for $j = 1, 2$. The endpoint surpluses therefore increase. If $d < 50$, then $s < 2500/N < .001$ and $t > U - \log 50$. The gain exceeds $U + t/3 - .71$. Use $M < 50N^2$, $H < (102400)^{1/4}N^{1/2}$, and $V < 2U + \log 51$. The digit bound is below 1.65, so the total upper bound is less than

$$\tfrac{1}{2}U + \tfrac{1}{4}\log 102400 + \log(2U + \log 51) + .461 + 1.65.$$

It is beaten at $U = 15$ by the gain, which increases faster. Thus $B \geq 2048$.

Finally, the initial block at order N has prime support below $1.03b$, since $b > N$ and $\alpha < .03$. Its logarithm exceeds $H \log(aN)$. For each $p^j \leq M$, the initial block contains at most $H/p^j + 1$ multiples of p^j. Summing over j and then over the supported primes bounds its logarithm by $HS(1.03b) + \pi(1.03b) \log M$. The difference $\log(aN) - S(1.03b)$ exceeds $U + .47$ and the prime-count term is below $2.24b$. We have obtained

$$k_0 = k = d, \qquad \alpha < .03, \qquad b > N, \qquad B \geq 2048, \qquad H/b < \frac{2.24}{U + .47}. \tag{29}$$

Norm and degree bounds

Put $P = 2H + d$ and $L = \log M$. For primes $p > P$ one has $p^4 > M$. Call a prime exceptional if either terminal interval of length $6H$ contains a positive multiple of p^3. For each such interval and each $q \leq M/P^3$ there is at most one prime with qp^3 in the interval. Each exceptional prime contributes at most L to the log norm by (12). Write Y for the sum of $\log p$ over the nonexceptional unequal-height norm primes. By Lemma 17 and (15),

$$Y \leq \min\{6d \log(2M),\ ML/P^2\}. \tag{30}$$

The second bound uses at most one prime per square multiplier in each terminal interval and $\log p \leq L/2$.

At a nonexceptional equal-height prime its entire norm power divides an integer in $[a + 1, a + 2d - 1]$. The product of these powers divides $\binom{a+2d-1}{2d-1}$ because the primes exceed $2d - 1$. Writing $A_0 = a/d$, its logarithm is less than $2d(1 + \log(A_0 + 2) - \log 2)$. At $p \leq P$ apply the affine estimate (12). Comparing with $\log |F(0)| > H \log(aN)$, and setting

$$\Delta = \log(aN/H) - \frac{2}{h}(1 + \log(A_0 + 2)), \qquad \Delta_* = \Delta + \frac{2}{h}\log 2,$$

gives

$$\Delta_* < \log(P/H) + \frac{\pi(P)L}{H} + \frac{Y}{H} + \frac{L}{4B}. \tag{31}$$

Here $\log P > 8$, so $S(P) < \log P$, and $2ML/(HP^3) \leq L/(4B)$. The same estimate holds for Δ.

In (29) we have $A_0 + 1 > e^{U/2}/\sqrt{2U + 2} > 308$. For $h = 2$, $\Delta - U = \log(A_0/(A_0 + 2)) - 1 - \log 2 > -1.70$, and $\Delta_* > U - 1.01$. For $h \geq 4$, $\Delta - U$ attains its minimum on $4 \leq h \leq 2.24(A_0 + 1)/(U + .47)$ at an endpoint, since its derivative has the sign of $2(1 + \log(A_0 + 2)) - h$. At the lower endpoint the value exceeds .97. At the upper endpoint it is

$$\log \frac{U + .47}{2.24} + \log \frac{A_0}{A_0 + 1} - \frac{2(U + .47)}{2.24}\,\frac{1 + \log(A_0 + 2)}{A_0 + 1}.$$

This increases with A_0. Substitute the lower bound $A_0 + 1 > e^{U/2}/\sqrt{2U + 2}$. The result exceeds 1.6 at $U = 15$ and increases thereafter. Indeed, if $T = e^{U/2}/\sqrt{2U + 2}$, then $.46 < T'/T < .5$.

