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A COUNTEREXAMPLE TO RANK LOG-CONCAVITY OF BRUHAT INTERVALS

Abstract. In this note, we disprove Brenti's conjecture that Bruhat intervals in Weyl groups are rank log-concave by constructing an explicit counterexample in a large symmetric group.

1. Introduction

The study of Bruhat rank sequences in a Weyl group connects enumeration combinatorics with the geometry of flag varieties. In 1980, Stanley used the hard Lefschetz theorem to prove rank symmetry, rank unimodality, and the strong Sperner property for finite Weyl-group parabolic quotients [Sta80, Theorem 3.1]. Here the quotient consists of the minimal length representatives of the cosets of a standard parabolic subgroup, ordered by Bruhat order.

For individual lower intervals, Björner and Ekedahl obtained shape inequalities using intersection cohomology [BE09]. More precisely, let (W, S) be a crystallographic Coxeter system, let $J \subseteq S$, and put $W_J = \langle J \rangle$. Let W^J be the minimal length representatives of W/W_J. For $v \in W^J$, let $f_i = \#\{w \in W^J : w \leq v,\ \ell(w) = i\}$. Their Theorem A gives

$$f_i \leq f_j \qquad \text{if } 0 \leq i < j \leq \ell(v) - i.$$

Thus the rank numbers increase through the lower half, and they are "top-heavy" in the sense that each rank number in the lower half is bounded by that in the corresponding opposite rank.

In the affine setting, Burrull, Gui, and Hu studied lower intervals in the minimal length representatives of right cosets in an affine Weyl group by the finite Weyl subgroup. They prove that the corresponding length-counting sequence is "asymptotically log-concave"(see [BGH25, Theorem 1.3 and Corollary 1.6] for the precise statements).

Now, let W be a finite Weyl group with Bruhat order $\leq$ and length function ℓ. For $u \leq v$, its rank generating function is

$$B_{u,v}(q) = \sum_{u \leq \pi \leq v} q^{\ell(\pi)-\ell(u)} = \sum_{j=0}^{\ell(v)-\ell(u)} a_j q^j.$$

Recall that a nonnegative sequence $(a_0, \ldots, a_d)$ is *log-concave* if $a_j^2 \geq a_{j-1}a_{j+1}$ for $1 \leq j < d$. We say that a Bruhat interval is *rank log-concave* if the sequence of the coefficients in its rank generating function is log-concave.

With some computational evidence, Brenti made the following conjecture.

Conjecture 1 (Brenti [Bre24, Conjecture 2.11])**.** Every Bruhat interval in a Weyl group is rank log-concave.

Brenti records computations supporting the conjecture in types A_d and D_d for $d \leq 5$, in type B_d for $d \leq 4$, in F_4, and in dihedral groups, as well as in type B_5 for intervals of length at least 20 [Bre24]. He also exhibits a noncrystallographic H_3 interval with rank sequence

$$(1, 3, 5, 7, 10, 10, 5, 1),$$

whose middle ranks satisfy $7^2 < 5 \cdot 10$.

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However, we construct a huge counterexample in type A in this note.

Theorem 2. *There exist a positive integer $n < 4 \cdot 10^{102}$ and permutations $u \le v$ in S_n such that $[u, v]$ is not rank log-concave. In particular, Conjecture 1 is false.*

Our construction uses seven consecutive position blocks and seven consecutive value blocks. Their intersection sizes form a 7×7 table. Bounds on its northwest partial sums specify a single Bruhat interval; the extremal double-coset representatives are described by Hohlweg and Skandera [HS05].

The remainder of the paper is organized as follows. In Section 2, we provide the required interval description using block tables and prove a length-generating formula. In Section 3, we give the explicit Bruhat interval using the table construction in Section 2. In Section 4, we derives a contradiction to the rank log-concavity of that interval. Appendix A contains the complete Sage codes for the finite arithmetic.

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2. Block tables and Bruhat intervals

For $\pi \in S_n$, recall its *length* $\ell(\pi)$ is the inversion number

$$\ell(\pi) = \#\{(h, k) : h < k,\ \pi(h) > \pi(k)\}.$$

We use $\le$ to denote the *(strong) Bruhat order*. We denote $[\pi, \tau]$ the *Bruhat interval* for $\pi \le \tau$. For a permutation π, let

$$R_\pi(p, q) := \#\{h \le p : \pi(h) \le q\}, \qquad 0 \le p, q \le n.$$

The classical rank-matrix criterion states that $\pi \le \tau$ if and only if $R_\pi(p, q) \ge R_\tau(p, q)$ for all p, q, see for example, [BB05, Theorem 2.1.5].

Fix positive integers $r_0, \dots, r_{s-1}$ and $c_0, \dots, c_{s-1}$, both with sum n. Partition positions into consecutive blocks of sizes r_i and values into consecutive blocks of sizes c_j. Consider the table K which has entry K_{ij} equal to the number of positions in block i whose values belong to block j. We call r and c the *margins* of K, where $r = (r_0, \dots, r_{s-1})$ and $c = (c_0, \dots, c_{s-1})$. For such a table, define

$$\sigma_{ij}(K) := \sum_{h<i,\, k<j} K_{hk} \quad (0 \le i, j \le s), \qquad R_i := \sum_{h<i} r_h, \qquad V_j := \sum_{k<j} c_k.$$

We call $\sigma_{i+1,j+1}$, $0 \le i, j < s-1$, the *proper northwest coordinates*.

We construct two permutations $\min K$ and $\max K$ in S_n for such a table K. For both constructions, process the position blocks in increasing order. For $\min K$, place the value-block labels in increasing order and assign the available values in each value block in increasing order. For $\max K$, use decreasing order for both the labels and the available values. For such a table K, we collect all the permutations such that for every $0 \le i, j < s$, exactly K_{ij} positions in position block i have values in value block j, and call it the *table fiber* of K. We denote $K(\pi)$ the corresponding table such that π is in its table fiber.

Lemma 3. *At $p = R_i + a$, $q = V_j + b$, where $0 \le a \le r_i$ and $0 \le b \le c_j$, the two permutations satisfy*

$$R_{\min K}(p, q) = \min\{\sigma_{ij} + a + b,\ \sigma_{i,j+1} + a,\ \sigma_{i+1,j} + b,\ \sigma_{i+1,j+1}\}, \tag{2.1}$$

$$R_{\max K}(p, q) = \max\{\sigma_{ij},\ \sigma_{i,j+1} - (c_j - b),\ \sigma_{i+1,j} - (r_i - a), \sigma_{i+1,j+1} - (r_i - a) - (c_j - b)\}. \tag{2.2}$$

They are, respectively, the unique minimum and maximum of their table fiber in Bruhat order.

Proof. For any permutation in the fiber, monotonicity of R_π and the fact that increasing either argument by h increases it by at most h give the four upper bounds in (2.1) and four lower bounds in (2.2). These bounds use only the four surrounding block corners, where the ranks equal the corresponding σ's.

