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CHARACTERIZATIONS OF E-FINITE ALGEBRAS AND CONJECTURES ON BRICK-FINITENESS

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Abstract. A finite-dimensional algebra over a field is called E-finite if every difference of classes of finitely generated projective modules lies in a two-term silting cone. Over an arbitrary field, we prove that an algebra is E-finite if and only if any two nonzero finite-dimensional modules with the same dimension vector admit a nonzero homomorphism in each direction. Using this criterion, we show that E-finiteness is also equivalent to finiteness of every semibrick, functorial finiteness of every wide subcategory of finite-dimensional modules, finiteness of the number of isomorphism classes of finite-dimensional bricks of each endolength, and nonexistence of generic bricks, along with further conditions on torsion classes, universal localizations, and moduli spaces. Consequently, several open problems become equivalent for each algebra, each asking whether every E-finite algebra is brick-finite; among them are Demonet's lattice point conjecture, the finite semibrick conjecture, the endolength form of the second brick-Brauer–Thrall conjecture and Pfeifer's stable variant, Mousavand and Paquette's generic brick conjecture, and the wide subcategory question of Angeleri Hügel and Sentieri. Finally, we prove that every E-finite algebra with at most two nonisomorphic simple modules is brick-finite over any field.

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1. Introduction

In this paper, we show that for any finite-dimensional algebra Λ over an arbitrary field, E-finiteness is equivalent to each of the following properties:

- any two nonzero finite-dimensional modules with the same dimension vector admit a nonzero homomorphism in each direction;
- every semibrick of finite-dimensional modules is finite;
- every wide subcategory of finite-dimensional modules is functorially finite;
- only finitely many finite-dimensional bricks of each endolength exist up to isomorphism;
- there is no generic brick.

Theorem A establishes further characterizations in terms of torsion classes and their Grothendieck monoids, universal localizations and ring epimorphisms, brick counts over infinite fields, and representation varieties and King moduli spaces over algebraically closed fields. As an immediate consequence, several open problems asking for sufficient conditions for brick-finiteness become equivalent for each individual algebra: among them are Demonet's lattice point conjecture, the finite semibrick conjecture, the second brick-Brauer–Thrall conjecture, Pfeifer's stable variant, Mousavand and Paquette's generic brick conjecture, and the wide subcategory question of Angeleri Hügel and Sentieri. Each of them asks whether every E-finite algebra is brick-finite, and in Theorem B we prove that this holds over any field when Λ has at most two nonisomorphic simple modules.

1.1. Background and the conjectures.

Let Λ be a finite-dimensional algebra over a field k, and let n be its rank, the number of isomorphism classes of simple right Λ-modules. Write $\mathsf{brick}\,\Lambda$ for the set of isomorphism classes of finite-dimensional bricks. Over any field, finiteness of $\mathsf{brick}\,\Lambda$ (*brick-finiteness*) is equivalent both to τ-tilting finiteness by Demonet, Iyama, and Jasso [26, Theorem 1.4] and to completeness of the g-vector fan by Asai [9, Theorem 4.7].

Write $\Sigma(\Lambda) \subseteq K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}}$ for the support of this fan. It is natural to ask whether completeness of the fan is already detected by its integral lattice points. We say that Λ is *E-finite* if $K_0(\mathsf{proj}\,\Lambda) \subseteq \Sigma(\Lambda)$, and *E-infinite* otherwise; for instance, the Kronecker algebra is E-infinite because the ray through $(1,-1)$ meets $\Sigma(\Lambda)$ only at the origin (Figure 2). Since the cones of the fan are rational, $K_0(\mathsf{proj}\,\Lambda) \subseteq \Sigma(\Lambda)$ holds precisely when $\Sigma(\Lambda)$ contains every rational vector, so that the fan is *rationally complete* in the terminology of Pfeifer [56, Conjecture 3.3]. By Lemma 2.1(1), each integral vector in $\Sigma(\Lambda)$ is represented by a two-term complex P with $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, P[1]) = 0$; over an algebraically closed field, our definition of E-finiteness therefore agrees with that of Asai and Iyama [12, Definition 6.3], reflecting the vanishing of the self E-invariant of Derksen and Fei [29, Section 3]. Demonet [25, Question 3.49] asked whether every algebra with a rationally complete g-vector fan is brick-finite; see [56, Conjecture 3.3] and [51, Conjecture 1.2] for formulations over algebraically closed fields, and Asai [11, Conjecture 1.7] for the following formulation over an arbitrary field.

Conjecture 1.1 (Demonet's lattice point conjecture [25, Question 3.49], [11, Conjecture 1.7])**.** *Let Λ be a finite-dimensional algebra over an arbitrary field. If Λ is E-finite, then it is brick-finite.*

Parallel to Demonet's polyhedral question, several module-theoretic and categorical problems ask whether brick-finiteness is forced by finiteness conditions on semibricks, wide subcategories, generic bricks, or bricks of bounded size, in analogy with the classical second Brauer–Thrall conjecture. Over an arbitrary field k, the appropriate size invariant of a brick B is its *endolength*, its length as a module over its endomorphism division ring $\mathrm{End}_\Lambda(B)$. Accordingly, we call Λ *brick-discrete* if for every integer $r \geq 1$ only finitely many members of $\mathsf{brick}\,\Lambda$ have endolength r, and *brick-continuous* otherwise. Over an algebraically closed field, every finite-dimensional

brick has endomorphism ring k, so endolength coincides with k-dimension as in [50, Section 2]; in this setting, brick-discreteness also agrees with the condition of Mousavand and Paquette [53, Definition 6.8] that every brick has an open orbit (Remark 6.2). Over a finite field, by contrast, counting bricks by k-dimension and by endolength yield inequivalent conditions (Example A.1).

We consider the following seven conjectures and questions over k:

(i) *Demonet's lattice point conjecture* (Conjecture 1.1). If Λ is E-finite (E-fin), then Λ is brick-finite.

(ii) *The finite semibrick conjecture* [30, Conjecture 5.12]. If every semibrick in $\mathsf{mod}\,\Lambda$ is finite (fin-sbrick), then Λ is brick-finite.

(iii) *The n-semibrick question of Mousavand and Paquette* [52, discussion following Theorem A]. If the simple modules form the unique n-element semibrick in $\mathsf{mod}\,\Lambda$ (n-sbrick), then Λ is brick-finite.

(iv) *The second brick-Brauer–Thrall conjecture* [48, Conjecture 6.6(2)], [52, Conjecture 2.1]. If Λ is brick-discrete (brick-discrete), then Λ is brick-finite.

(v) *Mousavand and Paquette's generic brick conjecture* [50, Conjecture 1.1]. If $\mathsf{Mod}\,\Lambda$ contains no generic brick (no-gen-brick), then Λ is brick-finite.

(vi) *Pfeifer's stable Brauer–Thrall* II′ *conjecture* [56, Conjecture 5.2]. If for each integral weight θ and each integer $r \geq 1$ only finitely many θ-stable members of $\mathsf{brick}\,\Lambda$ have endolength r (endo-stab), then Λ is brick-finite.

(vii) *The wide subcategory question of Angeleri Hügel and Sentieri* [6, Questions 5.11(1)]. If every wide subcategory of $\mathsf{mod}\,\Lambda$ is functorially finite (wide), then Λ is brick-finite.

Over an algebraically closed field, where the endolength of a finite-dimensional brick equals its k-dimension, (iv) recovers Kinser's question for Schur representations [45, Problem 4.3] (see also [63, Conjecture 2]). Over the same field, since the closed points of the stable locus in King's moduli space classify stable modules up to isomorphism [44, Proposition 4.2], the contrapositive of (vi) predicts that every brick-infinite algebra admits a positive-dimensional stable moduli space for some integral weight and dimension vector [56, Conjecture 5.2], [52, Conjecture 2.5].

1.2. **Main results.** Our first main result shows that E-finiteness is equivalent over an arbitrary field to all seven hypotheses in (i) to (vii) together with a family of homological, categorical, and ring-theoretic conditions, and over infinite or algebraically closed fields to further numerical and geometric conditions.

Theorem A. *Let Λ be a finite-dimensional k-algebra of rank n.*

(I) Over any field k, the following conditions are equivalent.

The projective lattice, dimension vectors, and torsion classes (Theorems 3.2 and 4.1).

(E-fin) *Λ is E-finite, that is, $K_0(\mathsf{proj}\,\Lambda) \subseteq \Sigma(\Lambda)$.*

(K_0-Hom) *For all $M, N \in \mathsf{mod}\,\Lambda$, if $[M] = [N] \neq 0$ in $K_0(\mathsf{mod}\,\Lambda)$, then $\mathrm{Hom}_\Lambda(M, N) \neq 0$.*

(mon-dim) *For every torsion class $\mathcal{T}$ in $\mathsf{mod}\,\Lambda$, the dimension-vector map $M(\mathcal{T}) \to K_0(\mathsf{mod}\,\Lambda)$ on its Grothendieck monoid is injective.*

(comp-tors) *Every compact torsion class is functorially finite.*

(num-disj) *Every torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathsf{mod}\,\Lambda$ is numerically disjoint.*

Wide subcategories and brick labels (Theorem 5.1).

(wide) *Every wide subcategory of $\mathsf{mod}\,\Lambda$ is functorially finite.*

(wide_1) *For every brick $B \in \mathsf{brick}\,\Lambda$, the subcategory $\mathsf{Filt}(B)$ is functorially finite.*

(brick-label) *For every $B \in \mathsf{brick}\,\Lambda$, there is a cover $\mathcal{T}' \subsetneq \mathcal{T}$ in the poset of functorially finite torsion classes with $\mathsf{Filt}(B) = \mathcal{T} \cap (\mathcal{T}')^{\perp}$.*

Semibricks and brick counts (Theorems 3.2 and 5.2, Proposition 5.3).

(n-sbrick) *The set $\{S_1, \ldots, S_n\}$ of simple modules is the unique n-element semibrick in $\mathsf{mod}\,\Lambda$.*

(fin-sbrick) *Every semibrick in $\mathsf{mod}\,\Lambda$ is finite.*

(brick-discrete) *Λ is brick-discrete, that is, for every $r \geq 1$, there are only finitely many members of $\mathsf{brick}\,\Lambda$ of endolength r.*

(endo-stab) *For every $\theta \in K_0(\mathsf{proj}\,\Lambda)$ and every $r \geq 1$, there are only finitely many θ-stable members of $\mathsf{brick}\,\Lambda$ of endolength r.*
(poly-brick) *The number of isomorphism classes of bricks of composition length at most m is bounded by a polynomial in m.*

Large modules, two-term complexes, and ring epimorphisms (Theorem 7.8).

(no-gen-brick) *There is no generic brick in $\mathsf{Mod}\,\Lambda$.*
(no-gen-grain) *There is no generic grain in $\mathsf{Mod}\,\Lambda$.*
(thick-gen) *Every thick subcategory of $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ is of the form $\mathsf{thick}(U) \cap \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ for some two-term presilting complex U, where $\mathsf{thick}(U)$ denotes the thick subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ generated by U.*
(univ-loc) *Every universal localization of Λ is finite-dimensional over k.*
(ring-epi) *Every ring epimorphism $\Lambda \to \Gamma$ with simple Artinian target satisfies $\dim_k \Gamma < \infty$.*

(II) If k is infinite, each of the following conditions is also equivalent to (E-fin) (Corollary 5.4).

(dim-brick) *For every $d \in \mathbb{N}^n$, there are only finitely many isomorphism classes of bricks of dimension vector d.*
(dim-stab) *For every $\theta \in K_0(\mathsf{proj}\,\Lambda)$ and every $d \in \mathbb{N}^n$, there are only finitely many isomorphism classes of θ-stable bricks of dimension vector d.*

(III) If k is algebraically closed, each of the following geometric conditions is also equivalent to (E-fin) (Theorem 6.1).

(moduli-pt) *For every $\theta \in K_0(\mathsf{proj}\,\Lambda)$ and every $d \in \mathbb{N}^n$, the King moduli space $\mathcal{M}^{\mathrm{ss}}_\theta(\Lambda, d)$ of θ-semistable representations has at most one closed point.*
(open-brick) *Every brick $B \in \mathsf{brick}\,\Lambda$ has a Zariski-open $\mathrm{GL}(d)$-orbit in $\mathsf{rep}(\Lambda, d)$, where $d = [B]$.*
(open-wide) *For every $d \in \mathbb{N}^n$ and every wide subcategory $\mathcal{W} \subseteq \mathsf{mod}\,\Lambda$, the locus $\{X \in \mathsf{rep}(\Lambda, d) \mid X \in \mathcal{W}\}$ is Zariski-open in $\mathsf{rep}(\Lambda, d)$.*

Theorem A(I) implies that for each finite-dimensional k-algebra Λ, the seven implications (i) to (vii) are equivalent; consequently, their universal forms over k are equivalent as well.

Our second main result proves this common conjecture in rank at most two over an arbitrary field.

Theorem B (= Theorem 8.4)**.** *Let Λ be a finite-dimensional algebra of rank $n \leq 2$ over an arbitrary field k. Then Λ is E-finite if and only if Λ is brick-finite (equivalently, τ-tilting finite).*

In particular, all seven implications (i) to (vii) hold for every finite-dimensional algebra of rank at most two over any field.

Over an algebraically closed field, Theorem A also resolves a conjecture of Weyman [21, Conjecture 20] relating Schur-representation-finiteness (dim-brick) to multiplicity-freeness.

Corollary C (Weyman's conjecture)**.** *Let Λ be a finite-dimensional algebra over an algebraically closed field. Then Λ is Schur-representation-finite* (dim-brick) *if and only if it is multiplicity-free.*

1.3. **Strategy of proof.** Let us outline the proofs of Theorems A and B. The proof of Theorem A centers on the equivalence between the polyhedral condition (E-fin) and the module-theoretic condition (K_0-Hom). At any weight lying in $\Sigma(\Lambda)$, the semistable subcategory is a τ-perpendicular category, so the dimension vectors of its stable bricks are linearly independent. Conversely, starting from any pair of nonzero modules $M, N \in \mathsf{mod}\,\Lambda$ with $[M] = [N]$ and $\mathrm{Hom}_\Lambda(M, N) = 0$, we extract a finite semibrick with linearly dependent dimension vectors whose members are simultaneously stable at a single integral weight; since no such weight can lie in $\Sigma(\Lambda)$, we obtain (E-fin) $\Rightarrow$ (K_0-Hom). In the opposite direction, an integral weight outside $\Sigma(\Lambda)$ produces families of bricks stable at a common weight, including two nonisomorphic bricks of the same dimension vector over any field, which gives (K_0-Hom) $\Rightarrow$ (E-fin).

Once (E-fin) and (K_0-Hom) are linked, the remaining conditions in Theorem A connect to this central equivalence along the routes displayed in Figure 1. Under (K_0-Hom) and (E-fin),

every finite semibrick stabilizes at a single integral weight, which embeds each wide subcategory as a Serre subcategory of a τ-perpendicular category and bounds bricks via stability chambers and normalized submodule polytopes, while pushouts along nonzero morphisms identify modules of equal dimension vector in the Grothendieck monoid of any torsion class. On the infinite-dimensional and homological side, E-finiteness rules out generic bricks by rank reduction to a τ-perpendicular category and provides presilting generators for thick subcategories of $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$, forcing universal localizations to be finite-dimensional. Conversely, when $\Sigma(\Lambda)$ omits an integral weight, the resulting families of stable bricks (descended to finite fields via Frobenius orbits), Zariski-open separation over $\overline{k}$, and a generic grain constructed over Cohn's free skew field refute the remaining conditions.

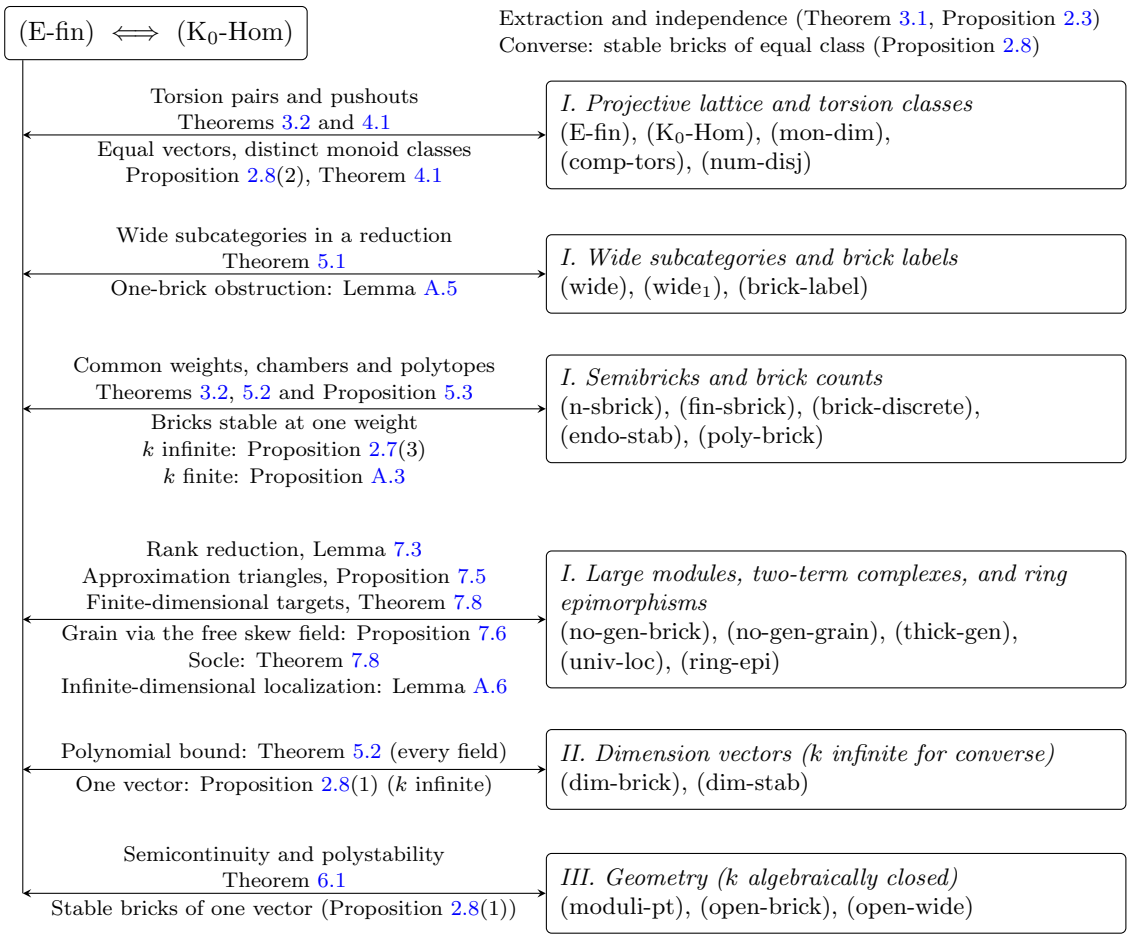

FIGURE 1. Structure of the proof of Theorem A, with forward implications above the arrows and converses below.

To prove Theorem B in rank two, we show that if $\Sigma(\Lambda)$ contains every rational ray in the plane, it cannot miss an irrational ray either. The positive and negative quadrants are covered by the projective and shifted projective cones, and we subdivide each of the two mixed quadrants into sectors whose boundary presilting complexes E and L are ordered with $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(E, L[1]) = 0$. When $\dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(L, E[1]) = 0$, the sector already lies in a single silting cone and the subdivision stops; when this dimension is positive, inserting the two-term presilting complex of class $[E]+[L]$ supplied by E-finiteness preserves the ordering and strictly decreases this nonnegative integer on both subsectors, so the subdivision terminates after finitely many steps with finitely many silting cones covering the plane.

1.4. **Relation to earlier work.**

Known cases of the conjectures. Although the conjectures (i) to (vii) remain open in rank three and higher, several special classes of algebras are known to satisfy them over an algebraically closed field. In [50, Conjecture 1.1], Mousavand and Paquette conjectured the equivalence of brick-infiniteness, brick-continuity, and the existence of a generic brick, of which (v) is one direction in contrapositive form; all three conditions are equivalent over an algebraically closed field for biserial algebras by Mousavand and Paquette [50, Theorem 1.3] and for algebras of defined Krull–Gabriel dimension by Schlegel [61, Theorem 5.8]. Over such a field, Tacchetti [65, Section 2.3] constructs both a generic brick and infinitely many bricks of a single dimension vector for the incidence algebra of every representation-infinite finite poset.

Still over an algebraically closed field, Mousavand and Paquette [52, Theorem C] prove that every connected representation-infinite algebra admitting a generalized standard Auslander–Reiten component has both a rational ray outside $\Sigma(\Lambda)$ and infinitely many bricks of one dimension stable at a common weight, verifying (i) and (vi) for these algebras. In addition, Haerizadeh and Yurikusa proved (i) for connected nondegenerate Jacobi-finite quivers with potential [39, Theorem 5.3] and (vi) for finite-dimensional nondegenerate Jacobian algebras [40, Theorem 5.1].

Earlier implications and equivalences. Several individual implications involving the conditions of Theorem A were known prior to this work, mostly over algebraically closed fields. Table 1 lists these earlier implications together with their sources and hypotheses.

TABLE 1. Earlier implications involving the conditions of Theorem A.

Known implication	Source and additional hypotheses
Over an algebraically closed field	
(dim-stab) $\Rightarrow$ (E-fin) and (K_0-Hom) $\Rightarrow$ (E-fin)	Mousavand and Paquette [54, Theorem A]: infinitely many bricks of one dimension vector stable at one weight for an E-infinite algebra
(E-fin) $\Rightarrow$ (dim-stab)	Pfeifer [56, proof of Proposition 5.6]: at each integral weight the stable modules are the simples of a finite-dimensional reduction algebra
(E-fin) $\Leftrightarrow$ multiplicity-free	Mousavand and Paquette [54, Theorem B] (multiplicity-free algebras are called *stably-discrete* in [53, Definition 6.7])
(fin-sbrick) $\Rightarrow$ (E-fin)	Asai [11, Theorems 1.8 and 5.4(1)]
(fin-sbrick) $\Rightarrow$ (dim-brick)	Mousavand and Paquette [52, Proposition 3.4]
brick-finite $\Rightarrow$ (n-sbrick)	Mousavand and Paquette [52, Theorem A(1)]
(dim-brick) $\Leftrightarrow$ (no-gen-brick)	Bautista, Pérez, and Salmerón [16, Corollary 1.3] and Schlegel [61, Theorem 5.6], for tame algebras
Over an arbitrary field	
τ-tilting finite $\Rightarrow$ (wide)	Every torsion class is functorially finite [26, Theorem 3.8], so every wide subcategory is left finite and hence functorially finite [8, Proposition 1.28]. Over an algebraically closed field see [47, Corollary 3.11].
(no-gen-brick) $\Rightarrow$ (no-gen-grain)	Schlegel [61, Lemma 5.2], for Artin algebras
τ-tilting finite $\Rightarrow$ every thick subcategory of two-term complexes has a presilting generator	Garcia [38, Theorem 4.14]

Among the conjectures themselves, Pfeifer [56, Main Theorem 1 and Proposition 5.6] showed for each algebra over an algebraically closed field that (vi) implies Demonet's conjecture (i), and proved the converse when the generic E-invariant $e_\Lambda(g,g)$ vanishes for every integral projective class g [56, Definition 6.1]. Mousavand and Paquette [54, Corollary C] proved the converse unconditionally over algebraically closed fields, establishing the equivalence of (i) and (vi), while Asai [11, Theorems 1.8 and 5.4] deduced the finite semibrick conjecture (ii) from Demonet's conjecture.

Over any field, Nasr-Isfahani [55, proof of Proposition 2.5] observed that under (wide) the extension closure of a semibrick is functorially finite and hence has finitely many simple objects, so (wide) implies (fin-sbrick) and (ii) implies an affirmative answer to (vii) for each algebra; Nasr-Isfahani [55, Theorem 3.2] also proved that (fin-sbrick) is equivalent to the ascending chain condition on wide subcategories. Theorem A(I) supplies the converse implication (fin-sbrick) $\Rightarrow$ (wide) and closes the full cycle of equivalences among (i) to (vii) over an arbitrary field.

Finally, Theorem A(I) shows over any field that Λ is brick-continuous if and only if $\mathsf{Mod}\,\Lambda$ admits a generic brick. Over an algebraically closed field, Ringel's specialization argument [59, Proposition 3.1 and §3.3] (see also [50, Proposition 4.3]) deduces brick-continuity from a generic brick whose endomorphism ring is the rational function field $k(x)$. Whether, conversely, every brick-continuous algebra over an algebraically closed field admits a generic brick with endomorphism ring $k(x)$, as asked by Bautista, Pérez, and Salmerón after [16, Corollary 1.3], remains open, since Theorem A(I) does not identify the endomorphism division ring of the generic brick it provides.

Questions answered in rank two. For ring epimorphisms $\Lambda \to \Gamma$ with $\mathrm{Tor}_1^\Lambda(\Gamma,\Gamma) = 0$, Angeleri Hügel, Marks, and Vitória [5, Theorem 4.7 and Remark 4.10] proved over an algebraically closed field that if Λ is τ-tilting finite, then every such epimorphism is a universal localization with finite-dimensional target. They asked whether Λ must be τ-tilting finite if every ring epimorphism $\Lambda \to \Gamma$ with $\mathrm{Tor}_1^\Lambda(\Gamma,\Gamma) = 0$ has finite-dimensional target. Angeleri Hügel and Sentieri [6, Questions 5.11(3)] asked the corresponding question for Artin algebras with Artin algebra targets. In Corollary 7.9, we prove in arbitrary rank that this hypothesis implies E-finiteness, which answers both questions affirmatively for finite-dimensional algebras of rank at most two by Theorem B and shows that Demonet's conjecture would suffice in higher rank. Similarly, Angeleri Hügel and Sentieri observe at the end of [6, Questions 5.11] that if every coproduct-closed wide subcategory of $\mathsf{Mod}\,\Lambda$ is closed under products, then (wide) holds; by Theorems A and B, this hypothesis in their Question 5.11(2) therefore implies τ-tilting finiteness in rank at most two.

Organization. In Section 2, we review the g-vector fan and numerical stability, prove linear independence of the classes of stable bricks at weights in $\Sigma(\Lambda)$, construct families of stable bricks at integral weights outside $\Sigma(\Lambda)$, and compare differentials on fixed projective terms. In Section 3, we extract jointly stable dependent semibricks from pairs of modules violating (K_0-Hom) and prove the equivalence of (E-fin) with (K_0-Hom), (num-disj), (comp-tors), (fin-sbrick), and (n-sbrick). In Section 4, we characterize E-finiteness via the Grothendieck monoids of torsion classes (mon-dim) and deduce the cancellation criterion for torsion and torsion-free classes. In Section 5, we realize wide subcategories of E-finite algebras as τ-perpendicular categories to prove (wide), (wide_1), and (brick-label), and use stability chambers and normalized submodule polytopes to establish (poly-brick), (brick-discrete), (endo-stab), (dim-brick), and (dim-stab). In Section 6, assuming k is algebraically closed, we prove Corollary C and establish the geometric conditions (moduli-pt), (open-brick), and (open-wide). In Section 7, we study generic bricks, generic grains, thick subcategories of $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$, universal localizations, and ring epimorphisms, proving Theorem 7.8 and Corollary 7.9. In Section 8, we prove Theorem B by Stern–Brocot subdivision and compute the fans of the Kronecker algebra, a two-vertex loop algebra, and Wang's algebra Ω_3. Appendix A establishes the finite-field brick counts via Frobenius descent and the descent lemmas for $\mathsf{Filt}(B)$ and universal localizations, completing the proof of Theorem A.

Conventions and notation. *Algebras, fields, and modules.* Throughout the paper, k denotes a field with algebraic closure $\overline{k}$, and Λ is a finite-dimensional k-algebra. We work over an arbitrary field unless a field hypothesis is explicitly stated; Section 6 assumes that k is algebraically closed. Unless stated otherwise, modules are right modules. We write $\mathsf{Mod}\,\Lambda$ for the category of all right Λ-modules, $\mathsf{mod}\,\Lambda$ for the full subcategory of finite-dimensional right Λ-modules, $\mathsf{proj}\,\Lambda$ for finitely generated projective right Λ-modules, and $\mathsf{inj}\,\Lambda$ for finitely generated injective right Λ-modules. We denote the k-duality by $D = \mathrm{Hom}_k(-,k)$ and the Nakayama equivalence by $\nu = D\,\mathrm{Hom}_\Lambda(-,\Lambda)\colon \mathsf{proj}\,\Lambda \xrightarrow{\sim} \mathsf{inj}\,\Lambda$.

