Streamlining Three Recent Proofs of the Reinhardt Conjecture

Abstract

The Reinhardt Conjecture states that, among centrally symmetric planar convex disks, the smoothed regular octagon minimizes the density of the densest packing, with equality precisely for its invertible affine images. Three long proofs were announced in November 2026, by The-Anh Vu-Le, Lucas Schmutz, and Fei Xue. Our purpose is to streamline these proofs by combining their ideas in a single argument. In the optimal-control formulation of Hales and Vajjha, a minimizer gives a finite walk on a triangle. The conserved momentum defines a quadratic form on the plane of the disk. Its values along the straight edges express stationarity of the area. In a walk with switches in both directions, a visit of least active margin forces a second reverse center nearby. Stationarity there excludes such a minimizer. In an all-forward walk, the form bounds the sum of two adjacent angles. Winding then forces exactly four visits. Among the resulting four-arc periods the smoothed octagon uniquely minimizes area up to cyclic offset. The accompanying verification package gives exact checks for the finite calculations.

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1 The packing theorem

In 1934 Reinhardt conjectured that the centrally symmetric planar convex disk of least maximal packing density is the regular octagon with its corners rounded by hyperbolic arcs [1]. Mahler related the problem to critical lattices and circumscribed hexagons [2, 3]. Fejes Tóth proved that lattice and unrestricted packing densities agree for these disks [4]. Hales developed a variational formulation [5] and an optimal-control formulation [6]. Vajjha confined minimizers to a compact region of the state space [7, Theorem 2.3.15]. Hales and Vajjha developed the geometric reduction and the analysis of optimal controls in *Packings of Smoothed Polygons* (PSP) [8].

In November 2026 Vu-Le, Schmutz, and Xue announced proofs of the conjecture [9, 11, 10]. From Schmutz we use the five-dimensional reduced Pontryagin system and the one-sided restriction to an edge of the control triangle. From Vu-Le's Lean development we take the description of a finite period as a walk, its signed area sum in adjacent margins, and the quadratic form obtained by measuring the conserved momentum along the straight edges of the disk. Vu-Le's analysis also pairs the reverse switches and marks a visit of least active margin. Both Vu-Le and Xue use area to bound the momentum determinant and compare areas throughout the family of closed four-arc motions. Xue's use of cost bounds on short windows also informs the local comparisons below.

Two further reductions account for much of the shortening. In a mixed walk the smaller neighbor of a least-margin visit determines which way to look for a second reverse center. Exactly two turns intervene, so one stationarity test at that center suffices. For a forward walk we measure the momentum in the moving tangent frame. The two visits incident to a switch then give a bound on the sum of their angles in the metric defined by the momentum. The winding identity converts this local bound into a count of four visits.

For a planar convex disk K let $\delta(K)$ be the supremum of the densities of packings by congruent copies of K. Write K_{oct} for the smoothed octagon: in the regular octagon of inradius one, with $\mathrm{dist}_r^{\mathrm{oct}}$ the inward distance to side r, the corner between sides $r, r+1$ is replaced by the arc of $\mathrm{dist}_{r-1}^{\mathrm{oct}}\,\mathrm{dist}_{r+2}^{\mathrm{oct}} = \sqrt{2}/4$ tangent to these two sides.

Theorem 1 (The Reinhardt conjecture)**.** *Every centrally symmetric planar convex disk K satisfies*

$$\delta(K) \geq \delta_{\mathrm{oct}} \coloneqq \frac{8 - 4\sqrt{2} - \log 2}{2\sqrt{2} - 1},$$

with equality exactly for the invertible affine images of K_{oct}.

Section 2 recalls the control formulation. Section 3 reduces the theorem to three propositions on walks, and Sections 4–10 prove them. The forward argument in Section 9 reads two consecutive edge angles in the switch metric and compares their sum with the winding of the period. Section 11 lists the exact claims used in the proof.

2 The control problem

2.1 Frame, state and margins

In the normalization of a least-density disk, a family of circumscribed centrally symmetric hexagons of least area has its edge midpoints on the boundary [8, Sections 2.2 and 2.4]. Distinguish the circumscribed hexagon from the inscribed hexagon formed by its midpoints. In reference position the latter has vertices $\mathbf{s}_r^* = R^r(1,0)^t$ for r mod 6, where R rotates through $\pi/3$. The contact frame $g(t) \in \mathrm{SL}_2(\mathbb{R})$ carries these six reference points to the six moving midpoints. Two successive

midpoints $\mathbf{u}, \mathbf{v}$ determine the area of the circumscribed hexagon: if its first three vertices are $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the polygon area formula gives

$$4\det(\mathbf{u}, \mathbf{v}) = \det(\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}) = \det(\mathbf{a}, \mathbf{b}) + \det(\mathbf{a}, \mathbf{c}) + \det(\mathbf{b}, \mathbf{c}).$$

Consequently the circumscribed hexagon always has area $4\det(\mathbf{s}_0^*, \mathbf{s}_1^*) = \sqrt{12}$. The state is $X = g^{-1}\dot{g}$. One period of duration t_f carries the reference midpoints to their next positions, $g(t_f) = R$ (Figure 1).

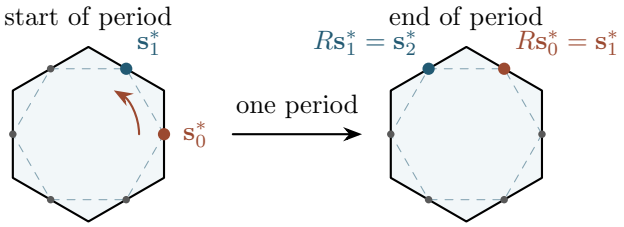

Figure 1: The solid outline is the circumscribed reference hexagon. Its six edge midpoints form the dashed inscribed hexagon. After a period the midpoint labels have advanced by one.

We use the coordinates of PSP [8, Sections 3.1–3.6 and 4.1–4.4]. Let $\langle A, B\rangle := \operatorname{tr}(AB)$ on $\mathfrak{sl}_2(\mathbb{R})$, $a_* := 1/\sqrt{3}$, and

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R := \exp(\pi J/3), \qquad \Phi(x,y) := \frac{1}{y}\begin{pmatrix} x & -x^2-y^2 \\ 1 & -x \end{pmatrix}.$$

Conjugation on the orbit of J is the fractional linear action on the upper half-plane via $X = \Phi(x,y)$. The state lies over the ideal triangle $\mathfrak{h}^* := \{x+iy : y>0,\ |x|<a_*,\ x^2+y^2>a_*^2\}$, whose center is $i = \Phi^{-1}(J)$ and whose sides are permuted by R. The control set is $U_T := \{\mathbf{u} \in \mathbb{R}^3 : u_i \geq 0,\ \sum u_i = 1\}$ with vertices $\mathbf{e}_i$ and control indices read mod 3. We label a vertex by the index of its nonzero coordinate, so $\mathbf{e}_i$ here is the vertex denoted $\mathbf{e}_{i+1}$ in PSP's one-based vertex notation. Put

$$Z_{\mathbf{u}} := \begin{pmatrix} a_*(u_1-u_2) & (u_0-2u_1-2u_2)/3 \\ u_0 & a_*(u_2-u_1) \end{pmatrix},$$
$$p_{\mathbf{u}}(X) := -\sqrt{3}\langle X, Z_{\mathbf{u}}\rangle, \qquad P_{\mathbf{u}} := -\sqrt{3}Z_{\mathbf{u}}/p_{\mathbf{u}}(X), \qquad P_i := P_{\mathbf{e}_i}.$$

The *margin* for control i is $p_i(X) := p_{\mathbf{e}_i}(X)$; we use the same formula for the linear coordinates p_i on all of $\mathfrak{sl}_2(\mathbb{R})$. Explicitly

$$p_0 = \frac{\sqrt{3}(x^2+y^2-a_*^2)}{y}, \qquad p_1 = \frac{2(a_*-x)}{y}, \qquad p_2 = \frac{2(a_*+x)}{y}, \tag{2.1}$$

so side 0 is the semicircle and sides 1, 2 are the right and left vertical lines. In PSP's star functions $(p_0, p_1, p_2) = 2(\rho_2^{\mathrm{PSP}}, \rho_1^{\mathrm{PSP}}, \rho_0^{\mathrm{PSP}})$ [8, Corollary 3.1.6].

Lemma 2.1 (Hyperbolic dictionary)**.** *Let $X = \Phi(z)$ with $z \in \mathfrak{h}^*$, and let $\operatorname{dist}_i^{\mathrm{hyp}}(z)$ be the hyperbolic distance from z to side i. Then*

$$\begin{aligned} p_i(X) &= 2\sinh \operatorname{dist}_i^{\mathrm{hyp}}(z), \qquad p_0p_1 + p_1p_2 + p_2p_0 = 4, \\ \tfrac{\sqrt{3}}{2}(p_0+p_1+p_2) &= -\tfrac{3}{2}\operatorname{tr}(JX) = 3\cosh \operatorname{dist}(z,i). \end{aligned} \tag{2.2}$$

On all of $\mathfrak{sl}_2(\mathbb{R})$ the quadric identity reads $p_0p_1 + p_1p_2 + p_2p_0 = 4\det X$.

Proof. For the vertical line $x = b$ the distance formula gives $\sinh(\mathrm{dist}_i^{\mathrm{hyp}}(z)) = |x - b|/y$. The isometry $z \mapsto (z + a_*)/(a_* - z)$ sends the circular side to the imaginary axis, and at the image w one finds $-\Re w/\Im w = (x^2 + y^2 - a_*^2)/(2a_* y)$. The quadric and the sum follow from (2.1), and $-\operatorname{tr}(JX)/2 = (x^2 + y^2 + 1)/(2y)$ is the hyperbolic cosine of the distance from z to i. □

A *vertex arc* is an interval of positive duration on which the control is one vertex $\mathbf{e}_i$ almost everywhere; its *active margin* is $p_i(X)$. It is constant on the arc because P_i is a multiple of $Z_{\mathbf{e}_i}$ and $\operatorname{tr}(Z_{\mathbf{e}_i}[P_i, X]) = 0$. Thus the state runs along a curve at fixed distance from side i (Figure 2), approaching side $i - 1$ and receding from side $i + 1$.

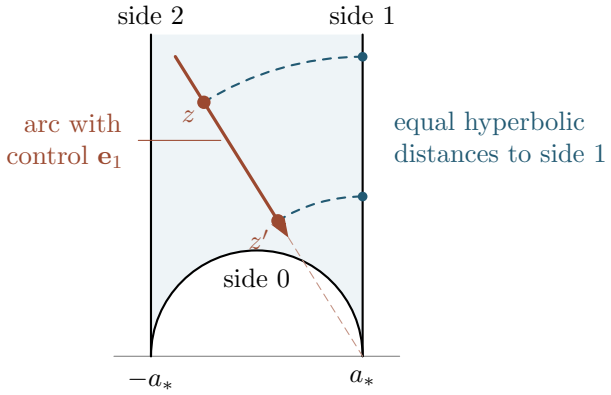

Figure 2: On a vertex arc with control $\mathbf{e}_1$, the state moves along a straight ray in the half-plane chart toward the ideal endpoint a_*. The two dashed perpendiculars have the same hyperbolic length. Their Euclidean sizes differ because the height above the real axis also decreases along the arc. The continuation of the ray is shown dashed.

For free terminal time $t_f > 0$, a measurable control determines an *admissible motion* if the absolutely continuous frame and state satisfy

$$\dot{g} = gX, \qquad \dot{X} = [P_{\mathbf{u}}, X], \qquad g(0) = I, \quad g(t_f) = R, \quad X(t_f) = R^{-1}X(0)R, \tag{2.3}$$

where the differential equations hold almost everywhere, $g(t) \in \mathrm{SL}_2(\mathbb{R})$, and X lies over $\mathfrak{h}^*$ [8, Sections 3.2 and 4.6]. Extend a period by $g(t + t_f) = g(t)R$ and $X(t + t_f) = R^{-1}X(t)R$. The conjugation of the state is the *twist*. It advances the control labels by one mod 3, which we call the *offset*. By Green's theorem, the area of a centrally symmetric planar convex disk with positively oriented boundary is $\frac{1}{2}\int_{\partial K}(x\,dy - y\,dx)$. For the normalized disk, the six oriented contact paths $g(t)\mathbf{s}_r^*$, $0 \le r < 6$, together trace this boundary over one period of the frame. Summing their contributions gives the area functional

$$A := \int_0^{t_f} \alpha(X(t))\,dt, \qquad \alpha(X) := -\tfrac{3}{2}\operatorname{tr}(JX) = \tfrac{\sqrt{3}}{2}(p_0 + p_1 + p_2) \ge 3, \tag{2.4}$$

by Lemma 2.1. Write $A_{\mathrm{oct}} := \sqrt{12}\,\delta_{\mathrm{oct}} < \pi$.

2.2 Polar angle and recovery of the disk

Definition 2.2 (Polar angle)**.** For a continuous path $g(t) \in \mathrm{SL}_2(\mathbb{R})$, take the polar decomposition $g(t) = \exp(\theta(t)J)B(t)$, where $B(t)$ is symmetric positive definite. A *polar angle* of the path is a

continuous real lift θ of the rotation angle. Any two lifts differ by a constant multiple of 2π, so the advance $\theta(t_2) - \theta(t_1)$ is well-defined along the specified path.

Lemma 2.2 (Polar speed and one boundary traversal)**.** *Fix $h \in \mathrm{SL}_2(\mathbb{R})$ independently of time. For $g' = h^{-1}g$ put $r = -\operatorname{tr}(JX)/2$ and $r' = -\operatorname{tr}(Jg'X(g')^{-1})/2$. Both are positive. The polar angle of g' satisfies, almost everywhere,*

$$0 < \dot\theta \le \frac{2rr'}{r + r'} < 2r.$$

If $A < \pi$, the contact curve $t \mapsto g(t)\mathbf{s}_0^*$ runs once around a centrally symmetric convex disk in time $6t_f$, with nondecreasing tangent angle. The disk has area A and packing density $A/\sqrt{12}$.

Proof. Write $g' = \exp(\theta J)B$. The matrix $-JX$ is positive definite by the formula for Φ. We have $2r = \operatorname{tr}(-JX)$ and $2r' = \operatorname{tr}(B^{-2}(-JX))$, since $BJB = J$. The equation $\dot g' = g'X$ gives $\dot\theta JB + \dot B = BX$. Take the trace after multiplication by $-J$. The symmetry of $\dot B$ and the relation $BJ = JB^{-1}$ give

$$\dot\theta = \frac{\operatorname{tr}(B^{-1}(-JX))}{\operatorname{tr} B}.$$

Cayley–Hamilton gives $I + B^{-2} = (\operatorname{tr} B)B^{-1}$. Cauchy–Schwarz in the inner product $(U, V) \mapsto \operatorname{tr}(U^tV(-JX))$, applied to I, B^{-1}, now yields

$$(r + r')\dot\theta = \tfrac{1}{2}[\operatorname{tr}(B^{-1}(-JX))]^2 \le 2rr'.$$

Take $h = I$. Since $g(0) = I$ and $g(t_f) = R$, the advance is congruent to $\pi/3$ modulo 2π. It is less than $2A/3 < 2\pi/3$, so it is exactly $\pi/3$. Contract the positive polar factor by $B \mapsto B^s$, $1 \ge s \ge 0$, fixing both endpoints. The frame thus belongs to the homotopy class of the rotation from I to R through $\pi/3$. The reconstruction theorem gives a centrally symmetric disk with these boundary contacts, critical determinant $\sqrt{3}/2$, and area A [6, Sections 2.2–2.3], [11, Proposition 2.3(c)]. The contact makes one circuit: its position angle increases since $\det(\mathbf{s}_0^*, X\mathbf{s}_0^*) > 0$, and the polar homotopy over six periods gives it degree one. Convexity gives the nondecreasing tangent angle. The packing formula gives $\delta = A/\sqrt{12}$ [8, Theorem 2.2.2]. □

2.3 Geometric reduction and compact core

We use the following facts [8, Chapters 2–5].

- A least-density disk K exists and admits the normalization by contacts above. Its frame and state satisfy (2.3), and its area (2.4) is $\sqrt{12}\,\delta(K)$.
- An admissible motion of area $A < \pi$ reconstructs a centrally symmetric disk of area A and packing density $A/\sqrt{12}$ (Lemma 2.2).
- At an area minimizer the state is confined to Vajjha's compact core $\mathfrak{h}^{**}$ [7, Theorem 2.3.15], [8, Definition 5.0.1, Theorem 5.0.2]. The numerical consequence used at the second reverse center in Section 8 is

  $$p_i(X) \ge 2/15 \qquad (i = 0, 1, 2). \tag{2.5}$$

  By (2.2) this keeps the state at least $\operatorname{arsinh}(1/15)$ from each side.
- The octagon is realized by four equal arcs with labels $2, 0, 1, 2$ starting at $\Phi(0, y_0)$, $y_0^2 = (2\sqrt{2} - 1)/3$, each of duration $\Delta t_{\mathrm{oct}} \coloneqq \log 2/(2\sqrt{2\sqrt{2} - 1})$, with area A_{oct} [8, Section 7.5].

Hence the normalized realization of a least-density disk is a global minimizer of (2.3), of area at most A_{oct}: a trajectory of smaller area would be below π and reconstruct a disk of smaller density.

2.4 Maximum principle

Apply the free-time maximum principle on $\mathrm{SL}_2(\mathbb{R}) \times \mathfrak{h}^*$ [8, Sections 6.2–6.7], with the state X on its two-dimensional orbit. In the trace pairing its covector has a unique representative Λ_2 orthogonal to X. Use $\Lambda_R \coloneqq -[X, \Lambda_2]$ as the state costate. The commutator is invertible on this tangent plane since its square is -4 there. The frame costate Λ_1 is left-trivialized. The area multiplier is $k_0 \geq 0$, equal to $-\lambda_{\mathrm{cost}}$ in PSP's sign convention. Write $L \coloneqq \Lambda_1 + \frac{3}{2} k_0 J$ for the shifted frame costate. The Hamiltonian and equations are

$$\mathcal{H}_{\mathbf{u}} \coloneqq \langle L, X \rangle - \langle \Lambda_R, P_{\mathbf{u}} \rangle, \tag{2.6}$$

$$\dot{\Lambda}_1 = [\Lambda_1, X], \qquad \dot{\Lambda}_R = [P_{\mathbf{u}}, \Lambda_R] - \langle \Lambda_R, P_{\mathbf{u}} \rangle [P_{\mathbf{u}}, X] - [L, X]. \tag{2.7}$$

The costates are absolutely continuous, $\langle \Lambda_R, X \rangle = 0$, and $(k_0, \Lambda_1, \Lambda_R)$ is nonzero. The state term of the Hamiltonian comes from $\operatorname{tr}(\Lambda_2 [P_{\mathbf{u}}, X]) = -\operatorname{tr}(\Lambda_R P_{\mathbf{u}})$. Differentiating the state field in a direction Y gives $[P_{\mathbf{u}}, Y] - \operatorname{tr}(P_{\mathbf{u}} Y)[P_{\mathbf{u}}, X]$. Its adjoint gives the equation for Λ_2 modulo the normal line spanned by X. Commuting with X and differentiating $\Lambda_R = -[X, \Lambda_2]$ gives (2.7) in the reduced variables.