Write $x_+ = \max\{x, 0\}$. For the increasing construction, let $x = \sigma_{i,j+1} - \sigma_{ij}$, $y = \sigma_{i+1,j} - \sigma_{ij}$, and $k = K_{ij}$. The earlier position blocks contribute $\sigma_{ij} + \min(b, x)$: their x values in value block j are its first x values. In position block i, the labels below j occupy its first y positions, contributing $\min(a, y)$. The cell (i, j) occupies the next k positions and uses the next k values after those x values. Its contribution is $\min\{k, (a-y)_+, (b-x)_+\}$. Hence

$$R_{\min K}(p, q) = \sigma_{ij} + \min(a, y) + \min(b, x) + \min\{k, (a-y)_+, (b-x)_+\}.$$

If $a \le y, b \le x$, this is $\sigma_{ij} + a + b$. If $a \le y, b > x$, it is $\sigma_{ij} + a + x$. If $a > y, b \le x$, it is $\sigma_{ij} + y + b$. In the remaining case it is $\sigma_{ij} + x + y + \min\{k, a-y, b-x\}$. In each case this is exactly the minimum of the four displayed upper bounds.

Reverse the columns of K to form K^* and let $\tau = \min K^*$. The permutation $p \mapsto n + 1 - \tau(p)$ is $\max K$, and its rank at (p, q) is $p - R_\tau(p, n-q)$. If $j^* = s - 1 - j$, then

$$\sigma^*_{i,j^*} = R_i - \sigma_{i,j+1}, \qquad \sigma^*_{i,j^*+1} = R_i - \sigma_{ij},$$

and the local value coordinate becomes $c_j - b$. Substitution in (2.1), and $p - \min = \max(p - \cdot)$, gives (2.2). The universal upper and lower bounds now show $\min K \le \pi \le \max K$ for every permutation π in the fiber. The rank characterization gives uniqueness. □

Proposition 4. *Let A, B have the same margins and suppose $\sigma_{ij}(A) \ge \sigma_{ij}(B)$ for all i, j. Then*

$$[\min A, \max B] = \{\pi : \sigma_{ij}(A) \ge \sigma_{ij}(K(\pi)) \ge \sigma_{ij}(B) \text{ for all } i, j\}.$$

Proof. Both four-corner expressions are increasing functions of each σ. Consequently the stated table inequalities imply $\min A \le \min K \le \pi \le \max K \le \max B$. Conversely, the rank inequalities for $\min A \le \pi \le \max B$, evaluated at the block boundaries, give exactly the stated inequalities for $K(\pi)$. □

For an integer $m \ge 0$, recall that we have the q-integer and q-factorial

$$[m]_q := \sum_{h=0}^{m-1} q^h, \qquad [m]_q! := \prod_{h=1}^{m} (1 + q + \cdots + q^{h-1}),$$

with the conventions $[0]_q := 0$ and $[0]_q! := 1$.

For a table K, we define the *length generating function* $W_K(q)$ of its table fiber by

$$W_K(q) := \sum_{K(\pi)=K} q^{\ell(\pi)}.$$

We define the number

$$I(K) := \sum_{i<h,\, j>k} K_{ij} K_{hk}. \tag{2.3}$$

Lemma 5. *For every nonnegative integer table K with margins r, c, we have*

$$W_K(q) = q^{I(K)} \frac{\prod_i [r_i]_q! \prod_j [c_j]_q!}{\prod_{i,j} [K_{ij}]_q!}. \tag{2.4}$$

Proof. First choose the word of value-block labels in each position block. Let $T(k_0, \ldots, k_{s-1})$ be the inversion generating polynomial of words with these multiplicities, and let $r := \sum_j k_j$. Classifying the words by their last label gives the recurrence

$$T(k_0, \ldots, k_{s-1}) = \sum_{j : k_j > 0} q^{\sum_{h>j} k_h} T(k_0, \ldots, k_j - 1, \ldots, k_{s-1}).$$

The quotient $[r]_q!/\prod_j [k_j]_q!$ satisfies this recurrence because

$$\sum_j [k_j]_q q^{\sum_{h>j} k_h} = [r]_q.$$

the consecutive exponent ranges partition $\{0,\dots,r-1\}$. Induction from the empty word proves the formula.

Next assign the distinct values in value block j to all occurrences of its label. Their inversion polynomial is $[c_j]_q!$: insertion of the largest value contributes any of $0,\dots,c_j-1$ inversions, and induction applies. Pairs in different position and value blocks contribute the fixed number $I(K)$. Pairs in the same position block and different value blocks contribute the word inversions. Pairs in the same value block contribute the last factorials. These three classes partition all pairs, and the choices specify each permutation exactly once. □

3. The explicit interval

All 7×7 matrices below have indices $0,\dots,6$. Fix the ordered list

$$\mathcal{S} = ((0,1),(1,0),(1,2),(2,1),(2,2),(3,0),(3,1),(3,3),(4,2),(4,3),(5,4)). \tag{3.1}$$

For $s=(i,j)\in\mathcal{S}$, let D_s have entries $+1$ at $(i,j),(i+1,j+1)$, entries -1 at $(i,j+1),(i+1,j)$, and zero elsewhere. Write $Dy=\sum_{s\in\mathcal{S}} y_s D_s$. Each D_s has zero row and column sums. Its proper northwest coordinates are 1 at s and 0 elsewhere.

Put

$$p=\frac{1}{32},\quad t_1=\frac{1089}{32},\quad t_2=\frac{36993}{32},\quad t_3=\frac{33}{35872},\quad t_4=\frac{1184865}{35872},\quad t_5=\frac{36993}{1148960},$$

$$t=\begin{pmatrix} p & p & p & p & p & p & p\\ p & p & t_1 & p & p & p & p\\ p & t_1 & p & p & p & p & p\\ p & p & p & t_1 & p & p & p\\ p & t_1 & t_2 & p & p & p & p\\ p & p & p & t_3 & t_4 & p & p\\ p & p & p & p & p & t_5 & p \end{pmatrix},\qquad M_{ij}=\log(1+1/t_{ij}).$$

Define the column vector

$$w=\frac{1}{100000}(-17048,-30702,17618,31695,38253,28244,29146,-38002,-29147,-32509,31476)^T. \tag{3.2}$$

The coordinates are ordered by (3.1), numbered $1,\dots,11$. Finally set

$$N=10^{100},\quad \varepsilon=10^{-12},\quad C_{ij}=\lfloor NM_{ij}\rfloor,\quad h_s=N\varepsilon|w_s|,\qquad A=C+Dh,\quad B=C-Dh. \tag{3.3}$$

Since $N\varepsilon/100000=10^{83}$, every h_s is an integer. Let r_i,c_j be the margins of C, let $n=\sum_{ij}C_{ij}$, and define the permutations

$$u:=\min A,\qquad v:=\max B, \tag{3.4}$$

using the explicit filling rules of Section 2.

We will use the following finite bounds. The appendix proves all their arithmetic assertions with exact rational operations and rigorous logarithm bounds.

Lemma 6. *The data above satisfy*

$$\frac{1}{1200} < M_{ij} < 8, \qquad \max_s |w_s| < \frac{2}{5}, \qquad \sum_s |w_s| < 4, \tag{3.5}$$

$$\max_{i,j} \sum_s |(D_s)_{ij} w_s| = \frac{49829}{50000} < 1, \tag{3.6}$$

$$\sum_{ij} t_{ij}(1+t_{ij})|(Dw)_{ij}|^3 = \frac{92117978523348864650859972 93}{44859469886720000000000000000} < \frac{21}{1000}. \tag{3.7}$$

All 49 *floors in* (3.3) *are uniquely determined by the rational logarithm intervals in the appendix, and* $n < 400N$.