Grothendieck groups and Euler pairing. Let $S_1, \dots, S_n$ be a complete set of representatives of the isomorphism classes of simple right Λ-modules, and let $P_1, \dots, P_n$ be their projective covers. We write $[M] \in K_0(\mathsf{mod}\,\Lambda) = \bigoplus_{i=1}^n \mathbb{Z}[S_i]$ for the class of $M \in \mathsf{mod}\,\Lambda$. We call $[M]$ the dimension vector of M; its coordinates are the composition multiplicities $[M : S_i]$, and their sum is the composition length $\ell(M)$. We identify $K_0(\mathsf{proj}\,\Lambda) = \bigoplus_{i=1}^n \mathbb{Z}[P_i]$, and equip $K_0(\mathsf{proj}\,\Lambda) \times K_0(\mathsf{mod}\,\Lambda) \to \mathbb{Z}$ with the Euler pairing

$$\langle [P], [M] \rangle := \dim_k \mathrm{Hom}_\Lambda(P, M) \qquad (P \in \mathsf{proj}\,\Lambda,\ M \in \mathsf{mod}\,\Lambda),$$

extended $\mathbb{R}$-bilinearly to $K_0(\mathsf{proj}\,\Lambda)_\mathbb{R} \times K_0(\mathsf{mod}\,\Lambda)_\mathbb{R} \to \mathbb{R}$. On the standard bases we have

$$\langle [P_i], [S_j] \rangle = \delta_{ij} \dim_k \mathrm{End}_\Lambda(S_j), \tag{1.1}$$

which equals δ_{ij} when $k = \overline{k}$.

Complexes and subcategories. We use cochain complexes with shift $(X[1])^i = X^{i+1}$. We denote by $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ the bounded homotopy category of $\mathsf{proj}\,\Lambda$, by $\mathsf{D}(\mathsf{Mod}\,\Lambda)$ the unbounded derived category of $\mathsf{Mod}\,\Lambda$, and by $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ the full subcategory of two-term complexes $U = (P^{-1} \xrightarrow{u} P^0)$ concentrated in degrees -1 and 0. For a morphism $p\colon P^{-1} \to P^0$ in $\mathsf{proj}\,\Lambda$, we write $C_p := (P^{-1} \xrightarrow{p} P^0) \in \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$; its g-vector is $[C_p] := [P^0] - [P^{-1}] \in K_0(\mathsf{proj}\,\Lambda)$. All subcategories are assumed to be full and closed under isomorphisms. For an abelian length category $\mathcal{A}$, we write $\mathsf{sim}(\mathcal{A})$ for the set of isomorphism classes of simple objects of $\mathcal{A}$. For a class $\mathcal{S} \subseteq \mathsf{mod}\,\Lambda$, we write $\mathsf{add}\,\mathcal{S}$ for its closure under finite direct sums and direct summands, $\mathsf{Fac}\,\mathcal{S}$ for quotients of objects in $\mathsf{add}\,\mathcal{S}$, and $\mathsf{Sub}\,\mathcal{S}$ for submodules of objects in $\mathsf{add}\,\mathcal{S}$. We write $\mathcal{S}^\perp = \{X \mid \mathrm{Hom}_\Lambda(\mathcal{S}, X) = 0\}$ and ${}^\perp\mathcal{S} = \{X \mid \mathrm{Hom}_\Lambda(X, \mathcal{S}) = 0\}$ in $\mathsf{mod}\,\Lambda$ (or in $\mathsf{Mod}\,\Lambda$ where indicated).

Formalization. Theorems A and B, Corollary C, and all theorems, propositions, lemmas, and corollaries in Sections 2–8 and Appendix A have been formalized in Lean 4 using Mathlib as the only external dependency. The source is available at https://github.com/haruhisa-enomoto/e-finite-characterizations, with a companion website and statement-by-statement concordance at https://haruhisa-enomoto.github.io/e-finite-characterizations/. Rather than assuming external results cited in the proofs as axioms, the development proves them directly. It uses self-contained alternative arguments at a few points: Lemma 2.2 establishes the rank count directly by linear algebra; Proposition 7.6 constructs its division ring via Malcev–Neumann well-ordered series on a bi-ordered free group rather than Cohn's universal skew field; Theorem 6.1 verifies (moduli-pt) directly on the graded ring of semi-invariants by splitting extensions rather than invoking King's polystable classification; Theorem 7.8 proves (E-fin) ⇒ (univ-loc) by direct rank induction along presilting pencils; and Theorem 8.4 deduces brick-finiteness directly from the finite sector cover. The worked examples (Examples 8.6–8.8 and A.1) are not formalized.

Use of AI. The author chose the questions and scope, directed the research and the writing, and is responsible for the mathematical content and final manuscript. The results grew out of several AI-assisted research threads. GPT-6 Astra developed the arguments for generic grains and localizations and an adaptation of Mousavand and Paquette's proof of openness. Claude Fable 5.2 found the rank-two proof. GPT-6.2 Astra proved the equal-class extraction and stated several of the equivalences of Theorem A over an algebraically closed field, but their significance was not recognized at the time. Separately, the author asked whether the Grothendieck monoid of every torsion class of a representation-finite algebra is cancellative. Gemini 4 Neon proved this for τ-tilting finite algebras, isolating the condition (K_0-Hom). When the author later asked it to survey the implications among the brick conjectures, it pointed out that these results already make several of them equivalent. Gemini 4 Neon then unified the equivalences, extended them to arbitrary fields, and wrote the first draft. GPT-6.2 Astra revised the manuscript, Claude Opus 5.5 organized and carried out the exposition reviews, and Gemini 4 Carbon and Gemini 4 Neon polished the exposition, using the author's paper-writing skills (https://github.com/haruhisa-enomoto/math-paper-skills). GPT-6.2 Astra, Claude Opus 5.5, Claude Fable 5.2, GPT-6.2 Sol and Gemini 4 Neon contributed proof simplifications, and GPT-6.2 Astra carried out most of the independent mathematical checks. The Lean 4 formalization was carried out by GPT-6.2 Astra under the author's direction, with documentation polished by Gemini 4 Carbon.

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2. Silting cones, torsion classes, and families of stable bricks

In this section, we identify the semistable subcategory at any weight in the g-vector fan with a τ-perpendicular category and prove that the classes of its simple objects are $\mathbb{Q}$-linearly independent (Proposition 2.3). Conversely, from an integral weight outside $\Sigma(\Lambda)$ we construct families of finite-dimensional bricks stable at a common weight (Proposition 2.8), which supply the witnesses for most of the converse implications in Theorem A. Subsection 2.3 gives a comparison of differentials on fixed projective terms used both for presilting generation of thick subcategories in Section 7 and for the rank-two subdivision in Section 8.

Before turning to two-term complexes, we fix terminology for bricks and subcategories of $\mathsf{mod}\,\Lambda$. A *brick* is a nonzero module whose endomorphism ring is a division ring, and a *semibrick* is a set of isomorphism classes of pairwise Hom-orthogonal bricks in $\mathsf{mod}\,\Lambda$. For a class of modules $\mathcal{S} \subseteq \mathsf{mod}\,\Lambda$, we write $\mathsf{Filt}(\mathcal{S})$ for the full subcategory of modules admitting a finite filtration with factors in $\mathcal{S}$ (including 0). An *exact abelian subcategory* of $\mathsf{mod}\,\Lambda$ is a full subcategory closed under finite direct sums, kernels, and cokernels in $\mathsf{mod}\,\Lambda$, and a *wide subcategory* is an exact abelian subcategory that is also closed under extensions. Finally, we write $\mathsf{T}(\mathcal{S})$ and $\mathsf{F}(\mathcal{S})$ for the smallest torsion and torsion-free classes containing $\mathcal{S}$, respectively, and denote by $\mathsf{tors}\,\Lambda$ and $\mathsf{torf}\,\Lambda$ the posets of torsion and torsion-free classes in $\mathsf{mod}\,\Lambda$, ordered by inclusion; for $\mathcal{T} \in \mathsf{tors}\,\Lambda$, the torsion radical $t_{\mathcal{T}}(M)$ is the largest submodule of M lying in $\mathcal{T}$.

2.1. **The g-vector fan and stability conditions.** A complex $U \in \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ is *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(U, U[1]) = 0$, and *silting* if in addition $\mathsf{thick}(U) = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. We denote by $\mathsf{2\text{-}psilt}\,\Lambda$ (respectively, $\mathsf{2\text{-}silt}\,\Lambda$) the set of isomorphism classes of basic two-term presilting (respectively, silting) complexes. Over any field, every basic two-term presilting complex $U = \bigoplus_{i=1}^{r} U_i$ extends to a two-term silting complex and is already silting precisely when $r = n$ by Adachi, Iyama, and Reiten [1, Proposition 3.3]. Moreover, the classes of the indecomposable summands of a two-term silting complex form a $\mathbb{Z}$-basis of $K_0(\mathsf{proj}\,\Lambda)$ [1, Proposition 1.6(b)], so $r \leq n$ and $[U_1], \dots, [U_r]$ extend to a $\mathbb{Z}$-basis of $K_0(\mathsf{proj}\,\Lambda)$.

For each $U = \bigoplus_{i=1}^{r} U_i \in \mathsf{2\text{-}psilt}\,\Lambda$, let

$$C(U) := \sum_{i=1}^{r} \mathbb{R}_{\geq 0}[U_i] \subseteq K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}}, \qquad C^{\circ}(U) := \sum_{i=1}^{r} \mathbb{R}_{>0}[U_i]$$

be its closed cone and relative interior (with $C(0) = C^{\circ}(0) = \{0\}$). A *silting cone* is a cone $C(T)$ for $T \in \mathsf{2\text{-}silt}\,\Lambda$. Over any field, two silting cones intersect in the face spanned by their common indecomposable summands by Demonet, Iyama, and Jasso [26, Corollary 6.7(b)]. Since every two-term presilting complex extends to a silting complex, the cones $C(U)$ form a fan, called the *g-vector fan* of Λ, with support

$$\Sigma(\Lambda) := \bigcup_{U \in \mathsf{2\text{-}psilt}\,\Lambda} C(U) \subseteq K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}}.$$

Integral points of the fan determine two-term presilting complexes uniquely, and duality identifies the fan of Λ^{op} with the negative of the fan of Λ.

Lemma 2.1.

(1) *For $U \in \mathsf{2\text{-}psilt}\,\Lambda$, the integral points of $C(U)$ are precisely the classes of complexes in $\mathsf{add}\,U$. The class of a two-term presilting complex lies in the relative interior of the cone spanned by its distinct indecomposable summands.*

(2) *Two two-term presilting complexes of the same class are isomorphic, whether or not they are basic.*

(3) *Under* $[P] \mapsto [P^*]$, *where* $P^* = \operatorname{Hom}_\Lambda(P, \Lambda)$, *we have* $\Sigma(\Lambda^{\mathrm{op}}) = -\Sigma(\Lambda)$. *In particular, E-finiteness passes to the opposite algebra.*

Proof. (1) Write $U = \bigoplus_{i=1}^r U_i$, where $U_1, \dots, U_r$ are pairwise nonisomorphic and indecomposable. Since $[U_1], \dots, [U_r]$ extend to a $\mathbb{Z}$-basis of $K_0(\mathsf{proj}\,\Lambda)$, a vector in $C(U) = \sum_{i=1}^r \mathbb{R}_{\geq 0}[U_i]$ belongs to $K_0(\mathsf{proj}\,\Lambda)$ if and only if its coefficients are nonnegative integers, that is, if and only if it is the class of $\bigoplus_{i=1}^r U_i^{m_i} \in \mathsf{add}\, U$. In particular, in any two-term presilting complex V, each distinct indecomposable summand occurs with multiplicity at least one, so $[V]$ lies in the relative interior of the cone spanned by those summands.

(2) Choose basic silting complexes T and T' containing the indecomposable summands of the first and second presilting complexes, respectively. The common class of the two presilting complexes lies in $C(T) \cap C(T')$, which is spanned by the classes of the common summands of T and T'. In either silting basis the coefficients on those summands are the same and the other coefficients vanish. By (1) these coefficients are the multiplicities in the two complexes.

(3) The shifted duality $\operatorname{Hom}_\Lambda(-, \Lambda)[1] \colon \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda^{\mathrm{op}})$ sends a two-term complex $P^{-1} \xrightarrow{u} P^0$ to $(P^0)^* \to (P^{-1})^*$ (with differential $\pm u^*$) in degrees -1 and 0. This equivalence preserves the presilting condition and maps $[P^0] - [P^{-1}]$ to $[(P^{-1})^*] - [(P^0)^*]$, so $\Sigma(\Lambda^{\mathrm{op}}) = -\Sigma(\Lambda)$. In particular, Λ is E-finite if and only if Λ^{op} is E-finite. □

We next express the Euler pairing of a projective class with a module in terms of two Hom spaces. For any complex $U = (P^{-1} \xrightarrow{u} P^0) \in \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ and any right module $M \in \mathsf{Mod}\,\Lambda$, applying $\operatorname{Hom}_\Lambda(-, M)$ to $P^{-1} \xrightarrow{u} P^0 \to H^0(U) \to 0$ yields the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_\Lambda(H^0(U), M) \to \operatorname{Hom}_\Lambda(P^0, M) \\ \xrightarrow{\operatorname{Hom}_\Lambda(u, M)} \operatorname{Hom}_\Lambda(P^{-1}, M) \to \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U, M[1]) \to 0. \end{aligned} \tag{2.1}$$

Since $P^{-1}, P^0 \in \mathsf{proj}\,\Lambda$, the Nakayama isomorphism $D\operatorname{Hom}_\Lambda(P, M) \cong \operatorname{Hom}_\Lambda(M, \nu P)$ holds for every $M \in \mathsf{Mod}\,\Lambda$ (even infinite-dimensional). Applying D to (2.1) therefore yields a functorial isomorphism

$$D\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U, M[1]) \cong \operatorname{Hom}_\Lambda(M, H^{-1}(\nu U)), \tag{2.2}$$

where $H^{-1}(\nu U) = \ker(\nu u) \in \mathsf{mod}\,\Lambda$. In particular, for $M \in \mathsf{mod}\,\Lambda$,

$$\begin{aligned} \langle [U], [M] \rangle &= \dim_k \operatorname{Hom}_\Lambda(H^0(U), M) - \dim_k \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U, M[1]) \\ &= \dim_k \operatorname{Hom}_\Lambda(H^0(U), M) - \dim_k \operatorname{Hom}_\Lambda(M, H^{-1}(\nu U)). \end{aligned} \tag{2.3}$$

When a weight lies in the relative interior of a presilting cone $C(U)$, the corresponding semistable subcategory turns out to be the τ-perpendicular category of U, which we now recall. For $U \in \mathsf{2\text{-}psilt}\,\Lambda$, its *$\tau$-perpendicular category* is

$$\mathcal{J}(U) := \{M \in \mathsf{mod}\,\Lambda \mid \operatorname{Hom}_\Lambda(H^0(U), M) = 0 = \operatorname{Hom}_\Lambda(M, H^{-1}(\nu U))\}.$$

Adachi, Iyama, and Reiten [1, Theorem 3.2] identify basic two-term silting complexes T with basic support τ-tilting modules $H^0(T)$. The bijection with functorially finite torsion classes is $T \mapsto \mathsf{Fac}\, H^0(T)$ [1, Theorem 2.7], in the arbitrary-field setting of [27, Section 4.1]. The basic *Bongartz completion* T_U of U is the completion whose associated torsion class is largest; by [27, Theorem 4.4],

$$\mathsf{Fac}\, H^0(T_U) = {}^{\perp}H^{-1}(\nu U). \tag{2.4}$$

The *reduction algebra* of U is

$$\Gamma_U = \operatorname{End}_\Lambda(H^0(T_U))/\mathfrak{a}_U, \tag{2.5}$$

where $\mathfrak{a}_U$ denotes the ideal of endomorphisms factoring through $\mathsf{add}\, H^0(U)$. We write $|U|$ for the number of distinct indecomposable summands of U.

An Ext-*projective generator* of a wide subcategory $\mathcal{W} \subseteq \mathsf{mod}\,\Lambda$ is an object $G \in \mathcal{W}$ such that every $X \in \mathcal{W}$ satisfies $\operatorname{Ext}^1_{\mathcal{W}}(G, X) = 0$ and admits an epimorphism $G^m \twoheadrightarrow X$ in $\mathcal{W}$ for some $m \geq 0$. For a wide subcategory of the module category of an Artin algebra, [32, Proposition 4.12] identifies the following equivalent properties of $\mathcal{W}$: covariant, contravariant or functorial finiteness, existence of an Ext-projective generator, and equivalence with the module category of an Artin

algebra. Over a finite-dimensional k-algebra Λ, the endomorphism algebra of an Ext-projective generator is a finite-dimensional k-algebra whose module category is equivalent to $\mathcal{W}$.

Lemma 2.2 ([42, Theorem 3.8], [27, Theorem 4.12(b)])**.** *For a basic two-term presilting complex U over any field, $\mathcal{J}(U)$ is a functorially finite wide subcategory of $\mathsf{mod}\,\Lambda$. There is an exact equivalence $\mathcal{J}(U) \simeq \mathsf{mod}\,\Gamma_U$, and $|\mathsf{sim}\,\mathcal{J}(U)| = n - |U|$.*

Proof. Represent U minimally in $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ as $U' \oplus P[1]$, where $P \in \mathsf{proj}\,\Lambda$ and U' has no summand of the form $Q[1]$. Putting $Y := H^0(U) = H^0(U')$, the pair (Y, P) is the support τ-rigid pair associated to U, and we have $H^{-1}(\nu U) \cong \tau Y \oplus \nu P$. Nakayama duality thus identifies $\mathcal{J}(U)$ with $Y^{\perp} \cap {}^{\perp}(\tau Y) \cap P^{\perp}$. When $P = 0$, Jasso [42, Theorem 3.8] identifies this category with $\mathsf{mod}\,\Gamma_U$. For a general pair (Y, P), Demonet et al. [27, Theorem 4.12(b)] apply Jasso's equivalence over $\Lambda/\Lambda e\Lambda$, where $\mathsf{add}\,P = \mathsf{add}(e\Lambda)$ and $P^{\perp} = \mathsf{mod}(\Lambda/\Lambda e\Lambda)$, and the resulting endomorphism quotient is precisely (2.5). The equivalence is exact with wide image, so the image of the regular Γ_U-module is an Ext-projective generator, giving functorial finiteness by [32, Proposition 4.12]. The indecomposable projective Γ_U-modules correspond to the $n-|U|$ complementary summands of the basic Bongartz completion. Buan and Hanson collect these properties in [19, Theorem 3.3]. □

We evaluate weights on modules by $\theta(M) := \langle \theta, [M] \rangle$. For any weight $\theta \in K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}}$, we follow Baumann, Kamnitzer, and Tingley [15, Section 3] in associating to θ the numerical torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ in $\mathsf{mod}\,\Lambda$ defined by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \mathsf{mod}\,\Lambda \mid \theta(Q) \geq 0 \text{ for all quotients } Q \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \mathsf{mod}\,\Lambda \mid \theta(Q) > 0 \text{ for all nonzero quotients } Q \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \mathsf{mod}\,\Lambda \mid \theta(L) \leq 0 \text{ for all submodules } L \subseteq M\},\\
\mathcal{F}_\theta &:= \{M \in \mathsf{mod}\,\Lambda \mid \theta(L) < 0 \text{ for all nonzero submodules } L \subseteq M\}.
\end{aligned}$$

The *θ-semistable subcategory* is the wide subcategory

$$\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta,$$

consisting of the modules $M \in \mathsf{mod}\,\Lambda$ such that $\theta(M) = 0$ and $\theta(L) \leq 0$ for every submodule $L \subseteq M$. A nonzero module $M \in \mathcal{W}_\theta$ is *θ-stable* if it is a simple object of $\mathcal{W}_\theta$, equivalently, if $\theta(L) < 0$ for every proper nonzero submodule $L \subsetneq M$. Every θ-stable module is a brick, and nonisomorphic θ-stable modules are Hom-orthogonal. For any brick $B \in \mathsf{brick}\,\Lambda$, its *stability wall* $\Theta_B = \{\theta \in K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}} \mid B \in \mathcal{W}_\theta\}$ is cut out by $\theta(B) = 0$ together with the inequalities $\theta(L) \leq 0$ for the finitely many classes of submodules $L \subseteq B$, so Θ_B is a closed rational polyhedral cone. For the wall-and-chamber geometry see Asai [9] over any field, and King [44] and Brüstle, Smith, and Treffinger [18] over algebraically closed fields.

2.2. **Classes of stable modules and families of bricks.** Over an algebraically closed field, the classes of the simple objects of $\mathcal{W}_{[U]}$ are linearly independent for every two-term presilting complex U by Asai and Iyama [12, Proposition 5.6]. For a basic algebra, pairing an indecomposable summand of a two-term silting complex with the corresponding simple object of its heart gives the endomorphism dimension of that simple [8, Theorem 2.17], which equals 1 over an algebraically closed field [36, Section 3]. For a cone of arbitrary dimension over any field, we compute the pairing with the complementary summands of a Bongartz completion using projective covers in $\mathcal{J}(U)$.

Proposition 2.3. *Let Λ be a finite-dimensional algebra over an arbitrary field k, let $U \in 2\text{-}\mathsf{psilt}\,\Lambda$, and let $\theta \in C^\circ(U)$. Then:*

1. *$\mathcal{T}_\theta = \mathsf{Fac}\,H^0(U)$, $\overline{\mathcal{T}}_\theta = {}^{\perp}H^{-1}(\nu U)$, $\overline{\mathcal{F}}_\theta = H^0(U)^{\perp}$, $\mathcal{F}_\theta = \mathsf{Sub}\,H^{-1}(\nu U)$, and $\mathcal{W}_\theta = \mathcal{J}(U)$.*
2. *The classes in $K_0(\mathsf{mod}\,\Lambda)$ of the $n - |U|$ simple objects of $\mathcal{W}_\theta$ are $\mathbb{Q}$-linearly independent.*

Proof. (1) These identities hold over any field by Yurikusa [66, Proposition 3.3], in the formulation via torsion pairs of Asai [9, Proposition 3.14].

(2) By (1) and Lemma 2.2, $\mathcal{W}_\theta = \mathcal{J}(U) \simeq \operatorname{mod}\Gamma_U$ has $r = n - |U|$ isomorphism classes of simple objects $E_1, \dots, E_r$. Write the Bongartz completion as $T_U = U \oplus V$, with basic complement $V = \bigoplus_{i=1}^{r} V_i$. The projective covers of $E_1, \dots, E_r$ in $\mathcal{J}(U)$ are

$$\overline{V}_i = H^0(V_i)/\operatorname{tr}_{H^0(U)}(H^0(V_i)) \qquad (1 \le i \le r),$$

where $\operatorname{tr}_{H^0(U)}$ denotes the trace of $H^0(U)$. Indeed, the torsion-free quotient with respect to $\operatorname{Fac} H^0(U)$ sends the Bongartz completion to the projective generator of $\mathcal{J}(U)$, with one indecomposable projective for each complementary summand [42, Section 3.1 and Theorem 3.8]. The same description applies to a support pair after passing to the idempotent quotient as in Lemma 2.2 [27, Theorem 4.12(b)].

To evaluate the Euler pairing $\langle [V_i], [E_j] \rangle$, we first verify that $\operatorname{Hom}_\Lambda(M, H^{-1}(\nu V_i)) = 0$ for every $M \in \mathcal{J}(U)$. Choose $\eta \in C^\circ(T_U)$. Since $|T_U| = n$, part (1) and Lemma 2.2 give $\mathcal{W}_\eta = \mathcal{J}(T_U) = 0$. For any $X \in \overline{\mathcal{T}}_\eta$, the quotient $X/t_{\mathcal{T}_\eta}(X)$ belongs to $\overline{\mathcal{T}}_\eta \cap \overline{\mathcal{F}}_\eta = \mathcal{W}_\eta = 0$, so $\mathcal{T}_\eta = \overline{\mathcal{T}}_\eta$. Applying (1) at η therefore gives $\operatorname{Fac} H^0(T_U) = {}^\perp H^{-1}(\nu T_U)$. Combining this with (2.4) and (1) at θ gives

$$\overline{\mathcal{T}}_\theta = {}^\perp H^{-1}(\nu U) = \operatorname{Fac} H^0(T_U) = {}^\perp H^{-1}(\nu T_U),$$

so $\mathcal{J}(U) \subseteq \overline{\mathcal{T}}_\theta \cap H^0(U)^\perp = {}^\perp H^{-1}(\nu T_U) \cap H^0(U)^\perp$. Thus for every $M \in \mathcal{J}(U)$ we have $\operatorname{Hom}_\Lambda(M, H^{-1}(\nu V_i)) = 0$, and every morphism $H^0(V_i) \to M$ vanishes on $\operatorname{tr}_{H^0(U)}(H^0(V_i))$. By (2.3),

$$\begin{aligned}\langle [V_i], [E_j] \rangle &= \dim_k \operatorname{Hom}_\Lambda(H^0(V_i), E_j) = \dim_k \operatorname{Hom}_\Lambda(\overline{V}_i, E_j) \\ &= \delta_{ij} \dim_k \operatorname{End}_\Lambda(E_j) \qquad (1 \le i, j \le r).\end{aligned}$$

Because the $r \times r$ pairing matrix $(\langle [V_i], [E_j] \rangle)$ is diagonal with positive diagonal entries, the classes $[E_1], \dots, [E_r]$ are $\mathbb{Q}$-linearly independent in $K_0(\operatorname{mod}\Lambda)$. □

The following sign rule for a full silting cone will be used in Theorem 3.2 and Lemma 7.3.

Lemma 2.4. *Let $\mathcal{T} = \operatorname{Fac} H^0(T)$ for $T \in \text{2-silt}\,\Lambda$ and let $\psi \in C^\circ(T)$. Then ψ is positive on every nonzero module in $\mathcal{T}$ and negative on every nonzero module in $\mathcal{T}^\perp$. In particular, no nonzero module in $\mathcal{T}$ has the same class as a module in $\mathcal{T}^\perp$.*

Proof. By Proposition 2.3, $\mathcal{T} = \mathcal{T}_\psi$ and $\mathcal{T}^\perp = \overline{\mathcal{F}}_\psi = \mathcal{F}_\psi$, since $\mathcal{W}_\psi = \mathcal{J}(T) = 0$. The signs follow from the definitions. □

Asai [11, Theorems 1.4 and 1.5] uses compactness to characterize weights in the fan and, together with numerical disjointness and cocompactness, to characterize functorially finite torsion classes. A torsion class is *compact* if it has the form $\mathsf{T}(M)$ for some $M \in \operatorname{mod}\Lambda$, *cocompact* if it has the form ${}^\perp N$ for some $N \in \operatorname{mod}\Lambda$, and *bicompact* if both hold. We write $\text{f-tors}\,\Lambda$ for the functorially finite torsion classes. A torsion pair $(\mathcal{T}, \mathcal{F})$ is numerically disjoint if $[X] \neq [Y]$ whenever $0 \neq X \in \mathcal{T}$ and $0 \neq Y \in \mathcal{F}$.

We recall Asai's test for a weight to lie in the fan. The invertibility of $\operatorname{Hom}_\Lambda(p, X)$ in (2) will give a single map to invert in Lemma A.6.

Proposition 2.5. *Let Λ be a finite-dimensional algebra over a field k.*

(1) (Asai [11, Theorem 1.4]) *Over any field, for a fixed weight $\theta \in K_0(\operatorname{proj}\Lambda)_\mathbb{R}$, the following are equivalent: $\theta \in \Sigma(\Lambda)$; $\mathcal{T}_\theta$ is functorially finite; $\mathcal{T}_\theta$ is compact; $\mathcal{T}_\theta$ is bicompact; $\overline{\mathcal{T}}_\theta$ is functorially finite; $\overline{\mathcal{T}}_\theta$ is cocompact; $\overline{\mathcal{T}}_\theta$ is bicompact.*

(2) (Asai and Iyama [12, Theorem 4.3(a)]) *Suppose $k = \overline{k}$ and $\theta \in K_0(\operatorname{proj}\Lambda)$. A module X belongs to $\mathcal{W}_\theta$ if and only if there exist an integer $s > 0$ and a morphism $p\colon V^{-1} \to V^0$ in $\operatorname{proj}\Lambda$ with $[V^0] - [V^{-1}] = s\theta$ such that $\operatorname{Hom}_\Lambda(p, X)$ is bijective. For the independent semi-invariant criterion of Fei, see* [34, Lemma 3.13].

To construct families of stable bricks when $\Sigma(\Lambda)$ omits an integral class, we first record a lemma in linear algebra valid over any field K (compare Flanders [35] and Atkinson and Lloyd [13, Lemma 4], who assume that $|K|$ exceeds the maximum rank): if a map in a linear space of homomorphisms already achieves the maximum rank attained over $\overline{K}$, then every map in the space sends its kernel into its image.