The frame costate obeys the same endpoint conjugation as X because $g \Lambda_1 g^{-1}$ is constant. Write $d \coloneqq \det \Lambda_1$ for its conserved determinant and $d_1 \coloneqq \sqrt{|d|}$ for its scale [8, Sections 7.5 and 10.1]. For a smooth endpoint variation, write ∂_λ for the derivative at $\lambda = 0$. Then $\partial_\lambda X(t_f) = R^{-1} \partial_\lambda X(0) R$. Transversality on this coupled endpoint condition gives $\Lambda_2(t_f) = R^{-1} \Lambda_2(0) R$ in the orthogonal representatives, hence the same twist for Λ_R. This is the reduced maximum principle used by Schmutz [11, Section 2], with the area multiplier sign chosen here as in (2.6).

A motion equipped with these costates and the area multiplier has a *Pontryagin lift*. The maximized Hamiltonian is zero and the control attains this maximum almost everywhere. In fact

$$P_{\mathbf{u}} = \sum_i \frac{u_i p_i(X)}{p_{\mathbf{u}}(X)} P_{\mathbf{e}_i},$$

so the Hamiltonian is a convex combination of its vertex values. Every vertex Hamiltonian is nonpositive at every time by continuity. The Hamiltonian of an active arc vanishes along the whole closed arc, including its two ends. A lift with $k_0 > 0$ can be rescaled to $k_0 = 1$ and is called normal. The case $k_0 = 0$ is abnormal. For the individual vertex values it is convenient to use

$$\widehat{\chi}_i \coloneqq p_i(X) \mathcal{H}_{\mathbf{e}_i}.$$

The weight is positive, so $\widehat{\chi}_i$ has the same sign and zeros as the vertex Hamiltonian. In PSP the switching function χ_{ij} is the difference $\mathcal{H}_{\mathbf{e}_i} - \mathcal{H}_{\mathbf{e}_j}$ along an arc of control $\mathbf{e}_i$ [8, Section 7.5].

3 The endgame as walks on a triangle

A vertex arc is a *visit* to a vertex of the triangle with labels $0, 1, 2$. Merge successive arcs with the same control and number the visits in time order. A *switch* is a directed edge from one visit to the next. We call $i \to i + 1$ *forward* and $i \to i - 1$ *reverse*, with labels read modulo three. A *run* is a maximal string of switches in the same direction. A *turn visit* has opposite incoming and outgoing directions, a *reverse center* has two reverse switches, and a walk is *mixed* if both directions occur.

Choose a switch as the period cut and let N be the number of visits in a period. Visit k has label i_k, starts at t_k, and its outgoing switch at t_{k+1} has sign $\epsilon_k = \pm 1$ (forward/reverse). By the offset return,

$$i_{k+N} = i_k + 1 \pmod 3, \qquad t_{k+N} = t_k + t_f, \tag{3.1}$$

so $\sum_{k<N} \epsilon_k \equiv 1 \pmod 3$. A *window* is a block of consecutive visits of the extended walk and can cross the cut. The octagon has walk `01201`. The last digit is the first label after the cut, so this word records four arcs. The duration, active margin and arc factor of visit k, and the position parameter within it, are

$$\Delta t_k \coloneqq t_{k+1} - t_k, \quad p[k] \coloneqq p_{i_k}(X(t_k)), \quad q_k \coloneqq \exp(2\Delta t_k / p[k]), \quad q(t) \coloneqq \exp(2(t - t_k)/p[k]). \tag{3.2}$$

In hyperbolic terms, visit k translates the state along side i_k through the distance $\log q_k$. The *arc area contribution* $A_k \coloneqq \int_{t_k}^{t_{k+1}} \alpha(X(t))\, dt$ is the contribution of visit k to A. Margins, durations, factors, arc area contributions and signs repeat with period N; only the labels are offset.

A walk is *normal* if its period has a normal Pontryagin lift ($k_0 = 1$) and *competitive* if its area is at most A_{oct}. Sections 4 and 5 show that every minimizer is a competitive normal walk in the compact core.

3.1 Reverse runs

Proposition 3.1 (Reverse steps)**.** *In a normal walk every reverse step belongs to a run of exactly two.*

For example, the walks `0101` and `0210` are disallowed.

3.2 Mixed periods

Proposition 3.2 (Mixed directions)**.** *A competitive normal walk whose active margins are all at least* $2/15$ *has only forward steps.*

3.3 Forward periods

Proposition 3.3 (Forward steps)**.** *A competitive normal walk with only forward steps has four arcs and realizes the smoothed octagon, up to cyclic offset.*

Three geometric facts organize the proof. The first is hyperbolic. The state is a point of the ideal triangle, its margins measure its distances to the three sides (Lemma 2.1), and a vertex arc is a translation along one side. So a walk is recorded by its active margins $p[k]$, and its area is the sum over the switches of one function of the two adjacent margins (Lemma 5.2). The second is the shape of the disk. Its boundary consists of straight edges and curved arcs. Each visit draws a pair of opposite arcs, and each switch lies on an edge (Section 5). The third is the *switch form* S, the conserved momentum of the frame read as a quadratic form on the plane of the disk and measured along the edges (Lemma 5.4). In these terms the maximum principle is a multiplier rule: the active margins are a critical point of the area under the condition that the disk closes up, with S as the multiplier, and inserting a visit of another control does not lower the area to first order (Lemmas 5.5–5.7). Finally S is positive definite, hence a Euclidean structure on the plane of the disk. The area bounds its determinant, and the total turning of the edges fixes a sum of angles in this metric (Sections 6 and 7).

The propositions are proved in Sections 6, 8 and 9–10. For Proposition 3.2 one marks a visit of least active margin. The preceding turn bounds the switch form at that center. On the side of the smaller neighboring turn, this bound forces a short connection to a second reverse center, where stationarity conflicts with the competitive scale bound. For Proposition 3.3, an all-forward period has $1, 4, 7, \ldots$ visits by (3.1). At seven or more visits winding requires some pair of adjacent angles

to sum to at least $4\pi/7$. The determinant cap and stationarity on the two incident visits put every such sum below $4\pi/7$. A single visit is excluded by the switch-vector return (Lemma 7.1).

Proof of Theorem 1. Realize a least-density disk as a global minimizer of (2.3) of area at most A_{oct} (Section 2). By Propositions 4 and 5 it is a competitive normal walk in the compact core, and Propositions 3.1–3.3 identify its trajectory with the four-arc octagon. Hence the least density is $A_{\mathrm{oct}}/\sqrt{12} = \delta_{\mathrm{oct}}$. For any disk attaining it, restart the normalized frame at a switch t_*. Proposition 3.3 identifies its four arcs after a cyclic relabeling by some integer r, so

$$g(t_* + t) = g(t_*)R^r g_{\mathrm{oct}}(t)R^{-r}.$$

The endpoint relation extends this equality to every period. Applying it to $\mathbf{s}_0^*$ takes the octagon contact with label $-r$ onto the boundary of the normalized disk. That label shifts time by $-rt_f$. Taking convex hulls and undoing the normalization yields an affine image of K_{oct}. Conversely an invertible affine map preserves lattice packing density, since its linear part multiplies the area and the lattice covolume by the same factor. Fejes Tóth's theorem identifies this density with $\delta(K)$. □

4 Finite switching

We first bound the area without a switching hypothesis; this rules out the circular configuration at which the maximum principle cannot distinguish the controls locally. The edge problem of PSP then gives finitely many arcs. Normality is proved in Section 5, with the switch form.

Proposition 4 (Finite switching)**.** *The control of a global minimizer of area at most A_{oct} is almost everywhere a finite concatenation of vertex controls.*

4.1 An area bound before switching

The elliptic sheet through $-J$ consists of the matrices conjugate in $\mathrm{SL}_2(\mathbb{R})$ to $-d_1 J$ for some $d_1 > 0$.

Lemma 4.1 (Elliptic area bound)**.** *Suppose an admissible period has a normal maximizing lift with frame costate on the elliptic sheet through $-J$. Then $A \geq (d_1 + 3/2)\pi/3$. In a competitive period, fix a time-independent $h \in \mathrm{SL}_2(\mathbb{R})$ with $\Lambda_1(0) = -d_1 h J h^{-1}$ and let θ be the polar angle of $h^{-1}g$. It advances by exactly $\pi/3$ and on every subinterval from t_- to t_+ satisfies*

$$(d_1 + \tfrac{3}{2})\big(\theta(t_+) - \theta(t_-)\big) \leq \int_{t_-}^{t_+} \alpha(X(t))\, dt. \tag{4.1}$$

Proof. Choose a fixed $h \in \mathrm{SL}_2(\mathbb{R})$ with $\Lambda_1(0) = -d_1 h J h^{-1}$ and write $g' = h^{-1}g$. Transport gives $\Lambda_1 = -d_1 (g')^{-1} J g'$. Use r, r' from Lemma 2.2. At every state there is a comparison control that holds the state fixed and satisfies $P_{\mathbf{u}} = -X/2$. Indeed the coefficients of $X = \Phi(x, y)$ in the basis $(Z_{\mathbf{e}_i})_{i=0}^2$ are

$$\frac{1}{y}, \qquad \frac{3(x + a_*)^2 + 3y^2}{4y}, \qquad \frac{3(x - a_*)^2 + 3y^2}{4y}.$$

Normalize them to sum to one. Then $Z_{\mathbf{u}}$ is proportional to X and $\mathrm{tr}(X^2) = -2$ gives the asserted $P_{\mathbf{u}}$. The state costate term vanishes for this control by orthogonality. Its Hamiltonian is nonpositive, so $2d_1 r' \leq 3r$.

Lemma 2.2 now gives

$$0 < \dot{\theta} \le \frac{2rr'}{r+r'} \le \frac{3r}{d_1 + 3/2}.$$

Since $3r = \alpha(X)$, integration proves (4.1). The relation $g'(t_f) = g'(0)R$ adds $\pi/3$ to the polar angle modulo 2π: right multiplication by R conjugates the positive polar factor by R^{-1} and rotates the orthogonal factor by R. The positive advance is therefore at least $\pi/3$. In a competitive period it is less than $2\pi/3$ by the interval estimate, hence exactly $\pi/3$. □

4.2 One-sided edges and finite switching

Lemma 4.2 (One-sided edge restriction)**.** *Suppose a maximizing lift avoids the configuration $X = J$ and $L = \Lambda_R = 0$. About any time, the control lies almost everywhere in a fixed edge on each sufficiently small one-sided interval.*

Proof. Fix a time t_0. A nonzero matrix V orthogonal to $X(t_0)$ gives at least two distinct vertex values $-\operatorname{tr}(VP_{\mathbf{e}_i})$ at that state. Equality of all three values makes them zero by testing at $P_{\mathbf{u}} = -X/2$, which is a convex combination of the three $P_{\mathbf{e}_i}$. These matrices span $\mathfrak{sl}_2(\mathbb{R})$, so the vanishing values would give $V = 0$. This observation shows that when $\Lambda_R(t_0) \neq 0$, one vertex is strictly below a competitor on both sides of t_0 in a sufficiently small neighborhood.

When $\Lambda_R(t_0) = 0$, its equation has the form $\dot{\Lambda}_R = \mathcal{L}(t)\Lambda_R - [L, X]$ with $\mathcal{L}(t)$ a locally bounded linear map. Use a subscript 0 for evaluation at t_0. Integration first gives

$$\Lambda_R(t_0 + \Delta t) = -\Delta t[L_0, X_0] + o(\Delta t).$$

If the coefficient vanishes, L_0 commutes with X_0. The traceless centralizer of X_0 is its span, and the zero Hamiltonian gives $\operatorname{tr}(L_0X_0) = 0$. Thus $L_0 = 0$ and nontriviality gives $k_0 > 0$. At this point $\dot{L}_0 = -\frac{3}{2}k_0[J, X_0]$. If $[J, X_0] = 0$, then $X_0 = J$ on the state orbit, which the hypothesis excludes. Hence $\dot{L}_0$ is a nonzero matrix orthogonal to X_0 and $[\dot{L}_0, X_0] \neq 0$. The second expansion is

$$\Lambda_R(t_0 + \Delta t) = -\tfrac{1}{2}(\Delta t)^2[\dot{L}_0, X_0] + o((\Delta t)^2).$$

Indeed $L(t_0+\Delta t) = \Delta t\dot{L}_0 + o(\Delta t)$, X is continuous, and the first expansion gives $\mathcal{L}(t_0+\Delta t)\Lambda_R(t_0+\Delta t) = o(\Delta t)$.

The leading coefficient in either expansion is orthogonal to X_0. The separation observation now gives a vertex below a competitor on each side of t_0. In the first-order case the sign changes across t_0, so the chosen edge can change as well. On each sufficiently short closed half-interval the maximizing control therefore lies almost everywhere in a fixed edge. This is the one-sided edge argument in Schmutz's correction of PSP Theorem 8.3.1 [11, Section 2]. □

Proof of Proposition 4. Extend a Pontryagin lift by the offset return. If $X = J$ and $L = \Lambda_R = 0$ at some time, nontriviality gives $k_0 > 0$. After normalization, $\Lambda_1 = -\frac{3}{2}J$ there. Its evolution stays on the same elliptic sheet. Lemma 4.1 with $d_1 = 3/2$ gives $A \ge \pi$. Thus Lemma 4.2 applies at every time. Use the chart for the edge between $\mathbf{e}_1$ and $\mathbf{e}_2$, applying a cyclic offset if necessary. On this edge $P_{\mathbf{u}}$ is upper triangular, $\dot{x} = y$, and the tangent frame gX satisfies $(gX)^{-1}\frac{d}{dt}(gX) = -2P_{\mathbf{u}}$. If the interval starts at (x_0, y_0) at time t_0, its tangent-frame transfer $h = (g(t_0)X(t_0))^{-1}g(t)X(t)$ is upper triangular and obeys

$$h_{11} = y_0/y, \qquad \frac{h_{12}}{y} = -y_0\left(\int_{t_0}^{t}\frac{ds}{y(s)} + \frac{x}{y^2} - \frac{x_0}{y_0^2}\right).$$

Thus x, y and the accumulated integral of $1/y$ fix the endpoint state and frame. A cheaper competitor with these data would splice into the original motion. On a sufficiently short interval all edge competitors with these data stay inside the open triangle: their x increases and the two vertex controls bound $d(\log y)/dx$ between $-1/(a_* - x)$ and $1/(a_* + x)$. The segment therefore minimizes the PSP edge problem with fixed endpoint data and free duration. PSP Theorem 8.1.4 gives a finite concatenation of vertex controls there. Applying this on both sides of each time and taking a finite subcover gives finitely many arcs. □

4.3 Finite walks and their reflection

Merge adjacent arcs of equal label, including across the cut in the extended walk. There is a genuine switch because the label offset at the return is one. Restart at a switch at time t_*, with frame $g(t_*)^{-1}g(t_* + t)$ and state $X(t_* + t)$. The cost, closing conditions and lift are preserved. The resulting walk satisfies (3.1).

We will also use reflection together with time reversal. For $D \coloneqq \operatorname{diag}(1, -1)$, this sends the state and both costates to $-DX(t_f - t)D$ and $-D\Lambda_j(t_f - t)D$ for $j \in \{1, R\}$. The frame becomes $DR^{-1}g(t_f - t)D$. The control label is reflected, interchanging 1 and 2. Substitution into the equations proves that the area, the core condition, and normality are preserved. The order of the signs is reversed and each sign keeps its value. Active margins are carried along with the visits. This is the symmetry used to choose the smaller neighbor on the right of a least-margin visit.

5 Normality and the switch form

To decide whether a finite vertex period admits a normal lift, we express the first variation of area through its straight edges. Arc evolution gives their tangent vectors. The conserved frame momentum pairs these vectors in a single quadratic form, and the maximum principle relates its values to the changes of area at a switch.

Proposition 5 (Normality)**.** *Every Pontryagin lift of a finite vertex period has positive area multiplier and can be normalized to $k_0 = 1$.*

5.1 Evolution of the margins

Lemma 5.1 (Evolution on a vertex arc)**.** *On a vertex arc the active margin stays fixed. Each of the two sums of the active margin and a neighboring margin is multiplied by the position parameter or its reciprocal, according to the cyclic order of the labels.*

Proof. At the start of an arc with control i, write a, b, c for the margins to sides $i, i + 1, i - 1$, in that order. At position parameter q the margins are

$$a, \qquad (a + b)q - a, \qquad (a + c)/q - a. \tag{5.1}$$

The trace pairing makes $p_i(X)$ constant on its own arc. Thus $P_i = -\sqrt{3}Z_{\mathbf{e}_i}/a$ is constant. With the time origin at the start of the arc, $X(t) = e^{tP_i}X(0)e^{-tP_i}$. The bracket acts on $p_i + p_{i\pm1}$ by the scalars $\pm 2/a$, proving (5.1). □

5.2 Arc factors and area contributions

Write $\tau_k \coloneqq p[k] + p[k+1]$ for the sum of the two active margins at the outgoing switch of visit k. At that switch the unused margin is $(4 - p[k]p[k+1])/\tau_k$. Its positivity gives $p[k]p[k+1] < 4$. For the arc formulas put $\mu_k \coloneqq p[k]^2 + 4$.

Lemma 5.2 (Arc factors and the area sum)**.** *The factor on visit k is determined by its incident switch signs and the three successive active margins:*

$$\begin{array}{c|cccc} (\epsilon_{k-1}, \epsilon_k) & ++ & -- & -+ & +- \\ \hline q_k & \tau_{k-1}\tau_k/\mu_k & \mu_k/(\tau_{k-1}\tau_k) & \tau_k/\tau_{k-1} & \tau_{k-1}/\tau_k. \end{array} \tag{5.2}$$

The area of a period is

$$A = \sum_{k=0}^{N-1} \epsilon_k A_{\mathrm{mar}}(p[k], p[k+1]),$$

where the symmetric margin area function A_{mar} is defined by

$$\frac{4}{\sqrt{3}} A_{\mathrm{mar}}(a,b) = 3ab - 4 - (a^2+b^2)\log(a+b) + \tfrac{1}{2}a^2\log(a^2+4) + \tfrac{1}{2}b^2\log(b^2+4). \tag{5.3}$$

Thus A_k belongs to visit k, while $\epsilon_k A_{\mathrm{mar}}(p[k], p[k+1])$ belongs to its outgoing switch.

Proof. Read the incoming and outgoing active margins in (5.1). At either switch the sum of the unused margin and the margin of the current visit is μ_k divided by the sum of the two active margins. This gives the four entries in (5.2). In particular, at a turn, $q_k > 1$ forces the outer active margins to be unequal.