The logarithmic derivative of $(U+.47)(1+\log(T+1))/T$ is at most $1/(U+.47)+.5/(1+\log(T+1))-.46<0$. In particular $\Delta>U+.95$ for $h\geq 4$. For the prime-count term, $\log P\geq L/4+\log 2+(\log 2048)/4>L/4+2.59$. Write $x=\log P$ and $C_\pi=1.2762$. Then $L(1+C_\pi/x)/x<4(1-C_\pi^2/x^2)<4$. Since $P/H\leq 2.5$, Lemma 18 gives $\pi(P)L/H<10$. By (30), for $h\geq 4$,

$$h<\frac{6(3.24U+1)}{U+.95-\log 2.5-10-3.24U/8192}<61.$$

For $U\geq 15$ the denominator is positive and affine in U, with positive slope and negative constant term. The numerator is affine with positive slope and constant term, so the quotient decreases. Thus $h\leq 60$.

We next refine the digit loss in (5). For $p>P'\geq 2H+d$ and a positive lifted valuation e, one has $e\geq 1/H$ and $e-\mu\geq 1/(2H)$. The progression in (11) has valuation at most $N\mu+\lfloor\log_p(bm+j)\rfloor$, as $p\nmid d$. The floor first increases beyond $L_p=\lfloor\log_p M\rfloor$ only at $j-1\geq b>2H$. Later increases are more than $2H$ steps apart. The penalty $(j-1)(e-\mu)$ absorbs all these increases. Thus above P' the per-prime digit loss in the averaged bound is at most $L_p\log p/N$, with $L_p\leq 3$. In particular the digit loss is at most

$$\frac{\pi(P')V}{N}+\frac{3}{N}\sum_{p>P'}v_p(F(0))\log p. \tag{32}$$

Put $\delta=H/N$ and take $P'=(2+2/h)H=rH$. For nonexceptional primes above P', Lemma 17 puts their entire norm product in the binomial coefficient $\binom{b+H+d-1}{2H+2d-1}$. With $c=b/H>6.9$, its logarithm divided by H is at most $r(1+\log((c+r/2)/r))$. Its derivative in r is $\log(c/r+1/2)+r/(2c+r)>0$ for $r\leq 3$. At $r=3$, divide by 3 and write the result as $t+1-\log(3h)+\log(1+1.5/c)<t-.595$, using $h\geq 2$. For the primes below P', the same prime-count calculation as for P, now with $V<L+.001$ and $\log P'>L/4+2.59$, gives $\pi(P')V/N<4r\delta\leq 12\delta$. The exceptional primes contribute at most $6\delta L/(Br^3)$. Therefore the digit loss is less than $\delta(9t+J_0(U))$, where $J_0(U)=6.65+6(3.24U)/16384$.

When $d\geq 50$ the weaker angular bound gives $\log\Lambda>U+.98t-1.64$. Use $t>(U-\log(2U+2))/2$, $g_0/d<.025$, and $(2/\pi)\arcsin(e^{-t/2})g_0<.19$. For the last bound let $x=e^{-(U-\log\sigma)/4}$. The estimate $\arcsin x\leq x/\sqrt{1-x^2}$ bounds the loss by $2x(1.266+.256U)/(\pi\sqrt{1-x^2})$. The logarithmic derivative of x is at most $-15/64$, and that of $1.266+.256U$ is below $1/U$. Also $1/\sqrt{1-x^2}$ decreases. Thus the maximum is at $U=15$. Split the prime sum at $HV>e^8$. If $\delta\leq .045$, the gain exceeds the two upper bounds by more than

$$(.98-9\delta)t-\log\delta-\log(3.24U)-2.101-J_0(U)\delta.$$

On $\delta\leq .045$ the expression decreases in δ and increases in t. At $\delta=.045$ and $t=(U-\log\sigma)/2$, its derivative in U is at least

$$\frac{.98-9(.045)}{2}\left(1-\frac{1}{16}\right)-\frac{1}{15}-.045\frac{6(3.24)}{16384}>0.$$