Lemma 2.6. *Let K be any field, let $\mathcal{V}, H_1, H_2$ be finite-dimensional K-vector spaces, and let $\Phi\colon \mathcal{V} \to \operatorname{Hom}_K(H_1, H_2)$ be a K-linear map. Let r be the maximum rank of $\Phi(p)$ as p varies over $\mathcal{V} \otimes_K \overline{K}$. If $p_0 \in \mathcal{V}$ satisfies $\operatorname{rank}\Phi(p_0) = r$, then $\Phi(h)(\ker\Phi(p_0)) \subseteq \operatorname{Im}\Phi(p_0)$ for every $h \in \mathcal{V}$.*

Proof. Choose bases of H_1 and H_2 in which $\Phi(p_0)$ has an $r \times r$ identity block in the top left corner and zeros elsewhere. For any $h \in \mathcal{V}$, every $(r+1)\times(r+1)$ minor of $\Phi(p_0) + t\Phi(h)$ vanishes for all $t \in \overline{K}$, and hence vanishes as a polynomial in t because $\overline{K}$ is infinite. Consider the $(r+1)\times(r+1)$ minor obtained by adjoining one row and one column outside the first r indices to the $r \times r$ identity block. Expanding along the adjoined row, the coefficient of t in this minor is precisely the entry of $\Phi(h)$ in the adjoined row and column, which must therefore vanish. Thus the induced map $\ker\Phi(p_0) \to H_2/\operatorname{Im}\Phi(p_0)$ is zero, meaning that $\Phi(h)(\ker\Phi(p_0)) \subseteq \operatorname{Im}\Phi(p_0)$. □

We now adapt two arguments of Mousavand and Paquette [54, Lemmas 2.2 and 2.3 and Proposition 2.4] to an arbitrary field: part (1) extracts a stable brick as a nonzero image of minimal dimension, while part (2) bounds the degree of a polynomial vanishing on the locus of presentations yielding a given module, which will also allow us to count such points over finite extensions of a finite field in Appendix A. Notice that any integral class $g \in K_0(\mathsf{proj}\,\Lambda)$ can be written as $[P^0]-[P^{-1}]$ with $P^0, P^{-1} \in \mathsf{proj}\,\Lambda$ sharing no indecomposable summand by separating the positive and negative coefficients of g in the indecomposable projective basis.

Proposition 2.7. *Let Λ be a finite-dimensional algebra over an arbitrary field k, and suppose $g = [P^0] - [P^{-1}] \in K_0(\mathsf{proj}\,\Lambda) \setminus \Sigma(\Lambda)$, where $P^0, P^{-1} \in \mathsf{proj}\,\Lambda$ share no indecomposable summand. Put $\mathcal{V} := \operatorname{Hom}_\Lambda(P^{-1}, P^0)$, and for $p \in \mathcal{V}$ set $X_p := \operatorname{Coker} p$ and $Y_p := \ker(\nu p)$.*

1. *For every $p \in \mathcal{V}$, we have $\operatorname{Hom}_\Lambda(X_p, Y_p) \neq 0$. Choose a nonzero morphism $X_p \to Y_p$ of minimal image dimension and denote its image by B_p. For every such choice, B_p is a brick of k-dimension at most $\dim_k P^0$, and B_p is θ_p-stable for the integral weight*
$$\theta_p := (\dim_k B_p)\, g - g(B_p)\,[\Lambda] \in K_0(\mathsf{proj}\,\Lambda).$$
2. *For each fixed module $M \in \mathsf{mod}\,\Lambda$, the set $\{p \in \mathcal{V} \mid X_p \twoheadrightarrow M \text{ and } \operatorname{Hom}_\Lambda(M, Y_p) \neq 0\}$ is contained in the zero locus of a single nonzero polynomial on $\mathcal{V}$ of degree at most $\dim_k P^{-1} \cdot \dim_k M$.*
3. (Mousavand and Paquette [54, Theorem A] for $k = \overline{k}$) *If k is infinite, then there exist an integral weight $\theta \in K_0(\mathsf{proj}\,\Lambda)$, a class $d \in K_0(\mathsf{mod}\,\Lambda)$, and an infinite set $\mathfrak{F}$ of pairwise nonisomorphic θ-stable bricks in $\mathsf{mod}\,\Lambda$ of class d.*

Proof. (1) For any $M \in \mathsf{mod}\,\Lambda$, define the k-linear map
$$\Phi_M\colon \mathcal{V} \longrightarrow \operatorname{Hom}_k\big(\operatorname{Hom}_\Lambda(P^0, M), \operatorname{Hom}_\Lambda(P^{-1}, M)\big), \qquad \Phi_M(p) := \operatorname{Hom}_\Lambda(p, M).$$
By (2.1), (2.2), and (2.3), we have $\ker\Phi_M(p) = \operatorname{Hom}_\Lambda(X_p, M)$, $D\operatorname{Coker}\Phi_M(p) \cong \operatorname{Hom}_\Lambda(M, Y_p)$, and
$$g(M) = \dim_k \operatorname{Hom}_\Lambda(X_p, M) - \dim_k \operatorname{Hom}_\Lambda(M, Y_p),$$
$$\operatorname{Hom}_{\mathsf{K}^b(\mathsf{proj}\,\Lambda)}(C_p, C_p[1]) \cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(C_p, X_p[1]) \cong \operatorname{Coker}\Phi_{X_p}(p) \cong D\operatorname{Hom}_\Lambda(X_p, Y_p),$$
where the first isomorphism in the second line follows from applying $\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(C_p, -)$ to the truncation triangle $H^{-1}(C_p)[1] \to C_p \to X_p \to H^{-1}(C_p)[2]$. Since $g \notin \Sigma(\Lambda)$, the complex C_p is not presilting, so $\operatorname{Hom}_\Lambda(X_p, Y_p) \neq 0$.

Choose a nonzero morphism $f\colon X_p \to Y_p$ whose image $B = B_p$ has minimal k-dimension; then $\dim_k B \leq \dim_k X_p \leq \dim_k P^0$. For any nonzero $e \in \operatorname{End}_\Lambda(B)$, the composite $X_p \twoheadrightarrow B \xrightarrow{e} B \hookrightarrow Y_p$ is nonzero with image $e(B) \subseteq B$, so $e(B) = B$ by minimality; hence e is an isomorphism and B is a brick. For any proper nonzero submodule $0 \neq L \subsetneq B$ with quotient $Q = B/L$, minimality of $\dim_k B$ gives $\operatorname{Hom}_\Lambda(X_p, L) = 0$ (otherwise composing with $L \hookrightarrow B \hookrightarrow Y_p$ would yield a smaller nonzero image) and $\operatorname{Hom}_\Lambda(Q, Y_p) = 0$ (otherwise composing with $X_p \twoheadrightarrow B \twoheadrightarrow Q$ would do the same). Since $L \hookrightarrow Y_p$ and $X_p \twoheadrightarrow Q$, we obtain $g(L) = -\dim_k \operatorname{Hom}_\Lambda(L, Y_p) < 0$ and $g(Q) = \dim_k \operatorname{Hom}_\Lambda(X_p, Q) > 0$. Because $\langle [\Lambda], [M] \rangle = \dim_k M$, the weight $\theta_p = (\dim_k B)\, g - g(B)\,[\Lambda]$ satisfies $\theta_p(B) = 0$ and
$$\theta_p(L) = (\dim_k B)\, g(L) - (g(L) + g(Q))\, \dim_k L = (\dim_k Q)\, g(L) - (\dim_k L)\, g(Q) < 0.$$

Hence B is θ_p-stable.

(2) Let r_M be the maximum rank of $\Phi_M(p)$ over $\mathcal{V}\otimes_k \overline{k}$. Suppose $p_0 \in \mathcal{V}$ satisfies $\operatorname{rank}\Phi_M(p_0) = r_M$ and $X_{p_0} \twoheadrightarrow M$. Then the composite epimorphism $q\colon P^0 \twoheadrightarrow X_{p_0} \twoheadrightarrow M$ lies in $\ker\Phi_M(p_0)$. Since q is surjective and P^{-1} is projective, every morphism in $\operatorname{Hom}_\Lambda(P^{-1}, M)$ lifts through q, so $\Phi_M(h)(q) = q\circ h$ ranges over all of $\operatorname{Hom}_\Lambda(P^{-1}, M)$ as h varies over $\mathcal{V}$. By Lemma 2.6, it follows that $\operatorname{Im}\Phi_M(p_0) = \operatorname{Hom}_\Lambda(P^{-1}, M)$, whence $\operatorname{Coker}\Phi_M(p_0) = 0$ and $\operatorname{Hom}_\Lambda(M, Y_{p_0}) = 0$. Consequently, every $p \in \mathcal{V}$ satisfying $X_p \twoheadrightarrow M$ and $\operatorname{Hom}_\Lambda(M, Y_p) \neq 0$ has $\operatorname{rank}\Phi_M(p) < r_M$. Choosing an $r_M \times r_M$ minor of Φ_M that does not vanish identically gives a nonzero polynomial vanishing at every such p, of degree at most

$$r_M \le \dim_k \operatorname{Hom}_\Lambda(P^{-1}, M) \le (\dim_k P^{-1})(\dim_k M).$$

(3) By (1), every $p \in \mathcal{V}$ produces a brick B_p satisfying $X_p \twoheadrightarrow B_p$ and $B_p \hookrightarrow Y_p$, so p lies in the zero locus of the nonzero polynomial in (2) for $M = B_p$. Because k is infinite, $\mathcal{V}$ cannot be covered by the zero loci of finitely many nonzero polynomials, so infinitely many isomorphism classes occur among $\{B_p \mid p \in \mathcal{V}\}$. Since $\dim_k B_p \le \dim_k P^0$, only finitely many classes $[B_p] \in K_0(\operatorname{mod}\Lambda)$ can occur, and hence only finitely many weights θ_p arise. By the pigeonhole principle, infinitely many pairwise nonisomorphic bricks in the family are stable for a single integral weight θ and share one dimension vector d. □

The converses in Theorem A that involve finite-dimensional bricks use the following two consequences of a missing lattice point.

Proposition 2.8. *Let Λ be E-infinite over a field k.*

1. *For some integral weight θ and some endolength r, there are infinitely many pairwise nonisomorphic θ-stable bricks of endolength r in $\operatorname{mod}\Lambda$. If k is infinite, they can be chosen to have the same Grothendieck class.*
2. *There are two nonisomorphic bricks in $\operatorname{mod}\Lambda$ stable at a common integral weight with the same class.*

Proof. (1) Over an infinite field, Proposition 2.7(3) yields infinitely many nonisomorphic bricks of one class stable at one integral weight. They have the same k-dimension, so their endolengths are bounded, and hence infinitely many share a single endolength. Over a finite field, Proposition A.3(1) gives infinitely many bricks of bounded endolength stable at a common integral weight, so again infinitely many have one endolength. The finite-field count follows from Proposition 2.7 and Frobenius descent (Lemma A.2).

(2) Over an infinite field, any two distinct isomorphism classes from the family in (1) suffice. Over $k = \mathbb{F}_q$, Proposition A.3 gives constants b_0, c_0 and a finite set of weights Ω independent of s. For each $s \ge 1$ it gives at least $q^s/(c_0 s)$ pairwise nonisomorphic bricks B of k-dimension at most sb_0, each stable at some $\theta \in \Omega$. Choose such a weight for each brick. Each coordinate of $[B]$ lies between 0 and sb_0, so there are at most $|\Omega|(sb_0+1)^n$ pairs $(\theta, [B])$. For sufficiently large s the number of bricks exceeds this bound, and two nonisomorphic bricks have the same pair. □

2.3. **Presilting representatives on fixed projective terms.** Both presilting generation of thick subcategories (Proposition 7.5) and the rank-two subdivision (Lemma 8.1) compare differentials on fixed projective terms. For the orbit of a projective presentation and its normal space, see Derksen and Fei [29, Sections 2 and 3]. We write $\Lambda^F = \Lambda\otimes_k F$ for extension of scalars to a field extension F/k.

Lemma 2.9. *Let $Q^{-1}, Q^0 \in \operatorname{proj}\Lambda$, and let U be a two-term presilting complex of class $[Q^0]-[Q^{-1}]$. Then U has a representative of the form $Q^{-1} \to Q^0$.*

Proof. Let $A \xrightarrow{u} B$ be a minimal representative of U, so that u lies in the radical. Since U is presilting, $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(U, U[1]) = 0$, which means $\operatorname{Hom}_\Lambda(A,B) = u\operatorname{End}_\Lambda(A) + \operatorname{End}_\Lambda(B)u$; hence every morphism $A \to B$ belongs to the radical, so A and B share no indecomposable direct summand. Comparing multiplicities of indecomposable projectives in $[B]-[A] = [Q^0]-[Q^{-1}]$ gives $Q^{-1} \cong A\oplus Q$ and $Q^0 \cong B\oplus Q$ for a common projective module Q. Adjoining the contractible summand $Q \xrightarrow{1} Q$ yields the desired representative. □

On the line joining a presilting differential to any other differential on the same terms, the complex over the rational function field is isomorphic to the presilting one. Specializing to the other differential can only increase the dimensions of Hom to and from each fixed two-term complex.

Lemma 2.10. *Let Λ be a finite-dimensional algebra over an arbitrary field k, let $Q^{-1}, Q^0 \in \mathsf{proj}\,\Lambda$, and let $u, \sigma \in \mathcal{V} = \mathrm{Hom}_\Lambda(Q^{-1}, Q^0)$ with C_u presilting. Put $p_s := u + s(\sigma - u) \in \mathcal{V} \otimes_k k[s]$. Then there exist automorphisms $\alpha \in \mathrm{Aut}_{\Lambda^{k(s)}}(Q^{-1} \otimes_k k(s))$ and $\beta \in \mathrm{Aut}_{\Lambda^{k(s)}}(Q^0 \otimes_k k(s))$ such that $\beta p_s = u\alpha$. Moreover, for every two-term complex $V \in \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ and every $i \in \mathbb{Z}$,*

$$\dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(C_u, V[i]) \le \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(C_\sigma, V[i]),$$
$$\dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(V, C_u[i]) \le \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(V, C_\sigma[i]).$$

Proof. For a differential p on the given projective terms, consider the map

$$\mathrm{End}_\Lambda(Q^{-1}) \oplus \mathrm{End}_\Lambda(Q^0) \longrightarrow \mathrm{Hom}_\Lambda(Q^{-1}, Q^0), \qquad (a, b) \longmapsto pa - bp.$$

Since a chain map $C_p \to C_p[1]$ is a morphism $Q^{-1} \to Q^0$ and the null-homotopic chain maps are precisely those of the form $pa - bp$, the cokernel of this map is $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(C_p, C_p[1])$ over any field (compare [29, Sections 2 and 3]). For $p = p_s$ over $k[s]$, specializing at $s = 0$ recovers the map for the presilting differential u, which is surjective. A full-rank minor therefore remains nonzero over $k(s)$, so C_{p_s} is presilting over $k(s)$. Since C_{p_s} has the same class in $K_0(\mathsf{proj}\,\Lambda^{k(s)})$ as the constant complex $C_u \otimes_k k(s)$, Lemma 2.1(2) shows that the two complexes are isomorphic in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda^{k(s)})$. Decomposing each complex over $k(s)$ as the direct sum of a minimal two-term complex and a contractible complex of the form $1_R\colon R \to R$, we obtain a chain isomorphism between the minimal summands from the homotopy equivalence. Their degree-zero projective terms are therefore isomorphic, so cancellation inside $Q^0 \otimes_k k(s)$ yields an isomorphism between the two contractible summands as well. Taking the direct sum of these chain isomorphisms gives automorphisms α and β with $\beta p_s = u\alpha$.

We use semicontinuity for projective presentations as in Derksen and Fei [29, Section 3]. To deduce the Hom inequalities, we extend V to $V_{k[s]} := V \otimes_k k[s]$ and form the Hom complexes $\mathrm{Hom}^\bullet(C_{p_s}, V_{k[s]}[i])$ and $\mathrm{Hom}^\bullet(V_{k[s]}, C_{p_s}[i])$ over $\Lambda \otimes_k k[s]$. In each complex, specializing at $s = 1$ (where $p_1 = \sigma$) cannot increase the ranks of the differential matrices relative to their ranks over $k(s)$; consequently, the cohomology dimension at C_σ is at least its dimension over $k(s)$. By the $k(s)$-isomorphism $C_{p_s} \cong C_u \otimes_k k(s)$ just proved and flat base change, the latter dimension equals the Hom dimension at C_u. □

3. Hom spaces, semibricks, and compact torsion classes

We now connect E-finiteness to the module-theoretic condition (K_0-Hom), semibricks, and compact torsion classes. In Subsection 3.1, we show that any pair of nonzero modules of equal dimension vector violating (K_0-Hom) yields a finite semibrick with linearly dependent dimension vectors whose members are simultaneously stable at one integral weight; combined with Proposition 2.3(2), this proves (E-fin) ⇒ (K_0-Hom) and controls finite semibricks under (K_0-Hom). Subsection 3.2 completes the proof of Theorem 3.2 and records two consequences.

3.1. Extraction and stabilization of semibricks.

We call a finite semibrick $\mathcal{S}$ *dependent* if its dimension vectors $([B])_{B \in \mathcal{S}}$ are linearly dependent over $\mathbb{Q}$, and *minimally dependent* if it is dependent and every proper subset has $\mathbb{Q}$-linearly independent dimension vectors. Over an algebraically closed field, Mousavand and Paquette [52, Proposition B] characterize the existence of infinitely many bricks of one dimension vector by the existence, for some $d \neq 0$, of two Hom-orthogonal orbits in a single irreducible component of $\mathsf{rep}(\Lambda, d)$. Theorem 3.1(1) below applies over any field to any two nonzero modules of equal dimension vector with a vanishing Hom space in one direction, without requiring them to lie in the same component.

Theorem 3.1. *Let Λ be a finite-dimensional algebra over a field k.*

(1) *If there exist nonzero $M, N \in \mathsf{mod}\,\Lambda$ with $[M] = [N]$ and $\mathrm{Hom}_\Lambda(M, N) = 0$, then there exist $\theta \in K_0(\mathsf{proj}\,\Lambda)$ and a finite dependent semibrick $\mathcal{S} \subseteq \mathsf{mod}\,\Lambda$ whose members are all θ-stable.*

(2) *Assume* (K_0-Hom). *Then every finite semibrick* $\mathcal{S} \subseteq \mathsf{mod}\,\Lambda$ *has* $\mathbb{Q}$*-linearly independent classes* $([B])_{B\in\mathcal{S}}$ *and admits a weight* $\theta \in K_0(\mathsf{proj}\,\Lambda)$ *for which every* $B \in \mathcal{S}$ *is* θ*-stable.*

Proof. (1) We divide the proof into two steps: first extracting a dependent semibrick, and then stabilizing a minimal one at a common integral weight.

Step 1: Extracting a dependent semibrick. Mousavand and Paquette extract a semibrick from pairwise Hom-orthogonal modules by taking images of endomorphisms of least nonzero rank [52, proof of Proposition 3.1]. Here a single Hom space vanishes, and we retain a relation among the dimension vectors. We show that any pair (M, N) of nonzero objects in $\mathsf{mod}\,\Lambda$ with $[M] = [N]$ and $\mathrm{Hom}_\Lambda(M, N) = 0$ yields a finite dependent semibrick whose members are subquotients of indecomposable summands of $M \oplus N$. Let $\mathcal{Q}$ be the collection of all ordered pairs (X, Y) of nonzero objects of $\mathsf{mod}\,\Lambda$ such that $[X] = [Y]$, $\mathrm{Hom}_\Lambda(X, Y) = 0$, and every indecomposable direct summand of X or Y is a subquotient of an indecomposable summand of M or N. Choose $(X, Y) \in \mathcal{Q}$ minimizing $\sum_Z \ell(Z)^2$, summed over the indecomposable summands of $X \oplus Y$ with multiplicities.

We claim that $\mathrm{Hom}_\Lambda(Y, X) = 0$ as well. Suppose to the contrary that there exists a nonzero morphism $f\colon C \to B$ from an indecomposable summand C of Y to an indecomposable summand B of X. Write $X = B \oplus X'$ and $Y = C \oplus Y'$. Since $\mathrm{Hom}_\Lambda(X, Y) = 0$, the objects X and Y share no indecomposable direct summand, so f is not an isomorphism. Thus $0 < \ell(\mathrm{Im}\, f) < \max(\ell(B), \ell(C))$. Set $X_1 := \mathrm{Coker}\, f \oplus X'$ and $Y_1 := \ker f \oplus Y'$. Every indecomposable summand of $X_1 \oplus Y_1$ remains a subquotient of an indecomposable summand of $M \oplus N$. Subtracting $[\mathrm{Im}\, f]$ from both sides of $[X] = [Y]$ gives $[X_1] = [Y_1]$. The canonical maps $X \twoheadrightarrow X_1$ and $Y_1 \hookrightarrow Y$ induce an injection $\mathrm{Hom}_\Lambda(X_1, Y_1) \hookrightarrow \mathrm{Hom}_\Lambda(X, Y) = 0$. If $X_1 = 0$ or $Y_1 = 0$, then $[X_1] = [Y_1] = 0$, so $X_1 = Y_1 = 0$ because only the zero module has zero dimension vector; hence $\ker f = \mathrm{Coker}\, f = 0$, contradicting that f is not an isomorphism. Thus $(X_1, Y_1) \in \mathcal{Q}$. Writing $a = \ell(\ker f)$, $b = \ell(\mathrm{Im}\, f) > 0$, and $c = \ell(\mathrm{Coker}\, f)$, we have $\ell(B) = b + c$ and $\ell(C) = a + b$, so

$$\ell(B)^2 + \ell(C)^2 = (b + c)^2 + (a + b)^2 > a^2 + c^2 = \ell(\ker f)^2 + \ell(\mathrm{Coker}\, f)^2,$$

and decomposing $\ker f$ and $\mathrm{Coker}\, f$ into indecomposable summands can only decrease the sum of squares further. Thus replacing (X, Y) by (X_1, Y_1) strictly decreases $\sum \ell(Z)^2$, contradicting minimality. Hence $\mathrm{Hom}_\Lambda(Y, X) = 0$.

Next, let $f\colon B \to B'$ be a nonzero nonisomorphism between indecomposable summands on the same side (allowing $B = B'$). By the symmetry of $\mathrm{Hom}_\Lambda(X, Y) = 0 = \mathrm{Hom}_\Lambda(Y, X)$, we may assume that B and B' are summands of X. There are two exhaustive cases:

- *Case 1:* f *is not an epimorphism.* We replace one copy of B' in X by $\mathrm{Im}\, f \oplus \mathrm{Coker}\, f$ to form X_1. Because $\mathrm{Im}\, f$ and $\mathrm{Coker}\, f$ are quotients of B and B', respectively, we retain $\mathrm{Hom}_\Lambda(X_1, Y) = 0$, so $(X_1, Y) \in \mathcal{Q}$.
- *Case 2:* f *is an epimorphism.* Then $\ker f \neq 0$, and we replace one copy of B in X by $\ker f \oplus B'$ to form X_1. Because $\ker f$ and B' are subobjects of B and B', respectively, we retain $\mathrm{Hom}_\Lambda(Y, X_1) = 0$, so $(Y, X_1) \in \mathcal{Q}$.

In either case, $[X_1] = [X] = [Y]$ and one indecomposable summand of length $a + b$ $(a, b \geq 1)$ is replaced by two objects of lengths a and b. Since $(a+b)^2 > a^2+b^2$, this strictly decreases $\sum \ell(Z)^2$ and contradicts minimality. Therefore the distinct indecomposable summands of $X \oplus Y$ form a finite semibrick $\mathcal{S}$. Since X and Y share no indecomposable summand, $[X] = [Y]$ is a nontrivial linear relation among $([B])_{B\in\mathcal{S}}$, so $\mathcal{S}$ is dependent.

Step 2: Joint stabilization. By Step 1 applied to (M, N), finite dependent semibricks exist in $\mathsf{mod}\,\Lambda$. Among all minimally dependent finite semibricks in $\mathsf{mod}\,\Lambda$, choose $\mathcal{S}$ minimizing lexicographically the pair

$$\lambda(\mathcal{S}) := \big(\max_{B\in\mathcal{S}} \ell(B),\ \#\{B \in \mathcal{S} \mid \ell(B) = m\}\big), \qquad m := \max_{B\in\mathcal{S}} \ell(B).$$

Pick one member $B^* \in \mathcal{S}$ of length m. Because $\mathcal{S}$ is minimally dependent, $[B^*]$ lies in $\mathrm{span}_\mathbb{Q}\{[B] \mid B \in \mathcal{S} \setminus \{B^*\}\}$, so

$$H := \mathrm{span}_\mathbb{Q}\{[B] \mid B \in \mathcal{S}\} = \mathrm{span}_\mathbb{Q}\{[B] \mid B \in \mathcal{S} \setminus \{B^*\}\}.$$

Let $\mathcal{L}$ be the finite set of vectors $[L] \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}$ as L ranges over all proper nonzero subobjects $0 \neq L \subsetneq B$ of members $B \in \mathcal{S}$ (finiteness holds because $0 \leq [L] \leq [B]$ coordinatewise). For each $v \in \mathcal{L}$, choose a representative subobject L_v with $[L_v] = v$. By (1.1), a positive multiple of any integral linear functional on dimension vectors is represented by an integral projective weight. It therefore suffices to find $\xi \in \mathbb{Z}^n$ with $\xi \cdot [B] = 0$ for all $B \in \mathcal{S}$ and $\xi \cdot v < 0$ for all $v \in \mathcal{L}$.

Applying the rational Farkas alternative to the images $\bar{v} \in \mathbb{Q}^n/H$ of the vectors $v \in \mathcal{L}$, either there exists a $\mathbb{Q}$-linear functional $\bar{\xi}$ on $\mathbb{Q}^n/H$ with $\bar{\xi}(\bar{v}) \leq -1$ for each $v \in \mathcal{L}$, in which case pulling $\bar{\xi}$ back to $\mathbb{Q}^n$ and clearing denominators produces the desired $\xi \in \mathbb{Z}^n$, or else there exist nonnegative rational numbers q_v, not all zero, with $\sum_{v \in \mathcal{L}} q_v \bar{v} = 0$. Lifting this relation to $\mathbb{Q}^n$ and clearing denominators gives nonnegative integers a_v ($v \in \mathcal{L}$), not all zero, and integers c_B ($B \in \mathcal{S} \setminus \{B^*\}$) such that

$$\sum_{v \in \mathcal{L}} a_v [L_v] = \sum_{B \in \mathcal{S} \setminus \{B^*\}} c_B [B].$$

Moving the terms with $c_B < 0$ to the left-hand side gives

$$\sum_{c_B > 0} c_B [B] = \sum_{v \in \mathcal{L}} a_v [L_v] + \sum_{c_B < 0} (-c_B)[B].$$

The right-hand side is nonzero because some $a_v > 0$ and $[L_v] \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}$. Put $M' := \bigoplus_{c_B > 0} B^{c_B}$ and $N' := (\bigoplus_{v \in \mathcal{L}} L_v^{a_v}) \oplus (\bigoplus_{c_B < 0} B^{-c_B})$. Then $[M'] = [N'] \neq 0$. Distinct members of $\mathcal{S}$ are Hom-orthogonal, and a nonzero morphism from $B_1 \in \mathcal{S}$ to a proper subobject of $B_2 \in \mathcal{S}$ would compose with the inclusion to give a nonzero nonisomorphism $B_1 \to B_2$, impossible even when $B_1 = B_2$. Hence $\mathrm{Hom}_\Lambda(M', N') = 0$.

Applying Step 1 to (M', N') yields a finite dependent semibrick $\mathcal{S}'$, from which we pass to a minimally dependent subset $\mathcal{S}'' \subseteq \mathcal{S}'$. Every indecomposable summand of $M' \oplus N'$ is either a member of $\mathcal{S} \setminus \{B^*\}$ or a submodule of some L_v of length less than m. Hence a subquotient of such a summand has length at most m, with equality only if it is isomorphic to a member of $\mathcal{S} \setminus \{B^*\}$ of length m. Thus either the maximum length drops in $\mathcal{S}''$ or fewer isomorphism classes attain it. This contradicts the minimality of $\lambda(\mathcal{S})$. Therefore the desired functional $\xi \in \mathbb{Z}^n$ exists, and since $\xi \cdot [L] < 0$ for every proper nonzero subobject L of each member of $\mathcal{S}$, every member of $\mathcal{S}$ is stable at the corresponding integral projective weight.

(2) Assume (K_0-Hom), and let $\mathcal{S} \subseteq \mathsf{mod}\,\Lambda$ be any finite semibrick. If $\sum_{B \in \mathcal{S}} c_B [B] = 0$ were a nontrivial integral relation, then both positive and negative coefficients would occur because $[B] \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}$, so $X := \bigoplus_{c_B > 0} B^{c_B}$ and $Y := \bigoplus_{c_B < 0} B^{-c_B}$ would satisfy $[X] = [Y] \neq 0$ and $\mathrm{Hom}_\Lambda(X, Y) = 0$, a contradiction. Thus $([B])_{B \in \mathcal{S}}$ is $\mathbb{Q}$-linearly independent.

To find a weight at which every member of $\mathcal{S}$ is stable, we apply the Farkas alternative of Step 2 with $H = \mathrm{span}_{\mathbb{Q}}\{[B] \mid B \in \mathcal{S}\}$ and all of $\mathcal{S}$ in place of $\mathcal{S} \setminus \{B^*\}$. If no $\xi \in \mathbb{Z}^n$ vanishes on $\{[B] \mid B \in \mathcal{S}\}$ and is strictly negative on all proper nonzero subobjects of members of $\mathcal{S}$, then the second alternative produces nonzero objects $M', N' \in \mathsf{mod}\,\Lambda$ with $[M'] = [N']$ and $\mathrm{Hom}_\Lambda(M', N') = 0$, contradicting the hypothesis. □

3.2. **(K_0-Hom) and compact torsion classes.** To pass from (K_0-Hom) to functorial finiteness of compact torsion classes, we use Asai's characterization [11, Theorem 1.5]: over any field, a torsion class $\mathcal{T} \in \mathsf{tors}\,\Lambda$ is functorially finite if and only if $\mathcal{T}$ is bicompact and $(\mathcal{T}, \mathcal{T}^\perp)$ is numerically disjoint. For a compact torsion class, (K_0-Hom) yields numerical disjointness, while the polyhedral cone spanned by the classes of quotients of a generator yields cocompactness.