For an arc starting with margins a, b, c as in Lemma 5.1, integrate $\alpha(X)$ using $dt = (a/2)\, dq/q$. Its arc area contribution from the start to position parameter q equals

$$\frac{\sqrt{3}}{4} a\big((a+b)(q-1) + (a+c)(1-q^{-1}) - a\log q\big). \tag{5.4}$$

In units of $\sqrt{3}/4$, the arc area contribution splits into an incoming and an outgoing contribution of sign ϵ_{k-1} and ϵ_k, respectively. The contribution for active margin a and neighboring margin b, before attaching its sign, is $a[b - (4-ab)/(a+b)] - a^2\log(a+b) + (a^2/2)\log(a^2+4)$, by (5.4) and (5.2). At each switch the sum of the two contributions is $4\epsilon_k A_{\mathrm{mar}}(p[k], p[k+1])/\sqrt{3}$. □

5.3 The tangent frame and the edges of the disk

Lemma 5.3 (Tangent transport and closure)**.** *Let $g(t)X(t)$ be the tangent frame. At time t in visit k,*

$$g(t)X(t) = g(t_k)X(t_k)h_{i_k}(q(t)), \qquad h_i(q) \coloneqq \tfrac{1}{2}(q+q^{-1})I + \tfrac{\sqrt{3}}{2}(q-q^{-1})Z_{\mathbf{e}_i}.$$

For a period of N visits starting at $t_0 = 0$, the endpoint conditions imply $h_{i_0}(q_0)\cdots h_{i_{N-1}}(q_{N-1}) = R$. Equivalently,

$$h_0(q_0)R^{-\epsilon_0}\, h_0(q_1)R^{-\epsilon_1}\cdots h_0(q_{N-1})R^{-\epsilon_{N-1}} = R^{1-\sum_{j=0}^{N-1}\epsilon_j}. \tag{5.5}$$

We use the closure equation for the mixed family in Remark 8.3 and for the four-arc classification in Lemma 10.1.

Proof. Let $D(t) \coloneqq g(t)X(t)$. On the arc, $D^{-1}\dot{D} = X + X^{-1}P_iX - P_i = -2P_i$, since $X^{-1} = -X$ and $\langle X, P_i\rangle = 1$. The eigenvalues of $-2P_i$ are $\pm 2/p_i(X)$. Exponentiating gives $h_i(q)$, using $Z_{\mathbf{e}_i}^2 = I/3$. The endpoint conditions give $D(t_f) = D(0)R$. Next $R^{-1}Z_{\mathbf{e}_i}R = Z_{\mathbf{e}_{i+1}}$, so Choose integer representatives for the labels, starting with i_0 and setting $i_{k+1} = i_k + \epsilon_k$. Then $h_{i_k}(q) = R^{-i_k}h_0(q)R^{i_k}$. The product of these matrices is the left side of (5.5) multiplied by R^{-i_0} on the left and by $R^{i_0+\sum_{j=0}^{N-1}\epsilon_j}$ on the right. The right side of (5.5) is $\pm I$, because $\sum_{j=0}^{N-1}\epsilon_j \equiv 1 \bmod 3$ and $R^3 = -I$. For calculations we conjugate by $D_0 \coloneqq \left(\begin{smallmatrix}\sqrt{3} & 1\\ \sqrt{3} & -1\end{smallmatrix}\right)$ and denote the conjugated matrices by primes:

$$h_0'(q) = \begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}, \qquad R' = \begin{pmatrix} 0 & 1 \\ -1 & 1\end{pmatrix}. \tag{5.6}$$

□

The vectors $\mathbf{s}_{1-i}^*$ and $\mathbf{s}_{2-i}^*$ are eigenvectors of $Z_{\mathbf{e}_i}$, hence of $h_i(q)$, with eigenvalues q and q^{-1}. The contact $g\mathbf{s}_r^*$ has velocity $gX\mathbf{s}_r^*$. So during a visit with label i the contacts with $r \not\equiv -i \bmod 3$ move along straight lines, and only the two opposite contacts $\pm g\mathbf{s}_{-i}^*$ bend. The boundary of the disk therefore consists of straight *edges* and curved arcs, with a pair of opposite arcs for each visit (Figure 3). At the switch from visit k to visit $k+1$, one pair of opposite contacts moves on the same two supporting lines through both visits. When we choose one of those contacts, its straight boundary segment is the *edge of the switch*. At a turn the incoming and outgoing switches share this pair of boundary edges.

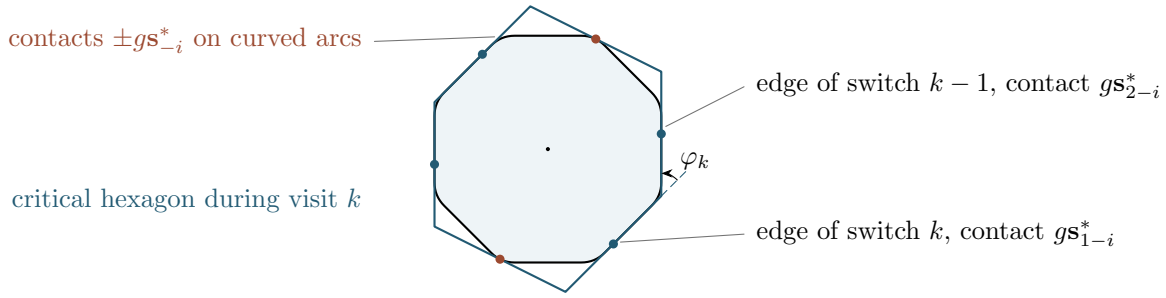

Figure 3: The smoothed octagon, drawn regular, during a visit k with label i and two forward switches. The angle φ_k is used in Section 7.

5.4 The switch form

Lemma 5.4 (Switch form)**.** *Fix a Pontryagin lift of a finite vertex period. There is a real symmetric 2×2 matrix S and a vector $\mathbf{y}_k \in \mathbb{R}^2$ for the outgoing switch of each visit k. In the basis specified below, $\mathbf{y}_k$ is the signed normalized velocity of a contact along that straight boundary edge. The vectors satisfy $\mathbf{y}_{k+N} = \pm\mathbf{y}_k$ with one common sign. For visit indices i, j, define*

$$\sigma_{i,j} \coloneqq \mathbf{y}_i^t S\mathbf{y}_j, \qquad \sigma_k \coloneqq \sigma_{k,k}.$$

These pairings are independent of the period cut when compared at the same switches of the extended walk. The matrix satisfies $12\det S = d$. If visit k is a turn, $\mathbf{y}_{k-1} = -\mathbf{y}_k$. When both switches at visit k have the same sign, $\det(\mathbf{y}_{k-1}, \mathbf{y}_k) = \epsilon_k/\mu_k$, so

$$\sigma_{k-1}\sigma_k - \sigma_{k-1,k}^2 = \frac{\det S}{\mu_k^2}. \tag{5.7}$$

If $k < k'$ are not turns and every intervening visit is a turn (there may be none), then

$$\mu_k \, \mathbf{y}_{k-1} + \mu_{k'} \, \mathbf{y}_{k'} = \left(\tau_k^2 - \tau_{k+1}^2 + \cdots \pm \tau_{k'-1}^2\right) \mathbf{y}_k. \tag{5.8}$$

Proof. The matrix $\Lambda \coloneqq g\Lambda_1 g^{-1}$ does not depend on time: its derivative is $g([X, \Lambda_1] + \dot{\Lambda}_1)g^{-1}$, which vanishes by (2.7). It is the conserved momentum of the frame, and $\Lambda = \Lambda_1(0)$. Since Λ is traceless, $\det(\Lambda\mathbf{u}, \mathbf{v})$ is a symmetric bilinear form on the plane of the disk, the *momentum form*. Its determinant in the standard basis is d. Let S be its matrix in the basis $\mathbf{s}_1^*/\sqrt{3}$, $\mathbf{s}_0^*/\sqrt{3}$, whose determinant is $-1/\sqrt{12}$. Then $12 \det S = d$. The vectors $\mathbf{y}_k$ are given in the coordinates of this basis, in which determinants are $-\sqrt{12}$ times the standard ones.

For a contact $\mathbf{c} = g\mathbf{s}_r^*$, the standard determinant of $\mathbf{c}$ and $\dot{\mathbf{c}}$ is $\det(\mathbf{s}_r^*, X\mathbf{s}_r^*)$. This is $\frac{\sqrt{3}}{4}$ times the sum of the two margins $p_i(X)$ with $i \not\equiv -r \bmod 3$: for $r = 0$ both are the lower left entry of X, and the general case follows by conjugation with R^r. The *edge vector* $\dot{\mathbf{c}}/(4 \det(\mathbf{c}, \dot{\mathbf{c}}))$ of the contact is constant while the contact moves on a line, since the velocity and the determinant are then both proportional to the speed.

Choose an integer r_0 representing one of the two contact labels modulo six on the straight edges at the outgoing switch of visit 0. Take $r_k = r_{k-1}$ when visit k is a turn and $r_k = r_{k-1} - \epsilon_k$ otherwise. This selects a contact at each outgoing switch: for three labels $i - \epsilon_k, i, i + \epsilon_k$, the incoming and outgoing straight-edge labels are $-i - \epsilon_k$ and $-i + \epsilon_k$ modulo three. Define $\mathbf{y}_k$ to be ϵ_k times the edge vector of $g\mathbf{s}_{r_k}^*$. If k is a turn, $\mathbf{y}_{k-1} = -\mathbf{y}_k$. Otherwise the labels $r_{k-1} = r_k + \epsilon_k$ and r_k select contacts on the two straight lines during visit k. The standard determinant of their velocities is $\det(\mathbf{s}_{r_k+\epsilon_k}^*, \mathbf{s}_{r_k}^*) = -\epsilon_k\sqrt{3}/2$, and their sums of margins have the product $(p_i + p_{i+1})(p_i + p_{i-1}) = \mu_k$ by the margin quadric. So $\det(\mathbf{y}_{k-1}, \mathbf{y}_k) = \epsilon_k/\mu_k$, and (5.7) follows from $(\mathbf{u}^t S\mathbf{u})(\mathbf{v}^t S\mathbf{v}) - (\mathbf{u}^t S\mathbf{v})^2 = \det S \, \det(\mathbf{u}, \mathbf{v})^2$.

Since $\frac{1}{2}(\epsilon_{k-1} + \epsilon_k)$ is ϵ_k at a visit that is not a turn and zero at a turn, $r_{k+N} = r_k - \sum_{j=0}^{N-1} \epsilon_j$. By $g(t + t_f)X(t + t_f) = g(t)X(t)R$, the velocity of the contact labeled $r_k - \sum_{j=0}^{N-1} \epsilon_j$ at $t + t_f$ is $R^{1-\sum_{j=0}^{N-1} \epsilon_j} = \pm I$ applied to that of contact r_k at t. Hence $\mathbf{y}_{k+N} = \pm\mathbf{y}_k$. Restarting the period at a switch at time t_* replaces g by $g(t_*)^{-1}g$ and transforms the momentum form and the edge vectors accordingly, and another choice of r_0 changes all $\mathbf{y}_k$ by one sign. The pairings do not change.

For (5.8), note that $\mathbf{s}_{r-1}^* + \mathbf{s}_{r+1}^* = \mathbf{s}_r^*$. So the velocities of the contacts with labels $r \pm 1$ add up to that with label r, and a velocity is $\sqrt{3}$ times the sum of margins times the edge vector. At the time of a switch j, the straight-edge contact labeled r has margin sum τ_j. The one of $r \pm 1$ that bends under i_{j+1} has the sum μ_j/τ_j, and the other, which bends under i_j, has the sum μ_{j+1}/τ_j, by the margin quadric. Hence

$$\mu_j \, \mathbf{v}_j + \mu_{j+1}\mathbf{v}_{j+1} = \tau_j^2 \, \mathbf{e}, \tag{5.9}$$

where $\mathbf{e}$ is the edge vector of contact r, and $\mathbf{v}_j$ and $\mathbf{v}_{j+1}$ are the edge vectors of the contacts with labels $r \pm 1$ that are on lines during visit j and during visit $j + 1$. Take the two visits $k < k'$ in the statement. The outgoing switches of visits $k, \ldots, k' - 1$ join the same two labels, so the common label $r = r_k$ selects the same straight-edge contact at each switch and $\mathbf{e} = \epsilon_k\mathbf{y}_k$. The alternating sum of (5.9) over these switches is $\mu_k\mathbf{v}_k + (-1)^{k'-k-1}\mu_{k'}\mathbf{v}_{k'} = (\tau_k^2 - \tau_{k+1}^2 + \cdots \pm \tau_{k'-1}^2)\,\mathbf{e}$. Here $\mathbf{v}_k$ and $\mathbf{v}_{k'}$ belong to the contacts $r_{k-1} = r + \epsilon_k$ and $r_{k'} = r - \epsilon_{k'}$, so $\mathbf{v}_k = \epsilon_k\mathbf{y}_{k-1}$ and $\mathbf{v}_{k'} = \epsilon_{k'}\mathbf{y}_{k'}$, and $\epsilon_{k'} = (-1)^{k'-k-1}\epsilon_k$ because the signs alternate at turns. This is (5.8). □

5.5 The multiplier rule

There are two useful variations of a walk. Varying its active margins preserves the sequence of controls and yields the multiplier rule below. Inserting a visit of initially zero duration tests the

Hamiltonian of an unused control. This gives the inequalities of Lemma 5.6.

Lemma 5.5 (Multiplier rule)**.** *Fix a Pontryagin lift of a finite vertex period, with area multiplier k_0. For every smooth variation of the active margins with fixed control sequence, the first variations at $\lambda = 0$ satisfy*

$$k_0\,\partial_\lambda A = 2\sqrt{3}\sum_{k=0}^{N-1}\epsilon_k\big(\sigma_k\,\partial_\lambda(\tau_k^2) - 2\sigma_{k-1,k}\,\partial_\lambda\mu_k\big), \tag{5.10}$$

where the pairings are those of the period and the term with $\partial_\lambda\mu_k$ is omitted when visit k is a turn. The right side equals $\langle\Lambda_1(0), \partial_\lambda g(t_f)\,R^{-1}\rangle$, for the final frame $g(t_f)$ of the motion with the given margins. For a normal lift this makes the period a critical point of the area under the condition $g(t_f) = R$, with the switch form as multiplier.

Proof. For margins near those of the period, start at $g(0) = I$ and at the state whose margins for the labels i_{-1} and i_0 are $p[N-1]$ and $p[0]$, and follow each control i_k until the margin of i_{k+1} equals $p[k+1]$, with $p[N] \coloneqq p[0]$. By Lemma 5.1 this motion depends smoothly on the margins. It satisfies the state return, since its final margins for the labels i_{N-1} and $i_0 + 1$ are again $p[N-1]$ and $p[0]$. Lemmas 5.2 and 5.3 apply to it, but its final frame $g(t_f)$ need not be R.

The costate equations (2.7) are adjoint to the linearized motion. Take a smooth family parametrized by λ, and write ∂_λ for its derivative at $\lambda = 0$. For solutions of the differential equations in (2.3) with a common control, write $\partial_\lambda X = [X, W]$ with $\langle W, X\rangle = 0$. Then

$$\frac{d}{dt}\big(\langle\Lambda_1, g^{-1}\partial_\lambda g\rangle + \langle\Lambda_R, W\rangle\big) = k_0\,\partial_\lambda\alpha(X). \tag{5.11}$$

Indeed $(g^{-1}\partial_\lambda g)^{\cdot} = [g^{-1}\partial_\lambda g, X] + \partial_\lambda X$, so the first term has derivative $\langle\Lambda_1, \partial_\lambda X\rangle$. Differentiating $\partial_\lambda X = [X, W]$ gives $\dot W = [P_{\mathbf u}, W] + \langle P_{\mathbf u}, \partial_\lambda X\rangle P_{\mathbf u}$ modulo X, and then (2.7) gives the derivative $-\langle L, \partial_\lambda X\rangle$ of the second term. The sum is $-\frac{3}{2}k_0\langle J, \partial_\lambda X\rangle = k_0\,\partial_\lambda\alpha(X)$. Take the derivative at $\lambda = 0$ along a smooth family with the same sequence of vertex controls and variable durations Δt_k. Use the costates of the given period. With W as in (5.11), $\langle\Lambda_1, g^{-1}\partial_\lambda g\rangle + \langle\Lambda_R, W\rangle$ grows over a visit of fixed duration by k_0 times the variation of its area. Lengthening the visit by $\partial_\lambda(\Delta t_k)$ adds $\partial_\lambda(\Delta t_k)$ times the velocity to its endpoint and $\alpha\,\partial_\lambda(\Delta t_k)$ to its area. For the velocity, $W = -P_{\mathbf u} - X/2$ and the expression is $\langle\Lambda_1, X\rangle - \langle\Lambda_R, P_{\mathbf u}\rangle = \mathcal H_{\mathbf u} + k_0\alpha$. Adding over the visits,

$$k_0\,\partial_\lambda A = \Big[\langle\Lambda_1, g^{-1}\partial_\lambda g\rangle + \langle\Lambda_R, W\rangle\Big]_0^{t_f} - \sum_k \mathcal H^{(k)}\,\partial_\lambda(\Delta t_k), \tag{5.12}$$

where $\mathcal H^{(k)}$ is the Hamiltonian of the k-th control at the end of its visit. Durations may vanish at the motion considered. For the family of the first paragraph, $\mathcal H^{(k)} = 0$ at the period, $\partial_\lambda g(0) = 0$, and the state terms cancel because $\partial_\lambda X$ and Λ_R have the same twist at the return. Since $g(t_f) = R$ and $\Lambda_1(t_f) = R^{-1}\Lambda_1(0)R$ at the period, $k_0\,\partial_\lambda A = \langle\Lambda, \partial_\lambda g(t_f)\,R^{-1}\rangle$ with the momentum $\Lambda = \Lambda_1(0)$.

By Lemma 5.3 and the state return, $g(t_f) = X(0)\,\Pi\,R^{-1}X(0)^{-1}R$, where Π is the product $h_{i_0}(q_0)\cdots h_{i_{N-1}}(q_{N-1})$, which is R at the period. Since $h_i(q) = \exp(\sqrt{3}\log q\,Z_{\mathbf e_i})$, the variation of $X(0)$ cancels and

$$\partial_\lambda g(t_f)\,R^{-1} = \sqrt{3}\sum_k \partial_\lambda(\log q_k)\,D_k Z_{\mathbf e_{i_k}} D_k^{-1}, \qquad D_k \coloneqq g(t_k)X(t_k).$$

Put $m_k \coloneqq \langle\Lambda, D_k Z_{\mathbf e_{i_k}} D_k^{-1}\rangle$. The velocities $\mathbf u, \mathbf v$ at t_k of the contacts $1 - i_k$ and $2 - i_k$ are eigenvectors of $D_k Z_{\mathbf e_{i_k}} D_k^{-1}$ with eigenvalues $\pm 1/\sqrt{3}$, and their standard determinant is $\sqrt{3}/2$, so

$m_k = \frac{4}{3}\det(\Lambda\mathbf{u}, \mathbf{v})$, also with the velocities at any other time of the visit. In terms of the edge vectors of the proof of Lemma 5.4, $m_k = 4\mu_k\sigma_{k-1,k}$ if visit k is not a turn. At every switch $m_k + m_{k+1} = \frac{4}{3}\det(\Lambda\mathbf{c}, \mathbf{c}) = 4\tau_k^2\sigma_k$, where $\mathbf{c}$ is the velocity of a contact of the switch, because the velocities of the other two contacts add up to $\mathbf{c}$. Finally $\log q_k = \epsilon_{k-1}\log\tau_{k-1} + \epsilon_k\log\tau_k - \frac{1}{2}(\epsilon_{k-1}+\epsilon_k)\log\mu_k$ by (5.2). So

$$k_0\,\partial_\lambda A = \sqrt{3}\sum_k m_k\,\partial_\lambda(\log q_k) = \sqrt{3}\sum_k \epsilon_k\big(4\tau_k^2\sigma_k\,\partial_\lambda(\log\tau_k) - 4\mu_k\sigma_{k-1,k}\,\partial_\lambda(\log\mu_k)\big),$$

with the last term omitted at turns. This is (5.10). □

5.6 The secant relation

For visit k put

$$\rho_k^- \coloneqq \frac{p[k]}{\tau_{k-1}}, \qquad \rho_k^+ \coloneqq \frac{p[k]}{\tau_k}, \qquad \gamma_k \coloneqq \frac{p[k]^2}{\mu_k}, \tag{5.13}$$

the shares of the active margin in the incoming and outgoing sums, and its scale. By (5.2), $\gamma_k/(\rho_k^-\rho_k^+) = q_k^{\epsilon_k}$ at a visit whose two switches have the same sign. Both shares at a switch are positive and $\rho_k^+ + \rho_{k+1}^- = 1$.