For a vector x, let $\mathrm{diag}(x)$ be the diagonal matrix with diagonal x. Every table in the integer box

$$\mathcal{K} = \{C + Dk : k_s \in \mathbb{Z},\ -h_s \le k_s \le h_s\} \tag{3.8}$$

is positive. Indeed, write $W := \mathrm{diag}(w)$. For $K = C + Dk \in \mathcal{K}$, put $z_s(K) := k_s/(N\varepsilon w_s)$ and $E := C/N - M$. The northwest coordinates of D_s make k unique. Thus $z(K) \in [-1,1]^{11}$ and

$$K/N = M + \varepsilon DWz(K) + E, \qquad -1/N < E_{ij} \le 0. \tag{3.9}$$

By (3.5)–(3.6), every entry here is greater than $1/1200 - \varepsilon - 1/N > 1/2400$. The margins do not depend on k.

The table is recovered from its northwest coordinates by $K_{ij} = \sigma_{i+1,j+1} - \sigma_{i,j+1} - \sigma_{i+1,j} + \sigma_{ij}$. It follows that the northwest inequalities between A and B fix every unselected coordinate and bound the selected coordinates by $[-h_s, h_s]$ around those of C. They therefore give exactly (3.8). Proposition 4 proves

$$[u, v] = \{\pi : K(\pi) \in \mathcal{K}\}. \tag{3.10}$$

There are at most $(N+1)^{11}$ tables, since $2h_s \le N$. The signed corner tables

$$K_\pm = C \pm N\varepsilon Dw$$

are integer tables in this box.

3.1. **A quadratic inequality.** For $\beta > 0$, $x \ge 0$, and nonnegative tables X, define

$$f_\beta(x) := \int_0^x \log \frac{e^{\beta s} - 1}{\beta}\, ds, \qquad F_\beta(X) := \beta I(X) - \sum_{ij} f_\beta(X_{ij}). \tag{3.11}$$

The integrand minus $\log s$ tends to zero as $s \downarrow 0$, so the integral is finite and $f_\beta(0) = 0$. Derivatives below are taken on positive tables. We vectorize a table by assigning (i,j) the coordinate $7i + j$; e_a denotes the corresponding standard basis vector. Let T be the symmetric 49×49 matrix with $T_{(i,j),(h,k)} = 1$ when $(i-h)(j-k) < 0$ and zero otherwise. Then $I(X) = \frac{1}{2} X^T T X$. The matrix D has the vectorized D_s as its columns, so Dy agrees with the preceding table notation. For a differentiable function F, we write $D_s F(X) := \frac{d}{dt} F(X + tD_s)\Big|_{t=0}$.

For a selected square, denote its entries by x_a, x_b, x_c, x_d, in upper-left, upper-right, lower-left, lower-right order. The crossing indicators give $TD_s = -e_b - e_c$, whence $D_s I(X) = X^T T D_s = -(x_b + x_c)$. The certificate checks every coordinate of this identity for the eleven selected squares. With $h_\beta(x) = (e^{\beta x} - 1)/\beta$, we obtain

$$g_s(X;\beta) := D_s F_\beta(X) = -\beta(x_b + x_c) - \log h_\beta(x_a) - \log h_\beta(x_d) + \log h_\beta(x_b) + \log h_\beta(x_c). \tag{3.12}$$

Write $g := (g_s)_{s \in \mathcal{S}}$. Since $e^{M_a} = (1 + t_a)/t_a$ and $h_1(M_a) = 1/t_a$, substitution gives

$$g_s(M;1) = \log \frac{t_a t_d}{(1+t_b)(1+t_c)}.$$

Thus $g_s(M;1)=0$ is equivalent to

$$t_a t_d = (1+t_b)(1+t_c). \tag{3.13}$$

All eleven identities hold. One may construct t by starting with every entry equal to $1/32$ and, in the order (3.1), assigning $t_{i+1,j+1} = (1+t_{i,j+1})(1+t_{i+1,j})/t_{ij}$. The appendix checks (3.13) on the completed table.

Since $f_1''(M_a) = e^{M_a}/(e^{M_a}-1) = 1+t_a$, the Hessian of $y \mapsto F_1(M+Dy)$ at 0 is the rational matrix

$$H := D^T T D - D^T \operatorname{diag}(1+t_{ij}) D. \tag{3.14}$$

Put $m=11$ and $c_* = 10^{-4}$. Let $\mathbf{1} \in \mathbb{R}^m$ be the all-ones vector, and let I_m be the identity matrix. Set

$$Q = WHW, \quad \alpha = Q\mathbf{1}, \quad L = \operatorname{diag}(\alpha) - Q, \quad P = I_m - \frac{1}{m}\mathbf{1}\mathbf{1}^T, \quad G = L - c_* P. \tag{3.15}$$

For symmetric matrices, we denote $U \succeq V$ to mean that $x^T(U-V)x \geq 0$ for every real vector x.

Lemma 7. *Every $\alpha_s > c_*$. The matrix G is symmetric, $G\mathbf{1}=0$, and its leading 10×10 principal submatrix is positive definite. Consequently $G \succeq 0$ and $L \succeq c_* P$.*

Proof. The finite rational assertions are verified in Appendix A, by constructing (3.14) and an exact LDL^T factorization with positive diagonal entries. For the final implication, take $z \in \mathbb{R}^{11}$ and put $y = z - z_{11}\mathbf{1}$. Then $y_{11}=0$ and $z^T G z = y^T G y$, since $G\mathbf{1}=0$. The latter is the quadratic form of the stated positive definite principal block on $(y_1,\dots,y_{10})$ and is nonnegative. This proves both conclusions. □

For vectors, denote $\|x\|_1 := \sum_i |x_i|$, $\|x\|_2 := (\sum_i |x_i|^2)^{1/2}$, and $\|x\|_\infty := \max_i |x_i|$.

Proposition 8. *For every $z \in [-1,1]^{11}$, we have*

$$w^T H w - z^T Q z \geq c_* \min_{s\in\{-1,1\}} \|z - s\mathbf{1}\|_1. \tag{3.16}$$

Proof. Let $\bar{z} = m^{-1}\sum_i z_i$. By (3.15), we have

$$\begin{aligned} w^T H w - z^T Q z &= \sum_i \alpha_i (1-z_i^2) + z^T L z \\ &\geq c_*(m - \|z\|_2^2 + \|z - \bar{z}\mathbf{1}\|_2^2) \\ &= m c_* (1-\bar{z}^2) \geq m c_* (1-|\bar{z}|) = c_* \min_{s\in\{-1,1\}} \|z - s\mathbf{1}\|_1. \end{aligned}$$

Here $1-z_i^2 \geq 0$ and $|\bar{z}| \leq 1$. □

3.2. **Corner estimates.** Set

$$M_\pm := M \pm \varepsilon D w, \qquad J := [1-2\varepsilon^2, 1+2\varepsilon^2].$$

Every entry of $M + \varepsilon DWz + \theta E$, with $z \in [-1,1]^{11}$ and $0 \leq \theta \leq 1$, lies in $(1/2400, 9)$, by Lemma 6. These tables have total mass less than 400, since $\sum M_{ij} < 392$, each D_s has zero total sum, and $E_{ij} \leq 0$.