Theorem 3.2. *Over any field, in the notation of Theorem A(I),* (E-fin) *is equivalent to each of* (K_0-Hom)*,* (num-disj)*,* (comp-tors)*,* (fin-sbrick)*, and* (n-sbrick)*.*

Proof. We first derive (K_0-Hom) and all remaining forward implications from E-finiteness, and then refute each condition for an E-infinite algebra using Proposition 2.8.

(E-fin) ⇒ (K_0-Hom): Suppose $M, N \in \mathsf{mod}\,\Lambda$ satisfy $[M] = [N] \neq 0$ and $\mathrm{Hom}_\Lambda(M, N) = 0$. Theorem 3.1(1) gives a finite dependent semibrick stable at one integral weight θ. Under E-finiteness, we have $\theta \in C^\circ(U)$ for some $U \in \mathsf{2\text{-}psilt}\,\Lambda$, contradicting the $\mathbb{Q}$-linear independence of the classes of simple objects of $\mathcal{W}_\theta$ in Proposition 2.3(2).

(K_0-Hom) $\Rightarrow$ (fin-sbrick) and (num-disj): By Theorem 3.1(2), (K_0-Hom) implies that every finite semibrick has $\mathbb{Q}$-linearly independent classes in $K_0(\mathsf{mod}\,\Lambda) \cong \mathbb{Z}^n$, so every semibrick has cardinality at most n; in particular, (fin-sbrick) holds. Moreover, if $(\mathcal{T}, \mathcal{F})$ is any torsion pair in $\mathsf{mod}\,\Lambda$ and $X \in \mathcal{T}$, $Y \in \mathcal{F}$ satisfy $[X] = [Y] \neq 0$, then $\mathrm{Hom}_\Lambda(X, Y) = 0$, contradicting ($\mathrm{K}_0$-Hom); hence (num-disj) holds.

(K_0-Hom) $\Rightarrow$ (n-sbrick): Suppose the semibrick $\mathcal{S} = \{B_1, \dots, B_n\}$ has n members, so by the linear independence just proved $\{[B_1], \dots, [B_n]\}$ is a $\mathbb{Q}$-basis of $K_0(\mathsf{mod}\,\Lambda)_\mathbb{Q}$. For any simple module $S \in \mathsf{mod}\,\Lambda$, write $m[S] = \sum_{j=1}^n c_j[B_j]$ with $m \in \mathbb{Z}_{>0}$ and $c_j \in \mathbb{Z}$, and set

$$X := \bigoplus_{c_j > 0} B_j^{c_j}, \qquad Y := S^m \oplus \bigoplus_{c_j < 0} B_j^{-c_j}.$$

Then $[X] = [Y] \neq 0$. Because the bricks B_j occurring in X are Hom-orthogonal to those occurring in Y, applying (K_0-Hom) to (X, Y) yields an index j with $c_j > 0$ and a nonzero (hence surjective) morphism $B_j \twoheadrightarrow S$. Similarly, applying (K_0-Hom) to (Y, X) yields an index j' with $c_{j'} > 0$ and a nonzero (hence injective) morphism $S \hookrightarrow B_{j'}$. The composite $B_j \twoheadrightarrow S \hookrightarrow B_{j'}$ is nonzero; since $\mathcal{S}$ is a semibrick, we must have $j = j'$, and this composite is then a nonzero endomorphism of the brick B_j, hence an isomorphism. Thus $B_j \cong S$. As S was arbitrary and $|\mathcal{S}| = n$, it follows that $\mathcal{S} = \{S_1, \dots, S_n\}$, proving (n-sbrick).

(E-fin) + (K_0-Hom) + (num-disj) $\Rightarrow$ (comp-tors): Let $\mathcal{T} = \mathsf{T}(M)$ be a compact torsion class ($M \in \mathsf{mod}\,\Lambda$). Following Asai's cone argument [11, Lemmas 4.5 and 4.6], consider the rational polyhedral cone

$$Q := \mathrm{cone}\{[X] \in K_0(\mathsf{mod}\,\Lambda)_\mathbb{R} \mid X \text{ is a quotient of } M\} \subseteq K_0(\mathsf{mod}\,\Lambda)_\mathbb{R}.$$

Since M has only finitely many quotient dimension vectors, Weyl–Minkowski duality allows us to write $Q = \bigcap_{i=1}^r \{v \in K_0(\mathsf{mod}\,\Lambda)_\mathbb{R} \mid \langle \eta_i, v\rangle \geq 0\}$ for finitely many integral weights $\eta_1, \dots, \eta_r \in K_0(\mathsf{proj}\,\Lambda)$, using (1.1) to realize positive multiples of rational facet normals as integral projective weights. We claim that

$$\mathcal{T} = \bigcap_{i=1}^r \overline{\mathcal{T}}_{\eta_i}. \tag{3.1}$$

The class of every quotient of M lies in Q, hence has nonnegative η_i-value for every i; because each $\overline{\mathcal{T}}_{\eta_i}$ is a torsion class, we have $\mathcal{T} = \mathsf{T}(M) \subseteq \bigcap_{i=1}^r \overline{\mathcal{T}}_{\eta_i}$. Conversely, let $X \in \bigcap_{i=1}^r \overline{\mathcal{T}}_{\eta_i}$ and set $Z := X/t_\mathcal{T}(X) \in \mathcal{T}^\perp$. Since each $\overline{\mathcal{T}}_{\eta_i}$ is closed under quotients, $Z \in \bigcap_{i=1}^r \overline{\mathcal{T}}_{\eta_i}$, so $[Z] \in Q$. If $Z \neq 0$, clearing denominators in a conic representation of $[Z]$ gives positive integers $m, c_1, \dots, c_s$ and quotients $Y_1, \dots, Y_s$ of M with $m[Z] = \sum_{j=1}^s c_j[Y_j]$. Then $Y := \bigoplus_{j=1}^s Y_j^{c_j} \in \mathcal{T}$ and $Z^m \in \mathcal{T}^\perp$ satisfy $[Y] = [Z^m] \neq 0$ and $\mathrm{Hom}_\Lambda(Y, Z^m) = 0$, contradicting ($\mathrm{K}_0$-Hom). Thus $Z = 0$ and $X \in \mathcal{T}$, proving (3.1).

By E-finiteness and Proposition 2.5(1), each $\overline{\mathcal{T}}_{\eta_i}$ is cocompact, so $\overline{\mathcal{T}}_{\eta_i} = {}^\perp N_i$ for some $N_i \in \mathsf{mod}\,\Lambda$. Then $\mathcal{T} = \bigcap_{i=1}^r {}^\perp N_i = {}^\perp(\bigoplus_{i=1}^r N_i)$ is cocompact as well, hence bicompact. Since $(\mathcal{T}, \mathcal{T}^\perp)$ is numerically disjoint by (num-disj), we conclude from Asai [11, Theorem 1.5] that $\mathcal{T} \in \mathsf{f\text{-}tors}\,\Lambda$.

(fin-sbrick), (n-sbrick), (K_0-Hom), (num-disj), (comp-tors) $\Rightarrow$ (E-fin): Suppose Λ is E-infinite. The pairwise nonisomorphic bricks stable at one weight in Proposition 2.8(1) form an infinite semibrick, refuting (fin-sbrick). Because there are only n simple modules, this semibrick contains a nonsimple brick; any n members including that brick form an n-element semibrick distinct from $\{S_1, \dots, S_n\}$, refuting (n-sbrick).

For the remaining conditions, Proposition 2.8(2) gives two nonisomorphic bricks B_1, B_2 with $[B_1] = [B_2]$ that are stable at a common integral weight θ. Being distinct simple objects of $\mathcal{W}_\theta$, they satisfy $\mathrm{Hom}_\Lambda(B_1, B_2) = 0$, which refutes ($\mathrm{K}_0$-Hom). Since $B_1 \in \mathsf{T}(B_1)$ and $B_2 \in B_1^\perp = \mathsf{T}(B_1)^\perp$, the torsion pair $(\mathsf{T}(B_1), B_1^\perp)$ refutes (num-disj). Finally, the compact torsion class $\mathsf{T}(B_1)$ cannot be functorially finite: otherwise $\mathsf{T}(B_1) = \mathsf{Fac}\,H^0(T)$ for some $T \in \mathsf{2\text{-}silt}\,\Lambda$ by [1, Theorems 2.7 and 3.2] and [27, Section 4.1], contradicting Lemma 2.4 for $B_1 \in \mathsf{T}(B_1)$ and $B_2 \in \mathsf{T}(B_1)^\perp$. □

For the chamber count in Theorem 5.2 we need a stabilizing weight for each finite-dimensional brick.

Corollary 3.3. *Assume* (K_0-Hom). *Then every* $B \in \mathsf{brick}\,\Lambda$ *is stable at some integral weight.*

Proof. Apply Theorem 3.1(2) to the singleton semibrick $\{B\}$. □

Over an algebraically closed field Tacchetti [65, Theorem 2.18(1)] transfers a family of bricks of one dimension vector along an exact fully faithful functor. Condition (K_0-Hom) gives the following restriction result over any field.

Corollary 3.4. *Let* $F\colon \mathsf{mod}\,\Gamma \to \mathsf{mod}\,\Lambda$ *be an exact fully faithful functor between the module categories of finite-dimensional algebras over* k. *If* Λ *is E-finite, then so is* Γ. *In particular, every quotient of an E-finite algebra is E-finite, by restriction of scalars along the quotient map.*

Proof. Suppose $X, Y \in \mathsf{mod}\,\Gamma$ satisfy $[X] = [Y] \neq 0$ in $K_0(\mathsf{mod}\,\Gamma)$. Since F is exact, it induces a homomorphism $K_0(\mathsf{mod}\,\Gamma) \to K_0(\mathsf{mod}\,\Lambda)$, so $[FX] = [FY]$, and because F is faithful, we have $FX \neq 0$. By Theorem 3.2, Λ satisfies (K_0-Hom), so $\mathrm{Hom}_\Lambda(FX, FY) \neq 0$; by fullness of F, we obtain $\mathrm{Hom}_\Gamma(X, Y) \neq 0$. Thus Γ satisfies (K_0-Hom), and hence Γ is E-finite by Theorem 3.2. □

4. Dimension vectors in Grothendieck monoids of torsion classes

Let $\mathcal{E} \subseteq \mathsf{mod}\,\Lambda$ be an extension-closed subcategory closed under finite direct sums and direct summands, equipped with the exact structure inherited from $\mathsf{mod}\,\Lambda$. Its *Grothendieck monoid* $M(\mathcal{E})$ is the commutative monoid generated by the isomorphism classes $\mu(X)$ of objects $X \in \mathcal{E}$ subject to $\mu(0) = 0$ and the relations $\mu(Y) = \mu(X) + \mu(Z)$ for every short exact sequence $0 \to X \to Y \to Z \to 0$ with all terms in $\mathcal{E}$. Its group completion is the Grothendieck group $K_0(\mathcal{E})$ (see [31, Definition 3.2]).

Two modules in a torsion class $\mathcal{T}$ with the same dimension vector need not represent the same element of $M(\mathcal{T})$. Using pushouts along the nonzero morphisms guaranteed by (K_0-Hom), we show that equality holds for every torsion class precisely when Λ is E-finite.

Theorem 4.1. *Over any field,* (E-fin) *holds if and only if for every torsion class* $\mathcal{T}$ *the map* $M(\mathcal{T}) \to K_0(\mathsf{mod}\,\Lambda)$, $\mu(X) \mapsto [X]$, *is injective* (mon-dim). *It suffices to test* $\mathcal{T} = \overline{\mathcal{T}}_\theta$ *for integral weights* θ.

Proof. (E-fin) ⇒ (mon-dim): Assume that Λ is E-finite, so (K_0-Hom) holds by Theorem 3.2. Fix a torsion class $\mathcal{T} \in \mathsf{tors}\,\Lambda$. Given $Y, L \in \mathcal{T}$ and a quotient $Y \twoheadrightarrow R$ with kernel K such that $[K] = [L]$ (note that $R \in \mathcal{T}$ because $\mathcal{T}$ is closed under quotients), we prove $\mu(Y) = \mu(L) + \mu(R)$ in $M(\mathcal{T})$ by induction on $\ell(L)$. For $L = 0$ we have $K = 0$ and $Y \cong R$, so this is immediate. Otherwise (K_0-Hom) gives a nonzero morphism $f\colon K \to L$. Taking the pushout of the inclusion $K \hookrightarrow Y$ along f in $\mathsf{mod}\,\Lambda$ yields a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \hookrightarrow & Y & \twoheadrightarrow & R & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & L & \hookrightarrow & Y_f & \twoheadrightarrow & R & \longrightarrow & 0. \end{array}$$

By the pushout property, we have $\ker g = \ker f$ inside Y and $\mathrm{Coker}\, g \cong \mathrm{Coker}\, f$. Thus $Y' := \mathrm{Im}\, g \cong Y/\ker f$ and $\mathrm{Coker}\, f$ are quotients of objects in $\mathcal{T}$ and hence lie in $\mathcal{T}$. The bottom row exhibits Y_f as an extension of R by L in $\mathcal{T}$, so $Y_f \in \mathcal{T}$, and both the bottom row and $0 \to Y' \to Y_f \to \mathrm{Coker}\, f \to 0$ give relations in $M(\mathcal{T})$. Since $[K] = [L]$, exactness of $0 \to \ker f \to K \xrightarrow{f} L \to \mathrm{Coker}\, f \to 0$ gives $[\ker f] = [\mathrm{Coker}\, f]$, and since $f \neq 0$ we have $\ell(\mathrm{Coker}\, f) < \ell(L)$. Applying the induction hypothesis to the quotient $Y \twoheadrightarrow Y'$ with kernel $\ker f$ and to $\mathrm{Coker}\, f \in \mathcal{T}$, we obtain

$$\mu(Y) = \mu(\mathrm{Coker}\, f) + \mu(Y') = \mu(Y_f) = \mu(L) + \mu(R).$$

Taking $R = 0$ shows that $[Y] = [L]$ implies $\mu(Y) = \mu(L)$ for all $Y, L \in \mathcal{T}$. Because every element of $M(\mathcal{T})$ is represented by a single module (as $\mu(X_1) + \mu(X_2) = \mu(X_1 \oplus X_2)$), the map $M(\mathcal{T}) \to K_0(\mathsf{mod}\,\Lambda)$ is injective.

(mon-dim) ⇒ (E-fin): If Λ is E-infinite, Proposition 2.8(2) gives two nonisomorphic θ-stable bricks B_1, B_2 of the same dimension vector at some integral weight θ. Both belong to $\overline{\mathcal{T}}_\theta$, while every proper nonzero submodule of either brick has strictly negative θ-weight and thus lies outside $\overline{\mathcal{T}}_\theta$.

Thus neither brick can be the middle term of a short exact sequence with two nonzero ends in $\overline{\mathcal{T}}_\theta$. In particular, B_1 and B_2 are nonisomorphic simple objects of the exact category $\overline{\mathcal{T}}_\theta$, so $\mu(B_1) \neq \mu(B_2)$ in $M(\overline{\mathcal{T}}_\theta)$ by [31, Proposition 3.6(2)], since $S \mapsto \mu(S)$ is injective on isomorphism classes of simple objects of an exact category. □

A commutative monoid is cancellative if $a + c = b + c$ implies $a = b$. To compare (mon-dim) with cancellativity we identify the group completion inside $K_0(\mathsf{mod}\,\Lambda)$.

Proposition 4.2. *Let Λ be a finite-dimensional algebra over any field. For a torsion class $\mathcal{T}$ or a torsion-free class $\mathcal{F}$, let*

$$I_{\mathcal{E}} := \{j \mid S_j \text{ is a composition factor of some } X \in \mathcal{E}\}, \qquad \mathcal{E} = \mathcal{T} \text{ or } \mathcal{F}.$$

The canonical map $\iota_{\mathcal{E}} \colon K_0(\mathcal{E}) \to K_0(\mathsf{mod}\,\Lambda)$, $[X]_{\mathcal{E}} \mapsto [X]$, identifies $K_0(\mathcal{E})$ with $\bigoplus_{j \in I_{\mathcal{E}}} \mathbb{Z}[S_j]$.

Proof. We first treat a torsion-free class $\mathcal{F} \in \mathsf{torf}\,\Lambda$. Put $\Gamma := \Lambda/\operatorname{ann}_\Lambda(\mathcal{F})$, where $\operatorname{ann}_\Lambda(\mathcal{F}) = \bigcap_{X \in \mathcal{F}} \operatorname{ann}_\Lambda(X)$. By the descending chain condition on subspaces of Λ, this intersection is already achieved by finitely many objects of $\mathcal{F}$. Their direct sum $V \in \mathcal{F}$ is faithful over Γ, so evaluating on a k-basis of V embeds Γ into a finite direct sum of copies of V in $\mathcal{F}$. Thus $\mathcal{F}$ contains Γ, every module in $\mathsf{proj}\,\Gamma$, and every submodule of a module in $\mathsf{proj}\,\Gamma$.

For $X \in \mathsf{mod}\,\Gamma$, choose a short exact sequence $0 \to L \to P \to X \to 0$ with $P \in \mathsf{proj}\,\Gamma$; then $L, P \in \mathcal{F}$. Schanuel's lemma gives independence of $[P]_{\mathcal{F}} - [L]_{\mathcal{F}}$ from the presentation, and the horseshoe lemma gives additivity. The resulting homomorphism $[X] \mapsto [P]_{\mathcal{F}} - [L]_{\mathcal{F}}$ is inverse to the map $K_0(\mathcal{F}) \to K_0(\mathsf{mod}\,\Gamma)$ induced by inclusion, as in the degree-zero resolution theorem [57, Section 4] (compare [31, Lemma 5.7(2)]). Since $\Gamma \in \mathcal{F} \subseteq \mathsf{mod}\,\Gamma$, the simple Γ-modules are precisely the simple Λ-modules S_j with $j \in I_{\mathcal{F}}$, so $K_0(\mathsf{mod}\,\Gamma) = \bigoplus_{j \in I_{\mathcal{F}}} \mathbb{Z}[S_j]$ inside $K_0(\mathsf{mod}\,\Lambda)$. The torsion case follows by k-duality. □

The description of the group completion in Proposition 4.2 turns the dimension-vector criterion into a cancellation criterion.

Corollary 4.3. *The algebra Λ is E-finite if and only if every torsion class in $\mathsf{mod}\,\Lambda$ has a cancellative Grothendieck monoid. The same equivalence holds for torsion-free classes.*

Proof. For any torsion class $\mathcal{T}$, Proposition 4.2 identifies its group completion $K_0(\mathcal{T})$ with a subgroup of $K_0(\mathsf{mod}\,\Lambda)$. Thus the dimension-vector map $M(\mathcal{T}) \to K_0(\mathsf{mod}\,\Lambda)$ is injective precisely when the canonical map $M(\mathcal{T}) \to K_0(\mathcal{T})$ is injective, that is, when $M(\mathcal{T})$ is cancellative. The torsion assertion therefore follows from Theorem 4.1, and k-duality gives the assertion for torsion-free classes since E-finiteness passes to Λ^{op} by Lemma 2.1(3). □

Remark 4.4. Over an algebraically closed field the Kronecker algebra has a torsion-free class with infinitely many indecomposables and a noncancellative monoid [31, Section 8.3.1]. Problem 9.2 of [31] asks whether a cotilting module U over an Artin algebra A can have a perpendicular category $\{X \in \mathsf{mod}\,A \mid \operatorname{Ext}^i_A(X, U) = 0 \text{ for all } i > 0\}$ with finitely many indecomposables and a noncancellative monoid, and whether such an example can be a torsion-free class. Corollary 4.3 excludes such a torsion-free class when the ambient algebra is E-finite, in particular when it is representation-finite.

For an ICE-closed or IKE-closed subcategory $\mathcal{E}$ of an E-finite algebra, the dimension-vector map $M(\mathcal{E}) \to K_0(\mathsf{mod}\,\Lambda)$ is likewise injective. Indeed, for such a subcategory (closed under images, extensions and, respectively, cokernels or kernels), [33, Corollary 3.5] and its dual realize $\mathcal{E}$ as a torsion or torsion-free class in a wide subcategory $\mathcal{W}$. Under E-finiteness, $\mathcal{W}$ is a module category by Theorem 5.1 below and inherits E-finiteness by Corollary 3.4. Hence $M(\mathcal{E}) \to K_0(\mathcal{W})$ is injective by Theorem 4.1 and duality, and $K_0(\mathcal{W}) \to K_0(\mathsf{mod}\,\Lambda)$ is injective by independence of its simple classes (Theorem 3.1(2)). Thus $M(\mathcal{E}) \to K_0(\mathsf{mod}\,\Lambda)$ is injective and $M(\mathcal{E})$ is cancellative.

5. Wide subcategories, brick labels, and brick counts

For an E-finite algebra, the simples of each wide subcategory are stable at a common weight by Theorem 3.1(2). We first describe the wide subcategory within the corresponding semistable category, then count bricks by composition length and endolength.

5.1. **Functorially finite wide subcategories.** Write $\mathcal{T} = \mathsf{Fac}\, H^0(T)$ and $\mathcal{T}' = \mathsf{Fac}\, H^0(T')$ for the functorially finite torsion classes associated to $T, T' \in \mathsf{2\text{-}silt}\,\Lambda$. A cover $\mathcal{T}' \subsetneq \mathcal{T}$ (a Hasse arrow $\mathcal{T} \to \mathcal{T}'$) corresponds to a mutation of two-term silting complexes by Adachi, Iyama, and Reiten [1, Corollary 2.34], with the arbitrary-field formulation in [27, Section 4.1]. By Demonet, Iyama, and Jasso [26, Corollary 6.7(b)], the cones $C(T)$ and $C(T')$ then share a codimension-one face $C(U)$ with $|U| = n - 1$. Among completions of U, the co-Bongartz completion has the smallest torsion class and the Bongartz completion the largest. These give the endpoints $\mathcal{T}' = \mathsf{Fac}\, H^0(U)$ and $\mathcal{T} = {}^{\perp}H^{-1}(\nu U)$ of the interval of torsion classes associated to U by Demonet et al. [27, Theorems 4.4 and 4.12].

The *brick label* is the unique brick in $\mathcal{T} \cap (\mathcal{T}')^{\perp} = \mathcal{J}(U)$ up to isomorphism, hence its unique simple object [27, Theorem 3.3(b) and Definition 3.5]. Under the correspondence with mutations this agrees with Asai's labeling [8, Section 1.2], [27, Proposition 4.9] and with the minimal extending module of Barnard, Carroll, and Zhu [14, Theorem 1.2]. Condition (brick-label) requires that every finite-dimensional brick occur as such a label.

Theorem 5.1. *Over any field, in the notation of Theorem A(I),* (E-fin) *is equivalent to* (wide)*, to* (wide$_1$)*, and to* (brick-label)*. When Λ is E-finite every wide subcategory has the form $\mathcal{J}(U')$ for a two-term presilting complex U'.*

Proof. We first realize every wide subcategory as a perpendicular category and specialize to $\mathcal{W} = \mathsf{Filt}(B)$ to obtain the brick labels. For the converse it suffices to test $\mathsf{Filt}(B)$, using Lemma A.5 in Appendix A.

(E-fin) $\Rightarrow$ (wide): Assume Λ is E-finite. The simple objects of a wide subcategory $\mathcal{W}$ form a semibrick $\mathcal{S}$, and $\mathcal{W} = \mathsf{Filt}(\mathcal{S})$ by Ringel [60, Section 1.2]. By Theorem 3.2, $\mathcal{S}$ is finite, so Theorem 3.1(2) gives an integral weight θ at which every member of $\mathcal{S}$ is stable. Choose $U \in \mathsf{2\text{-}psilt}\,\Lambda$ with $\theta \in C^{\circ}(U)$. In $\mathcal{W}_\theta = \mathcal{J}(U) \simeq \mathsf{mod}\,\Gamma_U$, the members of $\mathcal{S}$ are simple, and the modules filtered by them form a Serre subcategory: a module in $\mathsf{mod}\,\Gamma_U$ belongs to it precisely when all of its composition factors lie in $\mathcal{S}$.

Let $P = e\Gamma_U$ be a sum of projective covers of the simple Γ_U-modules outside $\mathcal{S}$, choosing one primitive idempotent for each such simple so that they are mutually orthogonal. Exactness of $\mathrm{Hom}_{\Gamma_U}(P, -)$ identifies this Serre subcategory with $P^{\perp} = \mathsf{mod}(\Gamma_U/\Gamma_U e\Gamma_U) = \mathcal{J}_{\Gamma_U}(P[1])$. Because the inclusion $\mathsf{mod}(\Gamma_U/\Gamma_U e\Gamma_U) \hookrightarrow \mathsf{mod}\,\Gamma_U$ has both a left and a right adjoint, $\mathcal{W}$ is functorially finite in $\mathcal{J}(U)$ and hence in $\mathsf{mod}\,\Lambda$ by Lemma 2.2. This proves (wide).

For the perpendicular description of $\mathcal{W}$ we give the argument of Buan and Hanson for lifting a Serre subcategory [19, Lemma 4.4] in two-term notation. To realize $\mathcal{W}$ as a perpendicular category over Λ, let V be the direct sum of the complementary summands of the Bongartz completion T_U corresponding to the indecomposable summands of P under Jasso's equivalence [42, Section 3.1 and Theorem 3.8], [27, Theorem 4.12(b)]. As in the proof of Proposition 2.3(2), for every $X \in \mathcal{J}(U)$ we have $\mathrm{Hom}_\Lambda(X, H^{-1}(\nu V)) = 0$, and every morphism $H^0(V) \to X$ factors through $\overline{V} := H^0(V)/\operatorname{tr}_{H^0(U)}(H^0(V))$, which corresponds to P under $\mathcal{J}(U) \simeq \mathsf{mod}\,\Gamma_U$. Hence $\mathrm{Hom}_\Lambda(H^0(V), X) \cong \mathrm{Hom}_{\mathcal{J}(U)}(\overline{V}, X)$, so the additional perpendicular conditions for $U \oplus V$ cut out precisely $P^{\perp}$ inside $\mathcal{J}(U)$; thus $\mathcal{W} = \mathcal{J}(U \oplus V)$.

(E-fin) $\Rightarrow$ (brick-label): For $\mathcal{W} = \mathsf{Filt}(B)$ with $B \in \mathsf{brick}\,\Lambda$, there is a single simple object, so $|U \oplus V| = n - 1$ by Lemma 2.2. The two silting completions of $U \oplus V$ yield a cover $\mathcal{T}' \subsetneq \mathcal{T}$ in $\mathsf{f\text{-}tors}\,\Lambda$ with $\mathcal{T} \cap (\mathcal{T}')^{\perp} = \mathcal{W}$ by [1, Corollary 2.34] and [27, Theorems 4.4 and 4.12].

(wide), (brick-label) $\Rightarrow$ (wide$_1$) $\Rightarrow$ (E-fin): The implication (wide) $\Rightarrow$ (wide$_1$) is immediate. For any cover $\mathcal{T}' \subsetneq \mathcal{T}$ in $\mathsf{f\text{-}tors}\,\Lambda$, the corresponding silting mutation has a common almost complete summand U, and $\mathcal{T} \cap (\mathcal{T}')^{\perp} = \mathcal{J}(U)$ is functorially finite by Lemma 2.2; hence (brick-label) $\Rightarrow$ (wide$_1$). Conversely, if Λ is E-infinite, Lemma A.5 gives a stable brick B for which $\mathsf{Filt}(B)$ is neither covariantly nor contravariantly finite. Hence (wide$_1$) implies (E-fin). □

5.2. **Stability walls and brick counts.** We count bricks of fixed dimension vector by stability chambers and bricks of fixed endolength by their submodule polytopes.

Theorem 5.2. *Over any field, in the notation of Theorem A(I),* (E-fin) *is equivalent to* (poly-brick)*. For an E-finite algebra of rank $n \geq 2$, the number of bricks of composition length at most m is at most*

$$2(n-1)(m+1)^{n(n-1)} \qquad (m \geq 1).$$

Proof. (K_0-Hom) $\Rightarrow$ (poly-brick): Under (E-fin), Theorem 3.2 gives (K_0-Hom). In rank zero there are no bricks. When $n = 1$, there is a unique simple module S up to isomorphism; any nonzero nonsimple module $M \in \mathsf{mod}\,\Lambda$ admits a nonzero composite $M \twoheadrightarrow S \hookrightarrow M$ that is not an isomorphism, so every brick in $\mathsf{mod}\,\Lambda$ is simple and $|\mathsf{brick}\,\Lambda| = 1$. Assume $n \geq 2$, and fix a nonzero vector $v \in \mathbb{N}^n$ of total length $\sum_{i=1}^n v_i \leq m$. By Corollary 3.3, every brick B with $[B] = v$ admits a weight $\theta \in v^\perp \subseteq K_0(\mathsf{proj}\,\Lambda)_\mathbb{R}$ at which B is θ-stable. Consider the central hyperplane arrangement in the $(n-1)$-dimensional space $v^\perp$ consisting of all hyperplanes $H_u := \{\theta \in v^\perp \mid \langle \theta, u\rangle = 0\}$ for integral vectors $0 \leq u \leq v$ $(u \neq 0, v)$ such that $H_u \neq v^\perp$. Let s be the number of distinct hyperplanes in this arrangement; then $s \leq (m+1)^n$.