For positive margins a, y define

$$f_a(y) = 1 - \frac{ay}{(a+y)^2} = \frac{3}{4} + \left(\frac{y-a}{2(y+a)}\right)^2.$$

We use angle brackets for the interval mean, with its continuous extension to the diagonal.

$$\langle f_a\rangle_{b,c} = \frac{1}{c-b}\int_b^c f_a(y)\,dy, \qquad \langle f_a\rangle_{b,b} = f_a(b). \tag{5.14}$$

For an arc it is convenient to express the same mean in its share and factor:

$$\langle f_a\rangle_{\tau-a,\tau/q-a} = \langle f\rangle_{\mathrm{arc}}(\rho, q) = 1 + \rho^2 q - \frac{\rho q\log q}{q-1}, \qquad \rho = a/\tau, \quad \langle f\rangle_{\mathrm{arc}}(\rho,1) = 1 - \rho + \rho^2. \tag{5.15}$$

Write $\partial_1 A_{\mathrm{mar}}$ for the derivative in its first argument. Direct differentiation of (5.3) and integration in the second argument give

$$\partial_1 A_{\mathrm{mar}}(a,c) - \partial_1 A_{\mathrm{mar}}(a,b) = \tfrac{\sqrt{3}}{2}(c-b)\langle f_a\rangle_{b,c}. \tag{5.16}$$

Lemma 5.6 (Secant relation)**.** *For a normal vertex period the interval means control the switch form as follows.*

(a) Switch bound. *With*

$$\sigma_k^* \coloneqq \frac{p[k]^2 + p[k]p[k+1] + p[k+1]^2}{8\tau_k^2} \geq \frac{3}{32}, \tag{5.17}$$

$\sigma_k \geq \sigma_k^*$ *at a forward switch and* $\sigma_k \leq \sigma_k^*$ *at a reverse switch.*

(b) Turns. *At a turn visit* k, $8\sigma_{k-1} = 8\sigma_k = \langle f_{p[k]}\rangle_{p[k-1],p[k+1]} \geq \frac{3}{4}$.

(c) Arc bound. *At a visit k with two forward switches,*

$$8\sigma_{k-1} \geq \langle f\rangle_{\mathrm{arc}}(\rho_k^-, q_k) \geq \tfrac{3}{4}, \qquad 8\sigma_k \geq \langle f\rangle_{\mathrm{arc}}(\rho_k^+, q_k) \geq \tfrac{3}{4}; \tag{5.18}$$

the right sides are the interval means from the neighboring active margin to the unused margin at the far switch of the visit.

The proof shows that $\widehat{\chi}_j$ is the first-order cost of inserting a visit of the control j. In (a), $8\sigma_k^*$ is twelve times the squared length of the edge vector of switch k in the Euclidean structure in which the hexagon at time t_{k+1} is regular. So the switch bound compares the momentum form with 3/2 times this structure, along the edge.

Proof. We first derive the identity for an arbitrary lift, including an abnormal one. Fix a Pontryagin lift with area multiplier k_0, and let visit k be entered by a switch of sign $\epsilon = \epsilon_{k-1}$ from the label $j = i_{k-1}$. At position parameter $q \in (1, q_k]$ of visit k the margin of label j is $s(q) = \tau_{k-1} q^{-\epsilon} - p[k]$, and the weighted vertex value satisfies

$$\widehat{\chi}_j = -\epsilon\, c\,\big(8\sigma_{k-1} - k_0 \langle f_{p[k]}\rangle_{p[k-1],s(q)}\big), \qquad c \coloneqq \sqrt{3}\,\tau_{k-1}^2 (q-1) q^{-(1+3\epsilon)/2} > 0. \tag{5.19}$$

Symmetrically, if visit k is left by a switch of sign $\epsilon = \epsilon_k$ to the label $j = i_{k+1}$, then at the point of visit k from which the remaining factor is q the margin of j is $s(q) = \tau_k q^{-\epsilon} - p[k]$, and (5.19) holds with σ_k, $p[k+1]$, τ_k in place of σ_{k-1}, $p[k-1]$, τ_{k-1}.

We treat the entry version. The exit version is the same with the two parts of the visit exchanged. Put $s \coloneqq s(q)$, which is the margin of j at the time t_* of position parameter q, by (5.1). Insert a visit of label j at t_*: consider the motions with the state return in which the label of visit k is replaced by the three labels i_k, j, i_k, with active margins a, s, b, and the other margins are those of the period. The period is the member $a = b = p[k]$, in which the new visit has duration zero. In general its duration is $\Delta t_{\mathrm{ins}} = \epsilon\, \frac{s}{2} \log \frac{s+b}{a+s}$ by (5.2), where a negative value means that the control is followed backward. The three switches between j and i_k have the signs $\epsilon, -\epsilon, \epsilon$, the edge of switch $k-1$, and the switch vectors $\mathbf{y}_{k-1}, -\mathbf{y}_{k-1}, \mathbf{y}_{k-1}$. The first part of visit k and the new visit are turns.

The proof of Lemma 5.5 applies to this family, with the term $-\mathcal{H}\,\partial_\lambda(\Delta t_{\mathrm{ins}})$ in (5.12) for the new visit, where $\mathcal{H} = \widehat{\chi}_j / s$ is the Hamiltonian of the control j at t_*. Take $\partial_\lambda b = 1$ and $\partial_\lambda a = 0$. Compared with the coefficient of $\partial_\lambda p[k]$ in (5.10) for the period, the switch into the second part of the visit has the margins s, b in place of $p[k-1], p[k]$, and $\partial(\Delta t_{\mathrm{ins}})/\partial b = \epsilon s/(2(p[k]+s))$. Hence

$$\epsilon k_0 \big(\partial_1 A_{\mathrm{mar}}(p[k], s) - \partial_1 A_{\mathrm{mar}}(p[k], p[k-1])\big) + \frac{\epsilon\, \widehat{\chi}_j}{2(p[k]+s)} = 4\sqrt{3}\,\epsilon\,\sigma_{k-1}\,(s - p[k-1]).$$

By (5.16) this says $\widehat{\chi}_j = \sqrt{3}\,(p[k]+s)(s-p[k-1])\big(8\sigma_{k-1} - k_0\langle f_{p[k]}\rangle_{p[k-1],s}\big)$, which is (5.19) because $p[k] + s = \tau_{k-1} q^{-\epsilon}$.

(a) Let $q \to 1$ in (5.19) for the visit entered by the switch: the interval mean tends to $1 - p[k]p[k-1]/\tau_{k-1}^2 = 8\sigma_{k-1}^*$. (b) At a turn the label j is also the outgoing label, so $\widehat{\chi}_j = 0$ at $q = q_k$, where $s(q_k) = \tau_k - p[k] = p[k+1]$ by (5.2); and $\sigma_k = \sigma_{k-1}$ since the switch vectors are opposite. (c) Take $q = q_k$ in both versions, with $\epsilon = +1$, and use (5.15). □

5.7 Stationarity at a visit

Lemma 5.7 (Stationarity)**.** *If both switches at visit k of a normal walk have sign ϵ_k, then*

$$16\sigma_{k-1,k} = 1 + \rho_k^- + \rho_k^+ - \gamma_k + \epsilon_k \log q_k - \frac{1 - 8\sigma_{k-1}}{\rho_k^-} - \frac{1 - 8\sigma_k}{\rho_k^+}. \tag{5.20}$$

Define the comparison pairing by evaluating the right side at the individual switch bounds:

$$16\sigma^*_{k-1,k} = -1 + 2\rho_k^- + 2\rho_k^+ - \gamma_k + \epsilon_k \log q_k. \tag{5.21}$$

Then $\sigma_{k-1,k} \geq \sigma^*_{k-1,k}$ *at a visit with two forward switches and* $\sigma_{k-1,k} \leq \sigma^*_{k-1,k}$ *at a reverse center.*

Proof. Subtracting the value at the two bounds gives the exact difference

$$2(\sigma_{k-1,k} - \sigma^*_{k-1,k}) = \frac{\sigma_{k-1} - \sigma^*_{k-1}}{\rho_k^-} + \frac{\sigma_k - \sigma^*_k}{\rho_k^+}.$$

The two switch inequalities of Lemma 5.6(a) give the stated signs. We derive the stationarity equation by varying just the active margin at this visit. In margin coordinates stationarity takes the form

$$\begin{gathered} 16p[k]\,\sigma_{k-1,k} = 8\tau_{k-1}\sigma_{k-1} + 8\tau_k\sigma_k - a_{\mathrm{stat}}(p[k-1], p[k], p[k+1]), \\ a_{\mathrm{stat}}(b_-, b_0, b_+) \coloneqq \tfrac{2}{\sqrt{3}}\big(\partial_1 A_{\mathrm{mar}}(b_0, b_-) + \partial_1 A_{\mathrm{mar}}(b_0, b_+)\big). \end{gathered} \tag{5.22}$$

For a lift with area multiplier k_0, (5.22) holds with $k_0 a_{\mathrm{stat}}$ in place of a_{stat}. By Lemma 5.2, the coefficient of $\partial_\lambda p[k]$ on the left of (5.10) is $k_0\epsilon_k\big(\partial_1 A_{\mathrm{mar}}(p[k], p[k-1]) + \partial_1 A_{\mathrm{mar}}(p[k], p[k+1])\big)$. On the right it is $2\sqrt{3}\,\epsilon_k\big(2\tau_{k-1}\sigma_{k-1} + 2\tau_k\sigma_k - 4p[k]\sigma_{k-1,k}\big)$. This is (5.22). The form (5.20) follows from $4\partial_1 A_{\mathrm{mar}}(a,b)/(\sqrt{3}a) = 2/\rho - 1 - 2\rho + \gamma + \log\big((a^2+4)/(a+b)^2\big)$ with $\rho = a/(a+b)$ and $\gamma = a^2/(a^2+4)$, and from (5.2). Since $8\sigma^*_{k-1} = \langle f\rangle_{\mathrm{arc}}(\rho_k^-, 1)$ and $8\sigma^*_k = \langle f\rangle_{\mathrm{arc}}(\rho_k^+, 1)$, the value at the two bounds is $16\sigma^*_{k-1,k}$. At a turn the same comparison of coefficients gives Lemma 5.6(b) once more, by (5.16). □

5.8 Excluding abnormal lifts

Lemma 5.8 (Signs of consecutive switch vectors)**.** *If* $\epsilon_k\sigma_{k-1,k} > 0$ *at every visit that is not a turn, then* $\det S > 0$.

Proof. There is a visit that is not a turn. Otherwise $i_{k+2} = i_k$ for all k, whereas the offset gives $i_{k+2N} \equiv i_k + 2$ modulo three. First suppose $\det S < 0$. Choose a basis $\mathbf{n}_1, \mathbf{n}_2$ of null vectors of S with $\mathbf{n}_1^t S \mathbf{n}_2 > 0$ and $\det(\mathbf{n}_1, \mathbf{n}_2) > 0$, and write $\mathbf{y}_k = x_k\mathbf{n}_1 + z_k\mathbf{n}_2$. At a visit that is not a turn, the sign hypothesis and $\epsilon_k \det(\mathbf{y}_{k-1}, \mathbf{y}_k) > 0$ read

$$\epsilon_k\,(x_{k-1}z_k + x_k z_{k-1}) > 0, \qquad \epsilon_k\,(x_{k-1}z_k - x_k z_{k-1}) > 0,$$

so $|x_{k-1}z_k| > |x_k z_{k-1}|$. At a turn the two sides are equal, since $\mathbf{y}_k = -\mathbf{y}_{k-1}$. Thus the ratio $|z_k| : |x_k|$ in $[0,\infty]$ never decreases along the walk and increases at every visit that is not a turn. This contradicts $\mathbf{y}_{k+N} = \pm\mathbf{y}_k$.

If $\det S = 0$, the strict sign at one visit excludes the zero form. A rank-one symmetric form is a limit of indefinite forms, made by perturbing its zero eigenvalue with sign opposite to its other eigenvalue. All the strict inequalities persist under a sufficiently small perturbation, because there are only finitely many pairings in a period. The indefinite case just proved excludes this possibility as well. □

Proof of Proposition 5. Suppose a lift has $k_0 = 0$. We shall obtain both $\det S < 0$ and the sign condition of Lemma 5.8.

The momentum is not zero. Otherwise $L = 0$, and at a switch the two vanishing Hamiltonians and $\langle \Lambda_R, X\rangle = 0$ make Λ_R orthogonal to X and to the matrices $Z_{\mathbf{e}_i}$ of the two controls. These

three span $\mathfrak{sl}_2(\mathbb{R})$, because X is a combination of all three $Z_{\mathbf{e}_i}$ with positive coefficients (proof of Lemma 4.1). So $\Lambda_R = 0$ there and the lift would be zero. Thus $S \neq 0$.

By (5.19) with $k_0 = 0$, $\epsilon_k \sigma_k \geq 0$ at every switch. Let visit k not be a turn, with sign ϵ. By Lemma 5.7 with $k_0 = 0$, $2p[k]\sigma_{k-1,k} = \tau_{k-1}\sigma_{k-1} + \tau_k\sigma_k$. So $\epsilon\sigma_{k-1,k} \geq 0$, and $p[k]^2\sigma_{k-1,k}^2 \geq \tau_{k-1}\tau_k\,\sigma_{k-1}\sigma_k \geq p[k]^2\sigma_{k-1}\sigma_k$ by the inequality of the means. Equality of the outer terms would force $\sigma_{k-1}\sigma_k = 0$, because $\tau_{k-1}\tau_k > p[k]^2$, then $\sigma_{k-1,k} = 0$ and $\sigma_{k-1} = \sigma_k = 0$, hence $S = 0$. So (5.7) gives

$$\det S < 0, \qquad \epsilon_k \sigma_{k-1,k} > 0 \quad \text{at every visit } k \text{ that is not a turn.} \tag{5.23}$$

This contradicts Lemma 5.8. Thus $k_0 > 0$. □

6 Reverse switches and positivity

Proposition 3.1 says that reverse switches come in pairs. The turn at each end puts the smallest margin inside a reverse run. Four successive arcs joined by reverse switches cannot have an interior minimum. Together these facts force the run length and give the signs needed for positivity of the form. All arguments here use the maximum principle with $k_0 = 1$ and apply to every normal vertex period; windows may cross the period cut.

6.1 Four arcs joined by reverse switches

Lemma 6.1 (A reverse window has no interior minimum)**.** *Number four consecutive visits* $0, 1, 2, 3$*, and suppose the three switches between them are reverse. Neither* $p[1]$ *nor* $p[2]$ *is a least margin among the four visits.*

Proof. Reflection with time reversal (Section 4) reverses the order of the four visits and keeps the switch signs. So it suffices to show that $p[2] \leq p[0]$ and $p[2] \leq p[3]$ cannot both hold. Write ω and ω_0 for the unused margins at t_2 and t_1, and $\tau := \tau_1$. By the margin quadric at t_2, $p[1]p[2] + \omega\tau = 4$, $\mu_1 = \tau(p[1] + \omega)$ and $\mu_2 = \tau(p[2] + \omega)$.

Eliminate the two cross pairings. Pair the edge relation $\mu_1\mathbf{y}_0 + \mu_2\mathbf{y}_2 = \tau^2\mathbf{y}_1$ of (5.8) with $\mathbf{y}_1$, and insert (5.22) for the cross pairings at visits 1 and 2. The result is

$$\begin{aligned} &8\omega\tau^2\sigma_1 + 8p[2](p[1] + \omega)\tau_0\,\sigma_0 + 8p[1](p[2] + \omega)\tau_2\,\sigma_2 \\ &\qquad = p[2](p[1] + \omega)\,a_{\mathrm{stat}}(p[0], p[1], p[2]) + p[1](p[2] + \omega)\,a_{\mathrm{stat}}(p[1], p[2], p[3]). \end{aligned} \tag{6.1}$$

By (5.1), during visit 1 the margin of the entry label grows from $p[0]$ to ω, and the margin of the exit label falls from ω_0 to $p[2]$. During visit 2 the margin of the exit label falls from ω to $p[3]$. So (5.19) at the far ends of these visits gives

$$8\sigma_0 \leq \langle f_{p[1]} \rangle_{p[0],\omega}, \qquad 8\sigma_1 \leq \langle f_{p[1]} \rangle_{p[2],\omega_0}, \qquad 8\sigma_2 \leq \langle f_{p[2]} \rangle_{p[3],\omega}.$$

Express the area derivatives as interval means. A direct computation from (5.3) gives $(a^2 + 4)\big(\partial_1 A_{\mathrm{mar}}(a, b) + \partial_1 A_{\mathrm{mar}}(a, c)\big) = 2\sqrt{3}(a + b + c)$ whenever $ab + bc + ca = 4$; the logarithms cancel because $a^2 + 4 = (a + b)(a + c)$. With the margins at t_2, the derivative difference (5.16) then yields

$$\begin{aligned} a_{\mathrm{stat}}(p[0], p[1], p[2]) &= 4(\tau + \omega)/\mu_1 - (\omega - p[0])\langle f_{p[1]} \rangle_{p[0],\omega}, \\ a_{\mathrm{stat}}(p[1], p[2], p[3]) &= 4(\tau + \omega)/\mu_2 - (\omega - p[3])\langle f_{p[2]} \rangle_{p[3],\omega}. \end{aligned}$$

The quadric also gives

$$4(\tau+\omega) = \omega\tau^2 f_{p[1]}(p[2]) + p[2](p[1]+\omega)^2 f_{p[1]}(\omega) \\ + p[1](p[2]+\omega)^2 f_{p[2]}(\omega).$$

Insert all this and the three bounds in (6.1). The result is $\Omega \le 0$, where

$$\Omega \coloneqq p[2](p[1]+\omega)^2\big(f_{p[1]}(\omega) - \langle f_{p[1]}\rangle_{p[0],\omega}\big) + \omega\tau^2\big(f_{p[1]}(p[2]) - \langle f_{p[1]}\rangle_{p[2],\omega_0}\big) \\ + p[1](p[2]+\omega)^2\big(f_{p[2]}(\omega) - \langle f_{p[2]}\rangle_{p[3],\omega}\big).$$

Each term compares the value of the kernel at one end of an interval with its mean over the interval.