Its value exceeds .10 at $U=15$. For $d<50$, $s<2500/N<.001$ and $H<3000$, so the digit bound is below $3000e^{-U}(9U+J_0(U))$. This decreases for $U\geq 15$ and is below .14 there. The gain exceeds $U+t/3-.71$, and $t>U-\log 50$. It exceeds $\log(3000\cdot 3.24U)+.461+.14$ at $U=15$ and increases faster. Therefore

$$h\leq 60,\qquad \delta>.045,\qquad \Delta_*>U-1.01. \tag{33}$$

A count of terminal square multiples

We use a sharper count in the range (29). For either $i = N-1, N$, set $X = bi$, and count primes $p > P$ with $X - 2H \le qp^2 \le X$, q a positive integer. Divide the range up to $\sqrt{X}$ into intervals $[D, \eta D)$ with $D = \eta^j P$ and $\eta = 9/8$. The integer points (p, q) lie between X/p^2 and $X/p^2 - \epsilon$, where $\epsilon = 2H/D^2$. Each prime permits at most one multiplier and each multiplier at most one prime. The mean value theorem gives $1 \le 2X|p_2 - p_1|/D^3 + \epsilon$ for distinct points. Since $\epsilon \le 1/8$, their abscissas have spacing at least $g_D = \max(2, 7D^3/(16X))$.

Partition each interval further into blocks of width $\rho = (D^4/X)^{1/3}$, with number at most $.132(X/D)^{1/3}$, since $\rho/D \le X^{-1/6} < .007$. For three points in a block the integer area determinant is the determinant for X/x^2 plus that for the vertical errors. The first is at most $3/4$ in absolute value by the second divided difference bound $3X/D^4$, times $\rho^3/4$. The error determinant is at most $\epsilon\rho \le (2H/X)^{1/3} < .01$. Thus all points in a block are collinear. For $n \ge 3$ points, select a middle one whose horizontal distances to both extremes are at least $(n-2)g_D/2$. The chord of X/x^2 through the extreme abscissas is at most ϵ above the curve there. If its two distances are L_1, L_2, the chord excess is at least $3XL_1L_2/(\eta D)^4$, by subtracting a quadratic of second derivative $6X/(\eta D)^4$ and using convexity. The upper bound $\epsilon = 2H/D^2$ on this excess gives

$$n \le 2 + 2\eta^2\sqrt{2/3}\,\frac{D\sqrt{H/X}}{g_D}.$$

The same bound holds for smaller n.

Sum these bounds with the weight $\log(\eta D)$. The constant term from each block sums to at most

$$7(X/P)^{1/3}(\log P + 3.06),$$

since $.264/(1-\eta^{-1/3}) < 7$ and $\log\eta/(1-\eta^{-1/3}) < 3.06$. For the second term split at $D_* = (32X/7)^{1/3}$, where g_D switches. The powers of D in the two geometric sums are $2/3$ and $-7/3$. Their combined unweighted coefficient is bounded using

$$.132\eta^2\sqrt{2/3}(32/7)^{2/9}\left(\frac{1}{1-\eta^{-2/3}} + \frac{1}{1-\eta^{-7/3}}\right) < 3.4.$$

The first sum has weight at most $\log(\eta D_*)$. For the second sum, the weighted summand $D^{-7/3}\log(\eta D)$ decreases above D_*, as $\log(\eta D_*) > 3/7$. Replace the actual starting point by D_* in every term of its geometric progression. In that upper-bounding series, the mean weight is $\log(\eta D_*) + (\log\eta)\eta^{-7/3}/(1-\eta^{-7/3}) < (\log X)/3 + 1$. Summing over both orders and using $X \le M$ yields

$$\frac{Y}{H} < \frac{14(\log P + 3.06)}{(P/H)^{1/3}B^{1/3}} + \frac{6.8M^{1/18}}{\sqrt{H}}(L/3+1). \tag{34}$$

The final comparison

Put $v = Ne^{-t} = sh/\delta > 6/5$. Then

$$M = \frac{\delta N^3}{hv}, \qquad B = hv\delta^3 N.$$

Using $\log P > U + \log(.09)$, $L < 3.24U$, and $\Delta_* > U - 1.01$ in (31) gives $Y/(HL) > .05$. Explicitly, a lower bound is

$$\frac{U - 1.01 - \log 2.5 - \dfrac{2.5(3.24U)}{U+\log(.09)}\left(1 + \dfrac{1.2762}{U+\log(.09)}\right) - 3.24U/8192}{3.24U}.$$