For a brick B of class v, θ-stability is defined by finitely many strict submodule inequalities inside $v^\perp$. Its stable locus is open in $v^\perp$ and nonempty by Corollary 3.3, so it meets the complement of the arrangement. No proper nonzero submodule $L \subsetneq B$ can have class proportional to v, for then $\theta(L)$ would vanish throughout $v^\perp$, contradicting stability at any weight in $v^\perp$. Every proper nonzero submodule of B therefore defines a hyperplane in the arrangement, so each defining inequality has constant sign on any chamber of the arrangement. The stable locus therefore contains an entire chamber. By (K_0-Hom), distinct bricks of class v cannot be stable at a common weight, so the number of bricks of class v is at most the number of chambers.

For integers $s, t \geq 1$, let $R(s,t)$ denote the maximum number of chambers determined by s central hyperplanes in $\mathbb{R}^t$ (allowing repetitions). Counting the regions cut by the last hyperplane gives $R(s,t) \leq R(s-1,t) + R(s-1,t-1)$ for $s, t \geq 2$. Together with $R(1,t) = R(s,1) = 2$ for $s, t \geq 1$, induction yields

$$R(s,t) \leq 2\sum_{j=0}^{t-1} \binom{s-1}{j} \leq 2ts^{t-1} \qquad (s,t \geq 1),$$

while for $s = 0$ there is a single chamber. Applying this with $t = n-1$ shows that each class v carries at most $2(n-1)(m+1)^{n(n-2)}$ bricks (which also bounds the case $s = 0$). Since there are at most $(m+1)^n$ nonzero vectors $v \in \mathbb{N}^n$ of total length at most m, the total number of bricks of composition length at most m is at most $2(n-1)(m+1)^{n(n-1)}$.

(poly-brick) $\Rightarrow$ (E-fin): Suppose Λ is E-infinite. If k is infinite, the family in Proposition 2.8(1) has constant composition length, refuting (poly-brick). Over $\mathbb{F}_q$, Proposition A.3(2) gives at least $\exp(cm)$ bricks up to isomorphism of composition length at most m for some $c > 0$ and all sufficiently large m, again refuting (poly-brick). □

To count bricks of bounded endolength over an arbitrary field, we pass to the submodule polytope

$$\mathcal{P}(B) = \mathrm{conv}\{[L] \mid L \subseteq B\} \subseteq K_0(\mathsf{mod}\,\Lambda)_\mathbb{R}. \tag{5.1}$$

For any brick $B \in \mathsf{brick}\,\Lambda$ over a field k, set $\Delta := \mathrm{End}_\Lambda(B)$, $a := \dim_k \Delta$, and $r := \mathrm{endolen}(B) = \dim_\Delta B$, and for each submodule $L \subseteq B$ put

$$w(L) := \big(\dim_k \mathrm{Hom}_\Lambda(P_1, L), \ldots, \dim_k \mathrm{Hom}_\Lambda(P_n, L)\big) \in \mathbb{N}^n.$$

By (1.1), we have $w_i(L) = (\dim_k \mathrm{End}_\Lambda(S_i))\,[L : S_i]$, so $w(L)$ differs from $[L]$ by a fixed invertible diagonal linear transformation. In the projective basis, identifying $\theta = \sum_{i=1}^n \theta_i [P_i] \in K_0(\mathsf{proj}\,\Lambda)_\mathbb{R}$ with $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ gives $\theta(L) = \theta \cdot w(L)$. Define the *normalized submodule polytope* of B by

$$\widetilde{P}(B) := \mathrm{conv}\left\{ \frac{1}{a} w(L) \;\middle|\; L \subseteq B \right\} \subseteq \mathbb{R}^n.$$

Reflection $x \mapsto [B] - x$ carries $\mathcal{P}(B)$ to the polytope of quotient classes studied by Cao [20, Theorem 3.2(ii) and (iii)].

Proposition 5.3. *Let Λ be a finite-dimensional algebra over an arbitrary field k.*

(1) *For every brick $B \in \mathsf{brick}\,\Lambda$ of endolength r, the vertices of $\widetilde{P}(B)$ lie in the integer grid $\{0,1,\dots,r\}^n$. In particular, at most $2^{(r+1)^n}$ normalized submodule polytopes occur at endolength r.*

(2) *The conditions* (E-fin)*,* (brick-discrete)*, and* (endo-stab) *are equivalent.*

Proof. (1) We use the largest subobject maximizing a supporting functional, as in Baumann, Kamnitzer, and Tingley [15, Proposition 3.2]. Fix any linear functional $\lambda\colon \mathbb{R}^n \to \mathbb{R}$, and let $L \subseteq B$ be a submodule of maximal composition length among those maximizing $\lambda(w(L))$. For any other maximizer $L' \subseteq B$, exactness of $0 \to L\cap L' \to L\oplus L' \to L+L' \to 0$ gives $w(L+L')+w(L\cap L') = w(L)+w(L')$, so both $L+L'$ and $L\cap L'$ also maximize $\lambda\circ w$. By the maximality of $\ell(L)$, we have $L+L' = L$, hence $L' \subseteq L$. Thus L is the unique largest maximizer of $\lambda\circ w$, so L is preserved by every automorphism of B and hence is invariant under $\Delta = \mathrm{End}_\Lambda(B)$ (as every nonzero element of Δ is an automorphism). Consequently, $\mathrm{Hom}_\Lambda(P_i, L)$ is a left Δ-subspace of $\mathrm{Hom}_\Lambda(P_i, B) \cong Be_i \subseteq B$, where $P_i = e_i\Lambda$. Thus $w_i(L)/a = \dim_\Delta \mathrm{Hom}_\Lambda(P_i, L)$ is an integer between 0 and $\dim_\Delta B = r$. Because $\widetilde{P}(B)$ and the convex hull of $\{w(L)/a \mid L \subseteq B \text{ is } \Delta\text{-invariant}\}$ have the same support function in every direction λ, they coincide. Every vertex of $\widetilde{P}(B)$ therefore lies in $\{0,1,\dots,r\}^n$; since a polytope is determined by its set of vertices, at most $2^{(r+1)^n}$ normalized submodule polytopes occur at endolength r.

(2) The coordinatewise maximum of $\widetilde{P}(B)$ is $w(B)/a$, and $\widetilde{P}(B)$ determines the stability wall via

$$\Theta_B = \Big\{\theta \in K_0(\mathsf{proj}\,\Lambda)_\mathbb{R} \;\Big|\; \theta\cdot\frac{w(B)}{a} = 0 \text{ and } \max_{x\in\widetilde{P}(B)} \theta\cdot x = 0\Big\}.$$

Suppose Λ is E-finite. Let $B_1, B_2 \in \mathsf{brick}\,\Lambda$ have the same normalized polytope, and put $a_i = \dim_k \mathrm{End}_\Lambda(B_i)$. Then their coordinatewise maxima $w(B_1)/a_1$ and $w(B_2)/a_2$ coincide, so $[B_1]$ and $[B_2]$ are positive rational multiples of one another, and their stability walls Θ_{B_1} and Θ_{B_2} coincide; write Θ for this common wall. By Corollary 3.3, each B_i admits a stabilizing weight $\theta_i \in \Theta$. For any proper nonzero submodule $L \subsetneq B_i$, the linear functional $\theta \mapsto \theta(L)$ is nonpositive on Θ and strictly negative at θ_i, so its zero locus meets Θ in a proper face of Θ. Consequently, every proper nonzero submodule of B_1 or B_2 has strictly negative weight throughout the relative interior of Θ, so B_1 and B_2 are simultaneously θ-stable at any θ in the relative interior of Θ. If $B_1 \not\cong B_2$, then $\mathrm{Hom}_\Lambda(B_1, B_2) = 0$; choosing positive integers c_1, c_2 with $c_1[B_1] = c_2[B_2]$ then gives $[B_1^{c_1}] = [B_2^{c_2}] \neq 0$ and $\mathrm{Hom}_\Lambda(B_1^{c_1}, B_2^{c_2}) = 0$, contradicting ($\mathrm{K}_0$-Hom). Thus each normalized submodule polytope arises from at most one brick up to isomorphism. By (1), there are at most $2^{(r+1)^n}$ bricks of endolength r, proving (brick-discrete) and hence (endo-stab).

Conversely, if Λ is E-infinite, Proposition 2.8(1) gives infinitely many nonisomorphic bricks of one endolength stable at the same weight, contradicting both (endo-stab) and (brick-discrete). □

The uniqueness of the brick realizing a normalized submodule polytope in the proof of Proposition 5.3(2) can also be deduced from a result of Cao [20, Theorem 3.2(ii) and proof of (iii)]. Indeed, if $B_1, B_2 \in \mathsf{brick}\,\Lambda$ have the same normalized polytope and $a_i = \dim_k \mathrm{End}_\Lambda(B_i)$, then undoing the diagonal coordinate change $[L] \mapsto w(L)$ and clearing denominators yields $\mathcal{P}(B_1^{a_2}) = \mathcal{P}(B_2^{a_1})$ because submodule polytopes add under direct sums. Since these two polytopes share the same coordinatewise maximum vector, reflection carries them to identical quotient polytopes. When Λ is E-finite, the compact torsion classes $\mathsf{T}(B_1^{a_2}) = \mathsf{T}(B_1)$ and $\mathsf{T}(B_2^{a_1}) = \mathsf{T}(B_2)$ are functorially finite by Theorem 3.2. Passing to a Morita equivalent basic algebra preserves subobjects and composition multiplicities, so [20, Theorem 3.2(ii)] gives $\mathsf{T}(B_1) = \mathsf{T}(B_2)$, and a left-finite brick is uniquely determined by the torsion class it generates.

Over an infinite field, Theorem 5.2 and Proposition 2.8(1) relate E-finiteness directly to counting bricks by dimension vector.

Corollary 5.4. *Over any field,* (E-fin) *implies* (dim-brick)*, which implies* (dim-stab)*. Over an infinite field these three conditions of Theorem A are equivalent.*

Proof. The polynomial bound of Theorem 5.2 gives (E-fin) ⇒ (dim-brick) ⇒ (dim-stab). If k is infinite and Λ is E-infinite, Proposition 2.8(1) gives infinitely many pairwise nonisomorphic bricks of a single class stable at one integral weight. □

Mousavand and Paquette [52, Conjecture 2.4 and the discussion following it] formulate Weyman's conjecture over an algebraically closed field in terms of stable bricks. This formulation extends to infinite fields.

Corollary 5.5. *Suppose k is infinite. If infinitely many nonisomorphic bricks in $\mathsf{mod}\,\Lambda$ have k-dimension r for some $r \geq 1$, then for some integral weight θ and some integer $s \geq 1$ there are infinitely many nonisomorphic θ-stable bricks of k-dimension s.*

Proof. Only finitely many dimension vectors occur in k-dimension r, so (dim-brick) fails. By Corollary 5.4, (dim-stab) fails as well. The common dimension vector of the resulting stable bricks determines their k-dimension. □

6. Representation varieties, open loci, and King moduli spaces

Throughout this section, let k be algebraically closed. By Morita equivalence we may assume that Λ is basic; then $\mathsf{rep}(\Lambda, d)$ parametrizes module structures on the vertex vector spaces k^{d_i}, and the orbits $\mathcal{O}_M$ of $\mathrm{GL}(d) = \prod_i \mathrm{GL}_{d_i}(k)$ correspond to isomorphism classes of modules of dimension vector d. Our stability inequality $\theta(L) \leq 0$ for submodules corresponds to King's parameter $-\theta$ in [44, Definition 1.1]. In this setting we deduce Corollary C and establish the geometric characterizations of E-finiteness in Theorem A(III).

Recall from [21, Definition 3] that Λ is *multiplicity-free* if, for every $d \in \mathbb{N}^n$ and every irreducible component of $\mathsf{rep}(\Lambda, d)$, the semi-invariants of each character span a space of dimension at most one. Chindriş, Kinser, and Weyman [21, Proposition 9 and Theorem 17] proved the forward implication of Corollary C in their setting and established the converse for tame algebras there.

Proof of Corollary C. By Corollary 5.4, Λ is Schur-representation-finite if and only if it is E-finite. Mousavand and Paquette [54, Theorem B] prove that E-finiteness is equivalent to multiplicity-freeness over every algebraically closed field. □

We next combine the τ-perpendicular description of wide subcategories in Theorem 5.1 with upper semicontinuity to characterize E-finiteness by openness of wide subcategories and brick orbits, and use the linear independence of stable classes in Proposition 2.3(2) to characterize it via the closed points of King's semistable moduli spaces.

Theorem 6.1. *Over an algebraically closed field, each of* (moduli-pt), (open-brick), *and* (open-wide) *in Theorem A is equivalent to* (E-fin).

Proof. We establish the implications involving Zariski-open loci first, using Theorems 3.2 and 5.1, and then prove the assertion on closed points.

(E-fin) ⇒ (open-wide): Mousavand and Paquette [49, Theorem 6.1] used perpendicular categories and upper semicontinuity to show that a minimal extending brick of a functorially finite torsion class has an open orbit. Under E-finiteness, the same argument applies to every wide subcategory. Fix $d \in \mathbb{N}^n$. If $\mathcal{W} \subseteq \mathsf{mod}\,\Lambda$ is a wide subcategory, then $\mathcal{W} = \mathcal{J}(U) = H^0(U)^{\perp} \cap {}^{\perp}H^{-1}(\nu U)$ for some $U \in \mathsf{2\text{-}psilt}\,\Lambda$ by Theorem 5.1. By upper semicontinuity of Hom dimensions on $\mathsf{rep}(\Lambda, d)$ (Crawley-Boevey and Schröer [24, Lemma 4.3]), the locus $\{X \in \mathsf{rep}(\Lambda, d) \mid \mathrm{Hom}_\Lambda(H^0(U), X) = 0 = \mathrm{Hom}_\Lambda(X, H^{-1}(\nu U))\}$ is Zariski-open in $\mathsf{rep}(\Lambda, d)$.

(open-wide) ⇒ (open-brick) ⇒ (dim-brick) ⇒ (dim-stab) ⇒ (E-fin): Let $B \in \mathsf{brick}\,\Lambda$ with $d = [B]$. Any module $X \in \mathsf{Filt}(B)$ with $[X] = d$ must have $\mathsf{Filt}(B)$-length one, so $X \cong B$. Hence

$$\{X \in \mathsf{rep}(\Lambda, d) \mid X \in \mathsf{Filt}(B)\} = \mathcal{O}_B. \tag{6.1}$$

Thus (open-wide) implies that $\mathcal{O}_B$ is Zariski-open in $\mathsf{rep}(\Lambda, d)$, giving (open-brick). There are only finitely many irreducible components of $\mathsf{rep}(\Lambda, d)$, and any two opens meeting the same component meet each other. Thus only finitely many disjoint nonempty open orbits can occur, so (open-brick) ⇒ (dim-brick). The implication (dim-brick) ⇒ (dim-stab) is immediate, and (dim-stab) ⇒ (E-fin) follows from Proposition 2.8(1).

(E-fin) ⇔ (moduli-pt): Assume that Λ is E-finite. By King [44, Propositions 3.2(i) and 4.2], closed points of $\mathcal{M}^{\mathrm{ss}}_\theta(\Lambda, d)$ correspond to isomorphism classes of θ-polystable modules (finite direct sums

of θ-stable modules) of dimension vector d. Since Λ is E-finite, the integral weight θ lies in $\Sigma(\Lambda)$. Because the classes of the θ-stable modules are $\mathbb{Q}$-linearly independent by Asai and Iyama [12, Proposition 5.6] (see Proposition 2.3(2)), the dimension vector d uniquely determines all summand multiplicities. Hence $\mathcal{M}^{\mathrm{ss}}_\theta(\Lambda, d)$ has at most one closed point, giving (moduli-pt) and in particular the finiteness in the GIT formulation of [45, Definition 4.4 and Conjecture 4.8]. Conversely, if Λ is E-infinite, Proposition 2.8(1) gives infinitely many pairwise nonisomorphic θ-stable modules of one dimension vector d, which give infinitely many distinct closed points of $\mathcal{M}^{\mathrm{ss}}_\theta(\Lambda, d)$. □

Remark 6.2. For a fixed nonzero dimension vector d, upper semicontinuity of endomorphism dimension [24, Lemma 4.3] implies that the brick locus $\{M \in \mathsf{rep}(\Lambda, d) \mid \dim_k \operatorname{End}_\Lambda(M) = 1\}$ is Zariski-open in $\mathsf{rep}(\Lambda, d)$. Because all brick orbits have the same dimension $\dim \mathrm{GL}(d) - 1$, each brick orbit is closed in the brick locus; consequently, if $\mathsf{rep}(\Lambda, d)$ contains only finitely many brick orbits, then each one is also open in the brick locus and hence in $\mathsf{rep}(\Lambda, d)$. Theorem 6.1 supplies the converse implication (open-brick) ⇒ (dim-brick). In [53, Definition 6.8], Mousavand and Paquette call Λ brick-discrete when every brick has an open orbit, which is condition (open-brick); over an algebraically closed field, this agrees both with (dim-brick) and with our endolength condition (brick-discrete), since endolength equals total k-dimension and each total dimension admits only finitely many dimension vectors. Finally, Asai's open bricks are those with open orbit in the whole representation variety, and [10, Theorem 1.1 and Corollary 1.3] already give (fin-sbrick) ⇒ (open-brick).

7. Generic modules, universal localizations, and ring epimorphisms

We now connect E-finiteness to infinite-dimensional modules, thick subcategories of two-term complexes, universal localizations, and ring epimorphisms.

A module of finite endolength is called *endofinite*. A *generic brick* is an infinite-dimensional endofinite brick. For $N \in \mathsf{Mod}\,\Lambda$, we write $\mathsf{cogen}\,N$ for the full subcategory of finite-dimensional modules that embed into a finite direct sum of copies of N.

Recall that a module is pure-injective if every pure embedding of it splits. Following Angeleri Hügel, Laking, and Sentieri [3, Proposition 5.1(2)], we call an indecomposable pure-injective module N a *grain* if its minimal injective copresentation $I^\bullet_N$ satisfies $\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(I^\bullet_N, I^\bullet_N[1]) = 0$. A grain is *generic* if it is infinite-dimensional and endofinite. For an infinite-dimensional indecomposable endofinite N we use the following equivalent criterion:

$$N \text{ is a } \textit{generic grain} \quad \Longleftrightarrow \quad \operatorname{Ext}^1_\Lambda(X, N) = 0 \text{ for every } X \in \mathsf{cogen}\,N.$$

Every endofinite module is pure-injective by Krause [46, Section 6.4, p. 46], and among indecomposable pure-injective modules the displayed Ext^1-vanishing characterizes grains by Schlegel [61, Remark 2.5(3), Theorem 3.5, and Section 5]. Although Schlegel defines $\mathsf{cogen}\,N$ using finite-length submodules of arbitrary products N^I, the two definitions agree: for any finite-length submodule $X \subseteq N^I$, the kernels of the coordinate projections intersect in zero, so finitely many of them already intersect in zero and embed X into a finite direct sum of copies of N.

For indecomposable modules, Crawley-Boevey [23, §9.6] characterized the existence of a generic module by the existence of infinitely many nonisomorphic indecomposable finite-length modules of one endolength. For bricks, Theorem A(I) gives the analogous equivalence. Subsection 7.1 excludes generic bricks over E-finite algebras by reducing the rank. Subsection 7.2 gives the converse and treats presilting generation and ring epimorphisms.

7.1. Generic bricks and reduction of rank.

For a generic brick, the action of its endomorphism division ring lets us use stability to reduce the rank of the algebra. For any division k-algebra Δ, let $\mathcal{C}_\Delta$ denote the category of (Δ, Λ)-bimodules M of finite left Δ-dimension $\dim_\Delta M < \infty$. In particular a generic brick G belongs to $\mathcal{C}_\Delta$ for $\Delta = \operatorname{End}_\Lambda(G)$, since $\dim_\Delta G$ is its endolength. Since every nonzero (Δ, Λ)-subbimodule has strictly positive left Δ-dimension, $\mathcal{C}_\Delta$ is a length abelian category. We equip $\mathcal{C}_\Delta$ with the additive class map

$$d_\Delta \colon K_0(\mathcal{C}_\Delta) \longrightarrow \mathbb{Z}^n, \qquad d_\Delta(M) := \big(\dim_\Delta \operatorname{Hom}_\Lambda(P_i, M)\big)_{i=1}^n \in \mathbb{Z}^n_{\geq 0}.$$

For every nonzero $M \in \mathcal{C}_\Delta$, at least one P_i admits a nonzero homomorphism to M, so $d_\Delta(M) \neq 0$. Each weight $\theta = \sum_{i=1}^n \theta_i[P_i] \in K_0(\mathsf{proj}\,\Lambda)_\mathbb{R}$ defines an additive function on $\mathcal{C}_\Delta$ by

$$\theta_\Delta(M) := \sum_{i=1}^n \theta_i \dim_\Delta \mathrm{Hom}_\Lambda(P_i, M).$$

A bimodule of weight zero is θ_Δ-semistable if all its subbimodules have nonpositive weight.

Let $U' = (Q^{-1} \xrightarrow{u'} Q^0)$ be a two-term presilting complex with basic version U. Its large perpendicular category is

$$\mathcal{X}_U := \{X \in \mathsf{Mod}\,\Lambda \mid \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U, X) = 0 = \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U, X[1])\}. \tag{7.1}$$

Over any field, the presilting presentation U of $H^0(U)$ defines a ring epimorphism whose fully faithful restriction of scalars has essential image $\mathcal{X}_U$ by Angeleri Hügel, Marks, and Vitória [4, Remark 3.1 and Proposition 3.3]. Let M_U be the module obtained in their Bongartz completion of $H^0(U)$ with respect to U; because the approximation used to form M_U is finite, the basic module associated with M_U is $H^0(T_U)$ by the silting correspondence and (2.4). By [4, Theorem 3.5], the target of the ring epimorphism is Morita equivalent to $\mathrm{End}_\Lambda(M_U)$ modulo the ideal of maps factoring through $\mathsf{Add}\,H^0(U)$. Passing from M_U to its basic version $H^0(T_U)$ preserves this quotient up to Morita equivalence (as the ideal on $H^0(T_U)$ consists of maps factoring through $\mathsf{add}\,H^0(U)$), and its basic algebra is the reduction algebra Γ_U in (2.5). Thus there is a k-linear exact equivalence $F\colon \mathcal{X}_U \xrightarrow{\sim} \mathsf{Mod}\,\Gamma_U$ that restricts to $\mathcal{J}(U) \xrightarrow{\sim} \mathsf{mod}\,\Gamma_U$, and the inclusion $\mathcal{X}_U \hookrightarrow \mathsf{Mod}\,\Lambda$ is exact.

Lemma 7.1. *Put $\theta = [U']$. Every θ_Δ-semistable $M \in \mathcal{C}_\Delta$ belongs to $\mathcal{X}_U$ as a right Λ-module.*

Proof. Consider the trace

$$L := \mathrm{tr}_{H^0(U')}(M) = \sum_{f \in \mathrm{Hom}_\Lambda(H^0(U'), M)} \mathrm{Im}\, f \subseteq M.$$

Because the left Δ-action commutes with the right Λ-action, L is a (Δ, Λ)-subbimodule of M. As a right Λ-module, L is a quotient of a direct sum $H^0(U')^{(I)}$. Applying $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U', -)$ to the truncation triangle $H^{-1}(U')[1] \to U' \to H^0(U') \to H^{-1}(U')[2]$ as in Proposition 2.7(1), the presilting condition gives

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U', H^0(U')[1]) = 0.$$

Since $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U', -[1]) = \mathrm{Coker}\,\mathrm{Hom}_\Lambda(u', -)$ is right exact and preserves direct sums, we obtain $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U', L[1]) = 0$. The exact sequence (2.1) for L is a sequence of finite-dimensional left Δ-vector spaces, so

$$\theta_\Delta(L) = \dim_\Delta \mathrm{Hom}_\Lambda(H^0(U'), L) \geq 0.$$

Since M is θ_Δ-semistable, we also have $\theta_\Delta(L) \leq 0$, whence $\mathrm{Hom}_\Lambda(H^0(U'), L) = 0$ and therefore $L = 0$; equivalently, $\mathrm{Hom}_\Lambda(H^0(U'), M) = 0$. Applying (2.1) to M together with $\theta_\Delta(M) = 0$ then yields

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(U', M[1]) = 0.$$

These two vanishings give $M \in \mathcal{X}_U$. □

To apply Lemma 2.4 in the proof of Lemma 7.3, we will prescribe finitely many torsion radicals.

Lemma 7.2. *Suppose Λ is E-finite. For every torsion class $\mathcal{T}$ and every finite family $X_1, \dots, X_s \in \mathsf{mod}\,\Lambda$, there is a functorially finite torsion class $\mathcal{T}_0 \subseteq \mathcal{T}$ with $t_{\mathcal{T}_0}(X_i) = t_\mathcal{T}(X_i)$ for each i.*

Proof. Set $\mathcal{T}_0 := \mathsf{T}(\bigoplus_{i=1}^s t_\mathcal{T}(X_i))$. Then $\mathcal{T}_0$ is compact, hence $\mathcal{T}_0 \in \mathsf{f\text{-}tors}\,\Lambda$ by Theorem 3.2, and since $t_\mathcal{T}(X_i) \in \mathcal{T}_0 \subseteq \mathcal{T}$, we have $t_{\mathcal{T}_0}(X_i) = t_\mathcal{T}(X_i)$ for every i. □

Lemma 7.3. *An E-finite algebra over any field has no generic brick.*

Proof. Suppose to the contrary that the assertion fails, and choose an E-finite algebra Λ of least rank n admitting a generic brick G. Put $\Delta := \mathrm{End}_\Lambda(G)$. Since G is a brick of finite endolength, it is finite-dimensional as a left Δ-space and hence belongs to $\mathcal{C}_\Delta$. Since rank zero gives the zero algebra, we have $n \geq 1$. We first separate the finite-dimensional quotients and submodules of G

by a weight, then find a nonzero rational weight for which G is semistable as a bimodule. The associated perpendicular category gives a counterexample of smaller rank.

Step 1: Separating finite-dimensional quotients and submodules. Let $Q_1, \dots, Q_a$ be nonzero finite-dimensional quotients of G, and let $N_1, \dots, N_b$ be nonzero finite-dimensional submodules of G. Any nonzero morphism $Q_i \to N_j$ would yield a nonzero endomorphism $G \twoheadrightarrow Q_i \to N_j \hookrightarrow G$ of finite-dimensional image, impossible because G is an infinite-dimensional brick. Thus each N_j lies in the right perpendicular of $\mathcal{T} := \mathsf{T}(\bigoplus_i Q_i)$. Applying Lemma 7.2 to $\mathcal{T}$ and the finite family $\{Q_i, N_j\}$, we obtain a functorially finite torsion class $\mathcal{T}_0 = \mathsf{Fac}\, H^0(T)$ ($T \in \mathsf{2\text{-}silt}\,\Lambda$) with $Q_i \in \mathcal{T}_0$ and $N_j \in \mathcal{T}_0^{\perp}$ for all i, j. Lemma 2.4 gives $\psi(Q_i) > 0$ and $\psi(N_j) < 0$ for every $\psi \in C^\circ(T)$.

On the unit sphere of $K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}}$, consider the closed subsets defined by $\eta(Q) \geq 0$ and $\eta(N) \leq 0$ as Q and N range over all finite-dimensional quotients and submodules of G, respectively. Every finite intersection of these closed subsets contains a normalized weight $\psi/\|\psi\|$ as above; by compactness, there exists a unit weight $\theta \in K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}}$ satisfying $\theta(Q) \geq 0$ and $\theta(N) \leq 0$ for all such Q and N.

Step 2: Passing to (Δ, Λ)-bimodules. We show that $\theta_\Delta(L) \leq 0$ for every (Δ, Λ)-subbimodule $L \subseteq G$ and $\theta_\Delta(G) = 0$. Take any rational weight $\alpha \in K_0(\mathsf{proj}\,\Lambda)_{\mathbb{Q}}$ with $\alpha_i \leq \theta_i$ for all i. Since $\dim_k \mathrm{Hom}_\Lambda(P_i, N) \geq 0$, we have

$$\alpha(N) \leq \theta(N) \leq 0$$

for every finite-dimensional submodule N of G. Clearing denominators and applying E-finiteness, we write a positive integer multiple $m\alpha = [U]$ for a two-term presilting complex $U \in \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$; by Lemma 2.1(1), α lies in the relative interior of the cone spanned by the distinct summands of U. Any nonzero morphism $H^0(U) \to L$ would have a nonzero finite-dimensional image $N \subseteq L \subseteq G$ lying in $\mathsf{Fac}\, H^0(U) = \mathcal{T}_\alpha$ (Proposition 2.3(1)), giving $\alpha(N) > 0$, a contradiction. Thus $\mathrm{Hom}_\Lambda(H^0(U), L) = 0$, and therefore the exact sequence (2.1) over Δ yields

$$m\alpha_\Delta(L) = -\dim_\Delta \mathrm{Hom}_{\mathsf{D(Mod}\,\Lambda)}(U, L[1]) \leq 0.$$

Dually, for any rational weight $\beta \in K_0(\mathsf{proj}\,\Lambda)_{\mathbb{Q}}$ with $\beta_i \geq \theta_i$ for all i, we have

$$\beta(Q) \geq \theta(Q) \geq 0$$

for every finite-dimensional quotient Q of G. Writing a positive integer multiple $m'\beta = [V]$ with $V \in \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ presilting, any nonzero morphism $G \to H^{-1}(\nu V)$ would have a nonzero finite-dimensional image $Q \in \mathsf{Sub}\, H^{-1}(\nu V) = \mathcal{F}_\beta$ with $\beta(Q) < 0$, a contradiction. By (2.2), we obtain $\mathrm{Hom}_{\mathsf{D(Mod}\,\Lambda)}(V, G[1]) = 0$, and therefore (2.1) yields

$$m'\beta_\Delta(G) = \dim_\Delta \mathrm{Hom}_\Lambda(H^0(V), G) \geq 0.$$

For each fixed subbimodule L, evaluation on its vector of Δ-dimensions is linear in the weight. Letting $\alpha \uparrow \theta$ and $\beta \downarrow \theta$ gives $\theta_\Delta(L) \leq 0$ for all (Δ, Λ)-subbimodules $L \subseteq G$ and $\theta_\Delta(G) = 0$.