Exclude an interior minimum. Suppose $p[2] \le p[0]$ and $p[2] \le p[3]$. The last term is positive, because the interval from $p[3]$ to ω lies in $[p[2], \infty)$, where $f_{p[2]}$ increases. For the first two terms write $p \coloneqq p[1]$, $p' \coloneqq p[2]$ and $q \coloneqq q_1$. By (5.1), $p+\omega = q(p+p[0])$ and $p+\omega_0 = q\tau$. So by (5.15) the two means are $\langle f\rangle_{\mathrm{arc}}(p/(p+\omega), q)$ and $\langle f\rangle_{\mathrm{arc}}(p/\tau, 1/q)$, and the two values are the same expressions with 1 in the second argument. We need two elementary bounds for their difference. For $q > 1$, the two logarithm differences $\eta_{\log}(q) \coloneqq q - 1 - \log q$ and $\zeta_{\log}(q) \coloneqq q\log q - q + 1$ are positive and satisfy

$$(2q+1)\eta_{\log}(q) < (q+2)\zeta_{\log}(q), \qquad \eta_{\log}(q) < \zeta_{\log}(q). \tag{6.2}$$

The differences $\eta_{\log}, \zeta_{\log}$ vanish at 1 and are increasing for $q > 1$. Subtract $3(q^2-1)/(q^2+4q+1)$ from $\log q$. The difference vanishes at 1 and has derivative

$$\frac{(q-1)^4}{q(q^2+4q+1)^2} > 0.$$

Clearing the positive denominator gives the first inequality in (6.2). The second follows from $(2q+1)/(q+2) > 1$.

The first two terms of Ω therefore have the sum

$$\frac{p}{q(q-1)}\Big(\omega\,\zeta_{\log}(q)\,(qp'+p) - qp'\,\eta_{\log}(q)\,(qp+\omega)\Big).$$

The expression in parentheses increases with ω, since $\eta_{\log} < \zeta_{\log}$, and $p[0] \ge p'$ means $\omega \ge qp' + (q-1)p$. At this value of ω replace $\zeta_{\log}(q)$ by its strict lower bound in (6.2). The resulting lower bound for $q+2$ times the expression is $(q-1)\,\eta_{\log}(q)\big((qp'-p)^2 + 2qp^2\big) > 0$. So $\Omega > 0$, a contradiction. □

6.2 Returning to the preceding control

Lemma 6.2 (Three-arc inequalities)**.** *If the control labels on visits $k, k+1, k+2$ of a normal walk are i, j, i with $i \ne j$, their active margins satisfy*

$$p[k]\,p[k+2] < p[k+1]^2.$$

If their switch signs are $-,+$, then $p[k] < p[k+2]$; if their signs are $+,-$, then $p[k+2] < p[k]$.

Proof. Visit $k+1$ is a turn. By (5.2) its factor is τ_{k+1}/τ_k or τ_k/τ_{k+1} according to the signs, and it exceeds one. This orders the outer margins as stated. Put $x \coloneqq p[k+1]$, and let $s_<$ and $s_>$ be the smaller and the larger outer margin. By Lemma 5.6(b), $8\sigma_k = 8\sigma_{k+1}$ is the mean of $f_x(y) = 1 - xy/(x+y)^2$ over $[s_<, s_>]$. The switch at the smaller outer margin is the reverse one, so by Lemma 5.6(a)

$$f_x(s_>) \le 8\sigma_k \le f_x(s_<).$$

The endpoint difference factors as

$$f_x(s_<) - f_x(s_>) = \frac{x(s_> - s_<)(x^2 - s_< s_>)}{(x+s_<)^2(x+s_>)^2}.$$

The switch inequalities make this nonnegative. If $s_< s_> = x^2$, the distinct endpoints straddle x and have the same value of f_x. Throughout the open interval f_x is strictly smaller than that common value, so its mean violates the lower switch inequality. Thus $s_< s_> < x^2$. □

6.3 The permitted reverse runs

Lemma 6.3 (A reverse run and its center)**.** *Every reverse switch of a normal period belongs to a run of exactly two. The active margin at the center of such a run is strictly below the active margins of the two visits before it and the two after it.*

Proof. If every switch were reverse, choose a least-margin visit k_*. The four visits from $k_* - 2$ to $k_* + 1$ contradict Lemma 6.1. The argument uses the extended visits, including any that cross the period cut.

Now consider a maximal reverse run of length n_{rev} bounded by forward switches. Its reverse switches are those out of visits $k, \dots, k + n_{\mathrm{rev}} - 1$. The switches out of $k-1$ and $k + n_{\mathrm{rev}}$ are forward. At the entry, Lemma 6.2 gives

$$p[k+1] < p[k-1], \qquad p[k+1] < p[k].$$

For the second inequality, if $p[k] \le p[k+1]$, then $p[k]$ would be at most both of its outer neighbors, contradicting the strict product inequality. The same argument at the exit gives

$$p[k + n_{\mathrm{rev}} - 1] < p[k + n_{\mathrm{rev}} + 1], \qquad p[k + n_{\mathrm{rev}} - 1] < p[k + n_{\mathrm{rev}}].$$

When $n_{\mathrm{rev}} = 1$, the two conclusions contradict each other. When $n_{\mathrm{rev}} \ge 3$, choose a smallest margin among the visits $k, \dots, k + n_{\mathrm{rev}}$. It lies at neither endpoint. If it is at $k+1$, use the four-arc window starting at k; otherwise, if it is at $k_* \ge k+2$, use the window starting at $k_* - 2$. In either case Lemma 6.1 is contradicted. We have proved

$$n_{\mathrm{rev}} = 2, \qquad p[k+1] < \min\{p[k-1], p[k], p[k+2], p[k+3]\}. \tag{6.3}$$

□

This proves Proposition 3.1.

6.4 The switch floor

Lemma 6.4 (Lower bound at every switch)**.** *At every switch of a normal period the switch form value satisfies*

$$\sigma_k \geq 3/32. \tag{6.4}$$

Proof. At a forward switch the claim is $\sigma_k \geq \sigma_k^* \geq 3/32$, by Lemma 5.6(a). Every reverse switch touches a turn by Lemma 6.3. At a turn the common incident form value is at least $3/32$ by Lemma 5.6(b). □

6.5 Positivity of the switch form

Lemma 6.5 (Positivity of the switch form)**.** *The switch form S of every normal vertex period is positive definite. In particular $d > 0$.*

Proof. Since $\sigma_0 \geq 3/32 > 0$, it suffices to exclude $\det S \leq 0$. Assume it.

Signs of the cross pairings. At a same-sign visit, (5.7) and (6.4) give $\sigma_{k-1,k}^2 \geq \sigma_{k-1}\sigma_k \geq (3/32)^2$. At a visit with two forward switches, $p[k-1]p[k] < 4$ gives $\gamma_k < \rho_k^-$, so $16\sigma_{k-1,k} \geq 16\sigma_{k-1,k}^* > -1$ by Lemma 5.7. At a reverse center both shares are below $1/2$ by (6.3), so $16\sigma_{k-1,k} \leq 16\sigma_{k-1,k}^* < 1$. As $3/32 > 1/16$,

$$\sigma_{k-1,k} > 0 \quad \text{at a visit with two forward switches,} \qquad \sigma_{k-1,k} < 0 \quad \text{at a reverse center.} \tag{6.5}$$

At every visit that is not a turn, (6.5) is precisely the sign condition of Lemma 5.8. It gives $\det S > 0$, a contradiction. □

6.6 The two shares at a reverse center

Lemma 6.6 (Data at a reverse center)**.** *At a reverse center both margin shares are below $1/2$. Their product bounds the center scale from above. Each incident form value lies between the universal floor and its reverse-switch bound.*

Proof. At a reverse center k,

$$\begin{gathered} 0 < \rho_k^{\pm} < \tfrac{1}{2}, \qquad 0 < \gamma_k < \rho_k^- \rho_k^+, \\ \tfrac{3}{4} \leq 8\sigma_{k-1} \leq 1 - \rho_k^- + (\rho_k^-)^2, \qquad \tfrac{3}{4} \leq 8\sigma_k \leq 1 - \rho_k^+ + (\rho_k^+)^2, \end{gathered} \tag{6.6}$$

and the cross pairing and determinant satisfy

$$\begin{aligned} 16\sigma_{k-1,k} &= 1 + \rho_k^- + \rho_k^+ - \gamma_k + \log \frac{\gamma_k}{\rho_k^- \rho_k^+} - \frac{1 - 8\sigma_{k-1}}{\rho_k^-} - \frac{1 - 8\sigma_k}{\rho_k^+}, \\ (1 - \gamma_k)^2 \det S &= 16(\sigma_{k-1}\sigma_k - \sigma_{k-1,k}^2). \end{aligned} \tag{6.7}$$

The strict minimum (6.3) gives $\rho_k^{\pm} < 1/2$, and $\gamma_k/(\rho_k^-\rho_k^+) = q_k^{-1} < 1$. The bounds on the form values are Lemma 6.4 and Lemma 5.6(a). The identities are (5.20) and (5.7) with $\mu_k = 4/(1 - \gamma_k)$. □

7 Area and winding bounds for the frame costate

Area controls the speed of the polar angle, hence the determinant of the switch form. Winding gives a second constraint: measured in the metric of that form, the edge directions make one turn around the boundary. The resulting angle sum will bound the length of a forward period in Section 9. A comparison across two forward visits also locates the least active margin relative to the circular state. In a mixed period it identifies where to mark the walk.

Lemma 7 (Elliptic sheet and determinant bound)**.** *In a normal vertex period the frame costate is on the elliptic sheet through $-J$. If the period is competitive, then*

$$\det S \le \frac{1}{12}\left(\frac{3A_{\mathrm{oct}}}{\pi} - \frac{3}{2}\right)^2 < \frac{9193}{50000}. \tag{7.1}$$

Proof. By Lemma 6.5 the momentum form $\det(\Lambda\mathbf{u}, \mathbf{u})$ of the proof of Lemma 5.4 is positive definite, where $\Lambda = \Lambda_1(0)$. A traceless matrix of determinant $d_1^2 > 0$ is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to $d_1 J$ or to $-d_1 J$, and $\det(-J\mathbf{u}, \mathbf{u}) = |\mathbf{u}|^2$. Consequently $\Lambda_1(0) = -d_1 h J h^{-1}$ for some $h \in \mathrm{SL}_2(\mathbb{R})$. Apply the area estimate of Lemma 4.1 with $A \le A_{\mathrm{oct}}$. The rational upper bound follows from $A_{\mathrm{oct}} < 3.12606$ and $\pi > 3.14159$, with exact comparisons in Claim V4. □

7.1 Winding in the switch metric

The positive definite switch form S defines a Euclidean metric on the plane of the disk, called the *switch metric*. Its determinant is d in standard coordinates and $d/12$ in the basis of Lemma 5.4. A point of the edge of switch k and its edge vector span a parallelogram of standard area $1/4$, so $16\sigma_k$ times the squared distance of the edge from the center is d, in the switch metric. For the octagon this is the metric of Figure 3. At a visit with two switches of the same sign, let $\varphi_k \in (0, \pi)$ be the angle between $\mathbf{y}_{k-1}$ and $\mathbf{y}_k$ in the switch metric,

$$\cos\varphi_k = \frac{\sigma_{k-1,k}}{\sqrt{\sigma_{k-1}\sigma_k}}. \tag{7.2}$$

Lemma 7.1 (Winding in the switch metric)**.** *The angles of a normal vertex period of area below π satisfy*

$$\sum_{\text{reverse centers}} (\varphi_k - \pi/3) + \sum_{\text{two forward switches}} (\pi/3 - \varphi_k) = \pi/3. \tag{7.3}$$

In particular, under this area bound an all-forward walk has $\sum_k \varphi_k = (N-1)\pi/3$.

Proof. First observe that $N > 1$. Indeed a single visit cannot be a turn, and $\mathbf{y}_{k+1} = \pm\mathbf{y}_k$ would contradict the nonzero determinant at that visit in Lemma 5.4. Thus every visit has duration strictly less than t_f. By Lemma 2.2 the contact curve $\mathbf{c}(t) \coloneqq g(t)\mathbf{s}_0^*$ runs once around the boundary of a convex disk in the time $6t_f$, and $\mathbf{c}(t + 3t_f) = -\mathbf{c}(t)$. Hence the direction of its velocity has a continuous nondecreasing argument $\varphi(t)$ in the switch metric, and $\varphi(t+3t_f) = \varphi(t)+\pi$. The other contacts are time shifts, $g(t)\mathbf{s}_r^* = \mathbf{c}(t + rt_f)$. With the contact labels r_k of the proof of Lemma 5.4, the vector with coordinates $\mathbf{y}_k$ is therefore a positive multiple of $\epsilon_k\dot{\mathbf{c}}(t'_k)$, where $t'_k \coloneqq t_{k+1} + r_k t_f$.

At a turn $r_k = r_{k-1}$, and the velocity keeps its direction from t'_{k-1} to t'_k. At a visit that is not a turn, $t'_k - t'_{k-1} = \Delta t_k - \epsilon_k t_f$. This lies between $-t_f$ and 0 at a visit with two forward switches, and between t_f and $2t_f$ at a reverse center. In both cases φ changes by at most π, so the change is the angle between the two velocities with its sign: $\varphi(t'_k) - \varphi(t'_{k-1}) = -\epsilon_k\varphi_k$. Over a period

$t'_N - t'_0 = (1 - \sum_{j=0}^{N-1} \epsilon_j) t_f$, because $r_N = r_0 - \sum_{j=0}^{N-1} \epsilon_j$, and $1 - \sum_{j=0}^{N-1} \epsilon_j$ is a multiple of 3. Hence the sum of φ_k over the reverse centers minus the sum over the visits with two forward switches is $(1 - \sum_{j=0}^{N-1} \epsilon_j)\pi/3$. By the proof of Lemma 5.4, $\sum_{j=0}^{N-1} \epsilon_j$ equals the number of visits with two forward switches minus the number of reverse centers. This gives (7.3). □

7.2 The form at a common forward switch

Lemma 7.2 (Forward switch comparison)**.** *At consecutive visits $k-1, k$ with forward switches at both ends, the outgoing value at visit k satisfies*

$$\frac{1}{8}\langle f \rangle_{\mathrm{arc}}(\rho_k^+, q_k) \;\le\; \sigma_k \;\le\; \sigma_k^{\max},$$

where $\sigma_k^{\max}$ is the upper bound transported from the preceding visit in (7.4)*.*

Proof. For this comparison measure all margins in units of $p[k]$. At the common switch the margins are $z, 1, m$, where $z = p[k-1]/p[k]$ and m is the unused margin. Write $w = p[k+1]/p[k]$ and $h = q_{k-1},\ q = q_k$. The margin quadric and the arc factors give

$$(1+z)(1+m) = 1 + 4/p[k]^2, \qquad q = (1+w)/(1+m).$$

At the preceding switch the unused margin in the same units is $p = (1+z)/h - z$. The margin of each label along the two visits is

label	entry to $k-1$	common switch	exit from k
i_{k-1}	z	z	$(1+z)/q - 1$
i_k	p	1	1
i_{k+1}	$(z+m)h - z$	m	w

The unchanged entries are the active margins. The means in (5.14) are invariant under the common change of scale.

At visit $k-1$ the outgoing arc bound gives $8\sigma_{k-1} \ge \langle f_z \rangle_{p,1}$; at visit k it gives $8\sigma_k \ge \langle f_1 \rangle_{m,w}$.

For the preceding arc use brackets for the ordinary mean in r on $[1, h]$. With $s = 1 + z$, the two stationarity equations give

$$\begin{gathered} \lambda\big(8\sigma_k - \langle f_1 \rangle_{m,w}\big) + 8\kappa(\sigma_k - \sigma_k^{\max}) = 0, \\ 8\sigma_k^{\max} = f_1(m) + \frac{\langle (1 - r^{-1})(z/r - m) \rangle_{1,h} - (ms^2/z)(8\sigma_{k-1} - f_1(z))}{(1+m)^2}, \\ \lambda = (q-1)[1 + s(h-1)], \quad \kappa = z(h-1). \end{gathered} \tag{7.4}$$

At the common switch the edge relation reads

$$(z+m)\mathbf{y}_{k-2} = s\mathbf{y}_{k-1} - (1+m)\mathbf{y}_k.$$

Use this vector in stationarity at $k-1$, and use stationarity at k to eliminate $\sigma_{k-1,k}$. The two logarithmic terms are accounted for by the following evaluations:

$$\begin{aligned} \langle f_1 \rangle_{m,w} &= 1 + \frac{1}{(1+m)(1+w)} - \frac{\log q}{(1+m)(q-1)}, \\ \langle (1 - r^{-1})(z/r - m) \rangle_{1,h} &= (z+m)\left(\frac{\log h}{h-1} - 1\right) + z(1 - 1/h). \end{aligned}$$

Substitution in (5.22) gives (7.4), as verified in Claim V9. The first difference in (7.4) is nonnegative by the arc bound. Since both weights are positive, $\sigma_k \le \sigma_k^{\max}$. □

7.3 The margin at a forward minimum

Lemma 7.3 (Forward minimum)**.** *If a normal period has four consecutive forward switches and the middle visit has the least margin among the five visits, its margin is greater than $2/\sqrt{3}$. In particular every active margin of a normal all-forward period is greater than $2/\sqrt{3}$.*

Proof. Number the five visits $k-2,\dots,k+2$, with k the middle visit. Use the coordinates of (7.4) there. Reflection and time reversal allow $1 \le z \le w$. Suppose $p[k] \le 2/\sqrt{3}$, so $(1+z)(1+m) \ge 4$. We distinguish which side of the active margin contains the unused margin m.

If $m \ge 1$, the next active margin w exceeds m, since $q > 1$. Hence $8\sigma_k \ge f_1(w) > f_1(m)$ by the switch bound. We show that the preceding arc forces the opposite inequality. Write $l = 1 - 1/h$ and

$$b_{\log}(h) = \frac{1+h^{-1}}{2} - \frac{\log h}{h-1} = \sum_{n\ge 2} \frac{l^n}{n(n+1)}.$$

Thus the preceding-arc mean equals $(z-m)l/2 - (z+m)b_{\log}(h)$. Insert $8\sigma_{k-1} \ge \langle f_z\rangle_{p,1} = \langle f\rangle_{\mathrm{arc}}(z/s,h)$ in (7.4) and collect this same logarithmic remainder. The result is

$$(1+m)^2(8\sigma_k^{\max} - f_1(m)) \le \frac{l}{2}\{z(1-m) - m(z-1)(h-1)\} - (z+m+msh)b_{\log}(h) < 0.$$

This contradicts $\sigma_k \le \sigma_k^{\max}$.

