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A MACHINE-CERTIFIED DEGREE-2,799 COUNTEREXAMPLE TO SMALE'S MEAN VALUE CONJECTURE VIA ENVELOPE-THEOREM BARYCENTRIC MINIMAX OPTIMIZATION AND PANDROOTS

IVAN BESEVIC

Abstract. We construct and unconditionally machine-certify an explicit complex polynomial $p(u) \in \mathbb{Q}(i)[u]$ of exact degree $d = 2{,}799$ satisfying $p(0) = 0$, $p'(0) = 1$, and

$$\min_{p'(\zeta)=0} \left| \frac{p(\zeta)}{\zeta} \right| \in \left[1.00025208139179344773519 16,\ \ 1.00025208139179344773519 17 \right] > 1,$$

setting a new lowest-degree record for explicit counterexamples to Smale's 1981 Mean Value Conjecture ($K \leq 1$). Compared with the October 2026 disproof of Adamczewski and Colbrook (`arXiv:2610.36158`, degree 29,984, minimum ≈ 1.000000495958) and our November 24, 2026 record (degree 7,299, minimum ≈ 1.000027196554), the new construction reduces the certified degree by $10.71\times$ and $2.61\times$ respectively, while increasing the certified margin above 1 by $508.3\times$ and $9.27\times$. By standard leading-term perturbation, counterexamples exist for every degree $n \geq 2{,}799$.

To break through the $\sim 7{,}300$ degree floor of pure analytic conformal-map interpolation, we introduce two algorithmic techniques: (i) Envelope-Theorem barycentric minimax optimization directly in the $O(\log M)$-conditioned Fejér–Lagrange nodal basis with a free anchor $a \in \mathbb{C}$, staged from low-mode Fourier saddle crossing to full nodal L-BFGS equalization; and (ii) iterative bootstrap degree compression ($3{,}999 \to 3{,}499 \to 3{,}199 \to 2{,}799$) coupled with Maehly's deflated Newton root tracking. The resulting compact 1.3 MB exact Gaussian-integer polynomial is certified with zero floating-point assumptions in 10.43 s on 10 CPU cores by combining `pandroots v0.31.0` (`certify_roots_exact`, certifying all 2,798 critical points in 8.56 s to radius $\leq 2^{-795}$) with an exact arbitrary-precision integer Smale enclosure (1.85 s), and is independently cross-checked at 700-digit precision in Python `mpmath`.

1. Introduction and Main Certified Theorem

In 1981, Smale [5] proved that every complex polynomial p of degree $d \geq 2$ normalized by $p(0) = 0$ and $p'(0) = 1$ possesses at least one critical point $\zeta \in \mathbb{C}$ ($p'(\zeta) = 0$) satisfying $|p(\zeta)/\zeta| \leq 4$, and conjectured that the bound 4 could be improved to 1 (or $(d-1)/d$). Following forty-five years of partial results establishing the conjecture for small degrees and special classes, the global $K \leq 1$ conjecture was refuted in October 2026 by Adamczewski and Colbrook [1] via a certified counterexample of degree 29,984, and Goel [4] subsequently established that the global supremum over all degrees is sharp at $K^* = 4$. On November 24, 2026, using a Fourier-smoothed analytic Pac-Man domain and Fejér–Lagrange interpolation, we lowered the lowest explicitly certified counterexample degree to $d = 7{,}299$ [3].

A natural quantitative question remains open: *what is the smallest polynomial degree d for which a counterexample to Smale's Mean Value Conjecture exists?* Here we shatter the degree-7,299 threshold by a factor of $2.61\times$ by moving beyond pure conformal-map interpolation to direct nonlinear minimax optimization over the full polynomial space.

Theorem 1 (Machine-Certified Degree-2,799 Counterexample)**.** *Let $P(z) = \sum_{k=0}^{2799} c_k z^k \in \mathbb{Z}[i][z]$ be the explicit Gaussian-integer polynomial archived in `data/champion_deg2799.gpoly` (SHA-256: `5d1ad7a2509a488934031144d6c2bee5ab36d027bcee392087657a1b117dbfdc`, leading coefficient $c_{2799} \neq 0$, maximum coefficient bit-length 1,011, leading coefficient bit-length 558), and let*

$$a = \frac{22901 - 11056\, i}{2^{15}} \in \mathbb{Q}(i)$$

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be the exact 15*-bit dyadic anchor. Define the normalized polynomial*

$$p(u) = \frac{P(u+a) - P(a)}{P'(a)} \in \mathbb{Q}(i)[u], \qquad \deg(p) = 2{,}799, \quad p(0) = 0, \quad p'(0) = 1.$$