Put $\tau(x) = 1/(e^x - 1)$ for $x > 0$. We first record the calculus bounds used below. For $k_\beta(x) = x/(e^{\beta x}-1)$ and $x>0$,

$$-1 \leq \partial_x k_\beta(x) \leq 0, \qquad |\partial_\beta k_\beta(x)| \leq \beta^{-2}. \tag{3.17}$$

Indeed, with $z = \beta x$, the first derivative is $[e^z - 1 - ze^z]/(e^z-1)^2$. Its numerator is nonpositive because $e^z(1-z) \leq 1$; adding its denominator gives $e^z(e^z - 1 - z) \geq 0$. The β-derivative in (3.17) has absolute value $\beta^{-2}[z/(2\sinh(z/2))]^2 \leq \beta^{-2}$. These inequalities follow also at zero by continuity.

Differentiating (3.12) gives

$$\partial_\beta g_s = -x_a - x_d - k_\beta(x_a) - k_\beta(x_d) + k_\beta(x_b) + k_\beta(x_c). \tag{3.18}$$

In our domain, $x < 9$, $\beta > 1/2$, and $0 < k_\beta(x) < 1/\beta < 2$. Consequently $|\partial_\beta g_s| < 30$.

Lemma 9. *For $\beta \in J$ and $z \in [-1,1]^{11}$, choose $s \in \{-1,1\}$ minimizing $\|z - s\mathbf{1}\|_1$. Then*

$$F_\beta(M_s) - F_\beta(M + \varepsilon DWz) \geq 4 \cdot 10^{-5}\varepsilon^2\|z - s\mathbf{1}\|_1. \tag{3.19}$$

Thus every global maximum on the box is one of M_+, M_-.

Proof. Write

$$F_\beta(M + \varepsilon DWz) := F_\beta(M) + \frac{\varepsilon^2}{2} z^T Q z + R_\beta(z).$$

We bound the gradient of the remainder. At $\beta = 1$, the second derivative of $\log(e^x - 1)$ in absolute value is $\tau(x)(1 + \tau(x))$, where $\tau(x) < 1/x < 2400$. Every cell displacement is less than ε by (3.6). In (3.12), each of four logarithms therefore has linear Taylor error at most $\frac{1}{2}\,2400 \cdot 2401\varepsilon^2$. The linear inversion term has no such error. Also changing β from 1 to an element of J changes g_s by less than $60\varepsilon^2$. As $\partial_{z_s} F_\beta = \varepsilon w_s g_s$, stationarity and (3.14) yield

$$\begin{aligned}\|\nabla R_\beta\|_\infty &< \frac{2}{5}\left(\frac{1}{2}\,4 \cdot 2400 \cdot 2401 + 60\right)\varepsilon^3 \\ &= 4609944\varepsilon^3 < 5 \cdot 10^6\varepsilon^3.\end{aligned}$$

Integrating this gradient bound along the segment from z to $s\mathbf{1}$ and using Proposition 8 gives

$$F_\beta(M_s) - F_\beta(M + \varepsilon DWz) \geq \left(\frac{c_*}{2} - 5 \cdot 10^6\varepsilon\right)\varepsilon^2\|z - s\mathbf{1}\|_1.$$

The parenthesis is $9/200000 > 4 \cdot 10^{-5}$. The right side is positive unless $z = s\mathbf{1}$, proving the last assertion. □

Lemma 10. *Put $\Delta(\beta) := F_\beta(M_+) - F_\beta(M_-)$. Then*

$$|\Delta(1)| < 0.014\varepsilon^3, \tag{3.20}$$

$$\Delta'(\beta) > 0.014\varepsilon \quad (\beta \in J), \tag{3.21}$$

$$\Delta(1 - 2\varepsilon^2) < -0.014\varepsilon^3, \qquad \Delta(1 + 2\varepsilon^2) > 0.014\varepsilon^3. \tag{3.22}$$

Proof. First let $\psi(s) = F_1(M + sDw)$. We have $\psi'(0) = 0$, and

$$\psi'''(s) = \sum_{ij} \tau(M_{ij} + s(Dw)_{ij})(1 + \tau(M_{ij} + s(Dw)_{ij}))(Dw)_{ij}^3.$$

For $0 \leq \delta \leq \min(x/10, 1/10)$, concavity of $1 - e^{-x}$ and $e^\delta \leq (1-\delta)^{-1} \leq 10/9$ imply

$$\frac{\tau(x-\delta)(1+\tau(x-\delta))}{\tau(x)(1+\tau(x))} = e^\delta\left(\frac{1-e^{-x}}{1-e^{-(x-\delta)}}\right)^2 \leq \frac{10/9}{(9/10)^2} < 2.$$

For positive displacements the ratio is at most 1. Our $\delta \leq \varepsilon$ satisfies these conditions, since $M_{ij} > 1/1200$. Equation (3.7) thus gives $|\psi'''(s)| < 0.042$ for $|s| \leq \varepsilon$. The two second-order Taylor remainders at $\pm\varepsilon$ are each bounded by $0.042\varepsilon^3/6$. The constant and quadratic terms cancel on subtraction, proving (3.20).

The exact logarithm certificate in the appendix gives

$$0.008 < a_* := w^T \partial_\beta g(M; 1) < 0.009. \tag{3.23}$$

Explicitly the s-th mixed derivative is

$$-(1 + t_a)M_a - (1 + t_d)M_d + t_b M_b + t_c M_c,$$

so this is a finite rational linear combination of six logarithms of rational numbers. By (3.17) and (3.18), each of the four cell derivatives of $\partial_\beta g_s$ has absolute value at most 1, and its β derivative has absolute value at most $4/\beta^2 \le 16$. Therefore, for $|s| \le \varepsilon$ and $\beta \in J$,

$$|w^T \partial_\beta g(M + sDw; \beta) - a_*| < 16\varepsilon + 128\varepsilon^2 < 0.001.$$

Here we used the displacement bound and $\sum_i |w_i| < 4$. Integrating in s proves

$$\Delta'(\beta) = \int_{-\varepsilon}^{\varepsilon} w^T \partial_\beta g(M + sDw; \beta)\, ds > 2\varepsilon(0.008 - 0.001) = 0.014\varepsilon.$$

Integration over either half of J, followed by (3.20), gives (3.22). □

4. The length distribution

4.1. **Table probabilities.** For the interval (3.10) and $\beta \in J$, set $q := e^{\beta/N}$ and

$$\mathbb{P}_\beta(\pi) := \frac{q^{\ell(\pi)}}{\sum_{\tau \in [u,v]} q^{\ell(\tau)}}. \tag{4.1}$$

The induced table probability is proportional to (2.4).