Now $m < 1$. The active margin at visit $k-2$, in the same units, is $(z+m)h - z \ge 1$, so $0 < p \le m$. At the outgoing switch of visit k the unused margin is $(4/p[k]^2 - w)/(1+w) \le m$, since $w \ge z$. Applying the same observation on the other side, the unused margin at the far switch of visit $k+1$ is also at most m. For $y \le 1 \le z \le w$, $f_z(y)$ decreases in y and increases in z. The two neighboring arc bounds and convexity of f_z on $[0,1]$ give

$$8\sigma_{k-1} \ge \langle f_z\rangle_{p,1} \ge u_0 \coloneqq f_z((1+p)/2), \qquad 8\sigma_k \ge \langle f_z\rangle_{m,1} \ge v_0 \coloneqq f_z((1+m)/2).$$

We may shorten the following interval $[m,w]$ to $[m,z]$ in the necessary comparison. Indeed

$$(q-1)\big(8\sigma_k - \langle f_1\rangle_{m,w}\big) = \frac{1}{1+m}\int_m^w (8\sigma_k - f_1(y))\,dy.$$

The removed integral is nonnegative because $8\sigma_k \ge f_1(w)$ and f_1 increases on $[z,w]$. On the remaining interval use $8\sigma_k \ge v_0$. In the second term of (7.4), use $8\sigma_{k-1} \ge u_0$, which decreases $\sigma_k^{\max}$ as $8\sigma_{k-1}$ increases. Thus the left side, evaluated at $w = z$, $8\sigma_{k-1} = u_0$, $8\sigma_k = v_0$, must be nonpositive.

Only a rational sign remains. With $b = (z-m)/(2+z+m)$, the series for $\log((1+b)/(1-b))$ gives

$$\langle f_1\rangle_{m,z} \le 1 + \frac{1}{(1+m)(1+z)} - \frac{2}{2+z+m}(1+b^2/3).$$

The preceding-arc mean is at most $(z-m)l/2 - (z+m)l^2/6$. Replacing both by these upper bounds gives a lower bound on the shortened left side. It is strictly positive by Claim V9. The exact check uses the nonnegative coordinates

$$\frac{m-p}{1-m}, \qquad \frac{p}{1-m}, \qquad (1+z)(1+m) - 4.$$

After substitution the numerator and denominator have positive constant terms and all their nonzero coefficients are positive. This contradicts stationarity and proves the lemma. □

Corollary 7.3 (The least active margin)**.** *The least active margin of a normal all-forward period exceeds $2/\sqrt{3}$, the margin at the circular state. In a normal mixed period it is smaller than $2/\sqrt{3}$, and every visit attaining it is a reverse center.*

Proof. The forward assertion is Lemma 7.3. In a mixed walk there is a reverse center by Proposition 3.1. A reverse center of margin p has larger neighbors by (6.3). Positive duration gives $4p^2 < 4 + p^2$, hence $p < 2/\sqrt{3}$. A turn has a smaller reverse center next to it. If the minimum had two forward switches, a reverse switch just beyond either one would again give a smaller center. Otherwise the surrounding four switches would be forward and Lemma 7.3 would place the minimum above $2/\sqrt{3}$. Thus every least-margin visit is a reverse center. □

8 Excluding mixed periods

Proposition 3.2. A competitive normal walk whose active margins are all at least $2/15$ has only forward steps.

Assume for contradiction that a period satisfying the hypotheses contains a reverse switch. An entirely reverse period has already been excluded in Lemma 6.3. In a mixed period mark a visit of least active margin and number it 0. It is a reverse center by Corollary 7.3. Reflection and time reversal allow us to take $p[1] \le p[-1]$. The incoming turn then gives a quantitative bound on the form at the marked center. On the other side that bound forces exactly two turns before the next reverse center. These first two steps apply to every normal mixed period. The final comparison uses the area bound and the lower bound $p[0] \ge 2/15$.

8.1 The turn preceding the minimum

Proposition 8.1 (Order of the intervening turns)**.** *Suppose two consecutive reverse centers are separated by exactly two turns. If the margin at the first center is at most that at the second, the same ordering holds for the two turn margins. Equal center margins give equal turn margins.*

Proof. Write the successive margins as a, e, x, y, with $a \le y$. Both centers are smaller than either turn by (6.3). Since the two turn visits share a switch, their means are equal: $\langle f_e \rangle_{a,x} = \langle f_x \rangle_{y,e}$. Suppose $e > x$. For $0 < \lambda < 1$ put $p = (1-\lambda)a + \lambda x$ and $z = (1-\lambda)y + \lambda e$. Then $e/p > z/x$ since $p < x$ and $z < e$. Also $ez > xp$, by linear interpolation between $ey > xa$ and $e^2 > x^2$. Thus $e/p > \max(z/x, x/z)$. The value $f_e(p)$ increases strictly with $|\log(e/p)|$ and therefore exceeds $f_x(z)$ at every interior point of these parametrizations. Integrating contradicts equality of the means. If $a = y$, read the same argument in the other direction. □

Lemma 8.1 (Loss at the preceding turn)**.** *At the incoming switch of the marked reverse center, the deficit per unit incoming share*

$$\frac{\sigma^*_{-1} - \sigma_{-1}}{\rho_0^-}$$

has a positive lower bound depending only on the outgoing shares ρ_0^+ and ρ_1^+.

Proof. Write the five successive active margins as

$$x \xrightarrow{+} e \xrightarrow{-} \underbrace{a}_{\text{marked}} \xrightarrow{-} b \xrightarrow{+} c, \qquad x, c > a, \quad e \ge b > a.$$

Write $\xi = 1 - 2\rho_0^+ = (b-a)/(b+a)$ and $\xi u = 1 - 2\rho_1^+ = (c-b)/(c+b)$. The normalization by $\xi > 0$ measures both changes in units of the marked turn. By the turn equation, $8\sigma_0 = \langle f_b \rangle_{a,c}$. In the integral defining this mean, set $y = (b-p)/(b+p)$. Since $4f_b(p) - 3 = y^2$, $32\sigma_0 - 3$ is the mean of y^2 on $[-\xi u, \xi]$ with density proportional to $(1+y)^{-2}$. The forward switch gives $32\sigma_0 - 3 \geq \xi^2 u^2$, the value of y^2 at the left endpoint. Thus y^2 minus its weighted mean changes sign only from negative to positive as y increases. Removing the decreasing weight raises the mean. Integrating with constant weight gives

$$u^2 \leq m = \frac{32\sigma_0 - 3}{\xi^2} \leq \frac{1 - u + u^2}{3}, \qquad -1 < u \leq \tfrac{1}{2}. \tag{8.1}$$

The lower endpoint follows from $c > a$ and the upper one from the two bounds on m.

Put $r = x/a - 1$ and $r_* = (7/8)\xi \min(1, 1+u)$. We claim $r > r_*$. If x is a turn, the next reverse center beyond it has margin at least a. Proposition 8.1, read from the marked center toward that center, gives $x \geq e \geq b$. Since $b/a - 1 = 2\xi/(1-\xi)$, this proves the claimed strict inequality in this case.

If x has two forward switches, transport across e yields $\nu \mathbf{y}_{-3} = -\mathbf{y}_0 - \omega \mathbf{y}_{-1}$, where $\nu = 1 + \gamma_0 r(2+r)$ and $\omega = \gamma_0 r(2/\rho_0^- + r)$. The margin y before x satisfies $(y+x)(x+e) > 4 + x^2$ by positive duration. The forward-switch floor applied to the squared norm of the transported vector gives

$$8(\sigma_0 + 2\omega\sigma_{-1,0} + \omega^2\sigma_{-1}) \geq \tfrac{3}{4}\nu^2 + \left[\nu/2 - \gamma_0(1+r)(1/\rho_0^- + r)\right]_+^2, \qquad [z]_+ = \max(z, 0). \tag{8.2}$$

The center equation (6.7) bounds the cross pairing using the two diagonal values. Claim V6 shows that their ceilings already contradict (8.2) for $r \leq r_*$. This proves the claim about x.

Now $8\sigma_{-1} = \langle f_e \rangle_{a,x} \geq f_e(x)$. The kernel decreases up to e and then increases, so its mean on any initial subinterval of $[a, x]$ is at least $8\sigma_{-1}$. On $[a, a(1+r_*)]$ it is convex, since $a(1+r_*) < e$. Its endpoint average bounds its mean from above. Subtracting from $f_e(a) = 8\sigma_{-1}^*$ and using $\rho_0^- \leq \rho_0^+$ gives

$$\frac{8(\sigma_{-1}^* - \sigma_{-1})}{\rho_0^-} \geq \eta, \qquad \eta = \frac{(1-\rho_0^+) r_* (\xi - r_*(\rho_0^+)^2)}{2(1 + r_*\rho_0^+)^2} > 0. \tag{8.3}$$

□

8.2 The forced short connection

Proposition 8.2 (The forced short connection)**.** *In a finite normal mixed period, mark a visit of least active margin. On the side of its smaller neighboring turn, exactly two turns separate the marked center from the next reverse center. In case of equality either side may be chosen.*

Proof. The switch entering visit 2 is forward. We claim that visit 2 is a turn. Suppose its outgoing switch is also forward. Throughout this proof measure margins and their sums $p[j], \tau_j$ in units of the original $\tau_0 = p[0] + p[1]$, and μ_j in units of its square. Form pairings, shares, scales and arc factors keep their original values. In these units $\tau_0 = 1$, $p[0] = \rho_0^+$, $\mu_0 > 1$, and $q_2 = \tau_1\tau_2/\mu_2$. The turn at visit 1 reverses its switch vector. With $\omega = \tau_1^2 - 1$, edge transport gives

$$\mu_2 \mathbf{y}_2 = -\mu_0 \mathbf{y}_{-1} - \omega \mathbf{y}_0, \qquad \mathbf{y}_1 = -\mathbf{y}_0.$$

These two identities determine $\sigma_{1,2}$ and σ_2 from the pairings at the marked center.

To use the loss (8.3), let x vary through values of the incoming share and y through values of the incoming form σ_{-1}. Keep the right-hand margins, γ_0, and σ_0 fixed. The center equation (6.7)

determines $\sigma_{-1,0}$ for each pair (x, y). First raise y to its ceiling $8y = 1 - (1+\eta)x + x^2$, then follow the ceiling down to $x_0 = \gamma_0/\rho_0^+ < \rho_0^-$. The marked arc has zero duration at this comparison point. We denote the resulting form entries by an overbar. For subsequent use,

$$\begin{aligned} 8\bar\sigma_{-1} &= 1 - (1+\eta)x_0 + x_0^2, \\ 16\bar\sigma_{-1,0} &= \rho_0^+ - (1-8\sigma_0)/\rho_0^+ + (2-\rho_0^+)x_0 - \eta. \end{aligned} \tag{8.4}$$

The logarithm in the center equation has vanished, while the loss η remains.

At visit 2 the stationarity residual $r_2(x, y)$ vanishes at the actual data. In the chosen units (5.20) gives

$$\begin{aligned} r_2 = \tau_1(1-8\sigma_0) + \tau_2(1-8\sigma_2) \\ + p[2]\big(16\sigma_{1,2} - 1 - p[2]/\tau_1 - p[2]/\tau_2 + p[2]^2/\mu_2 - \log q_2\big). \end{aligned} \tag{8.5}$$

Both parts of the comparison lower this residual, the second strictly. Indeed, with $y(x) = [1-(1+\eta)x+x^2]/8$ along the ceiling,

$$\frac{\partial_y r_2}{8} = \frac{\mu_0[p[2]\mu_2 - \tau_2(\omega+\mu_0 x)]}{x\mu_2^2} < 0, \qquad \frac{d}{dx} r_2(x, y(x)) = (2x-1)\frac{\partial_y r_2}{8} + \eta\tau_2(\mu_0/\mu_2)^2 > 0.$$

For the first sign use $\tau_2 \geq \mu_2/\tau_1$, $\mu_0 x \geq \rho_0^+$, and $\omega + \rho_0^+ - p[2]\tau_1 = (1-\rho_0^+)(p[2]-\rho_0^+) > 0$. Stationarity would therefore give $\bar r_2 < 0$.

The arc at visit 2 also bounds its own duration. Set $h = 1 - 1/q_2$ and use ξ, u, m from (8.1). Since $2\rho_2^- - 1 = \xi u$, the incoming secant bound gives

$$\xi^2 m \geq 4\langle f\rangle_{\mathrm{arc}}(\rho_2^-, q_2) - 3 = \frac{1}{h}\int_0^h \left(\frac{\xi u + z}{1-z}\right)^2 dz.$$

The integrand dominates $(\xi u + z)^2$, and for $u \geq 0$ it also dominates $[\xi u + (1+\xi u)z]^2$. Integrating and using the upper moment bound gives the first inequality below. The second is the positive unused margin condition at the outgoing switch (in original units, $p[2]p[3] < 4$):

$$h(1 + \max(0, \xi u)) \leq \xi(1-2u), \qquad h < (1-\xi u)/2. \tag{8.6}$$

On this range Claim V7 gives $\bar r_2 \geq 0$. The calculation is affine in μ_0 after denominators are cleared, and affine in the turn moment m. This contradicts the strict sign. Thus visit 2 turns, and its reverse switch starts a pair by Proposition 3.1. □

8.3 Stationarity at the second reverse center

We now use competitiveness to bound the scale at every reverse center. The lower margin bound enters only at the end of the proof of Proposition 3.2.

Lemma 8.3 (Scale of a reverse center)**.** *Every reverse center of a competitive mixed normal period has $\gamma_k < 3/20$, or equivalently $p[k]^2 < 12/17$.*

Proof. Suppose $\gamma_k \geq 3/20$. We seek a lower bound for $\det S$ on this range. First minimize the Gram expression in the incident form values at fixed shares and scale. Then minimize the remaining bound over $3/20 \leq \gamma \leq 1/4$. Let $\gamma = \gamma_k$ and $c = 16\sigma_{k-1,k}$. The Gram identity gives $16(1-\gamma)^2 \det S = 256\sigma_{k-1}\sigma_k - c^2$. For either share $r = \rho_k^-, \rho_k^+$ the diagonal ceilings imply $c < -1 + 2(\rho_k^- + \rho_k^+) - \rho_k^-\rho_k^+ < 3r/2$. By the center equation the derivative of c in the corresponding

diagonal entry is $8/r$. The other diagonal entry is at least $3/32$, so decreasing either entry lowers the Gram expression. Lower both to $3/32$.

Suppose $3/20 \le \gamma < 1/4$, and let c_0 be c at these lower values. Each share contributes $r - 1/(4r) - \log r$, an increasing function. Its upper limit at $r = 1/2$ gives $c_0 \le 1 - \gamma + \log(4\gamma)$. For a lower bound put $r_1 = \rho_k^-$ and reduce the other share to γ/r_1; then $r_1 + \gamma/r_1 \le 1/2 + 2\gamma$ and its coefficient is $1 - 1/(4\gamma) < 0$. Thus $c_0 \ge 1 + \gamma - 1/(8\gamma) > 0$. Using $\log z \le 2(z-1)/(z+1)$ for $0 < z \le 1$ now gives

$$\det S \ge b_{\mathrm{ctr}}(\gamma) := \frac{9/4 - [1 - \gamma + 2(4\gamma - 1)/(4\gamma + 1)]^2}{16(1-\gamma)^2}.$$

The last scalar minimization is explicit:

$$b'_{\mathrm{ctr}}(\gamma) = \frac{(1-4\gamma)^2(32\gamma^2 - 52\gamma + 65)}{32(1-\gamma)^3(4\gamma+1)^3} \ge 0, \qquad \min_{3/20 \le \gamma \le 1/4} b_{\mathrm{ctr}}(\gamma) = \frac{851}{4624}.$$

The minimum occurs at $3/20$ and exceeds the determinant cap $9193/50000$ in (7.1). □

Proof of Proposition 3.2. By Proposition 8.2 the second center is visit 3. Since $p[0] \le p[3] < \min(p[1], p[2])$, Proposition 8.1 gives $p[1] \le p[2]$, hence $u \ge 0$. The intervening turns carry the same value $8\sigma_0$. On $[p[3], p[1]]$ the kernel $f_{p[2]}$ is convex because $p[1] \le p[2]$. Its mean on each half is at least its value at the midpoint. Together with (8.1) this gives

$$\frac{f_{p[2]}((3p[3] + p[1])/4) + f_{p[2]}((p[3] + 3p[1])/4)}{2} \le 8\sigma_0 \le \tfrac{3}{4} + \xi^2(1 - u + u^2)/12. \tag{8.7}$$

For the transport from visit 0 to visit 3, measure every active margin in units of the original $p[0]$. Thus $p[0] = 1$ in the following calculation. The ratios of the μ_j, the shares, the scales and the pairings are unchanged by this choice of units. Put $\nu = \mu_3/\mu_0 = 1 + \gamma_0(p[3]^2 - 1)$ and $\omega = (1 + p[3])^2 - 2(p[2] - 1)(p[1] - p[3])$. The two turns restore the direction of the switch vector. Thus $\mathbf{y}_2 = \mathbf{y}_0$ and $\nu \mathbf{y}_3 = \gamma_0 \omega \mathbf{y}_0 - \mathbf{y}_{-1}$. In particular

$$\sigma_{2,3} = (\gamma_0 \omega \sigma_0 - \sigma_{-1,0})/\nu, \qquad \sigma_3 = [\sigma_{-1} + (\gamma_0\omega)^2 \sigma_0 - 2\gamma_0 \omega \sigma_{-1,0}]/\nu^2.$$

The incoming share is $\rho_3^- = p[3]/(p[2] + p[3])$. The outgoing share $r = \rho_3^+$ lies between $b_0 = \gamma_3/\rho_3^-$ and $1/2$, where $\gamma_3 = \gamma_0 p[3]^2/\nu$. The center equation requires the following residual to vanish at the actual data:

$$r_3 = 1 + \rho_3^- - (1 - 8\sigma_0)/\rho_3^- + r - \gamma_3 + \log(b_0/r) - (1 - 8\sigma_3)/r - 16\sigma_{2,3} = 0. \tag{8.8}$$

The ceiling $8\sigma_3 \le 1 - r + r^2$ and the decrease of $2r - \log r$ on this interval give $16\sigma_{2,3} \le \rho_3^- - (1 - 8\sigma_0)/\rho_3^- + 2b_0 - \gamma_3$.

Keep r fixed and make the same comparison (x, y) as in (8.4). Here the residual increases:

$$\frac{\partial_y r_3}{8} = \frac{x + r\nu - \gamma_0\omega}{r\nu^2 x} > 0, \qquad \frac{d}{dx} r_3(x, y(x)) = (2x - 1)\frac{\partial_y r_3}{8} - \frac{\eta}{r\nu^2} < 0.$$

For the first sign the numerator is at least $\gamma_0[1/\rho_0^+ + p[3]^2/\rho_3^- - \omega] = \gamma_0[p[1](p[2] - 1) + p[2](p[1] - p[3])] > 0$. The cross pairing $\sigma_{-1,0}$ increases by the marked-center equation, so $\sigma_{2,3}$ decreases. Both inequalities must therefore hold at the comparison point:

$$\bar{r}_3 \ge 0, \qquad 16\bar{\sigma}_{2,3} \le \rho_3^- - (1 - 8\sigma_0)/\rho_3^- + 2b_0 - \gamma_3.$$

Finally the physical lower bound $2/15$ on the marked margin gives $\gamma_0 \geq 1/226$; since $\gamma_0 < (\rho_0^+)^2$, this gives $\xi < 7/8$. The scale bound at visit 3, its positive duration, and the positive unused margin at the switch between visits 1 and 2 give

$$\frac{1}{226} \leq \gamma_0 \leq \min\left\{(\rho_0^+)^2, \frac{3}{17p[3]^2+3}, \frac{\rho_3^-}{\rho_3^- + (2-\rho_3^-)p[3]^2}, \frac{1}{1+p[1]p[2]}\right\}. \tag{8.9}$$

Claim V8 excludes the two inequalities on this scale interval, even allowing every form value between the two midpoint bounds (8.7). This proves Proposition 3.2. □

Remark 8.3 (Mixed areas near the octagon). Admissible mixed periods can have areas arbitrarily close to A_{oct} from above. Take six visits with signs $-,+,-,-,+,-$, with two reverse centers of margin a and four turns of margin $(1+\sqrt{2})a$. Put $\xi := 1+\sqrt{2}$ and $b := a^2$. Each turn has factor $\sqrt{2}$ and each center has factor $q_{\mathrm{rev}} := (b+4)/((1+\xi)^2 b)$. These arcs have positive duration for $0 < b < 4/((1+\xi)^2-1)$. The two possible adjacent margin products are ξb and $\xi^2 b$, both below four. Thus the switches have positive unused margins, and monotonicity on each arc keeps the state in the triangle. The repeating margins and the label offset give the state return.