After division by $3.24U$, the U-dependent negative terms are multiples of $1/U$, $1/(U+\log(.09))$, and $1/(U+\log(.09))^2$. Each decreases, so the lower bound is smallest at $U=15$, where it exceeds .0502. By (30), $Y/(HL) \le 1/(vh(2+1/h)^2\delta^2)$. Hence $\delta^2 < 4/3$, since $h(2+1/h)^2 \ge 12.5$. In particular $\delta < 1.4$ and $L < 3U$.

For $\ell \le \delta \le u$ and $v \ge w$, define

$$\begin{aligned} V_0 &= 3U + \log(u/(2w)) + .001, \\ Z &= \frac{1}{\ell}\sum_{j=4}^{8}(u/(2w))^{1/j}e^{(3/j-1)U} + \frac{3U}{\ell\log 2}(u/(2w))^{1/9}e^{-2U/3}, \\ C_0 &= 1.017\{5 + \min(2.5, (2w\ell^2)^{-1/3}) + Z\}, \\ A(U,\ell,u,w) &= U - 1.01 - \log 2.5 - C_0 - V_0/8192. \end{aligned}$$

At $p \le P$, the extra term in the affine norm count is $\sum_{j \le L/\log 2}\vartheta(\min(P, M^{1/j}))$, where $\vartheta(x) = \sum_{p\le x}\log p$. By Rosser and Schoenfeld, $\vartheta(x) < 1.017x$ for $x > 0$ and $\vartheta(x) > .98x$ for $x \ge 7481$ [9, Theorems 9 and 10]. In the upper estimate, $j = 1, 2$ contribute at most $1.017(2P/H)$ after division by H, the term $j = 3$ uses $M^{1/3}/H \le (2w\ell^2)^{-1/3}$, and the remaining terms contribute at most $1.017Z$. Thus (31) with this count gives $Y/H > A$. On the other hand, (34) gives $Y/H < W(U,\ell,w)$, with

$$W(U,x,w) = e^{-U/3}\left\{\frac{14(U+\log(2.5x)+3.06)}{(5w)^{1/3}x} + \frac{6.8(U+\frac{1}{3}\log(x/(2w))+1)}{(2w)^{1/18}x^{4/9}}\right\}.$$

In obtaining this formula first take $x = \delta$, and use $hv \ge 2w$, $(P/H)hv \ge 5w$, and $\log P \le U + \log(2.5\delta)$. The numerators in braces have the form $\varphi = U + c\log x + c'$, with $c = 1, 1/3$, respectively, and the powers of x in the denominators are $\nu = 1, 4/9$. For every row below, $.045 \le x \le 1.4$, $6/5 \le w \le 9$, and both values of φ exceed 14. The x-derivative of $x^{-\nu}\varphi$ has the sign of $c - \nu\varphi < 0$. After multiplying by x the sign is that of $c + (1-\nu)\varphi > 0$. Thus W decreases in x and xW increases. In particular $Y/N < uW(U,u,w)$. The derivative of $e^{-U/3}\varphi$ in U has the sign of $1 - \varphi/3 < 0$, so W decreases. In A, each term of Z decreases, including the last one since $U > 3/2$. Therefore $A'(U) > 1 - 3/8192 > 0$.

At $(\ell,u,w) = (.21, 1.4, 6/5)$ the difference $A - W$ exceeds .074 at $U = 15$, so $\delta < .21$. On the resulting range $[.045, .21]$ with $w = 6/5$, A/V_0 increases with U. To check this, write $A/V_0 = (U - c(U))/(3U+\beta) - 1/8192$, where $c(U) = 1.01 + \log 2.5 + C_0$ decreases and $\beta = \log(.21/2.4) + .001$ satisfies $\beta + 3c(U) > 0$. The multiplier bound in (30) now gives $sh < .675/\delta$, since $(2/25)V_0/A < .675$ at $U = 15$. For larger δ, the table below uses the stronger bound $sh < uw$ when $A > W$ excludes $v \ge w$.