Lemma 11. *For $\beta \in J$ and integers $0 \le m \le 400N$, we have*

$$|\log[m]_q! - m \log N - N f_\beta(m/N)| \le 1000 + \log(400N). \tag{4.2}$$

Proof. The case $m = 0$ is immediate. For $m \ge 1$, set $x := m/N$ and

$$b_\beta(s) := \log \frac{e^{\beta s} - 1}{\beta s}, \qquad b_\beta(0) := 0.$$

The inequalities $e^z \ge 1 + z$ and $e^{-z} \ge 1 - z$ give $0 \le b'_\beta(s) \le \beta$.Expanding each factor of $[m]_q!$, the error in (4.2) is exactly the sum of

$$\log(m!) - m \log m + m, \quad \sum_{j=1}^{m} b_\beta(j/N) - N \int_0^x b_\beta(s)\, ds, \quad -m b_\beta(1/N).$$

Integral comparison bounds the first term between 0 and $1 + \log m$. For an increasing function, comparison on each interval $[(j-1)/N, j/N]$ bounds the second between 0 and $b_\beta(x) - b_\beta(0) \le \beta x$. The third lies in $[-\beta x, 0]$. Since $\beta < 2$ and $x \le 400$, their sum lies in $[-800, 801 + \log m]$. This proves the claim. □

Put

$$\Gamma_N(\beta) := \sum_i f_\beta(r_i/N) + \sum_j f_\beta(c_j/N), \qquad X(z) = M + \varepsilon DWz.$$

There are $7 + 7 + 49 = 63$ factorials in (2.4).As $\log(400N) < 240$, Lemma 11 gives a total error less than 100000. This logarithm bound follows, for example, from $e > 27/10$ and $(27/10)^{240} > 400N$, both checked in the appendix. The $m \log N$ terms combine to $n \log N$.

On the floor-interpolation segments of (3.9),

$$\partial_{X_{ij}} F_\beta(X) = \beta (TX)_{ij} - \log h_\beta(X_{ij})$$

has absolute value less than 1000. Indeed $0 \le (TX)_{ij} < 400$, while $h_\beta(x) > x > 1/2400$ and $h_\beta(x) < 2e^{18} < e^{19}$ on the domain in question. Also $\log 2400 < 8$. Thus the absolute logarithm is less than 19. Summing the 49 cell perturbations shows

$$\left| \log W_K(e^{\beta/N}) - \{ n \log N + N\Gamma_N(\beta) + N F_\beta(X(z(K))) \} \right| < 150000. \tag{4.3}$$

For any collection of tables, comparison to one specified table has total logarithmic error, including the number of tables, less than

$$2 \cdot 150000 + 11 \log(N+1) < 10^6. \tag{4.4}$$

Let $\eta := 10^{-6}$ and define the disjoint sets

$$U_\pm = \{K \in \mathcal{K} : \|z(K) \mp \mathbf{1}\|_1 < \eta\}.$$

The signed corner tables belong to the corresponding sets. By (3.19), outside $U_+ \cup U_-$ the loss from the better corner is at least $4 \cdot 10^{-5}\varepsilon^2\eta = 4 \cdot 10^{-35}$. By (3.22), at the two ends $\beta_L = 1 - 2\varepsilon^2$ and $\beta_R = 1 + 2\varepsilon^2$, the less favored corner loses more than $1.4 \cdot 10^{-38}$. For a point in either U_s, its nearest corner is $s\mathbf{1}$, so (3.19) also bounds its value above by that of its corner.

Let $\delta := 10^{-100}$. Comparing weights in each of these sets to the favored corner table in (4.3) gives

$$\mathbb{P}_\beta(K \notin U_+ \cup U_-) < \delta \quad (\beta \in J), \tag{4.5}$$

$$\mathbb{P}_{\beta_L}(U_+) < \delta, \qquad \mathbb{P}_{\beta_R}(U_+) > 1 - \delta. \tag{4.6}$$

For the last bound, compare the entire complement of U_+ with the plus corner. Points in U_- lose by the corner preference; points outside both sets lose by the first bound. In every comparison the loss is greater than 10^{-40}, so (4.4) bounds the probability by

$$\exp(10^6 - N10^{-40}) < 10^{-100}.$$

Indeed, $N10^{-40} - 10^6 > 1000$, $e > 2$, and $2^{1000} > 10^{100}$.

Each table probability is continuous in β, since the interval is finite. Thus (4.6) and the intermediate value theorem give some $\widehat{\beta} \in J$ with

$$\mathbb{P}_{\widehat{\beta}}(U_+) = \frac{1}{2}, \qquad \mathbb{P}_{\widehat{\beta}}(U_-) > \frac{1}{2} - \delta. \tag{4.7}$$

4.2. **Conditional length moments.** Let $\mathcal{L} := \ell(\pi)$. Differentiation of the exact finite polynomial (2.4) gives

$$\partial_\beta \log W_K(e^{\beta/N}) = \mathbb{E}_\beta[\mathcal{L} \mid K]/N. \tag{4.8}$$

We estimate the mean by differentiating the factorials in (2.4).

Define

$$A_\beta(s) = \frac{s}{1 - e^{-\beta s}} - \frac{1}{\beta} = s + k_\beta(s) - \frac{1}{\beta}, \qquad A_\beta(0) = 0.$$

Equation (3.17) shows $0 \le A'_\beta(s) \le 1$ and $0 \le A_\beta(s) \le s$. For one factorial, with $x = m/N$, its length mean divided by N^2 is exactly

$$\frac{1}{N}\sum_{j=1}^{m} A_\beta(j/N) - xA_\beta(1/N). \tag{4.9}$$

To verify this, differentiate $\sum_{j=1}^m \log[(e^{\beta j/N} - 1)/(e^{\beta/N} - 1)]$ and divide the result by N. The constant $1/\beta$ terms cancel. The right Riemann sum in (4.9) differs from $\int_0^x A_\beta(s)\,ds = \partial_\beta f_\beta(x)$ by at most x/N. The subtracted term is at most x/N as well. Thus the error is at most $2x/N \le 800/N$. Using all 63 factorials yields

$$\left| \frac{\mathbb{E}_\beta[\mathcal{L} \mid K]}{N^2} - \{\Gamma'_N(\beta) + \partial_\beta F_\beta(K/N)\} \right| < 50400/N. \tag{4.10}$$

Set

$$\rho_\pm(\beta) := \Gamma'_N(\beta) + \partial_\beta F_\beta(M_\pm).$$

By (3.21),

$$\rho_+(\beta) - \rho_-(\beta) > 1.4 \cdot 10^{-14} \qquad (\beta \in J). \tag{4.11}$$

Within U_s, (3.18) and $\max|w_i| < 2/5$ bound the change in $\partial_\beta F_\beta$ from its corner by $30(2/5)\varepsilon\eta = 1.2 \cdot 10^{-17}$. Its cell derivative is

$$\partial_{X_{ij}} \partial_\beta F_\beta(X) = (TX)_{ij} - A_\beta(X_{ij}),$$

whose absolute value is less than 409. The floor perturbation therefore contributes less than $25000/N$. Since

$$1.2 \cdot 10^{-17} + \frac{25000 + 50400}{N} < 2 \cdot 10^{-17},$$

equation (4.10) gives

$$\left|\mathbb{E}_\beta[\mathcal{L} \mid K]/N^2 - \rho_s(\beta)\right| < 2 \cdot 10^{-17} \qquad (K \in U_s). \tag{4.12}$$