In (5.5) the product of the three transfer matrices for a center and its two turns has trace zero and determinant one, so its square is $-I$. The sum of the six switch signs is -2, and $R^3 = -I$. Thus (5.5) gives the frame return by Lemma 5.3 and the state return. The signed area sum (5.3) gives

$$A(b) = \frac{\sqrt{3}}{2}\big(4 + b(3 - \xi^2 \log 2 - \log q_{\mathrm{rev}})\big).$$

Its derivative is $\frac{\sqrt{3}}{2}(3 - \xi^2 \log 2 - \log q_{\mathrm{rev}} + 4/(b+4)) < 0$. Indeed $q_{\mathrm{rev}} > 1$ and $\xi^2 \log 2 > 4$, using $\xi^2 > 29/5$ and $\log 2 > 56/81$. At the limiting value $q_{\mathrm{rev}} = 1$ the centers have zero duration. The four remaining arcs are forward, with the octagon factor $\sqrt{2}$. Thus the mixed areas converge to A_{oct} from above, and there is no uniform positive area gap for all admissible mixed periods. Claim V5 records the exact identities for this example.

9 Forward periods

The preceding sections give a lower bound on every active margin of an all-forward period and an upper bound on the determinant of its switch form. At a common switch the two stationarity equations turn these bounds into a restriction on the sum of the incident angles. The total turning then determines the number of visits.

Proposition 9 (Length of a forward period)**.** *A competitive normal all-forward period has exactly four visits.*

Remark 9 (Regular walks). Consider the regular forward branch with $N = 3m+1$, $m \geq 1$, on which all $q_k = q$ and $p[k] = p$. In the coordinates (5.6) every transfer matrix in (5.5) is $h_0'(q)(R')^{-1}$, of trace q. The walk closes for $q = 2\cos\varphi$ with $\varphi = m\pi/N$, since then the product is $(-1)^m I = R^{1-N}$. The arc-factor formula gives $p^2 = 4q/(4-q)$. For a lift with the same symmetry, all diagonal form values equal σ_0, and (5.20) with $\sigma_{k-1,k} = \sigma_0 \cos\varphi$ and (5.7) give

$$8\sigma_0 = \frac{2 + q/4 - \log q}{4-q}, \qquad \det S = \mu_0^2 \sigma_0^2 \sin^2\varphi, \qquad A = \tfrac{\sqrt{3}}{4} N\,(3p^2 - 4 - p^2 \log q).$$

For $N = 4$ this is the octagon, with $\det S \approx 0.18020$ and $A = A_{\mathrm{oct}} \approx 3.12605$. For $N = 7$ one finds $A \approx 3.13866$ and $\det S \approx 0.18611$, and as $N \to \infty$ the values tend to π and $3/16$. In the seven-arc

walk every angle in the switch metric is $2\pi/7$, and its determinant exceeds the competitive cap (7.1). Section 9 shows that in a competitive forward period even the sum of two neighboring angles must be strictly smaller than $4\pi/7$.

9.1 Winding at a forward switch

Lemma 9.1 (Adjacent angles)**.** *At every switch of a competitive normal all-forward period, the two incident angles satisfy $\varphi_k + \varphi_{k+1} < 4\pi/7$.*

Proof. Write $a_j = p[j]/2$ for the active half-margin of the state. Measure the angle φ_j in the switch metric as in (7.2) and put $c_j = \cot\varphi_j$. Set

$$\kappa = \frac{1}{2\sqrt{\det S}}, \qquad \kappa_0 = \frac{583}{500}, \qquad r = \frac{2}{9}.$$

Lemma 7.3 gives $a_j > 1/\sqrt{3}$, and the determinant cap (7.1) gives $\kappa > \kappa_0$. Winding fixes the sum of the φ_j. We obtain a local bound by measuring both incident visits at their common switch.

First, every φ_j is acute. Indeed, $\gamma_j < \rho_j^-$ by the adjacent product bound, and $q_j > 1$, so (5.21) gives

$$16\sigma_{j-1,j} > -(1-\rho_j^-)(1-\rho_j^+).$$

The individual switch bounds (5.17) give $8\sigma_{j-1} \geq (1-\rho_j^-)^2$ and $8\sigma_j \geq (1-\rho_j^+)^2$. Thus $\cos\varphi_j > -1/2$. On the other hand, $\mu_j = 4(1+a_j^2) > 16/3$ and $\sigma_i \geq 3/32$ at every switch. The Gram identity gives

$$\sin\varphi_j = \frac{\sqrt{\det S}}{\mu_j\sqrt{\sigma_{j-1}\sigma_j}} < \frac{1}{\kappa} < \frac{\sqrt{3}}{2}.$$

These inequalities exclude $[\pi/2, 2\pi/3)$. In particular,

$$c_j > \sqrt{\kappa_0^2 - 1} > \frac{599}{1000}. \tag{9.1}$$

At the switch between visits k and $k+1$, write

$$\begin{gathered} a = a_k, \quad b = a_{k+1}, \quad x = c_k, \quad y = c_{k+1}, \\ u = a + b, \quad v = x + y, \quad z = \frac{1-ab}{u}, \quad h = \frac{1-xy}{v}. \end{gathered} \tag{9.2}$$

Here $z > 0$ is the unused state half-margin. The momentum read in the tangent frame in Lemma 5.5 satisfies $m_j = 4\mu_j\sigma_{j-1,j} = 4\sqrt{\det S}\,c_j$ by (5.7). Thus x, y are its normalized active coordinates. The third coordinate h is determined by $xy + h(x+y) = 1$, just as $ab + z(a+b) = 1$. The angle addition formula gives

$$h = -\cot(\varphi_k + \varphi_{k+1}), \qquad 0 < \varphi_k + \varphi_{k+1} < \pi.$$

The expressions for z and h are symmetric in the two visits.

Pair the forward edge relation $\mu_k\mathbf{y}_{k-1} + \mu_{k+1}\mathbf{y}_{k+1} = \tau_k^2\mathbf{y}_k$ with $\mathbf{y}_k$ using S. By (5.7), $\mu_j\sigma_{j-1,j} = \sqrt{\det S}\,c_j$. The central and outer measurements therefore satisfy

$$8\sigma_k = \frac{v}{\kappa u^2}, \qquad 8\sigma_{k-1} = \frac{x+h}{\kappa\tau_{\mathrm{lo}}^2}, \qquad \mu = 1+a^2, \quad \tau_{\mathrm{lo}} = a + z = \frac{\mu}{u}. \tag{9.3}$$

The first equality uses $\tau_k = 2u$. The second follows by solving the Gram identity for σ_{k-1}, using $\mu_k = 4\mu$ and $v(x+h) = 1+x^2$. At the central switch $8\sigma_k \geq 1 - ab/u^2$ by (5.17), hence

$$v \geq \kappa(a^2 + ab + b^2). \tag{9.4}$$

Let $p = a_{k-1}$ and $\tau_{\mathrm{hi}} = a + p$. The arc factor of visit k is $q_k = \tau_{\mathrm{hi}}/\tau_{\mathrm{lo}} > 1$, and $ap < 1$ at its outer switch, so $z < p < 1/a$. Define the common quantity

$$\chi = u + z - \frac{a(y+h) + b(x+h) + zv}{\kappa}.$$

Stationarity on visit k gives the integral identity

$$\frac{\chi}{u\tau_{\mathrm{lo}}} = \int_{\tau_{\mathrm{lo}}}^{\tau_{\mathrm{hi}}} \left[8\sigma_{k-1} - \left(1 - \frac{a}{w} + \frac{a^2}{w^2}\right)\right] dw. \tag{9.5}$$

Indeed, substituting the shares $a/\tau_{\mathrm{hi}}, a/u$ into (5.20) and collecting the terms in the two form measurements gives

$$8\sigma_{k-1}\tau_{\mathrm{hi}} + 8\sigma_k u - \frac{2ax}{\kappa\mu} = u + \tau_{\mathrm{hi}} - \frac{a^2}{u} - \frac{a^2}{\tau_{\mathrm{hi}}} - \frac{a}{\mu} - a\log(\tau_{\mathrm{hi}}/\tau_{\mathrm{lo}}).$$

Insert (9.3) and evaluate the integral as $(8\sigma_{k-1} - 1)(\tau_{\mathrm{hi}} - \tau_{\mathrm{lo}}) + a\log(\tau_{\mathrm{hi}}/\tau_{\mathrm{lo}}) + a^2(1/\tau_{\mathrm{hi}} - 1/\tau_{\mathrm{lo}})$. This yields (9.5) (Claim V10). Interchanging (a, x) and (b, y) gives the identity on visit $k+1$, with the same χ: the stationarity row is symmetric in its two incident measurements and shares.

The quadratic in (9.5) equals $3/4 + (a/w - 1/2)^2$. At the outer forward switch $8\sigma_{k-1} \geq 3/4$. Thus the integral is at most $(p - z)(8\sigma_{k-1} - 3/4)$, and $p < 1/a$ gives

$$\kappa\chi \leq u\tau_{\mathrm{lo}}(1/a - z)\left(\frac{x+h}{\tau_{\mathrm{lo}}^2} - \frac{3\kappa}{4}\right). \tag{9.6}$$

This bound and its counterpart on visit $k+1$ now use only the four coordinates at the central switch.

Feasibility at one switch. It suffices to exclude the simultaneous inequalities $h \geq r$, the central switch bound (9.4), and the two incident bounds (9.6). We express all four on a single rational box. Put $z_0 = 1 - ab$ and define

$$\begin{aligned} b_{\mathrm{ang}}(a,b,x,y) = {} & bu^2(1+x^2) - \tfrac{3}{4}\kappa_0 b(1+a^2)^2 v - \kappa_0 av(u^2 + z_0) \\ & + au\{a(1+y^2) + b(1+x^2)\} + az_0v^2. \end{aligned} \tag{9.7}$$

In (9.6), subtract the left side from the right and multiply by $auv > 0$. Using $1/a - z = b(1+a^2)/(au)$ gives b_{ang} with κ in place of κ_0. Both terms multiplying $-\kappa$ have positive factors. Replacing κ by its lower bound κ_0 therefore gives $b_{\mathrm{ang}}(a,b,x,y) \geq 0$. Reflection gives $b_{\mathrm{ang}}(b,a,y,x) \geq 0$.

Assume $h \geq r$. The switch and incident-visit bounds require all four expressions

$$1 - xy - rv, \qquad v - \kappa_0(a^2 + ab + b^2), \qquad b_{\mathrm{ang}}(a,b,x,y), \qquad b_{\mathrm{ang}}(b,a,y,x) \tag{9.8}$$

to be nonnegative. We claim that the switch coordinates lie in the single box

$$(a,b) \in [577/1000, 81/100]^2, \qquad (x,y) \in [599/1000, 11/10]^2. \tag{9.9}$$

To see this, put $m = 599/1000$. Then $(x-m)(y-m) \geq 0$ by (9.1); together with $xy + rv \leq 1$ this gives $v \leq (1+m^2)/(m+r) < 5/3$ and $x, y < 5/3 - m < 11/10$. Both $a, b > 1/\sqrt{3} > 577/1000$. If $a \geq 81/100$, then already at $a = 81/100,\ b = 577/1000$ the quantity $\kappa_0(a^2 + ab + b^2)$ exceeds $5/3$, contrary to (9.4). The same bound holds for b.

Claim V11 gives four nonnegative weights on (9.9) whose weighted sum of (9.8) is strictly negative throughout the box. This contradicts $h \geq r$. Finally $-\cot$ increases on $(0, \pi)$, and $-\cot(4\pi/7) = \tan(\pi/14) > \pi/14 > 2/9$, as required. □

Proof of Proposition 9. Since $A \leq A_{\mathrm{oct}} < \pi$, (7.3) gives $\sum_k \varphi_k = (N-1)\pi/3$. The average sum of two adjacent angles is therefore $2(N-1)\pi/(3N)$. If $N \geq 7$, this is at least $4\pi/7$, in contradiction to Lemma 9.1. Thus $N < 7$, and (3.1) gives $N \equiv 1 \pmod 3$. The one-visit case was excluded in the proof of Lemma 7.1. Hence $N = 4$. □

10 Four-arc closure and area

Frame closure forces the four arc factors to alternate. Their area then depends on a single parameter. The equality case fixes the state at one switch and hence the whole motion.

Proposition 10 (Identification of the four-arc walk)**.** *Among admissible periods of four forward visits, the minimum area is A_{oct}. Up to cyclic offset its unique normalized trajectory is the smoothed octagon.*

10.1 The closure equations

Consider a competitive normal all-forward walk. Proposition 9 gives $N = 4$. We now solve the closure equations for every admissible walk of this length.

Lemma 10.1 (Four-arc closure)**.** *The tangent transports for four consecutive forward visits with $q_0, \dots, q_3 > 1$ have product R exactly when*

$$q_0 = q_2, \qquad q_1 = q_3, \qquad q_0^{-2} + q_1^{-2} = 1. \tag{10.1}$$

Each such factor sequence determines a unique admissible state path once the initial control label is fixed.

Proof. Choose the label offset so that the four controls are `2012`. In the coordinates (5.6) each transfer matrix in (5.5) is $T(q) := h_0'(q)(R')^{-1} = \left(\begin{smallmatrix} q & -q \\ q^{-1} & 0 \end{smallmatrix}\right)$. All four switches are forward, so $R^{1-4} = -I$. Thus the return condition is $T(q_0)T(q_1)T(q_2)T(q_3) = -I$, equivalently $T(q_0)T(q_1) = -(T(q_2)T(q_3))^{-1}$. In entries this reads

$$\begin{pmatrix} q_0q_1 - q_0/q_1 & -q_0q_1 \\ q_1/q_0 & -q_1/q_0 \end{pmatrix} = \begin{pmatrix} q_3/q_2 & -q_2q_3 \\ q_3/q_2 & q_2/q_3 - q_2q_3 \end{pmatrix}.$$

The off-diagonal entries give $q_0q_1 = q_2q_3$ and $q_1/q_0 = q_3/q_2$, hence $q_0 = q_2$ and $q_1 = q_3$. Either diagonal entry then reads $q_0q_1 - q_0/q_1 = q_1/q_0$, which is $q_0^{-2} + q_1^{-2} = 1$.

Write $\bar{q}_0 := 1/q_0$ and $\bar{q}_1 := 1/q_1$. The factor pairs thus lie on the open quarter circle $\bar{q}_0^2 + \bar{q}_1^2 = 1$. Put

$$\begin{gathered} \bar{q}_{\mathrm{sum}} := \bar{q}_0 + \bar{q}_1, \qquad \omega_{\mathrm{rad}} := (\bar{q}_{\mathrm{sum}} - 1)(\bar{q}_{\mathrm{sum}} + 3), \\ \widehat{p}_0 := 2(\bar{q}_{\mathrm{sum}} - 1), \qquad \widehat{p}_1 := 2(1 - \bar{q}_0 + \bar{q}_1), \qquad \widehat{p}_2 := 2(1 + \bar{q}_0 - \bar{q}_1). \end{gathered} \tag{10.2}$$

The numerators $\widehat{p}_0, \widehat{p}_1, \widehat{p}_2$ and the scale $\sqrt{\omega_{\mathrm{rad}}}$ are positive. The initial state has margins

$$(p_0(X(0)), p_1(X(0)), p_2(X(0))) = \omega_{\mathrm{rad}}^{-1/2}(\widehat{p}_0, \widehat{p}_1, \widehat{p}_2). \tag{10.3}$$

Indeed, apply the one-arc map four times and require the final state to be the twisted copy of the initial state. This gives

$$(1 + \bar{q}_0 - \bar{q}_1)p_0(X(0)) = (\bar{q}_{\mathrm{sum}} - 1)p_2(X(0)), \qquad (1 + \bar{q}_0 - \bar{q}_1)p_1(X(0)) = (1 - \bar{q}_0 + \bar{q}_1)p_2(X(0)).$$

These equations fix the ratios. The state quadric $p_0p_1 + p_1p_2 + p_2p_0 = 4$ at $X(0)$ fixes the positive scale and gives (10.3).

The calculation also verifies admissibility of every member of the family. Since

$$\widehat{p}_0 + \widehat{p}_2 = 4\bar{q}_0, \qquad \widehat{p}_1 + \widehat{p}_2 = 4, \qquad \widehat{p}_0 + \widehat{p}_1 = 4\bar{q}_1,$$

the update (5.1) exchanges the two inactive margins at the ends of each arc and fixes its active margin. For the word `2012` this gives

$$(p[0], p[1], p[2], p[3]) = (p_2(X(0)), p_1(X(0)), p_2(X(0)), p_1(X(0))). \tag{10.4}$$

The unordered triple is the same at every switch. The four exchanges give the required label twist. Inside each arc one inactive margin grows and the other decreases, so endpoint positivity suffices. The three margins determine the state at the cut. The arc factors and active margins then determine the durations. The tangent return and the state return together imply the frame return, since $g(t_f)X(t_f) = g(0)X(0)R$. □

10.2 The area minimum

Figure 4 shows the closure parameters and the octagon.

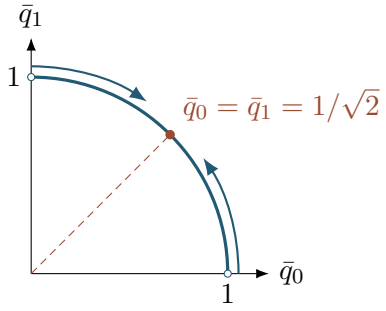

Figure 4: The closure parameters for four forward arcs. The arrows point toward lower area and the marked point gives the octagon.