Step 3: Reducing the rank. Because only finitely many integer vectors $d_\Delta(L) \in \mathbb{Z}^n_{\geq 0}$ occur as L ranges over (Δ, Λ)-subbimodules of G, the rational polyhedral cone

$$\left\{\eta \in K_0(\mathsf{proj}\,\Lambda)_{\mathbb{R}} \;\middle|\; \eta_\Delta(G) = 0 \text{ and } \eta_\Delta(L) \leq 0 \text{ for all } L \subseteq G\right\}$$

contains $\theta \neq 0$, and hence contains a nonzero integral weight $\eta \in K_0(\mathsf{proj}\,\Lambda) \setminus \{0\}$. Write $\eta = [U']$ for a nonzero two-term presilting complex U' with basic version $U \in \mathsf{2\text{-}psilt}\,\Lambda$. Since G is η_Δ-semistable, Lemma 7.1 gives $G \in \mathcal{X}_U$.

Since $U \neq 0$, Lemma 2.2 shows that Γ_U has rank $n - |U| < n$. Applying Corollary 3.4 to $\mathsf{mod}\,\Gamma_U \simeq \mathcal{J}(U) \hookrightarrow \mathsf{mod}\,\Lambda$ shows that Γ_U is E-finite.

Put $G' = F(G)$ and $P_\mathcal{J} = F^{-1}((\Gamma_U)_{\Gamma_U}) \in \mathcal{J}(U)$. Since F is a k-linear equivalence, it identifies $\mathrm{End}_{\Gamma_U}(G')$ with Δ and yields isomorphisms of left Δ-vector spaces

$$G' \cong \mathrm{Hom}_{\Gamma_U}(\Gamma_U, G') \cong \mathrm{Hom}_\Lambda(P_\mathcal{J}, G).$$

Evaluating on a finite set of Λ-generators of $P_\mathcal{J}$ embeds the last space into a finite direct sum of copies of G. Thus G' has finite left Δ-dimension. The k-dimension of G' is infinite because

the inverse equivalence preserves finite-dimensional modules. Thus G' is a generic brick over Γ_U, contradicting the minimality of n. □

7.2. **Generic grains, two-term thick subcategories, localizations, and ring epimorphisms.** We use presilting generators of thick subcategories to prove that universal localizations of E-finite algebras are finite-dimensional. For an E-infinite algebra we construct a generic grain from a lattice point outside the fan. A full additive subcategory of $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ is *thick* if it is closed under direct summands and satisfies two-out-of-three for every triangle in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ whose three terms lie in $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$. For U two-term presilting, let $\mathsf{thick}(U)$ be its thick closure in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. We write $\mathsf{thick}_2(U)$ for the intersection of this closure with $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$.

Following Garcia [38, Lemma 3.13 and Proposition 4.10], we prove presilting generation of a thick subcategory $\mathcal{H} \subseteq \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ by using approximation triangles to reduce to objects orthogonal to a maximal presilting complex in $\mathcal{H}$, extracting presilting summands from iterated self-extensions via the line of differentials in Subsection 2.3, and splitting a presentation triangle by nilpotence of a radical endomorphism. We first isolate a criterion for membership in $\mathsf{add}\,P$ based on the ordered extension decomposition of Aihara and Iyama [2, Propositions 2.13 and 2.17].

Lemma 7.4. *Let P be presilting in $K = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, and let $L \in \mathsf{thick}(P)$. If $\mathrm{Hom}_K(P, L[i]) = \mathrm{Hom}_K(L, P[i]) = 0$ for every $i > 0$, then $L \in \mathsf{add}\,P$.*

Proof. By Aihara and Iyama [2, Proposition 2.17], there is a retraction $L \xrightarrow{s} D \xrightarrow{r} L$ in K where D is an iterated extension of shifts of objects in $\mathsf{add}\,P$ in increasing order of shift, with the negative shifts first, then shift zero, then the positive shifts. Grouping these shifts yields distinguished triangles

$$D_{<0} \longrightarrow D \xrightarrow{q} D_{\geq 0} \longrightarrow D_{<0}[1], \qquad P_0 \longrightarrow D_{\geq 0} \longrightarrow D_{>0} \longrightarrow P_0[1]$$

with $P_0 \in \mathsf{add}\,P$, where the subscripts indicate the shifts occurring in each iterated extension. Since $\mathrm{Hom}_K(D_{<0}, L) = 0$, the morphism r factors through q. Since $\mathrm{Hom}_K(L, D_{>0}) = 0$, the composite qs factors through P_0. Hence $rs = 1_L$ factors through P_0, proving $L \in \mathsf{add}\,P$. □

Proposition 7.5. *Let Λ be an E-finite algebra over any field, and let $\mathcal{H}$ be a thick subcategory of $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$. Choose a basic presilting complex $U \in \mathcal{H}$ with the largest possible number of indecomposable summands. Then $\mathcal{H} = \mathsf{thick}_2(U)$.*

Proof. Write $K = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, $\mathcal{C} = \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$, and $E(X, Y) = \mathrm{Hom}_K(X, Y[1])$. Such a U exists because $0 \in \mathcal{H}$ and every basic two-term presilting complex has at most n indecomposable summands. We divide the proof into four steps.

Step 1: Reducing to $\mathcal{H} \cap \mathcal{Z} = \mathsf{add}\,U$. Put $\mathcal{Z} = \{Y \in \mathcal{C} \mid E(U, Y) = E(Y, U) = 0\}$. By the mapping cone construction, any distinguished triangle extending two objects of $\mathcal{C}$ has a two-term middle term, and the associated long exact Hom sequences show that $\mathcal{Z}$ is closed under extensions. Both $\mathcal{C}$ and $\mathcal{Z}$ are also closed under direct summands. For each $X \in \mathcal{C}$, we form distinguished triangles

$$X \longrightarrow X_1 \longrightarrow U^{d_1} \longrightarrow X[1], \qquad U^{d_2} \longrightarrow Y \longrightarrow X_1 \longrightarrow U^{d_2}[1],$$

whose connecting morphisms have components forming k-bases of $E(U, X)$ and $E(X_1, U)$, respectively. From the first triangle we obtain $E(U, X_1) = 0$ by the choice of basis and $E(U, U) = 0$; from the second we obtain $E(Y, U) = 0$ while preserving $E(U, Y) = 0$. Hence $Y \in \mathcal{Z}$. In any thick subcategory of $\mathcal{C}$ containing U, the two-out-of-three property shows that X belongs to the subcategory if and only if Y does.

For objects in $\mathcal{C}$, all Hom spaces in shifts $i \geq 2$ vanish. Applying Lemma 7.4 with $P = U$ therefore yields $\mathsf{thick}_2(U) \cap \mathcal{Z} = \mathsf{add}\,U$. For both $\mathcal{H}$ and $\mathsf{thick}_2(U)$, the triangles above reduce membership of X to membership of the same $Y \in \mathcal{Z}$. Thus it suffices to prove that $\mathcal{H} \cap \mathcal{Z} = \mathsf{add}\,U$.

Step 2: Realizing the class of Y in $\mathsf{add}\,U$. Let $Y \in \mathcal{H} \cap \mathcal{Z}$. If $[Y] = 0$, set $V = 0$; otherwise E-finiteness provides a two-term presilting complex V of class $[Y]$. Writing $Y = C_\sigma$, Lemma 2.9 gives a representative $V = C_u$ on the same projective terms. We show that V is a direct summand of an iterated extension of finitely many copies of Y over k.

Setting $q_t = \sigma + t(u-\sigma)$ (that is, $t = 1-s$ in Lemma 2.10), the complex C_{q_t} over $k(t)$ is isomorphic to $C_u \otimes_k k(t)$ as a complex over $\Lambda^{k(t)}$. Clearing denominators in this isomorphism and its inverse yields polynomial chain maps $f\colon C_u \otimes_k k[t] \to C_{q_t}$ and $h\colon C_{q_t} \to C_u \otimes_k k[t]$ such that $hf = t^j u_0(t)\, 1_{C_u \otimes_k k[t]}$ for some $j \geq 0$ and $u_0(t) \in k[t]$ with $u_0(0) \neq 0$. In the composite

$$C_u \longrightarrow C_u \otimes_k k[t]/(t^{j+1}) \xrightarrow{\bar{f}} C_{q_t}/t^{j+1}C_{q_t} \xrightarrow{\bar{h}} C_u \otimes_k k[t]/(t^{j+1}) \longrightarrow C_u,$$

where the first map includes constants and the last extracts the coefficient of t^j, the composite equals $u_0(0)\, 1_{C_u}$, which is invertible. Compare the finite thickening in the degeneration argument of Jensen, Su, and Zimmermann [43, Section 3] used in [38, proof of Proposition 4.10]. Here the coefficient map gives the retraction over k. The t-adic filtration of $C_{q_t}/t^{j+1}C_{q_t}$ has factors isomorphic to Y and splits in each degree over Λ, so it exhibits $C_{q_t}/t^{j+1}C_{q_t}$ as an iterated extension of copies of Y in $\mathcal{C}$. Dividing the coefficient map by $u_0(0)$ therefore exhibits $C_u \simeq V$ as a retract of this iterated extension. Consequently $V \in \mathcal{H} \cap \mathcal{Z}$. Because $U \oplus V$ is then presilting in $\mathcal{H}$, maximality of $|U|$ gives $V \in \operatorname{\mathsf{add}} U$ (which also holds when $V = 0$).

Step 3: A presentation in the Bongartz completion of U. As in the positive completion of Derksen and Fei [29, Section 5], we take a right $\operatorname{\mathsf{add}} U$-approximation $U_\Lambda \to \Lambda[1]$ and complete it to a distinguished triangle $\Lambda \to C \to U_\Lambda \to \Lambda[1]$. Let T be basic with $\operatorname{\mathsf{add}} T = \operatorname{\mathsf{add}}(U \oplus C)$. The approximation property gives $E(U, C) = 0$, and applying $\operatorname{Hom}_K(-, U[1])$ and $\operatorname{Hom}_K(-, C[1])$ to the triangle gives $E(C, U) = E(C, C) = 0$, using $E(\Lambda, X) = 0$ for all $X \in \mathcal{C}$. Since $\Lambda \in \operatorname{\mathsf{thick}}(T)$, the complex T is two-term silting. To identify T with the Bongartz completion T_U, observe that for any module M, applying $\operatorname{Hom}(-, M[1])$ to the triangle shows that $\operatorname{Hom}(U, M[1]) = 0$ implies $\operatorname{Hom}(C, M[1]) = 0$, since $\operatorname{Hom}(\Lambda, M[1]) = 0$. Conversely, $\operatorname{Hom}(T, M[1]) = 0$ implies $\operatorname{Hom}(U, M[1]) = 0$ since $U \in \operatorname{\mathsf{add}} T$. Thus T has the torsion class in (2.4). Applying $\operatorname{Hom}_K(-, Y[1])$ to the same triangle together with $E(U, Y) = 0$ also gives $E(T, Y) = 0$.

Now choose a right $\operatorname{\mathsf{add}} T$-approximation $T_0 \to Y$ and complete it to a distinguished triangle $T_1 \to T_0 \to Y \to T_1[1]$. The resulting long exact Hom sequences give $\operatorname{Hom}_K(T, T_1[i]) = 0$ for all $i > 0$ (using the approximation property for $i = 1$ and $\operatorname{Hom}_K(T, Y[j]) = 0$ for $j > 0$) as well as $\operatorname{Hom}_K(T_1, T[i]) = 0$ for all $i > 0$ (using $\operatorname{Hom}_K(T_0, T[i]) = 0 = \operatorname{Hom}_K(Y, T[i+1])$). Applying Lemma 7.4 with $P = T$ yields $T_1 \in \operatorname{\mathsf{add}} T$.

Canceling any isomorphism blocks of $T_1 \to T_0$ (splitting off summands of the form $X \xrightarrow{\sim} X$), we may assume that $T_1 \to T_0$ lies in the radical of $\operatorname{\mathsf{add}} T$ while keeping its cone isomorphic to Y. Because $[T_0] = [T_1] + [Y] = [T_1] + [V]$ and the classes of the indecomposable summands of T form a $\mathbb{Z}$-basis of $K_0(\operatorname{\mathsf{proj}} \Lambda)$, the multiplicities of all indecomposable summands of T in T_0 and $T_1 \oplus V$ coincide; thus $T_0 \cong T_1 \oplus V$, and setting $W := T_1 \in \operatorname{\mathsf{add}} T$ brings the triangle to the form

$$W \xrightarrow{(a,b)} W \oplus V \longrightarrow Y \longrightarrow W[1]. \tag{7.2}$$

Step 4: Extracting W and splitting (7.2). As in the case $[Y] = 0$ of Garcia's proof of [38, Proposition 4.10], we use nilpotence in $\operatorname{End}_K(W)$. Since (a, b) lies in the radical, a belongs to the radical of the finite-dimensional algebra $\operatorname{End}_K(W)$, so $a^m = 0$ for some $m \geq 1$. We compose m copies of (a, b), extending each successive morphism by the identity on the copies of V already present:

$$W \longrightarrow W \oplus V \longrightarrow W \oplus V^2 \longrightarrow \cdots \longrightarrow W \oplus V^m.$$

Each map $W \oplus V^j \to W \oplus V^{j+1}$ has cone isomorphic to Y. By the octahedral axiom, the cone of the m-fold composite is an iterated extension of copies of Y and therefore belongs to $\mathcal{H}$. On the other hand, the first component $W \to W$ of this composite is $a^m = 0$ in K, so the composite equals $(0, c)\colon W \to W \oplus V^m$ for some morphism $c\colon W \to V^m$. Its cone is isomorphic to $W \oplus \operatorname{cone}(c)$, so W is a direct summand of this cone and hence $W \in \mathcal{H}$. Because $W \in \operatorname{\mathsf{add}} T$, the direct sum $U \oplus W$ is presilting in $\mathcal{H}$, so maximality of $|U|$ gives $W \in \operatorname{\mathsf{add}} U$.

Finally, since $Y \in \mathcal{Z}$ and $W \in \operatorname{\mathsf{add}} U$, we have $E(Y, W) = 0$, so the triangle (7.2) splits. Thus Y is a direct summand of $W \oplus V$, and hence $Y \in \operatorname{\mathsf{add}} U$. □

Both the construction of a generic grain and the proof of (univ-loc) rely on the module category of a universal localization. For a ring R and a set Φ of morphisms between finitely generated projective right R-modules, let R_Φ denote the universal localization: $R \to R_\Phi$ is initial among ring maps $R \to S$ for which every $p \otimes_R S$, $p \in \Phi$, is invertible. Write $R_p := R_{\{p\}}$. By Schofield [62, p. 57, proof of Theorem 4.7], restriction of scalars is fully faithful and identifies

$$\operatorname{Mod} R_\Phi = \{X \in \operatorname{Mod} R \mid \operatorname{Hom}_R(p, X) \text{ is bijective for every } p \in \Phi\}. \tag{7.3}$$

Writing $P^* = \operatorname{Hom}_R(P, R)$ for $P \in \operatorname{proj} R$, we have $\operatorname{Hom}_R(P, X) \cong X \otimes_R P^*$ naturally in P. Thus $\operatorname{Hom}_R(p, X)$ is bijective precisely when $X \otimes_R p^*$ is bijective.

To detect E-infiniteness over an arbitrary field, we construct a generic grain from any integral weight outside the g-vector fan.

Proposition 7.6. *Let k be any field, and let $\gamma \in K_0(\operatorname{proj}\Lambda) \setminus \Sigma(\Lambda)$. There exists an endofinite module $N \in \operatorname{Mod}\Lambda$ such that $(\overline{\mathcal{T}}_\gamma, \mathcal{F}_\gamma) = ({}^\perp N, \operatorname{cogen} N)$, and some indecomposable summand of N is a generic grain.*

Proof. We divide the proof into three steps: constructing N over Cohn's universal division ring, proving $\operatorname{Ext}^1_\Lambda(X, N) = 0$ for all $X \in \operatorname{cogen} N$, and identifying the torsion pair to extract a generic grain.

Step 1: Constructing N over the universal division ring. Write $\gamma = [P^+] - [P^-]$ with $P^+, P^- \in \operatorname{proj}\Lambda$ sharing no indecomposable direct summand. We construct N as the kernel of a generic homomorphism of injectives with independent noncommuting coefficients. Put $I^0 := \nu P^-$ and $I^1 := \nu P^+$ in $\operatorname{inj}\Lambda$, and choose a k-basis $h_1, \dots, h_m$ of $\operatorname{Hom}_\Lambda(I^0, I^1)$. Let $R := k\langle x_1, \dots, x_m\rangle$ be the free associative algebra. Recall that an $r \times r$ matrix over R is *full* if it does not factor through a free R-module of rank $r-1$. Over any field k, universally localizing R at all full square matrices produces its free skew field (the universal division ring of fractions) $\mathbb{K}$ by Cohn [22, Corollary 2.5.2, Theorem 7.5.13(e), and Corollary 7.5.14] (for $m = 0$, take $\mathbb{K} = k$). Viewing each $\mathbb{K} \otimes_k I^i$ as a $(\mathbb{K}, \Lambda)$-bimodule, right multiplication ρ_{x_j} by x_j on $\mathbb{K}$ is left $\mathbb{K}$-linear, so

$$\delta := \sum_{j=1}^{m} \rho_{x_j} \otimes h_j \colon \mathbb{K} \otimes_k I^0 \longrightarrow \mathbb{K} \otimes_k I^1$$

is a $(\mathbb{K}, \Lambda)$-bimodule homomorphism. Set $N := \ker\delta \in \operatorname{Mod}\Lambda$.

Step 2: Lifting maps through δ and vanishing of Ext^1. We first express any tuple of k-linear operators $b_1, \dots, b_m \in \operatorname{End}_k(\mathbb{K})$ simultaneously as commutators with ρ_{x_j}. We endow $V := \mathbb{K} \oplus \mathbb{K}$ with the right R-module structure $(u, v) \cdot x_j := (ux_j + b_j(v), vx_j)$. Every full square matrix over R acts on V in block upper-triangular form with identical invertible diagonal blocks on the two copies of $\mathbb{K}$, and therefore acts invertibly on V. By (7.3), V is a right $\mathbb{K}$-module and the exact sequence $0 \to \mathbb{K} \to V \to \mathbb{K} \to 0$ is $\mathbb{K}$-linear, so it splits via a section $v \mapsto (\sigma(v), v)$ for some $\sigma \in \operatorname{End}_k(\mathbb{K})$. Right R-linearity of this section reads $b_j = \sigma\rho_{x_j} - \rho_{x_j}\sigma$ for all $j = 1, \dots, m$.

Each $\mathbb{K} \otimes_k I^i$ is injective, being a direct sum of injectives over the right Noetherian ring Λ. By adjunction between tensor product and Hom and finite presentation of I^0,

$$\operatorname{Hom}_\Lambda(\mathbb{K} \otimes_k I^0, \mathbb{K} \otimes_k I^1) \cong \operatorname{Hom}_k(\mathbb{K}, \mathbb{K} \otimes_k \operatorname{Hom}_\Lambda(I^0, I^1)).$$

Expanding in the basis (h_j), every $b \in \operatorname{Hom}_\Lambda(\mathbb{K} \otimes_k I^0, \mathbb{K} \otimes_k I^1)$ has the form $b = \sum_{j=1}^m b_j \otimes h_j$ with $b_j \in \operatorname{End}_k(\mathbb{K})$. Choosing σ as above and setting $\sigma_i := \sigma \otimes \mathrm{id}_{I^i}$ yields $b = \sigma_1\delta - \delta\sigma_0$. Given any right Λ-module homomorphism $f \colon N \to \mathbb{K} \otimes_k I^1$, since $\mathbb{K} \otimes_k I^1$ is injective, f extends to a morphism $b \colon \mathbb{K} \otimes_k I^0 \to \mathbb{K} \otimes_k I^1$; its restriction to $N = \ker\delta$ equals $(\sigma_1\delta - \delta\sigma_0)|_N = -\delta\sigma_0|_N$ and therefore factors through δ. Consequently, $\operatorname{Hom}_\Lambda(N, \mathbb{K} \otimes_k I^0) \to \operatorname{Hom}_\Lambda(N, \mathbb{K} \otimes_k I^1)$ is surjective.

Next we pass from N to $\operatorname{cogen} N$. Since $\dim_{\mathbb{K}} N \le \dim_k I^0 < \infty$, the canonical map $\mathbb{K}^J \otimes_{\mathbb{K}} N \to N^J$ is an isomorphism of right Λ-modules for every index set J; because $\mathbb{K}^J$ is a free right $\mathbb{K}$-vector space, N^J is isomorphic to a direct sum of copies of N. For any submodule $X \subseteq N^J$, extending maps $X \to \mathbb{K} \otimes_k I^1$ to N^J shows that $\operatorname{Hom}_\Lambda(X, \mathbb{K} \otimes_k I^0) \to \operatorname{Hom}_\Lambda(X, \mathbb{K} \otimes_k I^1)$ is surjective, so every map $X \to \operatorname{Im}\delta$ lifts through δ. Since $\mathbb{K} \otimes_k I^0$ is injective, the exact sequence $0 \to N \to \mathbb{K} \otimes_k I^0 \to \operatorname{Im}\delta \to 0$ yields $\operatorname{Ext}^1_\Lambda(X, N) = 0$.

Step 3: Identifying the torsion pair and extracting a generic grain. Given a short exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in \mathsf{cogen}\, N$, choose embeddings $X \hookrightarrow N^r$ and $Z \hookrightarrow N^s$. Because $\mathrm{Ext}^1_\Lambda(Z, N^r) = 0$, the first embedding extends to a map $Y \to N^r$, which together with $Y \twoheadrightarrow Z \hookrightarrow N^s$ embeds Y into N^{r+s}. Thus $\mathsf{cogen}\, N$ is closed under submodules and extensions, so $({}^\perp N, \mathsf{cogen}\, N)$ is a torsion pair in $\mathsf{mod}\,\Lambda$. For every $X \in \mathsf{mod}\,\Lambda$, we have $\mathrm{Hom}_\Lambda(X, \mathbb{K} \otimes_k I^i) \cong \mathbb{K} \otimes_k \mathrm{Hom}_\Lambda(X, I^i)$ as left $\mathbb{K}$-spaces, and Nakayama duality yields

$$\gamma(X) = \dim_k \mathrm{Hom}_\Lambda(X, I^1) - \dim_k \mathrm{Hom}_\Lambda(X, I^0) = \dim_{\mathbb{K}} \mathrm{Coker}\, \mathrm{Hom}_\Lambda(X, \delta) - \dim_{\mathbb{K}} \mathrm{Hom}_\Lambda(X, N).$$

For every nonzero $X \in \mathsf{cogen}\, N$, we have $\mathrm{Coker}\, \mathrm{Hom}_\Lambda(X, \delta) = 0$ by Step 2, while $\mathrm{Hom}_\Lambda(X, N) \neq 0$; hence $\gamma(X) < 0$. Since $\mathsf{cogen}\, N$ is closed under submodules, $\mathsf{cogen}\, N \subseteq \mathcal{F}_\gamma$. Moreover, every $Y \in {}^\perp N$ satisfies $\gamma(Y) \geq 0$; since ${}^\perp N$ is closed under quotients, ${}^\perp N \subseteq \overline{\mathcal{T}}_\gamma$. Taking right perpendiculars of the second inclusion gives $\mathcal{F}_\gamma = \overline{\mathcal{T}}_\gamma^\perp \subseteq ({}^\perp N)^\perp = \mathsf{cogen}\, N$ (as $({}^\perp N, \mathsf{cogen}\, N)$ is a torsion pair), whence $(\overline{\mathcal{T}}_\gamma, \mathcal{F}_\gamma) = ({}^\perp N, \mathsf{cogen}\, N)$.

Finally, because the left $\mathbb{K}$-action commutes with the right Λ-action, every $\mathrm{End}_\Lambda(N)$-submodule of N is a left $\mathbb{K}$-subspace. Hence $\ell_{\mathrm{End}_\Lambda(N)}(N) \leq \dim_{\mathbb{K}} N \leq \dim_k I^0 < \infty$, so N is endofinite. By Crawley-Boevey's decomposition of endofinite modules over an arbitrary ring (see Krause [46, Proposition 6.15]), N is a direct sum of copies of finitely many indecomposable endofinite modules $N_1, \ldots, N_r$. If every N_i were finite-dimensional, then setting $M := \bigoplus_{i=1}^r N_i \in \mathsf{mod}\,\Lambda$ would give $\overline{\mathcal{T}}_\gamma = {}^\perp N = {}^\perp M$, so $\overline{\mathcal{T}}_\gamma$ would be cocompact and Proposition 2.5(1) would imply $\gamma \in \Sigma(\Lambda)$, a contradiction. Therefore some summand N_i is infinite-dimensional; since it inherits $\mathrm{Ext}^1_\Lambda(\mathsf{cogen}\, N_i, N_i) = 0$, it is a generic grain. □

For a ring epimorphism $\Lambda \to \Gamma$, we also consider targets detected by their finite-dimensional modules. The algebra Γ is *residually finite-dimensional* if its finite-dimensional right modules have zero common annihilator; we write $\mathsf{mod}\,\Gamma$ for the category of finite-dimensional right Γ-modules even when Γ is infinite-dimensional.

For any ring epimorphism $\Lambda \to \Gamma$, restriction identifies $\mathsf{mod}\,\Gamma$ with an exact abelian subcategory of $\mathsf{mod}\,\Lambda$. For a residually finite-dimensional target we will use a left approximation of ${}_\Lambda\Lambda$ by this subcategory.

Lemma 7.7. *If Λ is E-finite, every exact abelian subcategory of $\mathsf{mod}\,\Lambda$ is functorially finite.*

Proof. Let $\mathcal{E} \subseteq \mathsf{mod}\,\Lambda$ be an exact abelian subcategory. Every object of $\mathcal{E}$ has finite length in $\mathsf{mod}\,\Lambda$ and hence in $\mathcal{E}$, so $\mathcal{S} := \mathsf{sim}(\mathcal{E})$ is a semibrick in $\mathsf{mod}\,\Lambda$ satisfying $\mathcal{E} \subseteq \mathcal{W} := \mathsf{Filt}(\mathcal{S})$. By Theorem 5.1, $\mathcal{W}$ is functorially finite, so [32, Proposition 4.12] yields an exact equivalence $\mathcal{W} \simeq \mathsf{mod}\, A$ for a finite-dimensional k-algebra A.

For any $E \in \mathcal{E}$ and any subobject $X \subseteq E$ in $\mathcal{W}$, we prove that $X, E/X \in \mathcal{E}$ by induction on $\ell_{\mathcal{W}}(X)$. When $\ell_{\mathcal{W}}(X) = 0$, we have $X = 0$ and the assertion is immediate. When $\ell_{\mathcal{W}}(X) > 0$, choose a simple subobject $S \subseteq X$ in $\mathcal{W}$. Since $S \in \mathcal{S} \subseteq \mathcal{E}$ and $\mathcal{E}$ is closed under cokernels and kernels in $\mathsf{mod}\,\Lambda$ (and hence in $\mathcal{W}$), we have $E/S \in \mathcal{E}$. Applying the induction hypothesis to $X/S \subseteq E/S$ yields $E/X \cong (E/S)/(X/S) \in \mathcal{E}$, and taking the kernel of $E \twoheadrightarrow E/X$ then gives $X \in \mathcal{E}$.