For the variance, write $\theta := \log q$. Each $\log(1 + q + \cdots + q^{j-1})$ has second θ derivative equal to the variance of a variable supported on $\{0, \ldots, j-1\}$. This variance is at most $(j-1)^2/4$: subtracting the midpoint of the range bounds the second moment by this quantity. In the second logarithmic derivative of (2.4), the denominator factorials have nonnegative variances and occur with a minus sign. Dropping them and using $\sum_{j=1}^{m}(j-1)^2 \le m^3$ proves

$$\mathrm{Var}_\beta(\mathcal{L} \mid K) \le \frac{1}{4}\left(\sum_i r_i^3 + \sum_j c_j^3\right) \le \frac{n^3}{2} < 3.2 \cdot 10^7 N^3. \tag{4.13}$$

Put $r_* := 10^{-16}$. The mean displacement in (4.12) is less than $r_*/2$. Chebyshev's inequality and (4.13) therefore give, for each $K \in U_s$,

$$\mathbb{P}_\beta\left(\left|\mathcal{L}/N^2 - \rho_s(\beta)\right| \ge r_* \mid K\right) < \frac{1.28 \cdot 10^{40}}{N} < 10^{-50}. \tag{4.14}$$

At the parameter $\widehat{\beta}$ from (4.7), define

$$J_s = \{j \in \mathbb{Z} : |j/N^2 - \rho_s(\widehat{\beta})| < r_*\}.$$

Averaging (4.14) over the tables in each U_s gives

$$\mathbb{P}_{\widehat{\beta}}(\mathcal{L} \in J_s) \ge \mathbb{P}_{\widehat{\beta}}(U_s)(1 - 10^{-50}) > (1/2 - \delta)(1 - 10^{-50}) > 0.49.$$

Thus

$$\mathbb{P}_{\widehat{\beta}}(\mathcal{L} \in J_s) > 0.49 \qquad (s = \pm 1). \tag{4.15}$$

The two windows are disjoint by (4.11).

Proof of Theorem 2. The endpoints u and v define a full Bruhat interval by (3.10). Suppose its rank numbers are log-concave. Let $q := e^{\widehat{\beta}/N}$ and $p_j := \mathbb{P}_{\widehat{\beta}}(\mathcal{L} = j)$. Since the rank numbers of a Bruhat interval have no internal zeros, these probabilities are positive for $\ell(u) \le j \le \ell(v)$. Multiplication by q^j and normalization give, at each interior rank,

$$\frac{p_j^2}{p_{j-1}p_{j+1}} = \frac{a_{j-\ell(u)}^2}{a_{j-\ell(u)-1}a_{j-\ell(u)+1}} \ge 1.$$

For a positive sequence, log-concavity makes the ratios p_{j+1}/p_j nonincreasing. Thus, for $i < j < k$, the average of $\log(p_{h+1}/p_h)$ over $i \le h < j$ is at least its average over $j \le h < k$. Rearranging gives

$$\log p_j \ge \frac{k-j}{k-i}\log p_i + \frac{j-i}{k-i}\log p_k \ge \min\{\log p_i, \log p_k\}.$$

Consequently,

$$i < j < k \quad \Longrightarrow \quad p_j \ge \min(p_i, p_k) \tag{4.16}$$

whenever the three indices are between the endpoints of the support.

Each window has at most $2r_*N^2 + 2 < 3r_*N^2$ integer points. By (4.15), it contains a point i_s with

$$p_{i_s} \geq \frac{0.49}{3r_*N^2}.$$

Between the two windows there are more than

$$(1.4 \cdot 10^{-14} - 2r_*)N^2 - 2 > 10^{-14}N^2$$

integer points. They all lie between i_- and i_+, and hence in the support. Equation (4.16) gives the same lower bound for the probability at every one of these points. Their total probability is consequently greater than

$$\frac{0.49 \cdot 10^{-14}N^2}{3 \cdot 10^{-16}N^2} = \frac{49}{3} > 1.$$

This contradicts total probability one. This contradiction shows that the rank sequence is not log-concave. □

Appendix A. The complete arithmetic certificate

The following Sage program verifies the arithmetic in Lemmas 6, 7, and 10, together with the numerical margins in the proof. It uses the exact rings $\mathbb{Z}$ and $\mathbb{Q}$, implemented as `ZZ` and `QQ`. To run it online, open `https://sagecell.sagemath.org/`, select `Sage`, and evaluate the complete listing as one cell. Assertions must be enabled. The listing was executed successfully with SageMath 10.10 in both the `Sage` and `Python` modes of SageMathCell. It also runs as `sage note_certificate.sage` or `sage -python note_certificate.sage`.

We first explain its two algorithms. If $1 \leq x \leq 2$ and $z = (x-1)/(x+1)$, then

$$2\sum_{j=0}^{L-1} \frac{z^{2j+1}}{2j+1} \leq \log x \leq 2\sum_{j=0}^{L-1} \frac{z^{2j+1}}{2j+1} + \frac{2z^{2L+1}}{(2L+1)(1-z^2)}. \tag{A.1}$$

This follows by integrating the geometric series for $2/(1-z^2)$; the omitted denominators are all at least $2L+1$. For $x \geq 1$, divide it by powers of 2 until it lies in $[1,2)$ and add the same rigorous interval for the required multiple of $\log 2$. We use $L = 200$. The lower and upper rational bounds for $N\log(1+1/t_{ij})$ lie strictly inside the same unit interval; these rational bounds determine each floor. Linear combinations of the logarithms use the lower bound for a positive coefficient and the upper bound for a negative coefficient, and conversely for their upper bound.

For a symmetric matrix V, the factorization routine constructs a unit lower triangular matrix U and diagonal entries p_j by

$$p_j = V_{jj} - \sum_{k<j} U_{jk}^2 p_k, \qquad U_{ij} = \frac{V_{ij} - \sum_{k<j} U_{ik}p_k U_{jk}}{p_j} \quad (i > j).$$

It checks $p_j > 0$ at every step and then verifies every entry of $V = U\,\mathrm{diag}(p)U^T$. Positivity of the pivots proves positive definiteness, since U is invertible. Together with $G\mathbf{1} = 0$, Lemma 7 gives the required semidefinite inequality in 11 dimensions.

For a short record of the exact output, the following integers are the floors of $10^9\alpha_i$ and $10^6 p_i$, respectively. Each entry certifies the enclosing interval between that integer divided by the indicated scale and the next integer divided by the scale.

i	$\lfloor 10^9 \alpha_i \rfloor$	$\lfloor 10^6 p_i \rfloor$
1	617350	1050442
2	2066148	3407161
3	732633	68058
4	2300462	220455
5	3259872	5142902
6	1769927	2883418
7	3207152	98341934
8	3261046	177706
9	625838	132082
10	2398200	3482076
11	2192866	—