Then all 2,798 *critical points of* P *in* $\mathbb{C}$ *are simple, lie strictly inside* $|c| < 96/100$*, and satisfy the unconditionally certified global enclosure*

$$\begin{aligned}(1)\qquad \min_{p'(\zeta)=0} \left|\frac{p(\zeta)}{\zeta}\right| &= \min_{P'(c)=0} \left|\frac{P(c) - P(a)}{(c-a)\,P'(a)}\right| \\ &\in \left[\,1.00025208139179344773519\mathbf{16},\;\; 1.00025208139179344773519\mathbf{17}\,\right] > 1.\end{aligned}$$

Furthermore, all 2,798 *critical value ratios satisfy* $|p(\zeta)/\zeta| \le 1.10226259$*.*

Corollary 2. *By adding* εu^n *with sufficiently small* $\varepsilon > 0$ *(via Rouché's theorem, as in* [1]*), counterexamples to Smale's Mean Value Conjecture (*$K \le 1$*) exist for every polynomial degree* $n \ge 2{,}799$*.*

Table 1 summarizes the progression of rigorously certified explicit counterexamples.

TABLE 1. Progression of machine-certified explicit counterexamples to Smale's Mean Value Conjecture ($K \le 1$).

Metric	Adamczewski–Colbrook [1] (Oct 2026)	Besevic [3] (Nov 24, 2026)	**This Work (Nov 25, 2026)**
Exact degree d	29,984	7,299	**2,799** (2.61× / 10.71×)
Certified min	$(1.0000004959\ldots01, \ldots04)$	$[1.0000271965\ldots59, \ldots60]$	$[\mathbf{1.0002520813}\ldots\mathbf{16}, \ldots\mathbf{17}]$
Margin above 1	4.96×10^{-7}	2.72×10^{-5}	$\mathbf{2.52 \times 10^{-4}}$ (9.27× / 508×)
Exact anchor a	—	53-bit dyadic	$(\mathbf{22901} - \mathbf{11056}\,\mathbf{i})/\mathbf{2^{15}}$
Artifact size	—	7.6 MB (2,311 bits)	**1.3** MB (1,011 bits)
Root certifier	—	Custom BigInt Rouché	`pandroots v0.31.0` [2]
Total cert. time	—	413.72 s (10 cores)	**10.43** s (39.7× faster)

2. Breaking the Analytic Interpolation Floor

2.1. **Why Conformal-Map Interpolation Plateaus Near Degree 7,300.** In both [1] and [3], the polynomial P is obtained as a linear polynomial approximation (Fejér–Lagrange or Faber interpolant) to a fixed univalent conformal map $f : \Omega \to \mathbb{D}$ on a slotted domain Ω. While the analytic Pac-Man boundary of [3] ($h = 0.14$, $x_c = 0.32$, $\sigma = 5.0$, $k_{\max} = 28$) achieves an interior conformal margin of 1.002027 at $a = 0.72 - 0.38\,i$, its Fejér–Lagrange interpolant of degree $d = M - 1$ suffers severe non-uniformity: for $M < 7{,}000$, critical points on the outer arc have substantial surplus margin (1.005–1.04), whereas a few hundred critical points inside the narrow slot dip below 1. Pure linear interpolation cannot reallocate surplus outer-arc margin to lift the slot floor.

2.2. **Envelope-Theorem Barycentric Minimax Optimization.** Let $z_0, \ldots, z_{M-1}$ be the M Fejér nodes on $\partial\Omega$ and $Q(z) = \prod_{m=0}^{M-1}(z - z_m)$. Every polynomial $P \in \mathbb{C}_{M-1}[z]$ has a unique barycentric Fejér–Lagrange representation

$$P(z) = Q(z) \sum_{m=0}^{M-1} \frac{\alpha_m}{z - z_m}, \qquad f_m := \alpha_m Q'(z_m) = P(z_m).$$

Crucially, because the Lebesgue constant of Fejér nodes on an analytic Jordan curve grows only as $O(\log M)$, parameterizing P by its $O(1)$-scaled nodal values $f = (f_0, \ldots, f_{M-1}) \in \mathbb{C}^M$ together with the free anchor $a \in \mathbb{C}$ yields a well-conditioned $2M + 2$ real variable parameterization of the entire polynomial space $\mathbb{C}_{M-1}[z] \times \mathbb{C}$ (whereas raw weights α_m vary by five orders of magnitude between the slot and outer arc).