Under $\mathcal{W} \simeq \mathsf{mod}\, A$, $\mathcal{E}$ corresponds to a full subcategory of $\mathsf{mod}\, A$ closed under finite direct sums and subquotients. Let $I := \bigcap_{E \in \mathcal{E}} \mathrm{ann}_A(E) \trianglelefteq A$. As in the proof of Proposition 4.2, finite dimensionality gives a finite sum $V \in \mathcal{E}$ with $I = \mathrm{ann}_A(V)$, and evaluation on a basis embeds A/I in $V^{\dim_k V}$. Closure under subquotients in $\mathsf{mod}\, A$ therefore implies $A/I \in \mathcal{E}$, and hence $\mathcal{E} = \mathsf{mod}(A/I)$. The adjoints of the inclusion $\mathsf{mod}(A/I) \hookrightarrow \mathsf{mod}\, A$ show that $\mathcal{E}$ is functorially finite in $\mathcal{W}$, and composing with the approximations for $\mathcal{W} \subseteq \mathsf{mod}\,\Lambda$ shows that $\mathcal{E}$ is functorially finite in $\mathsf{mod}\,\Lambda$. □

Theorem 7.8. *For a finite-dimensional algebra Λ over any field, the conditions* (E-fin), (no-gen-brick), (no-gen-grain), (thick-gen), (univ-loc), *and* (ring-epi) *of Theorem A(I) are equivalent.*

Condition (univ-loc) *is also equivalent to requiring $\dim_k \Lambda_p < \infty$ for every morphism p in $\mathsf{proj}\,\Lambda$. In* (ring-epi) *one can instead test all residually finite-dimensional targets.*

Proof. We first establish the equivalences for generic modules, next pass from thick subcategories to universal localizations, and finally treat the epimorphic targets.

(E-fin) $\Rightarrow$ (no-gen-brick) $\Rightarrow$ (no-gen-grain) $\Rightarrow$ (E-fin): Lemma 7.3 gives (E-fin) $\Rightarrow$ (no-gen-brick). If $N \in \mathsf{Mod}\,\Lambda$ is a generic grain, then Schlegel [61, Lemma 5.2] shows that its socle as an $\mathrm{End}_\Lambda(N)$-module is a generic brick; hence (no-gen-brick) $\Rightarrow$ (no-gen-grain). Conversely, if Λ is E-infinite, Proposition 7.6 constructs a generic grain from any integral weight outside $\Sigma(\Lambda)$, giving (no-gen-grain) $\Rightarrow$ (E-fin).

(E-fin) $\Rightarrow$ (thick-gen) $\Rightarrow$ (univ-loc): Proposition 7.5 gives (E-fin) $\Rightarrow$ (thick-gen) over any field. Now assume (thick-gen), and let Φ be any set of morphisms in $\mathsf{proj}\,\Lambda$. By (thick-gen), the thick subcategory of $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ generated by $\{C_p \mid p \in \Phi\}$ equals $\mathsf{thick}_2(U)$ for some $U \in \mathsf{2\text{-}psilt}\,\Lambda$. A right Λ-module $M \in \mathsf{Mod}\,\Lambda$ inverts p if and only if $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(C_p, M) = 0 = \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\Lambda)}(C_p, M[1])$. For fixed M, the two-term complexes satisfying these vanishings form a thick subcategory of $\mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda)$ by the long exact Hom sequence. It contains all C_p if and only if it contains U, since they generate the same thick subcategory. Hence (7.3) and (7.1) give $\mathsf{Mod}\,\Lambda_\Phi = \mathcal{X}_U$. By (7.3), $\mathcal{X}_U$ is also the essential image of localization at the differential u of U (see [4, Remark 3.1]). By uniqueness of ring epimorphisms with a given essential image [37, Theorem 1.2], $\Lambda_\Phi \cong \Lambda_u$. The latter algebra is Morita equivalent to the finite-dimensional algebra Γ_U by [4, Theorem 3.5].

$\neg$(E-fin) $\Rightarrow$ $\neg$(univ-loc) and $\neg$(residually finite-dimensional case of (ring-epi)): If Λ is E-infinite, Lemma A.6 gives a morphism p in $\mathsf{proj}\,\Lambda$ such that both Λ_p and its quotient R by the common annihilator of its finite-dimensional modules are infinite-dimensional. The composite $\Lambda \to \Lambda_p \twoheadrightarrow R$ is a ring epimorphism, and R is residually finite-dimensional by construction.

(E-fin) $\Leftrightarrow$ (ring-epi): Note first that for any ring epimorphism $f\colon \Lambda \to \Gamma$, fullness of restriction $\mathsf{Mod}\,\Gamma \hookrightarrow \mathsf{Mod}\,\Lambda$ implies $f(Z(\Lambda)) \subseteq Z(\Gamma)$, so Γ is automatically a k-algebra and f is a k-algebra epimorphism.

(1) *Simple Artinian* Γ: Ringel's correspondence [60, Section 1.1, Example 2] identifies simple Artinian epimorphic targets $\Lambda \to \Gamma$, taken up to isomorphism over Λ, with isomorphism classes of bricks $G \in \mathsf{Mod}\,\Lambda$ of finite endolength by restriction of the unique simple right Γ-module to Λ. Conversely, given G, take $\Gamma = \mathrm{End}_\Delta(G)^{\mathrm{op}}$ where $\Delta = \mathrm{End}_\Lambda(G)$. Since $\Gamma_\Gamma \cong G^r$ for $r = \dim_\Delta G$, we have $\dim_k \Gamma < \infty$ if and only if $\dim_k G < \infty$. Thus (ring-epi) for simple Artinian targets is equivalent to (no-gen-brick) for each fixed Λ.

(2) *Residually finite-dimensional* Γ: Suppose Λ is E-finite. The category $\mathsf{mod}\,\Gamma \subseteq \mathsf{mod}\,\Lambda$ is an exact abelian subcategory, hence functorially finite by Lemma 7.7. Choose a left $\mathsf{mod}\,\Gamma$-approximation $\Lambda \to E$. For every $M \in \mathsf{mod}\,\Gamma$, a Λ-epimorphism $\Lambda^a \twoheadrightarrow M$ factors through E^a, yielding an epimorphism $E^a \twoheadrightarrow M$, which is Γ-linear by fullness of restriction. Hence $\mathrm{ann}_\Gamma(E) = \bigcap_{M \in \mathsf{mod}\,\Gamma} \mathrm{ann}_\Gamma(M) = 0$, and the faithful action of Γ on E gives $\dim_k \Gamma \leq (\dim_k E)^2 < \infty$. The converse was established above in the proof of $\neg$(E-fin) $\Rightarrow$ $\neg$(univ-loc), using the infinite-dimensional residually finite-dimensional quotient from Lemma A.6.

□

Question 5.11(3) of Angeleri Hügel and Sentieri [6] recalled in Subsection 1.4 has the following partial answer.

Corollary 7.9. *If every ring epimorphism $\Lambda \to \Gamma$ with $\mathrm{Tor}_1^\Lambda(\Gamma, \Gamma) = 0$ has an Artin algebra as its target, then Λ is E-finite. If Λ has rank at most two, then it is also τ-tilting finite.*

Proof. Suppose Λ is E-infinite. Lemma A.6 gives a universal localization Λ_p with an infinite-dimensional residually finite-dimensional quotient R. Schofield [62, Theorems 4.7 and 4.8] gives $\mathrm{Tor}_1^\Lambda(\Lambda_p, \Lambda_p) = 0$. If Λ_p were an Artin algebra, then R would be right Artinian; by the descending chain condition on right ideals, the zero intersection $\bigcap_{M \in \mathsf{mod}\,R} \mathrm{ann}_R(M) = 0$ would reduce to a finite intersection of ideals of finite k-codimension, contradicting $\dim_k R = \infty$. In rank at most two, E-finiteness implies τ-tilting finiteness by Theorem B. □

8. Rank at most two over arbitrary fields

We now prove Theorem B by showing that in rank at most two, every g-vector fan containing all integral lattice points is complete. In rank one the two opposite projective rays cover the line. In rank two, the silting cones $C(P_1 \oplus P_2)$ and $C(P_1[1] \oplus P_2[1])$ already cover the positive and negative quadrants; using the Hom inequalities of Lemma 2.10, we subdivide the two mixed quadrants $\mathcal{Q}_{+-} = \operatorname{cone}([P_1], -[P_2])$ and $\mathcal{Q}_{-+} = \operatorname{cone}([P_2], -[P_1])$ into finitely many sectors, each contained in a silting cone.

Over an algebraically closed field, the τ-convergence criterion of Mousavand and Paquette [51, Definition 6.3 and Theorem 1.3] implies E-infiniteness for certain minimal brick-infinite algebras. For Wang's minimal τ-tilting infinite algebra Ω_3 [67, Proposition 3.8], Mousavand and Paquette [54, Example 4.1] identify $(2,-1)$ as the unique missing direction, although the criterion does not apply. We compute its fan in Example 8.8. The subdivision below gives a missing integral ray for every brick-infinite algebra of rank two.

For $V, W \in \mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, write $e(V,W)$ for the dimension of the morphism space defining the E-invariant of Derksen and Fei [29, Section 3]:

$$e(V,W) := \dim_k \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(V, W[1]).$$

Thus V is presilting if and only if $e(V,V) = 0$. For any two classes $v, w \in K_0(\operatorname{proj}\Lambda)$, we call their sum $v + w$ the *mediant* of (v,w).

Lemma 8.1. *Let $E, L \in \mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be presilting complexes such that $e(E,L) = 0$ and $e_0 := e(L,E) \geq 1$. If $Z \in \mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is any presilting complex whose class is the mediant $[Z] = [E] + [L]$ in $K_0(\operatorname{proj}\Lambda)$, then*

$$e(E,Z) = 0, \qquad e(Z,L) = 0, \qquad e(Z,E) \leq e_0 - 1, \qquad e(L,Z) \leq e_0 - 1.$$

Proof. For two complements Derksen and Fei [29, Section 5] use the universal extension. To decrease $e(L,E)$ we choose one nonzero morphism $w \in \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(L, E[1])$ and form the distinguished triangle

$$E \longrightarrow Z_0 \longrightarrow L \xrightarrow{w} E[1]$$

in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$. Explicitly, Z_0 has terms $Q^{-1} := E^{-1} \oplus L^{-1}$ and $Q^0 := E^0 \oplus L^0$ with block upper-triangular differential $\sigma = \begin{pmatrix} d_E & w \\ 0 & d_L \end{pmatrix} \in \mathcal{V} = \operatorname{Hom}_\Lambda(Q^{-1}, Q^0)$. Applying $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(E,-)$ and $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(-, L[1])$ to this triangle gives $e(E,Z_0) = 0$ and $e(Z_0,L) = 0$, since $e(E,E) = e(L,L) = e(E,L) = 0$. Applying $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(-, E[1])$ and $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(L,-)$ gives the exact sequences

$$\operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(E) \xrightarrow{a \mapsto a[1]\circ w} \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(L, E[1]) \longrightarrow \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(Z_0, E[1]) \longrightarrow 0,$$

$$\operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(L) \xrightarrow{b \mapsto w\circ b} \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(L, E[1]) \longrightarrow \operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)}(L, Z_0[1]) \longrightarrow 0.$$

Both connecting maps take the identity to $w \neq 0$, so we obtain $e(Z_0,E) \leq e_0 - 1$ and $e(L,Z_0) \leq e_0 - 1$.

Since $[Z] = [Q^0] - [Q^{-1}]$, Lemma 2.9 provides a representative C_u of Z with $u \in \mathcal{V}$. Applying Lemma 2.10 to u and σ with $V = E$ and $V = L$ yields all four inequalities for Z. □

We use the subdivision at a cone with primitive generators v, w that inserts the ray through $v + w$ [7, Definition 2.8]. For $v, w \in K_0(\operatorname{proj}\Lambda)$, let $\mathrm{SB}_j(v,w)$ denote the set of Stern–Brocot mediants between v and w through depth j, defined recursively by $\mathrm{SB}_0(v,w) = \varnothing$ and

$$\mathrm{SB}_j(v,w) = \{v + w\} \cup \mathrm{SB}_{j-1}(v, v+w) \cup \mathrm{SB}_{j-1}(v+w, w) \qquad (j \geq 1).$$

Thus $|\mathrm{SB}_j(v,w)| \leq 2^j - 1$, with equality whenever v and w are linearly independent.

Lemma 8.2. *In any rank, let E, L be two-term presilting complexes with linearly independent classes and $e(E,L) = 0$, and put $j = e(L,E)$. If $\mathrm{SB}_j([E],[L]) \subseteq \Sigma(\Lambda)$, then the sector $\operatorname{cone}([E],[L])$ is a union of at most 2^j sectors $\operatorname{cone}([E'],[L'])$ with $E' \oplus L'$ presilting; in particular, each such sector lies in a silting cone.*

Proof. We induct on j. When $j = 0$, the direct sum $E \oplus L$ is presilting and extends, after removing repeated summands, to a two-term silting complex. If $j > 0$, then the mediant $[E] + [L]$ lies in $\Sigma(\Lambda)$ by hypothesis and therefore equals $[Z]$ for a two-term presilting complex Z by Lemma 2.1(1). Lemma 8.1 gives $e(E, Z) = e(Z, L) = 0$ and $e(Z, E), e(L, Z) \leq j - 1$. Both new pairs (E, Z) and (Z, L) have linearly independent classes, and by the recursion for SB_j, the hypothesis holds at the required depth on each of the two subsectors $\mathrm{cone}([E], [Z])$ and $\mathrm{cone}([Z], [L])$. Since these subsectors cover $\mathrm{cone}([E], [L])$, the induction hypothesis gives at most $2 \cdot 2^{j-1} = 2^j$ sectors in all. □

Proposition 8.3. *Let P_1, P_2 be the two indecomposable projectives of a rank-two algebra Λ over a field k. Put $x = \dim_k \mathrm{Hom}_\Lambda(P_2, P_1)$ and $y = \dim_k \mathrm{Hom}_\Lambda(P_1, P_2)$. Then the g-vector fan covers the plane if and only if*

$$\mathrm{SB}_x([P_1], -[P_2]) \cup \mathrm{SB}_y([P_2], -[P_1]) \subseteq \Sigma(\Lambda).$$

In this case there are at most $2^x + 2^y + 2$ basic two-term silting complexes.

Proof. Necessity is immediate. Conversely, suppose that the indicated mediants lie in $\Sigma(\Lambda)$. For the first mixed quadrant $\mathcal{Q}_{+-}$, take $E = P_1$ and $L = P_2[1]$. Then

$$e(E, L) = \dim_k \mathrm{Hom}_{\mathsf{K}^\mathrm{b}(\mathsf{proj}\,\Lambda)}(P_1, P_2[2]) = 0, \qquad e(L, E) = \dim_k \mathrm{Hom}_\Lambda(P_2, P_1) = x,$$

so Lemma 8.2 covers $\mathcal{Q}_{+-}$ by at most 2^x silting cones. Interchanging P_1 and P_2 covers $\mathcal{Q}_{-+}$ by at most 2^y silting cones, and the two remaining quadrants are the silting cones $C(P_1 \oplus P_2)$ and $C(P_1[1] \oplus P_2[1])$.

These cones cover the plane. Since distinct maximal cones of a fan have disjoint interiors, every two-term silting cone occurs in this collection. □

Proposition 8.3 therefore gives a finite test for τ-tilting finiteness: it suffices to test at most $2^x + 2^y - 2$ specified lattice points for membership in the fan.

Theorem 8.4 (Theorem B)**.** *Let Λ be a finite-dimensional algebra of rank $n \leq 2$ over an arbitrary field k. Then Λ is E-finite if and only if Λ is brick-finite (equivalently, τ-tilting finite).*

Proof. For a brick-finite algebra the fan is complete by [26, Theorem 1.4] and [9, Theorem 4.7], so it covers the lattice. Conversely, suppose that Λ is E-finite. In rank two, all mediants of Proposition 8.3 lie in $\Sigma(\Lambda)$, so finitely many silting cones cover the plane. In rank one, the two cones $C(P_1)$ and $C(P_1[1])$ cover the line, and rank zero is immediate. By Theorem 3.2 and Corollary 3.3, each brick $B \in \mathsf{brick}\,\Lambda$ is stable at some integral weight θ. This weight lies in the relative interior $C^\circ(U)$ of one of the finitely many faces $C(U)$ of the fan, so B is a simple object of $\mathcal{W}_\theta = \mathcal{J}(U)$ by Proposition 2.3(1). By Lemma 2.2, each $\mathcal{J}(U)$ has only finitely many simple objects. Therefore Λ is brick-finite, which is equivalent to τ-tilting finiteness by Demonet, Iyama, and Jasso [26, Theorem 1.4]. □

Remark 8.5 (Planar sectors in arbitrary rank)**.** If Λ is E-finite of arbitrary rank, then the mediant hypothesis of Lemma 8.2 holds for every pair (E, L) of two-term presilting complexes with linearly independent classes and $e(E, L) = 0$. Hence the planar sector spanned by $[E]$ and $[L]$ admits a finite subdivision into sectors each contained in a silting cone. Let V be any further two-term complex and choose Z presilting with $[Z] = [E] + [L]$. By Lemma 2.9, Z can be represented on the projective terms of $E \oplus L$. Comparing its differential with the split differential of $E \oplus L$ by Lemma 2.10 gives

$$e(Z, V) \leq e(E, V) + e(L, V), \qquad e(V, Z) \leq e(V, E) + e(V, L).$$

The induction in Lemma 8.2 terminates because $e(L, E)$ decreases strictly in both planar subsectors by Lemma 8.1; the two bounds above give no such strict decrease when $[V]$ spans a third ray outside the plane of $[E]$ and $[L]$.

Example 8.6 (The Kronecker fan)**.** Write $P_i = e_i\Lambda$ for the two vertex projectives of $\Lambda = k(1 \rightrightarrows 2)$ and use $[P_1], [P_2]$ as coordinates on $K_0(\mathsf{proj}\,\Lambda)_\mathbb{R}$. Because $[P_1] = (1, 2)$ and $[P_2] = (0, 1)$ in $K_0(\mathsf{mod}\,\Lambda)$, the minimal projective resolution of a module M of dimension vector (p, q) has class

$(p, q-2p)$. In the Kronecker normal form for pairs of linear maps, the rectangular indecomposable blocks M_m and N_m ($m \geq 0$) have dimension vectors $(m, m+1)$ and $(m+1, m)$, with arrows given by the two inclusions in consecutive coordinates and by their transposes, respectively. Any endomorphism commuting with both maps is scalar. Since Λ is hereditary, $\dim_k \operatorname{Hom}_\Lambda(M,N) - \dim_k \operatorname{Ext}^1_\Lambda(M,N)$ equals the Euler form $pr + qs - 2ps$ on the dimension vectors (p,q) of M and (r,s) of N. Pairing either rectangular dimension vector with itself gives 1, so all M_m and N_m are rigid; on any regular block of dimension vector (p,p) ($p \geq 1$), by contrast, the Euler form vanishes, so $\dim_k \operatorname{Ext}^1_\Lambda(M,M) = \dim_k \operatorname{End}_\Lambda(M) \geq 1$ and no regular block is rigid. Thus the minimal projective resolutions of M_m and N_m, together with the shifted projectives, account for all rays of $\Sigma(\Lambda)$. Since $M_0 = P_2$ has class $(0,1)$, the remaining preprojective modules M_{j+1} ($j \geq 0$, of dimension $(j+1, j+2)$) and the preinjective modules N_{j-1} ($j \geq 1$, of dimension $(j, j-1)$), together with $P_2[1]$ at $j = 0$, yield all rays in the mixed quadrant $\mathcal{Q}_{+-}$:

$$(j+1, -j), \qquad (j, -j-1) \qquad (j \geq 0).$$

For two consecutive inclusion blocks M_m and M_{m+1}, commutation with the two inclusions determines any morphism $M_m \to M_{m+1}$ by the two scalar coefficients of the coordinate shifts, whereas any morphism $M_{m+1} \to M_m$ must vanish starting from the boundary coordinates. Hence $\dim_k \operatorname{Hom}_\Lambda(M_m, M_{m+1}) = 2$ and $\operatorname{Hom}_\Lambda(M_{m+1}, M_m) = 0$; comparing with the Euler form values 2 and 0 shows that both $\operatorname{Ext}^1_\Lambda$ spaces vanish, so consecutive rays $(j+1, -j)$ and $(j+2, -j-1)$ bound a silting cone. Transposition gives the same conclusion for consecutive preinjective blocks N_{j-1} and N_j ($j \geq 1$), while at $j = 0$ the vanishing $\operatorname{Hom}_\Lambda(P_2, S_1) = 0$ shows that $P_2[1]$ and the resolution of $N_0 = S_1$ span the boundary cone between $(0,-1)$ and $(1,-2)$ (compare [9, Example 5.4]). The two quadrants where the coordinates have equal signs are the silting cones $C(P_1 \oplus P_2)$ and $C(P_1[1] \oplus P_2[1])$, and the remaining quadrant $\mathcal{Q}_{-+}$ is the single cone $C(P_1[1] \oplus P_2)$ since $\operatorname{Hom}_\Lambda(P_1, P_2) = 0$. Hence the ray through $(1,-1)$ is the unique ray outside $\Sigma(\Lambda)$.

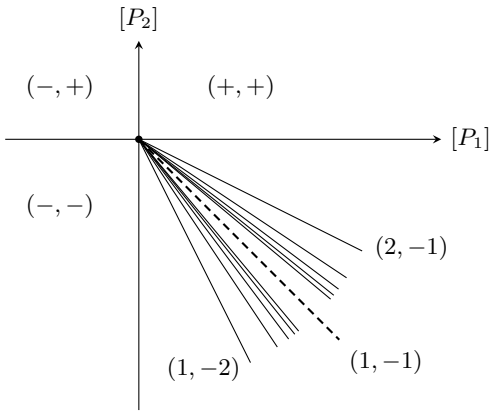


FIGURE 2. The Kronecker fan in projective coordinates. Each quadrant marked by signs is one silting cone. The dashed ray is missing.

The regular bricks are the simple regular blocks. On the finite affine chart of the pencil they are

$$R_f = (K_f \rightrightarrows K_f; 1, t), \qquad K_f = k[t]/(f), \quad f \text{ monic irreducible};$$

indeed, an endomorphism of the regular block with elementary divisor f^h commutes with multiplication by t, so its endomorphism ring is $k[t]/(f^h)$, which is a field precisely when $h = 1$. Together with $(k \rightrightarrows k; 0, 1)$ from the point at infinity and the rectangular blocks M_m, N_m, these exhaust $\mathsf{brick}\,\Lambda$. For instance, in any object of $\mathsf{Filt}(R_t)$ the first arrow acts invertibly, and identifying the two spaces via that arrow identifies $\mathsf{Filt}(R_t)$ with the category of finite-dimensional nilpotent $k[t]$-modules. This wide subcategory is not functorially finite: if Λ admitted a left $\mathsf{Filt}(R_t)$-approximation, finite direct sums of it would surject onto every object of $\mathsf{Filt}(R_t)$, whereas $k[t]/(t^h)$

has unbounded nilpotency index. Over a finite field, the degrees of the irreducible polynomials f are unbounded while every R_f has endolength two (Example A.1).

The corresponding generic brick is $G = (k(t) \rightrightarrows k(t); 1, t)$: any endomorphism commutes with 1 and with multiplication by t, hence is $k(t)$-linear, so $\mathrm{End}_\Lambda(G) = k(t)$ and G has endolength two. We can also see G directly inside the construction of Proposition 7.6 at the missing weight $\gamma = (1, -1) = [P_1] - [P_2]$. Here $I^0 = \nu P_2$ has vertex spaces (k^2, k) with arrows the two coordinate projections, while $I^1 = \nu P_1 = S_1$ is concentrated at vertex 1. Over the free skew field $\mathbb{K}$ on x and y, the map $\delta\colon \mathbb{K} \otimes_k I^0 \to \mathbb{K} \otimes_k I^1$ vanishes at vertex 2 and sends $(u, v) \in \mathbb{K}^2$ to $ux + vy \in \mathbb{K}$ at vertex 1. Writing $z := -xy^{-1}$, its kernel $N = \ker \delta$ consists of pairs (u, uz) at vertex 1 mapped to $\mathbb{K}$ at vertex 2 by the two projections, so its two arrows act as 1 and right multiplication by z. Because z is transcendental over k (any polynomial relation in $-xy^{-1}$ would specialize at $y = 1$ to the same relation for an indeterminate $-x$), choosing a right $k(z)$-basis of $\mathbb{K}$ decomposes N into a direct sum of copies of G, with $\mathrm{End}_\Lambda(N) = \mathrm{End}_{k(z)}(\mathbb{K})$, so the indecomposable generic grain produced by Proposition 7.6 is G itself.

Example 8.7 (Two subdivisions)**.** Consider the loop algebra of [7, Section 4.3], given over any field k by an arrow $\gamma\colon 1 \to 2$ and a loop u at 2 with $u^2 = 0$. Here $\mathrm{End}_\Lambda(P_2) = k[u]/(u^2)$, $\mathrm{End}_\Lambda(P_1) = k$, and $H := \mathrm{Hom}_\Lambda(P_2, P_1)$ has k-basis $\gamma, \gamma u$. Starting from $E = P_1$ and $L = P_2[1]$, the first mediant $(1, -1)$ is represented by $Z = (P_2 \xrightarrow{\gamma} P_1)$, and the second mediant $(2, -1)$ between E and Z by $Y = (P_2 \xrightarrow{(\gamma, \gamma u)^{\mathsf{t}}} P_1^2)$. For morphisms $f\colon A \to B$ and $g\colon A' \to B'$ in $\mathsf{proj}\,\Lambda$, the space of chain maps modulo null-homotopies gives

$$e(C_f, C_g) = \dim_k \frac{\mathrm{Hom}_\Lambda(A, B')}{g\,\mathrm{Hom}_\Lambda(A, A') + \mathrm{Hom}_\Lambda(B, B')f}.$$

Precomposing γ with $\mathrm{End}_\Lambda(P_2)$ spans H, while postcomposing $(\gamma, \gamma u)^{\mathsf{t}}$ with $\mathrm{End}_\Lambda(P_1^2) = \mathrm{Mat}_2(k)$ spans H^2, so $e(Z, Z) = e(Y, Y) = 0$. Both Z and Y are indecomposable with local endomorphism rings: an endomorphism of Z is determined on P_2 by its scalar action on P_1, and an endomorphism of Y is determined on P_1^2 by the matrix of right multiplication by an element of $k[u]/(u^2)$ in the basis $\gamma, \gamma u$.

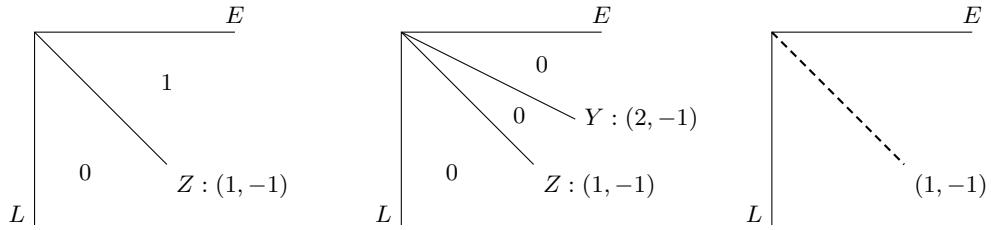
Initially $e(L, E) = 2$. Inserting Z yields

$$e(Z, E) = \dim_k(H/k\gamma) = 1, \qquad e(L, Z) = \dim_k\big(H/\gamma\,\mathrm{End}_\Lambda(P_2)\big) = 0,$$

and inserting Y between E and Z reduces the remaining reverse dimension to zero on both subsectors by Lemma 8.1. Figure 3 displays the resulting rays $(1, 0)$, $(2, -1)$, $(1, -1)$, and $(0, -1)$; since $\mathrm{Hom}_\Lambda(P_1, P_2) = 0$, the other mixed quadrant requires no subdivision, so the fan is complete and Λ is brick-finite.

The same subdivision pattern occurs for the triangular algebra $\left(\begin{smallmatrix} k & K \\ 0 & K \end{smallmatrix}\right)$ of a quadratic field extension $[K : k] = 2$ in [7, Section 5]: here $H = K = \mathrm{End}_\Lambda(P_2)$ and $\mathrm{End}_\Lambda(P_1) = k$, and replacing $\gamma, \gamma u$ by a k-basis $1, \xi$ of K gives the same sequence of reverse dimensions 2, then $1, 0$, and finally $0, 0, 0$. By contrast, for the Kronecker algebra, H again has dimension two, but both $\mathrm{End}_\Lambda(P_1)$ and $\mathrm{End}_\Lambda(P_2)$ consist of scalars; for any $C_f = (P_2 \xrightarrow{f} P_1)$ the quotient above is H/kf, so $e(C_f, C_f) = 1$ if $f \neq 0$ and $e(C_0, C_0) = 2$, showing by Lemma 2.9 that the first mediant $(1, -1)$ admits no presilting representative.