In particular every $\alpha_i > 10^{-4}$, and all ten pivots exceed $1/100$. The code reconstructs these values from the displayed data. For the mixed derivative, its coefficient vector is $TDw - \mathrm{diag}(1+t)Dw$, so

$$a_* = \sum_a \left((TDw)_a - (1+t_a)(Dw)_a\right) M_a.$$

Indeed $\partial_\beta F_\beta(X)|_{\beta=1} = I(X) - \sum_a \int_0^{X_a} A_1(s)\, ds$; differentiating along Dw gives the stated formula because $A_1(M_a) = (1+t_a)M_a - 1$ and $\sum_a (Dw)_a = 0$. The code checks this coefficient vector both by polarizing I and by the local four-cell expression following (3.23).

```
# Complete exact SageMath certificate for the Bruhat interval construction.
# Paste this entire file into https://sagecell.sagemath.org/ and select Sage.
# The same source also runs with: sage -python note_certificate.sage
# Assertions must be enabled. No files or additional packages are required.

from sage.all import QQ, ZZ, matrix, vector, diagonal_matrix, identity_matrix
from sage.version import version as sage_version
from builtins import sum as builtin_sum

if not __debug__:
    raise RuntimeError("Assertions must be enabled; do not use Python -O.")

def rat(a, b=1):
    return QQ(a) / ZZ(b)

def qsum(values):
    return builtin_sum(values, QQ(0))

def zsum(values):
    return builtin_sum(values, ZZ(0))

def exact_floor(x):
    return ZZ(QQ(x).floor())

# All rational divisions and empty sums are exact in either input mode.
assert rat(1, 3).parent() == QQ
assert qsum([]).parent() == QQ and qsum([]) == 0
assert zsum([]).parent() == ZZ and zsum([]) == 0

size, dim = 7, 11
ncells = size * size
```

```
cells = range(ncells)
N = ZZ(10)**100
eps = rat(1, ZZ(10)**12)
c = rat(1, 10000)
eta = rat(1, ZZ(10)**6)
radius = rat(1, ZZ(10)**16)
sites = [(0, 1), (1, 0), (1, 2), (2, 1), (2, 2),
         (3, 0), (3, 1), (3, 3), (4, 2), (4, 3), (5, 4)]
b = vector(ZZ, [-17048, -30702, 17618, 31695, 38253, 28244,
                29146, -38002, -29147, -32509, 31476])
w = vector(QQ, [rat(x, 100000) for x in b])
assert len(sites) == dim == len(b) == len(w)
assert len(set(sites)) == dim
assert all(0 <= i < size-1 and 0 <= j < size-1 for i, j in sites)
assert all(x != 0 for x in w)

# Construct the stationary table, then compare every entry with the paper.
p = rat(1, 32)
t_matrix = matrix(QQ, size, size, [p] * ncells)
for i, j in sites:
    t_matrix[i+1, j+1] = ((1+t_matrix[i, j+1])
                          * (1+t_matrix[i+1, j]) / t_matrix[i, j])
t1, t2, t3 = rat(1089, 32), rat(36993, 32), rat(33, 35872)
t4, t5 = rat(1184865, 35872), rat(36993, 1148960)
t_displayed = matrix(QQ, [
    [p, p, p, p, p, p, p],
    [p, p, t1, p, p, p, p],
    [p, t1, p, p, p, p, p],
    [p, p, p, t1, p, p, p],
    [p, t1, t2, p, p, p, p],
    [p, p, p, t3, t4, p, p],
    [p, p, p, p, p, t5, p]])
assert t_matrix == t_displayed
for i, j in sites:
    assert (t_matrix[i, j] * t_matrix[i+1, j+1]
            == (1+t_matrix[i, j+1]) * (1+t_matrix[i+1, j]))
t = vector(QQ, t_matrix.list())
assert all(t[size*i+j] == t_matrix[i, j]
           for i in range(size) for j in range(size))
assert min(t) > 0
assert set(t) == {p, t1, t2, t3, t4, t5}
assert len(set(t)) == 6

# D has the vectorized elementary square directions as its columns.
D = matrix(ZZ, ncells, dim, 0)
for s, (i, j) in enumerate(sites):
    for h, k, sign in [(i, j, 1), (i+1, j+1, 1),
                       (i, j+1, -1), (i+1, j, -1)]:
        D[size*h+k, s] = sign
Dcols = [D.column(s) for s in range(dim)]

def margins(x):
    assert len(x) == ncells
    rows = tuple(qsum(x[size*i+j] for j in range(size))
```

```
                for i in range(size))
    cols = tuple(qsum(x[size*i+j] for i in range(size))
                for j in range(size))
    return rows, cols

zero_margins = ((QQ(0),) * size, (QQ(0),) * size)
assert all(margins(d) == zero_margins for d in Dcols)
for i in range(size):
    for j in range(size):
        for s, site in enumerate(sites):
            nw = zsum(D[size*h+k, s]
                      for h in range(i+1) for k in range(j+1))
            assert nw == ZZ(site == (i, j))
assert D.rank() == dim

# Direct crossing quadratic and its symmetric matrix representation.
def inv(x):
    assert len(x) == ncells
    return qsum(x[size*i+j] * x[size*h+k]
                for i in range(size) for h in range(i+1, size)
                for j in range(size) for k in range(j))

T = matrix(ZZ, ncells, ncells,
           lambda a, b: ZZ((a//size-b//size) * (a%size-b%size) < 0))
assert T.is_symmetric()
assert all(T[a, a] == 0 for a in cells)
for a in cells:
    for d in cells:
        i, j = divmod(a, size)
        h, k = divmod(d, size)
        assert T[a, d] == ZZ((i < h and j > k) or (h < i and k > j))
for s, (i, j) in enumerate(sites):
    expected = vector(ZZ, [0] * ncells)
    expected[size*i+j+1] = -1
    expected[size*(i+1)+j] = -1
    assert T * Dcols[s] == expected

K = T - diagonal_matrix(QQ, [1+x for x in t])
H = D.transpose() * K * D
H_polarized = matrix(QQ, dim, dim,
    lambda i, j: inv(Dcols[i]+Dcols[j])-inv(Dcols[i])-inv(Dcols[j])
        -qsum((1+t[a])*D[a, i]*D[a, j] for a in cells))
assert H == H_polarized and H.is_symmetric()
W = diagonal_matrix(QQ, w)
Q = W * H * W
one = vector(QQ, [1] * dim)
alpha = Q * one
assert min(alpha) > c
P = matrix(QQ, dim, dim, lambda i, j: QQ(i == j)-rat(1, dim))
G = diagonal_matrix(QQ, alpha) - Q - c*P
assert H.base_ring() == Q.base_ring() == G.base_ring() == QQ
assert G.is_symmetric()
assert G * one == vector(QQ, [0] * dim)
assert all(qsum(G.row(i)) == 0 for i in range(dim))
```

```
def positive_ldl(A):
    A = matrix(QQ, A)
    nrows = A.nrows()
    assert A.ncols() == nrows and A.is_symmetric()
    U = identity_matrix(QQ, nrows)
    pivots = []
    for j in range(nrows):
        pivot = A[j, j]-qsum(U[j, k]**2*pivots[k] for k in range(j))
        assert pivot > 0
        pivots.append(pivot)
        for i in range(j+1, nrows):
            U[i, j] = (A[i, j]-qsum(U[i, k]*U[j, k]*pivots[k]
                                    for k in range(j))) / pivot
    assert all(U[i, i] == 1 for i in range(nrows))