Lemma 10.2 (Minimum area of a four-arc period)**.** *A four-arc all-forward period has area at least A_{oct}, with equality exactly when its four arc factors are all $\sqrt{2}$.*

Proof. Use the reciprocal factors and the margin numerators in (10.2). By (10.4) and the arc area formula (5.4), the first and third arcs contribute, in units of $\sqrt{3}/4$,

$$\frac{8\widehat{p}_2(1 - \bar{q}_0) + \widehat{p}_2^2 \log \bar{q}_0}{\omega_{\mathrm{rad}}} \qquad \text{each,}$$

and the other two contribute

$$\frac{8\widehat{p}_1(1-\bar{q}_1)+\widehat{p}_1^2\log\bar{q}_1}{\omega_{\mathrm{rad}}} \qquad \text{each.}$$

For example, the first arc has initial margins $\omega_{\mathrm{rad}}^{-1/2}(\widehat{p}_2,\widehat{p}_0,\widehat{p}_1)$ in active, growing and decreasing order. The factor is $\bar{q}_0^{-1}$ and the two initial pair sums are $4\bar{q}_0/\sqrt{\omega_{\mathrm{rad}}}$ and $4/\sqrt{\omega_{\mathrm{rad}}}$. On the quarter circle,

$$\widehat{p}_2^2 = 8(1+\bar{q}_0)(1-\bar{q}_1), \qquad \widehat{p}_1^2 = 8(1-\bar{q}_0)(1+\bar{q}_1),$$

and

$$(1+\bar{q}_0-\bar{q}_1)(1-\bar{q}_0)+(1-\bar{q}_0+\bar{q}_1)(1-\bar{q}_1) = \bar{q}_{\mathrm{sum}}(\bar{q}_{\mathrm{sum}}-1).$$

Adding the four terms gives the period area

$$A = 2\sqrt{3}\,A_{\mathrm{norm}}(\bar{q}_0,\bar{q}_1),$$
$$A_{\mathrm{norm}}(\bar{q}_0,\bar{q}_1) \coloneqq \frac{4\bar{q}_{\mathrm{sum}}(\bar{q}_{\mathrm{sum}}-1)+2(1+\bar{q}_0)(1-\bar{q}_1)\log\bar{q}_0+2(1-\bar{q}_0)(1+\bar{q}_1)\log\bar{q}_1}{(\bar{q}_{\mathrm{sum}}-1)(\bar{q}_{\mathrm{sum}}+3)}. \tag{10.5}$$

It remains to solve the constrained minimization problem

$$\min_{\substack{\bar{q}_0,\bar{q}_1>0\\ \bar{q}_0^2+\bar{q}_1^2=1}} A_{\mathrm{norm}}(\bar{q}_0,\bar{q}_1).$$

By symmetry it suffices to prove strict decrease along the lower half of the quarter circle up to its diagonal. At $\bar{q}_0=\bar{q}_1=1/\sqrt{2}$ the area is A_{oct}, using

$$\delta_{\mathrm{oct}} = \frac{8-4\sqrt{2}-\log 2}{2\sqrt{2}-1}, \qquad A_{\mathrm{oct}} = 2\sqrt{3}\,\delta_{\mathrm{oct}}. \tag{10.6}$$

To locate the minimum, we use a logarithmic bound.

For $0<x\le 1$,

$$\log x \ge \frac{x-x^{-1}}{2}. \tag{10.7}$$

With $x=e^{-y}$ this is $\sinh y \ge y$, obtained by integrating $\cosh y - 1 \ge 0$. For a fixed $c\ge 2$, define

$$w_c(x) \coloneqq \frac{(2x+c)\log x}{1-x}, \qquad 0<x<1.$$

Its derivative is

$$w_c'(x) = \frac{(2+c)\log x+(2x+c)(1-x)/x}{(1-x)^2} \ge \frac{c-2}{2x} \ge 0,$$

where the first inequality is (10.7). Thus w_c is nondecreasing.

Swapping the two factors exchanges $\bar{q}_0$ and $\bar{q}_1$, and the normalized area in (10.5) is invariant. It suffices to follow the half of the quarter circle from $\bar{q}_1=0$ toward the symmetric point. Assume $0<\bar{q}_1\le\bar{q}_0=\sqrt{1-\bar{q}_1^2}$, and differentiate the function $\psi(\bar{q}_1)\coloneqq A_{\mathrm{norm}}(\sqrt{1-\bar{q}_1^2},\bar{q}_1)$. With $c\coloneqq 2+2\bar{q}_0\bar{q}_1$ and $\bar{q}_{\mathrm{sum}}=\bar{q}_0+\bar{q}_1$, cancellation using $\bar{q}_0^2+\bar{q}_1^2=1$ gives

$$\psi'(\bar{q}_1) = \frac{2\big((\bar{q}_{\mathrm{sum}}+1)(w_c(\bar{q}_1)-w_c(\bar{q}_0))-\bar{q}_{\mathrm{sum}}(\bar{q}_{\mathrm{sum}}-1)(\bar{q}_0-\bar{q}_1)\big)}{\bar{q}_0(\bar{q}_{\mathrm{sum}}+1)(\bar{q}_{\mathrm{sum}}+3)^2}. \tag{10.8}$$

For $\bar{q}_1<\bar{q}_0$, the first term in the numerator is nonpositive because $c\ge 2$, and the second is strictly negative because $\bar{q}_{\mathrm{sum}}>1$. Therefore ψ decreases strictly up to $\bar{q}_1=1/\sqrt{2}$. Equations (10.5) and (10.6) prove

$$A \ge A_{\mathrm{oct}}, \qquad A = A_{\mathrm{oct}} \quad\Longleftrightarrow\quad \bar{q}_0=\bar{q}_1=1/\sqrt{2}.$$

□

10.3 Identifying the minimizing trajectory

Proof of Proposition 10. By Lemma 10.2, at equality every factor is $\sqrt{2}$. The active margin on each of the four arcs is

$$p_{\mathrm{oct}} = \frac{2}{\sqrt{2\sqrt{2}-1}}.$$

The initial margins in label order $(0,1,2)$, for the word `2012`, are

$$\frac{1}{\sqrt{2\sqrt{2}-1}}\big(2(\sqrt{2}-1),2,2\big).$$

A traceless state matrix is determined by its three margins. Thus these are the initial state and the control sequence of the smoothed octagon after a cyclic offset [8, Section 7.5]. Each arc has duration

$$\Delta t_{\mathrm{oct}} = \frac{p_{\mathrm{oct}}}{2}\log\sqrt{2} = \frac{\log 2}{2\sqrt{2\sqrt{2}-1}}.$$

The one-arc solutions are unique, including for controls specified only almost everywhere. With the initial frame normalized to the identity, the entire state and frame therefore agree with those of the smoothed octagon after conjugation by the power of R for the label offset. The prescribed frame therefore traces the boundary of K_{oct}, as identified in [8, Section 7.5]. □

Together with Proposition 9, this proves Proposition 3.3.

11 Verification claims

The claims below record the finite calculations used in the proofs and the supplementary identities for reverse runs. Each claim names the equations it checks. The verification guide gives the corresponding programs and the coordinates used in them. The marked-center notebook checks Claims V6–V8. Interval and Bernstein checks use closed boxes, including their common faces.

Claim V1 (Reduction)**.** The reduced costate equations follow by differentiation in the trace pairing. The polar trace identities in Lemma 2.2 give the speed bound used in Lemma 4.1. The exclusion of abnormal lifts is in Section 5.

Claim V2 (Switch relations)**.** The exact arc transport of Lemmas 5.1 and 5.2, the Gram identity (5.7), and the secant and stationarity identities (5.19) and (5.20) hold as stated. For example, the two entry signs in (5.19) give the secant arguments $\tau_{k-1}/q - p[k]$ and $\tau_{k-1}q - p[k]$.

Claim V3 (Reverse-window elimination)**.** The supplementary calculation for Lemma 6.1 verifies two weighted identities for the vertex values at consecutive reverse switches. This calculation provides a separate check on the proof of that lemma.

Claim V4 (Determinant bounds)**.** The exact stationarity normalization and the rational comparisons in Lemmas 7 and 11.1 give the stated determinant thresholds.

Claim V5 (Illustrative periods)**.** In Remark 9 the regular-arc transfer has trace q and the area tends to π along the stated branch. The half-period trace, frame closure, area formula, and limit in Remark 8.3 hold by exact algebra.

Claim V6 (Loss at the preceding turn)**.** Use the marked center and ξ, u, r of Lemma 8.1. Put $v = \min(1, 1+u)$, $m_0 = 1 - v + v^2/3$. Bound the three entries on the left of (8.2) by

$$8\sigma_0 \le (3 + \xi^2 m_0)/4, \qquad 8\sigma_{-1} \le 1 - \rho_0^- + (\rho_0^-)^2,$$
$$16\sigma_{-1,0} \le (2 - \rho_0^+)\rho_0^- - \xi[2 - \xi(1 + m_0)]/(4\rho_0^+).$$

The last inequality follows from (6.7) by raising the diagonal entries and increasing γ_0 to $\rho_0^-\rho_0^+$. For $r \le 7\xi v/8$ even these bounds violate (8.2). To certify the strict gap the notebook sets $y = r/\xi$, $z = \gamma_0/(\rho_0^-\rho_0^+)$. The expression in the positive part is at least $\xi(1/2 - \rho_0^+ zy - \gamma_0\xi y^2/2) > 0$. After division by $\xi^2 v$, the resulting polynomial gap has least Bernstein coefficient $23/768$ on the cube in $(\xi, \rho_0^-/\rho_0^+, 8y/(7v), z, v)$.

Claim V7 (Stationarity at the forward visit)**.** Use the units of Proposition 8.2, in which $\tau_0 = 1$. Use the comparison data (8.4) in (8.5). Replacing $\log q_2$ by its upper bound $h + h^2/2 + h^3/[3(1-h)]$ and multiplying by $\mu_2\tau_1 q_2$ gives a lower bound $b(\mu_0, m)$ for $\mu_2\tau_1 q_2 \bar{r}_2$. It is affine in its two arguments separately. At each endpoint of the moment interval (8.1), $b(1, m) \ge 0$ and $b(2, m) - b(1, m) \ge 0$, so it is nonnegative for $\mu_0 \ge 1$. The inequalities hold for $0 < \xi < 1$, $-1 < u \le 1/2$, and $0 < h < 1$ subject to (8.6). One cube covers $u \ge 0$. For $u < 0$ the duration bounds coincide at $\xi = 1/(2 - 3u)$; a cube on each side completes the check.

Claim V8 (Stationarity at the second reverse center)**.** Use the units of the transport between the two centers in Section 8, so that $p[0] = 1$. Let $e_0 = \rho_3^- - (1 - 8\sigma_0)/\rho_3^- + 2b_0 - \gamma_3 - 16\bar{\sigma}_{2,3}$ and $v_0 = 8\bar{\sigma}_3 - 3/4$. The necessary inequalities after (8.8) give

$$e_0 \ge 0, \qquad e_0 - \frac{(1 - 2b_0)^2}{1 + 2b_0} + \max\{2v_0, v_0/b_0\} \ge 0.$$

Indeed $r - \log(2r) - 1/(4r) \le 0$ for $r \le 1/2$, and $\log(2b_0) \le 2(2b_0 - 1)/(2b_0 + 1)$. Use these in $\bar{r}_3$ and bound v_0/r at the endpoints of $b_0 < r < 1/2$. The joint inequalities are incompatible with (8.7) and (8.9) for $0 < \xi < 7/8$, $0 \le u \le 1/2$, $1 \le p[3] \le p[1]$. Writing $v = (p[2] - p[3])/[\xi(p[2] + p[3])]$ gives $u \le v \le (1 + u)/(1 + \xi^2 u) \le 1 + u$. The notebook ranges over $v - u \in [0, 1]$ and the full midpoint interval, excluding the two inequalities on 87,404 terminal boxes by outward-rounded integer arithmetic.

Claim V9 (Forward switch comparison)**.** The two stationarity equations give the weighted identity (7.4). The rational comparison at the end of the proof of Lemma 7.3 is strictly positive in the three nonnegative coordinates specified there.

Claim V10 (Two incident forward visits)**.** The stationarity equation (5.20) gives (9.5) in the coordinates (9.2). Clearing (9.6) gives the polynomial in (9.7) with κ in place of κ_0. The symbolic check uses a formal variable for $\log(\tau_{\mathrm{hi}}/\tau_{\mathrm{lo}})$.

Claim V11 (Adjacent-angle certificate)**.** Let $b_1, \dots, b_4$ be the four polynomials in (9.8), in order. Rescale each coordinate of (9.9) affinely to $[0, 1]$, in the order (a, b, x, y). Define ω_i by multiaffine interpolation of the following integer values at the sixteen corners. Each row gives the values at

the two corners with the indicated final three bits.

	$(0,i,j,k)$				$(1,i,j,k)$			
(i,j,k)	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4
$(0,0,0)$	6	35	49	49	0	43	59	54
$(0,0,1)$	27	10	70	38	25	3	91	39
$(0,1,0)$	27	10	38	70	30	25	48	90
$(0,1,1)$	69	8	60	60	83	0	79	70
$(1,0,0)$	0	43	54	59	0	70	77	77
$(1,0,1)$	30	25	90	48	6	8	112	52
$(1,1,0)$	25	3	39	91	6	8	52	112
$(1,1,1)$	83	0	70	79	107	0	102	102

The largest tensor Bernstein coefficient of $\sum_{i=1}^{4} \omega_i b_i$ in multidegree $(5,5,3,3)$ is

$$-\frac{12977579752000390297}{84375000000000000000}.$$

Indeed, `checks/forward/check_winding.py` evaluates the polynomial on the tensor grid $x_j = i_j/n_j$, $0 \le i_j \le n_j$, $n = (5,5,3,3)$, and inverts the four univariate Bernstein evaluation matrices over the rationals. The displayed multidegree bounds the degree in each variable, so interpolation recovers all $6 \cdot 6 \cdot 4 \cdot 4 = 576$ coefficients exactly. The Bernstein basis is nonnegative and sums to one, as do the multiaffine interpolation weights. Thus every ω_i is nonnegative on the box and the weighted polynomial is strictly negative.

Claim V12 (Closure and geometric identities)**.** The four-arc closure and area identities in Section 10 and the margin and midpoint identities used in the geometric reduction hold with the stated orientations and scales.

11.1 A supplementary bound for mixed periods

Lemma 11.1 (Lower determinant bound for mixed periods)**.** *For a competitive normal period with both directions of switch,* $\det S > 27/256$.

Proof. Suppose $\det S \le 27/256$. By (5.7), $\mu_k > 4$ and $\sigma_{k-1}, \sigma_k \ge 3/32$ we get $\sin^2 \varphi_k < 3/4$, that is $|\cos \varphi_k| > 1/2$, at every same-sign visit. We determine the sign from Lemma 5.7.

At a reverse center, write $-\log q_k = \log(\gamma_k/(\rho_k^- \rho_k^+))$ in $16\sigma^*_{k-1,k}$. At fixed shares this expression increases with $\gamma_k < \rho_k^- \rho_k^+$, and at $\gamma_k = \rho_k^- \rho_k^+$ it is $-1 + 2\rho_k^- + 2\rho_k^+ - \rho_k^- \rho_k^+ < 3/4$ since both shares are below $1/2$. Thus $16\sigma_{k-1,k} < 3/4 \le 8\sqrt{\sigma_{k-1}\sigma_k}$, so $\cos \varphi_k < 1/2$ and therefore $\varphi_k > 2\pi/3$.

At a visit with two forward switches, $\log q_k > 0$ and $\gamma_k < \rho_k^-$ give $16\sigma^*_{k-1,k} > -(1-\rho_k^-)(1-\rho_k^+)$. By Lemma 5.6(a), $8\sigma_{k-1} \ge \langle f \rangle_{\mathrm{arc}}(\rho_k^-, 1) \ge (1-\rho_k^-)^2$ and likewise $8\sigma_k \ge (1-\rho_k^+)^2$. Thus $16\sigma_{k-1,k} > -8\sqrt{\sigma_{k-1}\sigma_k}$, so $\cos \varphi_k > -1/2$ and therefore $\varphi_k < \pi/3$.

Since $A \le A_{\mathrm{oct}} < \pi$, (7.3) applies. Every summand is now positive, and a mixed walk has a reverse center, which alone contributes more than $\pi/3$. □

12 Guide to notation

This guide collects notation that is shared across proofs. Time derivatives use a dot. A variation along a family of motions is written ∂_λ at $\lambda = 0$. The polar angle θ belongs to a frame path and

φ_k is an angle in the switch metric. The symbol $\delta(K)$ denotes packing density. Angle brackets $\langle f_a \rangle_{b,c}$ denote the mean of f_a on the interval with endpoints b, c.

The control and side labels are taken mod 3, and reference contacts use indices mod 6. The visit index runs along the extended walk. In $\sigma_{i,j}$, both subscripts are visit indices, whereas $p_i(X)$ uses a side label. Dot denotes time derivative, and a superscript t on a vector or matrix denotes transpose.

The families used in calculations with successive visits come from the margins, from the arc factors, and from the form. Margin sums, shares, and ratios use Greek letters. The factors use q, and the form pairings use σ. The switch bound σ_k^* is a function of two margins because it is compared directly with σ_k. Real scalar variables are set in lower case; A and its variants denote areas. Vectors are bold, with their scalar components in ordinary type.

Symbol	Meaning
$\mathrm{SL}_2(\mathbb{R}), \mathfrak{sl}_2(\mathbb{R}), \langle \cdot, \cdot \rangle$	Determinant-one matrices, traceless matrices, and the trace pairing.
g, h, X	Contact frame, auxiliary determinant-one matrix, and state $g^{-1}\dot{g}$.
$J, R, \Phi, \mathfrak{h}^*, \mathfrak{h}^{**}$	Complex structure, rotation by $\pi/3$, upper-half-plane state chart, ideal triangle, and compact core.
$\mathbf{s}_r^*, g(t)\mathbf{s}_r^*$	Reference midpoint and its image at time t.
$U_T, \mathbf{e}_i, Z_{\mathbf{u}}, P_{\mathbf{u}}$	Control simplex, control vertex, control matrix and its trace normalization.
$p_i(X), p[k]$	Side margin and active margin of visit k. The side margin is twice the hyperbolic sine of the distance to its side.
$t, t_f, t_k, \Delta t_k$	Time, period duration, start of a visit, and duration of that visit.
$N, i_k, \epsilon_k, q(t), q_k$	Visit count, label, outgoing switch sign, position parameter, and terminal factor.
$A, A_k, A_{\mathrm{oct}}, A_{\mathrm{mar}}$	Period area, arc area contribution, octagon area, and signed-sum kernel in margin coordinates (before attaching the switch sign).
a_j, c_j, h in Section 9	Active half-margin, cotangent of the switch angle, and negative cotangent of two adjacent angles' sum.
$\Lambda_1, \Lambda_R, k_0, L$	Frame and reduced state costates, area multiplier, and shifted frame costate.
$\mathcal{H}_{\mathbf{u}}, \widehat{\chi}_i$	Hamiltonian and weighted vertex value $p_i(X)\mathcal{H}_{\mathbf{e}_i}$.
d, d_1	Frame costate determinant and $\sqrt{\lvert d \rvert}$.
τ_k, μ_k	Adjacent margin sum $p[k] + p[k+1]$ and $p[k]^2 + 4$.
$\rho_k^{\pm}, \gamma_k$	Active margin shares in the two adjacent sums, and scale $p[k]^2/\mu_k$.
$S, \mathbf{y}_k, \sigma_{i,j}, \sigma_k$	Switch form, switch vectors, pairing $\mathbf{y}_i^t S \mathbf{y}_j$, and its diagonal value. Here $12 \det S = d$.
$\sigma_k^*, \sigma_{k-1,k}^*$	Individual switch bound and the cross pairing evaluated at both incident bounds.
$\langle f_a \rangle_{b,c}, \langle f \rangle_{\mathrm{arc}}(\rho, q)$	Interval mean of f_a and its expression in margin-share and arc-factor coordinates.
$\alpha(X)$	Area density of a normalized trajectory.

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