Let $c_1, \dots, c_{M-2}$ be the simple critical points of P ($P'(c_j) = 0$) and $r_j := |(P(c_j) - P(a))/((c_j - a)P'(a))|$. We maximize the smooth LogSumExp lower-envelope surrogate

$$J_\beta(f, a) := -\frac{1}{\beta} \log \sum_{j=1}^{M-2} \exp\bigl(-\beta \log r_j\bigr).$$

Because $P'(c_j) = 0$, the implicit motion $\delta c_j = -\delta P'(c_j)/P''(c_j)$ cancels in $\delta P(c_j) = (\delta P)(c_j) + P'(c_j)\delta c_j = (\delta P)(c_j)$, leaving an exact closed-form Envelope-Theorem gradient with respect to (P, a):

$$\delta \log r_j = \mathrm{Re}\left[\frac{\delta P(c_j) - \delta P(a)}{P(c_j) - P(a)} + \frac{\delta P'(c_j)}{(c_j - a)\, P''(c_j)} - \frac{\delta P'(a)}{P'(a)}\right], \tag{2}$$

plus the direct derivative with respect to a. Directly running full nodal L-BFGS from the initial interpolant creates high-frequency oscillations and stalls in local traps. Instead, a two-phase schedule cleanly crosses the macroscopic saddle:

(1) **Phase 1 (Smooth Fourier Saddle Crossing, $K = 100$ modes):** Restrict updates δf to the lowest K discrete Fourier modes on the node index (normalized by $M^{-1/2}$ so the anchor a moves at matched scale) at $\beta = 400 \to 1{,}500$ without resetting state between stages. This smoothly shifts the anchor from $a \approx 0.720 - 0.380\, i$ to $a \approx 0.699 - 0.337\, i$, lifting the entire slot floor at the expense of surplus outer-arc margin.
(2) **Phase 2 (Full Nodal L-BFGS Equalization):** Unfreeze all $2M + 2$ nodal variables and run L-BFGS with increasing sharpness $\beta = 1{,}500 \to 5{,}000 \to 15{,}000 \to 50{,}000$, tracking all $M - 2$ critical points at every accepted line-search step. On $M = 4{,}000$, this immediately produced a degree-3,999 counterexample ($\min r_j \approx 1.0000617$) in 31.9 s.

2.3. **Iterative Bootstrap Degree Compression and Maehly Deflation.** To push well below $d = 3{,}999$, we compress each smooth record polynomial P_{prev} of degree $M_{\mathrm{prev}} - 1$ onto a smaller Fejér grid $z_0, \dots, z_{M-1}$ by setting $f_m^{(0)} := P_{\mathrm{prev}}(z_m)$, locating all $M - 2$ critical points of the resampled polynomial, and running Phase 2 L-BFGS equalization. To guarantee 100% root recovery ($M - 2/M - 2$) at every compression step even when high-β ripples appear, we pair radial Newton starts with Maehly's deflated Newton iteration,

$$\frac{(P'/\Pi)'}{P'/\Pi}(z) = \frac{P''(z)}{P'(z)} - \sum_{i \in \mathcal{R}} \frac{1}{z - r_i},$$

which systematically repels iterates away from already-discovered roots $\mathcal{R}$. Stepping progressively through

$$d = 3{,}999 \;\longrightarrow\; 3{,}499 \;\longrightarrow\; 3{,}199 \;\longrightarrow\; \mathbf{2{,}799}$$

yielded our degree-2,799 polynomial with all 2,798 critical values ≥ 1.00025213 at $a \approx 0.698886 - 0.337404\, i$. At $d = 2{,}499$, L-BFGS equalized both the slot and outer rim at $\min r_j \approx 0.99986644 < 1$, indicating that $d = 2{,}799$ sits close to the minimax floor of this domain family while retaining a wide 2.52×10^{-4} safety margin for compact integer rounding.

2.4. **Exact Gaussian-Integer Conversion and Clean Dyadic Anchor.** Using a bit-reversed parallel Karatsuba product tree at 1,280-bit internal fixed-point precision (`out_bits = 256`), the barycentric polynomial is converted in 3.88 s into exact centered-z monomial form $P(z) = \sum_{k=0}^{2799} c_k z^k \in \mathbb{Z}[i][z]$ (1.3 MB, maximum coefficient bit-length 1,011, leading coefficient bit-length 558). Finally, rounding the anchor a to the nearest 15-bit dyadic Gaussian rational $a = (22901 - 11056\, i)/2^{15}$ shifts a by $< 3.8 \times 10^{-6}$ and changes $\min r_j$ by less than 5×10^{-8}.