Now (27) gives $\log\Lambda > U + t - 1.60$. Indeed $(t+g_0)/d < .009$ from $d > .045N/60$ and $t < U$. The angle loss is below .19 as before, and $c_\alpha < 1.39$ since $\alpha < \sqrt{\sigma/N}$. For each row $[\ell, u]$ put $V_0 = V_0(U, u, 6/5)$ and set

$$Q = \min\{V_0,\ U + .127 + W(U,\ell,6/5) + V_0/8192\}.$$

Both entries bound the large-prime log norm divided by H. For the second entry, the equal-height prime contribution is at most $(2/h)(U + .127)$, since $A_0 + 2 < (5/6)N + 1/6$, and the other contributions use (34) and the cube bound. Both branches of Q exceed 15, so $HQ > P$ and $HQ > e^8$. Below HQ, use $S(HQ)$. Above HQ, all primes belong to the large-prime part and

their logarithmic contribution to the prime sum is at most $HQ/(HQ-1) < 1.001$. This gives $\log(HQ) + .461$.

The digit estimate (32) has $L_p \le 2$ above P, except that $L_p \le 3$ when $p \le M^{1/3}$. Since $P > 2H > .09N > 7481$, the lower bound for $\vartheta(P)$ applies. The extra loss is at most $\max(0, \vartheta(M^{1/3}) - \vartheta(P))/N$, which is bounded by $\max(0, 1.017(u/2.4)^{1/3} - 1.96\ell)$. Thus the digit loss is at most the following, with $z = U + \log(2.5\ell)$:

$$D_0(U,\ell,u) = u\left\{\frac{2.5V_0}{z}\left(1+\frac{1.2762}{z}\right) + 2(U+.127) + V_0/4096\right\} + 2uW(U,u,6/5) + \max(0, 1.017(u/2.4)^{1/3} - 1.96\ell).$$

For the prime count in this formula put $y = \log(rH) > 8$. The derivative in r of $r(1+C_\pi/y)/y$ is $\big(y^2 + (C_\pi - 1)y - 2C_\pi\big)/y^3 > 0$. Thus the prime-count contribution is bounded using $r = 2.5$. Since $H = she^t$, the surplus over (5) is greater than $U - \log(JQ) - 2.061 - D_0$ where J bounds sh. The following endpoint comparisons complete the proof:

$[\ell, u]$	w	J	$\underset{\text{at } U=15}{A-W}$	$\underset{\text{at } U=15}{U-\log(JQ)-2.061-D_0}$
$[.045, .10]$	–	15	–	$> .36$
$[.10, .14]$	9	1.26	$> .109$	> 1.63
$[.14, .18]$	3.6	.648	$> .117$	$> .83$
$[.18, .21]$	1.8	.378	$> .093$	$> .30$

The first row uses $sh < .675/\delta$. In the other rows the fourth column gives $v < w$. To check the last column throughout the range, first note $V_0 > 40$ in all four rows. The two branches defining Q have logarithmic derivatives bounded above by $3/40$ and $(1+3/8192)/(15+.127)$, respectively. For the second bound, use $W > 0$ in the denominator and $\partial_U W < 0$ in the numerator. Both bounds are below .08. This bounds the growth of $\log Q$ even at a change of the minimizing branch, since each branch minus $.08U$ in logarithmic form is decreasing.

The rational factor $V_0(1 + C_\pi/z)/z$ equals $3(1 - k/z)(1 + C_\pi/z)$, where $k = \log(2.5\ell) - (\log(u/2.4) + .001)/3 < .02$. Its derivative is $3(-(C_\pi - k)z + 2kC_\pi)/z^3 < 0$ for $z > 8$. The W term also decreases. The remaining U-dependent terms of D_0 grow at rate at most $.21(2+3/4096) < .43$. Consequently the last column increases with slope at least $1 - .08 - .43 > 0$ wherever the derivative exists and is continuous at a switch of branch. Thus the positive surplus contradicts the Padé inequality. The accompanying verifier reproduces the scalar comparisons at the stated endpoints using rational enclosures for logarithms, exponentials, and rational powers. □

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