    assert all(U[i, j] == 0
               for i in range(nrows) for j in range(i+1, nrows))
    assert A == U * diagonal_matrix(QQ, pivots) * U.transpose()
    return vector(QQ, pivots)

pivots = positive_ldl(G.submatrix(0, 0, dim-1, dim-1))
assert min(pivots) > rat(1, 100)
dw = D * w
assert 2*inv(dw) == dw.dot_product(T*dw)
assert qsum(dw) == 0
cell_bound = max(qsum(abs(w[j]*D[a, j]) for j in range(dim))
                 for a in cells)
third = qsum(t[a]*(1+t[a])*abs(dw[a])**3 for a in cells)
assert cell_bound == rat(49829, 50000) < 1
assert third == rat(92117978523348864650859972 93,
                    44859469886720000000000000000)
assert third < rat(21, 1000)
assert max(abs(x) for x in w) < rat(2, 5)
assert qsum(abs(x) for x in w) < 4

# Rational logarithm enclosures; no floating-point logarithms are used.
def unit_log(x):
    x = QQ(x)
    assert 1 <= x <= 2
    z, terms = (x-1)/(x+1), 200
    low = 2*qsum(z**(2*j+1)/ZZ(2*j+1) for j in range(terms))
    tail = 2*z**(2*terms+1)/(ZZ(2*terms+1)*(1-z*z))
    return low, low+tail

log2 = unit_log(QQ(2))

def log_bounds(x):
    x = QQ(x)
    assert x >= 1
    k = 0
    while x >= 2:
        x = x/ZZ(2)
        k += 1
    low, high = unit_log(x)
```

```
    return low+k*log2[0], high+k*log2[1]

logs = {x: log_bounds(1+QQ(1)/x) for x in set(t)}
low = vector(QQ, [logs[x][0] for x in t])
high = vector(QQ, [logs[x][1] for x in t])
assert min(low) > rat(1, 1200)
assert max(high) < 8
assert rat(1, 1200)-eps-rat(1, N) > rat(1, 2400)

# Mixed derivative: compare crossing polarization with the local formula.
coeffs = K * dw
local_coeffs = vector(QQ, [0] * ncells)
for s, (i, j) in enumerate(sites):
    a, d = size*i+j, size*(i+1)+j+1
    bcell, ccell = size*i+j+1, size*(i+1)+j
    local_coeffs[a] -= w[s]*(1+t[a])
    local_coeffs[d] -= w[s]*(1+t[d])
    local_coeffs[bcell] += w[s]*t[bcell]
    local_coeffs[ccell] += w[s]*t[ccell]
assert coeffs == local_coeffs
a_low, a_high = QQ(0), QQ(0)
for h in cells:
    e = vector(QQ, [ZZ(k == h) for k in cells])
    coeff = inv(e+dw)-inv(e)-inv(dw)-(1+t[h])*dw[h]
    assert coeff == coeffs[h]
    lo, hi = (low[h], high[h]) if coeff >= 0 else (high[h], low[h])
    a_low += coeff*lo
    a_high += coeff*hi
assert rat(8, 1000) < a_low < a_high < rat(9, 1000)
assert a_high-a_low < rat(1, ZZ(10)**100)

# Determine all integer data and verify the entire box is positive.
C = vector(ZZ, [exact_floor(N*x) for x in low])
assert C == vector(ZZ, [exact_floor(N*x) for x in high])
assert all(C[a] < N*low[a] < N*high[a] < C[a]+1 for a in cells)
widths = vector(ZZ, [abs(x)*ZZ(10)**83 for x in b])
assert all(QQ(h) == N*eps*abs(x) for h, x in zip(widths, w))
A, B = C+D*widths, C-D*widths
assert A.base_ring() == B.base_ring() == C.base_ring() == ZZ
assert all(QQ(x).denominator() == 1 and x > 0 for x in list(A)+list(B))
assert margins(A) == margins(C) == margins(B)
box_radius = vector(ZZ, [zsum(abs(D[a, j])*widths[j]
                              for j in range(dim)) for a in cells])
assert all(C[a]-box_radius[a] > 0 for a in cells)
signed_widths = vector(ZZ, [N*eps*x for x in w])
assert vector(ZZ, [abs(x) for x in signed_widths]) == widths
corner_plus, corner_minus = C+D*signed_widths, C-D*signed_widths
assert margins(corner_plus) == margins(C) == margins(corner_minus)
assert all(x > 0 for x in list(corner_plus)+list(corner_minus))
n = zsum(C)
assert n == ZZ('153963043329233877797390678129903324642647289999 3'
               '279945157628816060796748419102920384729488295947520156')
assert n < 400*N
assert all(2*h <= N for h in widths)
```

```
# All numerical margins in the finite analytic argument.
assert qsum(rat(1, x) for x in [1, 1, 2, 6, 24]) > rat(27, 10)
assert rat(27, 10)**240 > 400*N
assert rat(27, 10)**8 > 2400
assert 63*(1000+240) < 100000
assert 49*1000 < 50000 and 49*409 < 25000
assert 2*150000+11*240 < ZZ(10)**6
rem = rat(2, 5)*(rat(1, 2)*4*2400*2401+60)
assert rem == 4609944 < 5*ZZ(10)**6
assert c/ZZ(2)-5*ZZ(10)**6*eps > rat(4, ZZ(10)**5)
assert 16*eps+128*eps**2 < rat(1, 1000)
assert rat(10, 9)/(rat(9, 10)**2) < 2
assert eps < rat(1, 12000) and eps < rat(1, 10)
assert rat(4, ZZ(10)**5)*eps**2*eta > rat(1, ZZ(10)**40)
assert rat(14, 1000)*eps**3 > rat(1, ZZ(10)**40)
assert N*rat(1, ZZ(10)**40)-ZZ(10)**6 > 1000
assert ZZ(2)**1000 > ZZ(10)**100
assert 12*eps*eta+rat(100000, N) < rat(2, ZZ(10)**17) < radius/2
failure = rat(4*400**3, 2*N)/radius**2
assert failure < rat(1, ZZ(10)**50)
assert rat(1, 2)-rat(1, ZZ(10)**100)-failure > rat(49, 100)
assert 2*radius*N**2+2 < 3*radius*N**2
assert (rat(14, 1000)*eps-2*radius)*N**2-2 > rat(N**2, ZZ(10)**14)
assert rat(49, 100)*rat(1, ZZ(10)**14)/(3*radius) == rat(49, 3) > 1

alpha_floors = [exact_floor(ZZ(10)**9*x) for x in alpha]
pivot_floors = [exact_floor(ZZ(10)**6*x) for x in pivots]
assert alpha_floors == [617350, 2066148, 732633, 2300462, 3259872,
                        1769927, 3207152, 3261046, 625838, 2398200, 2192866]
assert pivot_floors == [1050442, 3407161, 68058, 220455, 5142902,
                        2883418, 98341934, 177706, 132082, 3482076]

print('ALL EXACT CHECKS PASSED')
print('SageMath version:', sage_version)
print('floor(10^9 alpha):', alpha_floors)
print('floor(10^6 LDL pivots):', pivot_floors)
print('cell bound =', cell_bound)
print('third derivative sum =', third)
print('8/1000 < mixed derivative < 9/1000')
print('six distinct logarithms; all 49 floors certified')
for label, value in zip(['p', 't1', 't2', 't3', 't4', 't5'],
                        [p, t1, t2, t3, t4, t5]):
    print('floor for '+label+' =', exact_floor(N*logs[value][0]))
print('n =', n)
print('contradictory probability lower bound = 49/3')
```

The successful output begins with `ALL EXACT CHECKS PASSED`. It reports the Sage version, the two lists of integers above, the exact cell and cubic bounds in Lemma 6, the interval $0.008 < a_* < 0.009$, certification of all 49 floors using six logarithms, all six distinct integer floor values, the integer n, and the final lower bound $49/3$.

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