Example 8.8 (Wang's algebra Ω_3)**.** Over an algebraically closed field, consider Wang's minimal τ-tilting infinite algebra Ω_3 [67, Proposition 3.8], given by an arrow $\gamma\colon 1 \to 2$ and two loops u, v at 2 with all products of two loops equal to zero. Here $\mathrm{Hom}_\Lambda(P_2, P_1) = \gamma R$ for $R = k \oplus ku \oplus kv = \mathrm{End}_\Lambda(P_2)$, and $\mathrm{Hom}_\Lambda(P_1, P_2) = 0$. Following Mousavand and Paquette [54, Example 4.1], we classify presilting differentials $P_2^b \to P_1^a$ by factoring γ out of the $a \times b$ matrix over R; since presilting differentials have open orbits, it suffices to consider matrices whose constant part has maximal rank. When $a \leq b$, full row rank modulo $\mathrm{rad}\,R$ reduces the matrix to $[I_a\ 0]$ by R-linear column operations, so the sector from $(0, -1)$ to $(1, -1)$ is a single silting cone. When $a > b$, scalar row operations and R-linear column operations bring any matrix of full column rank modulo $\mathrm{rad}\,R$



Loop: first mediant Loop: second mediant Kronecker: first mediant missing

FIGURE 3. Both initial sectors have $e(L, E) = 2$. The numbers inside the subdivided sectors give their reverse E-dimensions.

to the normal form

$$\begin{pmatrix} I_b \\ uA + vB \end{pmatrix}, \qquad A, B \in \mathrm{Mat}_{(a-b)\times b}(k).$$

Any scalar row transformation preserving the constant column space is block upper triangular with diagonal blocks $D \in \mathrm{GL}_b(k)$ and $F \in \mathrm{GL}_{a-b}(k)$, giving the upper block constant part D; multiplying on the right first by D^{-1} and then by the radical column correction restores the upper block to I_b while leaving the lower block unchanged because $(\operatorname{rad} R)^2 = 0$. The residual action is therefore $(A, B) \mapsto (FAD^{-1}, FBD^{-1})$, that is, simultaneous base change on (A, B), compatible with block direct sums. As in Example 8.6, the open indecomposable orbits correspond to the rectangular Kronecker blocks of sizes $(a - b, b) = (j + 1, j)$ and $(j, j + 1)$, which yield the rays

$$(2j + 1, -j), \qquad (2j + 1, -j - 1) \qquad (j \geq 0),$$

with consecutive rays bounding silting cones, and both sequences converge to the unique missing direction $(2, -1)$, where the residual blocks are square. The remaining three quadrants are single cones.

In Proposition 8.3 we have $x = 3$ and $y = 0$, which tests $(1, -1)$ at depth one, $(2, -1)$ and $(1, -2)$ at depth two, and $(3, -1), (3, -2), (2, -3), (1, -3)$ at depth three. As shown in Figure 4, six of these seven test vectors belong to $\Sigma(\Lambda)$ (lying on rays or inside the cone from $(0, -1)$ to $(1, -1)$), while the missing vector $(2, -1)$ at depth two detects brick-infiniteness.

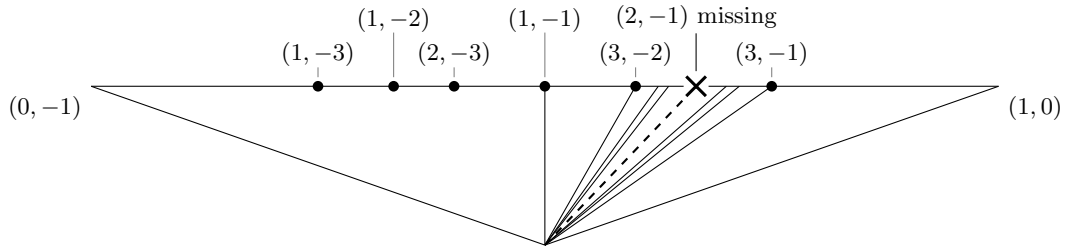


FIGURE 4. The mixed quadrant for Ω_3, cut at $a + b = 1$. The six dots lie in the fan. The cross marks the missing depth-two direction.

APPENDIX A. FINITE-FIELD COUNTS AND DESCENT FOR WIDE SUBCATEGORIES AND LOCALIZATIONS

This appendix proves the finite-field assertions used in Proposition 2.8 and the exponential lower bound used in Theorem 5.2, and the descent lemmas for subcategories $\mathsf{Filt}(B)$ generated by a brick and universal localizations at a single morphism. At the end of Subsection A.2 we combine these results with those of the body to prove Theorem A.

A.1. **Stable bricks and brick counts over finite fields.** The Kronecker algebra over $\mathbb{F}_q$ exhibits the difference between counting by endolength and counting by dimension vector.

Example A.1. Let $k = \mathbb{F}_q$ and let $\Lambda = k(1 \rightrightarrows 2)$ be the Kronecker algebra, with path multiplication from left to right so that right representations consist of linear maps $V_1 \to V_2$. For every integer $s \geq 1$, choose a generator α of the field extension $K = \mathbb{F}_{q^s}/\mathbb{F}_q$, and let B_s be the representation $(K \rightrightarrows K)$ whose two arrows act as id_K and multiplication by α. The missing ray in Figure 2 has primitive generator $\theta = [P_1] - [P_2]$, which evaluates as $\theta(W) = \dim_k W_1 - \dim_k W_2$. For any subrepresentation $W = (W_1, W_2) \subseteq B_s$, the action of the first arrow as id_K gives $W_1 \subseteq W_2$ and hence $\theta(W) = \dim_k W_1 - \dim_k W_2 \leq 0$. If $\theta(W) = 0$, then $W_1 = W_2$ is an α-invariant $\mathbb{F}_q$-subspace of K, so $W = 0$ or $W = B_s$. Thus B_s is θ-stable with $\mathrm{End}_\Lambda(B_s) \cong \mathbb{F}_{q^s}$, ordinary dimension $2s$, and constant endolength $\mathrm{endolen}(B_s) = 2$ for every $s \geq 1$. Consequently both (endo-stab) and (brick-discrete) fail, while (dim-brick) and (dim-stab) hold because a fixed dimension vector allows only finitely many pairs of arrow matrices over $\mathbb{F}_q$. By Theorem A(I) this algebra is E-infinite, so the infinite-field hypothesis in part (II) is necessary.

For a field extension F/k, put $X_F = X \otimes_k F$ and let θ_F be the extension of the projective class θ. For quiver representations, Hoskins and Schaffhauser [41, Proposition 2.4(2)] prove ascent of semistability under arbitrary extensions. Børve and Kaipel [17, Lemma 2.20] give the semistability equivalence for finite separable field extensions of finite-dimensional algebras. Over finite fields the orbit decomposition of an indecomposable and the degree of its residue endomorphism field are given by Deng and Du [28, Theorem 5.1]; Reineke [58, Section 4] describes the orbit of absolutely stable quiver representations. We use restriction of scalars and the socle in the semistable category to obtain the two assertions for the projective weight θ_F.

Lemma A.2. *Let Λ be a finite-dimensional algebra over a field k, and let $\theta \in K_0(\mathsf{proj}\,\Lambda)$.*

1. *Over any field k, semistability ascends to $\overline{k}$: if $X \in \mathcal{W}_\theta(\Lambda)$, then $X_{\overline{k}} \in \mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$.*
2. *If $k = \mathbb{F}_q$ is finite, sending $S \in \mathsf{sim}(\mathcal{W}_\theta(\Lambda))$ to the $\mathrm{Gal}(\overline{k}/\mathbb{F}_q)$-orbit of simple constituents of $S_{\overline{k}}$ in $\mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$ is a bijection between isomorphism classes of θ-stable Λ-modules and Frobenius orbits of $\theta_{\overline{k}}$-stable $\Lambda^{\overline{k}}$-modules. If the orbit has size e and each constituent B has $\overline{k}$-dimension r, then $S_{\overline{k}}$ is the direct sum of the e distinct conjugates of B, with $\dim_k S = er$, $\dim_k \mathrm{End}_\Lambda(S) = e$, and $\mathrm{endolen}(S) = r$.*

Proof. (1) A destabilizing submodule of $X_{\overline{k}}$ has a model $L_F \subseteq X_F$ over a finite extension F/k, with $\theta_F(L_F) > 0$. Restriction of scalars embeds $\mathrm{Res}_{F/k}(L_F)$ in $\mathrm{Res}_{F/k}(X_F) \cong X^{[F:k]}$. Its weight is $[F:k]\,\theta_F(L_F) > 0$, contradicting the semistability of the direct sum $X^{[F:k]}$.

(2) Let $S \in \mathsf{sim}(\mathcal{W}_\theta(\Lambda))$. By (1), we have $S_{\overline{k}} \in \mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$, so its socle in $\mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$ is nonzero and Frobenius-invariant. Working in coordinates with respect to a k-basis of S, the Frobenius automorphism sends the reduced row-echelon basis matrix of any Frobenius-invariant $\overline{k}$-subspace to another reduced row-echelon matrix for the same subspace; by uniqueness of the reduced row-echelon form, its entries lie in k, so the subspace descends to a k-subspace of S that is closed under the Λ-action if the original subspace was a $\Lambda^{\overline{k}}$-submodule. Applying this to the socle of $S_{\overline{k}}$ in $\mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$ produces a nonzero Λ-submodule of S of θ-weight zero, which must equal S by θ-stability; hence $S_{\overline{k}}$ is semisimple in $\mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$. Because Frobenius permutes the isotypic components of $S_{\overline{k}}$ while preserving multiplicities, the sum over any single Frobenius orbit likewise descends to a nonzero Λ-submodule of S of weight zero and therefore exhausts S. Thus $S_{\overline{k}}$ consists of a single Frobenius orbit of e pairwise nonisomorphic $\theta_{\overline{k}}$-stable modules $B_1, \dots, B_e$, each occurring with the same multiplicity m.

By Wedderburn's little theorem, $\mathrm{End}_\Lambda(S)$ is a finite field, so $\mathrm{End}_{\Lambda^{\overline{k}}}(S_{\overline{k}}) \cong \mathrm{End}_\Lambda(S) \otimes_k \overline{k}$ is commutative; hence $m = 1$ and $\dim_k \mathrm{End}_\Lambda(S) = e$, giving $\dim_k S = er$ and $\mathrm{endolen}(S) = r$.

To prove surjectivity, let B be a $\theta_{\overline{k}}$-stable $\Lambda^{\overline{k}}$-module, and choose a model B_F of B over a finite extension $F = \mathbb{F}_{q^s}$. Then $(\mathrm{Res}_{F/k} B_F) \otimes_k \overline{k} \cong \bigoplus_{\sigma \in \mathrm{Gal}(F/k)} B^\sigma$ is $\theta_{\overline{k}}$-semistable. Any destabilizing submodule of $\mathrm{Res}_{F/k} B_F$ over k would remain destabilizing after extending scalars, so $\mathrm{Res}_{F/k} B_F \in \mathcal{W}_\theta(\Lambda)$. Any simple subobject $S \subseteq \mathrm{Res}_{F/k} B_F$ in $\mathcal{W}_\theta(\Lambda)$ then has $S_{\overline{k}}$ equal to the

multiplicity-free sum of the Frobenius orbit of B. For injectivity, if two θ-stable Λ-modules S, S' share a simple constituent in $\mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$, then since $S_{\overline{k}}$ and $S'_{\overline{k}}$ are semisimple in $\mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$ they share a direct summand, so $\mathrm{Hom}_{\Lambda^{\overline{k}}}(S_{\overline{k}}, S'_{\overline{k}}) \neq 0$; hence $\mathrm{Hom}_{\Lambda}(S, S') \neq 0$, and therefore $S \cong S'$. □

Proposition A.3. *Suppose Λ is E-infinite over $\mathbb{F}_q$. Choose an integral class g outside $\Sigma(\Lambda)$ and write $g = [P^0] - [P^{-1}]$ with P^0 and P^{-1} sharing no indecomposable direct summand. Set $b_0 = \dim_k P^0$ and $c_0 = \max(1, b_0 \dim_k P^{-1})$. There exists a finite subset $\Omega \subseteq K_0(\mathsf{proj}\,\Lambda)$ such that for every integer $s \geq 1$, Λ has at least $q^s/(c_0 s)$ pairwise nonisomorphic bricks $B \in \mathsf{brick}\,\Lambda$ that are θ-stable for some $\theta \in \Omega$, with $\mathrm{endolen}(B) \leq b_0$ and $\dim_k B \leq s b_0$. In particular:*

1. *some weight $\theta \in \Omega$ and endolength $r \leq b_0$ admit infinitely many pairwise nonisomorphic θ-stable bricks in $\mathsf{mod}\,\Lambda$;*
2. *there exists $c > 0$ such that for all sufficiently large m, Λ has at least $\exp(cm)$ pairwise nonisomorphic bricks of k-dimension (hence also of composition length) at most m.*

Proof. Let $\Omega \subseteq K_0(\mathsf{proj}\,\Lambda)$ be the finite set of all weights of the form $a\,g - b\,[\Lambda]$ with integers $1 \leq a \leq b_0$ and $|b| \leq b_0 \cdot (\dim_k P^0 + \dim_k P^{-1})$. Fix any integer $s \geq 1$ and put $F := \mathbb{F}_{q^s}$. We establish the lower bound in three steps.

Step 1: Ascent of E-infiniteness to F. We check that $g_F \notin \Sigma(\Lambda^F)$. Suppose to the contrary that $U \in \mathsf{K}^{[-1,0]}(\mathsf{proj}\,\Lambda^F)$ were presilting of class g_F. Since g_F is $\mathrm{Gal}(F/k)$-invariant, every Galois twist U^σ ($\sigma \in \mathrm{Gal}(F/k)$) would be presilting of the same class g_F, so $U^\sigma \cong U$ by Lemma 2.1(2). Then $(\mathrm{Res}_{F/k} U) \otimes_k F \cong \bigoplus_\sigma U^\sigma \cong U^s$ would be presilting, so $\mathrm{Res}_{F/k} U$ would be presilting over Λ of class $s\,g$, contradicting $g \notin \Sigma(\Lambda)$ because $\Sigma(\Lambda)$ is a union of cones.

Step 2: Counting stable bricks over F. Put $\mathcal{V} = \mathrm{Hom}_\Lambda(P^{-1}, P^0)$ and $h = \dim_k \mathcal{V}$. We have $h \geq 1$, since otherwise the zero differential on $P^{-1} \to P^0$ would be presilting. The extended projectives P_F^{-1} and P_F^0 still share no indecomposable direct summand, because any common summand would restrict to a nonzero common summand of $(P^{-1})^s$ and $(P^0)^s$. By Proposition 2.7(1) and (2) applied over F, each $p \in \mathcal{V}_F \cong F^h$ yields a $(\theta_p)_F$-stable F-brick B_p of F-dimension at most b_0, where $\theta_p = (\dim_F B_p)\,g - g_F(B_p)\,[\Lambda]$ has scalar extension $(\theta_p)_F = (\dim_F B_p) g_F - g_F(B_p)[\Lambda^F]$. Bounding each projective Hom space gives $|g_F(B_p)| \leq (\dim_F B_p)(\dim_k P^0 + \dim_k P^{-1})$, so $\theta_p \in \Omega$. A nonzero polynomial of total degree d on F^h has at most $d|F|^{h-1}$ zeros by the Schwartz–Zippel bound [64, Lemma 1]. By Proposition 2.7(2), each isomorphism class of B_p accounts for at most $c_0|F|^{h-1}$ points of F^h. Hence at least $|F|^h/(c_0|F|^{h-1}) = q^s/c_0$ pairwise nonisomorphic F-bricks arise across Ω.

Step 3: Descending to Λ-bricks. For each such F-brick B (say θ_F-stable for $\theta \in \Omega$), Lemma A.2(2) identifies the simple constituents of $B_{\overline{k}}$ in $\mathcal{W}_{\theta_{\overline{k}}}(\Lambda^{\overline{k}})$ with a single $\mathrm{Gal}(\overline{k}/F)$-orbit of $\theta_{\overline{k}}$-stable $\Lambda^{\overline{k}}$-modules of $\overline{k}$-dimension $\mathrm{endolen}(B) \leq b_0$. Its $\mathrm{Gal}(\overline{k}/k)$-orbit then corresponds to a single θ-stable Λ-module S of the same endolength. As in the surjectivity proof of Lemma A.2(2), S occurs as a simple subobject of $\mathrm{Res}_{F/k}(B)$ in $\mathcal{W}_\theta$, so $\dim_k S \leq s \dim_F B \leq s b_0$.

To bound the fibers of $B \mapsto S$, note that by Krull–Schmidt, S determines the $\mathrm{Gal}(\overline{k}/k)$-orbit as the set of isomorphism classes of indecomposable summands of $S_{\overline{k}}$, independently of θ. This $\mathrm{Gal}(\overline{k}/k)$-orbit splits into at most $[F : k] = s$ orbits under $\mathrm{Gal}(\overline{k}/F)$. Each $\mathrm{Gal}(\overline{k}/F)$-orbit determines $B_{\overline{k}}$ as the direct sum of its members, which by flat base change determines $\dim_F B$ and $g_F(B)$ and hence the weight θ_p used in Step 2; at that weight, Lemma A.2(2) gives at most one F-brick for the orbit. Consequently, at most s of the F-bricks counted in Step 2 yield the same Λ-brick S, leaving at least $q^s/(c_0 s)$ pairwise nonisomorphic Λ-bricks with the stated bounds.

(1) The lower bound $q^s/(c_0 s)$ is unbounded as $s \to \infty$. Since Ω and $\{1, \ldots, b_0\}$ are finite, some pair (θ, r) occurs for infinitely many pairwise nonisomorphic bricks.

(2) Given $m \geq b_0$, set $s := \lfloor m/b_0 \rfloor$; then $s b_0 \leq m$ and $q^s/(c_0 s) \geq \exp(cm)$ for a constant $c > 0$ and all sufficiently large m. □

A.2. **Descent for wide subcategories and localizations.** Lemma A.4 separates a family of representations by Zariski-open subsets over an algebraically closed field. We apply it after scalar

extension to $\overline{k}$ to find a stable brick B whose subcategory $\mathsf{Filt}(B)$ fails both approximation properties, and Lemma A.6 uses this brick to construct an infinite-dimensional localization at a single morphism.

Lemma A.4. *Let Λ be a basic finite-dimensional algebra over $k = \overline{k}$, let $d \in \mathbb{N}^n$, and let $\{B_i\}_{i\in I}$ be a family of modules in $\mathsf{rep}(\Lambda, d)$. Suppose that for each $i \in I$ there exists $G_i \in \mathsf{mod}\,\Lambda$ such that $\operatorname{Hom}_\Lambda(G_i, B_i) \neq 0$, $\operatorname{Ext}^1_\Lambda(G_i, B_i) = 0$, and $\operatorname{Hom}_\Lambda(G_i, B_j) = 0$ for all $j \neq i$. Then I is finite.*

Proof. Fix $i \in I$ and choose a short exact sequence

$$0 \longrightarrow K_i \longrightarrow \Lambda^{a_i} \longrightarrow G_i \longrightarrow 0$$

in $\mathsf{mod}\,\Lambda$. Applying $\operatorname{Hom}_\Lambda(-, X)$ for any $X \in \mathsf{mod}\,\Lambda$ yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(G_i, X) \longrightarrow \operatorname{Hom}_\Lambda(\Lambda^{a_i}, X) \longrightarrow \operatorname{Hom}_\Lambda(K_i, X) \longrightarrow \operatorname{Ext}^1_\Lambda(G_i, X) \longrightarrow 0,$$

and hence

$$\begin{aligned}\dim_k \operatorname{Hom}_\Lambda(G_i, X) &= a_i \dim_k X - \dim_k \operatorname{Hom}_\Lambda(K_i, X) + \dim_k \operatorname{Ext}^1_\Lambda(G_i, X)\\ &\geq a_i \dim_k X - \dim_k \operatorname{Hom}_\Lambda(K_i, X).\end{aligned}$$

Equality holds for $X = B_i$ because $\operatorname{Ext}^1_\Lambda(G_i, B_i) = 0$. Since the function $X \mapsto \dim_k \operatorname{Hom}_\Lambda(K_i, X)$ is upper semicontinuous (Crawley-Boevey and Schröer [24, Lemma 4.3]), the subset

$$\mathcal{O}_i := \{X \in \mathsf{rep}(\Lambda, d) \mid \dim_k \operatorname{Hom}_\Lambda(K_i, X) \leq \dim_k \operatorname{Hom}_\Lambda(K_i, B_i)\}$$

is Zariski-open in $\mathsf{rep}(\Lambda, d)$ and contains B_i. For every $X \in \mathcal{O}_i$, we have

$$\dim_k \operatorname{Hom}_\Lambda(G_i, X) \geq a_i \dim_k B_i - \dim_k \operatorname{Hom}_\Lambda(K_i, B_i) = \dim_k \operatorname{Hom}_\Lambda(G_i, B_i) > 0,$$

so $\operatorname{Hom}_\Lambda(G_i, X) \neq 0$. Thus $\mathcal{O}_i$ contains B_i and omits B_j for every $j \neq i$. Hence $\{B_i \mid i \in I\}$ is a discrete subspace of the Noetherian topological space $\mathsf{rep}(\Lambda, d)$, so I is finite. □

Lemma A.5. *If Λ is E-infinite, then there exist an integral weight $\theta \in K_0(\mathsf{proj}\,\Lambda)$ and a θ-stable brick $B \in \mathsf{brick}\,\Lambda$ such that $\mathsf{Filt}_\Lambda(B)$ is neither covariantly nor contravariantly finite.*

Proof. We distinguish two cases according to whether k is infinite or finite.

- *Case 1: k is infinite.* Proposition 2.8(1) yields an integral weight θ and infinitely many pairwise nonisomorphic θ-stable modules M_i of a single class in $K_0(\mathsf{mod}\,\Lambda)$. Because scalar extension to $\overline{k}$ is exact, it induces a homomorphism $K_0(\mathsf{mod}\,\Lambda) \to K_0(\mathsf{mod}\,\Lambda^{\overline{k}})$, so the modules $N_i := (M_i)_{\overline{k}}$ all share the same dimension vector over $\Lambda^{\overline{k}}$. For $i \neq j$, flat base change gives $\operatorname{Hom}_{\Lambda^{\overline{k}}}(N_i, N_j) = 0$ from $\operatorname{Hom}_\Lambda(M_i, M_j) = 0$, so the N_i are pairwise nonisomorphic. Suppose every $\mathsf{Filt}_\Lambda(M_i)$ admitted an Ext-projective generator G_i. Setting $H_i := (G_i)_{\overline{k}}$, flat base change gives

  $$\operatorname{Hom}_{\Lambda^{\overline{k}}}(H_i, N_i) \neq 0, \qquad \operatorname{Ext}^1_{\Lambda^{\overline{k}}}(H_i, N_i) = 0, \qquad \operatorname{Hom}_{\Lambda^{\overline{k}}}(H_i, N_j) = 0 \quad (j \neq i),$$

  where the last equality holds because $G_i \in \mathsf{Filt}_\Lambda(M_i)$ and $\operatorname{Hom}_\Lambda(M_i, M_j) = 0$. Passing via Morita equivalence to a basic algebra preserves these Hom and Ext spaces as well as equality of dimension vectors, so Lemma A.4 yields a contradiction.
- *Case 2: $k = \mathbb{F}_q$ is finite.* Proposition 2.8(1) yields $\theta \in K_0(\mathsf{proj}\,\Lambda)$, $r \geq 1$, and infinitely many pairwise nonisomorphic θ-stable Λ-modules $\{M_i\}$ of endolength r. By Lemma A.2(2), each $(M_i)_{\overline{k}}$ is a multiplicity-free direct sum of a single Frobenius orbit of $\theta_{\overline{k}}$-stable $\Lambda^{\overline{k}}$-modules of $\overline{k}$-dimension r, and distinct M_i have disjoint orbits. For each i, choose one such constituent C_i. Because all C_i have $\overline{k}$-dimension r, only finitely many dimension vectors over $\Lambda^{\overline{k}}$ occur, so infinitely many of the C_i share a single dimension vector; we restrict to those indices. Suppose every $\mathsf{Filt}_\Lambda(M_i)$ admitted an Ext-projective generator G_i, and set $H_i := (G_i)_{\overline{k}}$. Some finite direct sum of copies of H_i maps onto $(M_i)_{\overline{k}}$ and hence onto C_i, so $\operatorname{Hom}_{\Lambda^{\overline{k}}}(H_i, C_i) \neq 0$. Since C_i is a direct summand of $(M_i)_{\overline{k}}$, base change gives $\operatorname{Ext}^1_{\Lambda^{\overline{k}}}(H_i, C_i) = 0$ from $\operatorname{Ext}^1_\Lambda(G_i, M_i) = 0$. For $j \neq i$, every stable constituent of $(M_i)_{\overline{k}}$ is Hom-orthogonal to C_j, so the filtration of G_i by M_i implies $\operatorname{Hom}_{\Lambda^{\overline{k}}}(H_i, C_j) = 0$.

After passing to a Morita equivalent basic algebra, Lemma A.4 applied to H_i and C_i again gives a contradiction.

In either case there is an index i such that $\mathsf{Filt}_\Lambda(M_i)$ has no Ext-projective generator; by [32, Proposition 4.12], it is neither covariantly nor contravariantly finite. Setting $B := M_i$ completes the proof. □

We now descend the invertibility criterion of Proposition 2.5(2) to obtain an infinite-dimensional localization at a single morphism of projectives over k.

Lemma A.6. *If Λ is E-infinite, then there exists a morphism p in $\operatorname{proj}\Lambda$ such that both Λ_p and its quotient by the common annihilator of its finite-dimensional modules are infinite-dimensional.*

Proof. Choose $\theta = [Q^0] - [Q^{-1}]$ and B by Lemma A.5. By Lemma A.2(1), $B_{\overline{k}}$ is $\theta_{\overline{k}}$-semistable, so Proposition 2.5(2) yields an integer $a > 0$ and a morphism $p_0 \colon V^{-1} \to V^0$ in $\operatorname{proj}\Lambda^{\overline{k}}$ of class $a\theta_{\overline{k}}$ such that $\operatorname{Hom}_{\Lambda^{\overline{k}}}(p_0, B_{\overline{k}})$ is bijective.

To descend the map, we first arrange that both projective terms are defined over k. Choose $m \geq 1$ and a complement $Q_c \in \operatorname{proj}\Lambda^{\overline{k}}$ with $V^{-1} \oplus Q_c \cong (\Lambda^{\overline{k}})^m$. Since $[V^0] + [Q_c] + a[Q_{\overline{k}}^{-1}] = a[Q_{\overline{k}}^0] + m[\Lambda^{\overline{k}}]$ in $K_0(\operatorname{proj}\Lambda^{\overline{k}})$, the Krull–Schmidt theorem yields an isomorphism $V^0 \oplus Q_c \oplus (Q_{\overline{k}}^{-1})^a \cong (Q_{\overline{k}}^0)^a \oplus (\Lambda^{\overline{k}})^m$. Taking the direct sum of p_0 with $\mathrm{id}_{Q_c \oplus (Q_{\overline{k}}^{-1})^a}$ produces a morphism $\widetilde{p} \colon \widetilde{Q}_{\overline{k}}^{-1} \to \widetilde{Q}_{\overline{k}}^0$ between the scalar extensions of $\widetilde{Q}^{-1} := (Q^{-1})^a \oplus \Lambda^m$ and $\widetilde{Q}^0 := (Q^0)^a \oplus \Lambda^m$ (with $[\widetilde{Q}^0] - [\widetilde{Q}^{-1}] = a\theta$) such that $\operatorname{Hom}_{\Lambda^{\overline{k}}}(\widetilde{p}, B_{\overline{k}})$ is bijective.

Choose a finite extension F/k inside $\overline{k}$ over which the matrices of $\widetilde{p} \colon \widetilde{Q}_F^{-1} \to \widetilde{Q}_F^0$ are defined; then $\operatorname{Hom}_{\Lambda^F}(\widetilde{p}, B_F)$ is bijective. Set $p := \operatorname{Res}_{F/k}(\widetilde{p}) \colon (\widetilde{Q}^{-1})^{[F:k]} \to (\widetilde{Q}^0)^{[F:k]}$ in $\operatorname{proj}\Lambda$, of class $[F:k]a\theta$. Choosing any nonzero k-linear functional $\lambda \colon F \to k$, the nondegenerate pairing $(x, y) \mapsto \lambda(xy)$ identifies $\operatorname{Hom}_k(F, B) \cong B \otimes_k F = B_F$ as right Λ^F-modules (even when F/k is inseparable). By adjunction between restriction and coinduction,

$$\operatorname{Hom}_\Lambda(p, B) \cong \operatorname{Hom}_{\Lambda^F}(\widetilde{p}, \operatorname{Hom}_k(F, B)) \cong \operatorname{Hom}_{\Lambda^F}(\widetilde{p}, B_F)$$

is bijective, so B is a Λ_p-module by (7.3). For any Λ_p-submodule $L \subseteq B$, bijectivity of $\operatorname{Hom}_\Lambda(p, L)$ implies $[F:k]a\theta(L) = 0$; by θ-stability of B, we have $L = 0$ or $L = B$, so B is simple over Λ_p.

Suppose now that the quotient R of Λ_p by the common annihilator of its finite-dimensional modules were finite-dimensional. Let e be the sum of the primitive idempotents corresponding to the simple R-modules other than B in a decomposition of 1_R. Since $\operatorname{Mod}\Lambda_p$ is extension-closed in $\operatorname{Mod}\Lambda$ by Schofield [62, Theorem 4.7] and $\operatorname{mod} R = \operatorname{mod}\Lambda_p$, restriction of scalars identifies $\operatorname{mod}(R/ReR)$ with $\mathsf{Filt}_\Lambda(B)$, whose objects are precisely the iterated extensions of B. By [32, Proposition 4.12], $\mathsf{Filt}_\Lambda(B)$ would then be functorially finite, contradicting the choice of B. Therefore both R and Λ_p are infinite-dimensional. □

We now assemble the results of Sections 2 to 7 and Appendix A to complete the proof of Theorem A.

Completion of the proof of Theorem A. (I) Combine Theorems 3.2, 4.1, 5.1, 5.2, and 7.8 with Proposition 5.3(2).

(II) This is Corollary 5.4.

(III) This is Theorem 6.1. □

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