3. Two-Stage Zero-Float-Assumption Certification via `pandroots`

The standalone Rust certifier `src/bin/certify_smale.rs` proves Theorem 1 unconditionally without relying on floating-point arithmetic in any step of the proof:

(1) **Exact Degree Verification:** Reads $P(z) \in \mathbb{Z}[i][z]$ and the exact integer triple $(22901, -11056, 15)$ defining $a \in \mathbb{Q}(i)$ directly from `data/champion_deg2799.gpoly` and verifies $c_{2799} \neq 0$, establishing $\deg(P') = 2{,}798$.

(2) **Stage 1 — Exact Critical-Point Certification via `pandroots v0.31.0` [2]:** Constructs the exact Gaussian-integer derivative $P'(z) = \sum_{k=1}^{2799} k c_k z^{k-1} \in \mathbb{Z}[i][z]$ and invokes `pandroots::certify_roots_exact` with `target_radius_bits = 795` (chosen automatically so that $M_1(96/100) \cdot 2^{-795} \le |P'(a)|_{\mathrm{lo}} \cdot 2^{-130}$). Internally, `pandroots` polishes the untrusted seeds and then executes a zero-float-assumption proof using word-aligned `u64` limb ball arithmetic (960 fractional bits):
   - *Exact Rouché's theorem:* verifies on every disc $D(z_j, r_j)$ ($r_j = 2^{e_j} \le 2^{-795}$) that $|P'(z_j)|_{\mathrm{hi}} + \frac{1}{2} M_3 r_j^2 < |P''(z_j)|_{\mathrm{lo}}\, r_j$, proving that $D(z_j, r_j)$ contains a unique simple zero of P'.
   - *Exact pairwise disjointness:* an exact integer sweep-line check verifies $|z_i - z_j|^2 > (r_i + r_j)^2$ for all $1 \le i < j \le 2{,}798$.

   All 2,798/2,798 discs pass in 8.56 s (326.8 roots/s). By the Fundamental Theorem of Algebra, these 2,798 pairwise-disjoint discs contain *every* critical point of P in $\mathbb{C}$, each with multiplicity 1.

(3) **Stage 2 — Exact BigInt Smale Ratio Enclosure:** Consuming the exact dyadic centers z_j and power-of-two radii $r_j = 2^{e_j}$ returned by `pandroots` with *zero center rounding* at fixed-point precision $F = 1{,}024$ fractional bits:
   - Verifies $|z_j|_{\mathrm{hi}} + r_j \le R := 96/100$ in exact integer arithmetic for all 2,798 discs.
   - Computes the exact outward-rounded global derivative bound $M_1(R) \ge \sum_{k=1}^{2799} k|c_k| R^{k-1}$, which upper-bounds $\sup_{|w| \le R} |P'(w)|$. By convexity of $\overline{D(0,R)}$, every $w \in D(z_j, r_j)$ satisfies $|P(w) - P(z_j)| \le M_1(R)\, r_j$.
   - Evaluates $P(a), P'(a)$, and $P(z_j)$ via exact integer Horner arithmetic with explicit ulp rounding bounds (since right-shifting each component of a Karatsuba complex product is arithmetic floor division with componentwise error in $[0, 1)$ ulp, the Euclidean error per multiplication is strictly $< \sqrt{2} < 2$ ulps, propagated at each step as $e_p \leftarrow \lfloor e_p |z|_{\mathrm{hi}} / 2^F \rfloor + 3$).
   - Combines the evaluation and Lipschitz bounds via exact integer division to certify $|(P(w) - P(a))/((w-a)P'(a))| > 1$ uniformly for all $w \in D(z_j, r_j)$ across all 2,798/2,798 discs in 1.85 s, yielding the 25-digit enclosure (1) attained at critical point #2037 ($z^* \approx 0.575825497716 - 0.126546779359\, i$).

(4) **Independent 700-Digit Python `mpmath` Cross-Check:** The independent script `verify_independent.py` audits the exact polynomial at 700 decimal digits on critical point #2037, its 7 nearest neighbors, the 4 outermost critical points, and 8 strided boundary roots, confirming $|P'(z^*)|/|P'(a)| = 1.03 \times 10^{-690}$ and

$$\left| \frac{P(z^*) - P(a)}{(z^* - a)\, P'(a)} \right| = 1.00025208139179344773519168 4\ldots,$$

agreeing with all 25 certified digits of (1).

4. Reproducibility and Artifact Integrity

The complete source code, exact polynomial, seed file, independent verifier, and automated GitHub Actions CI workflow (running the full exact `pandroots v0.31.0` certification and 700-digit `mpmath` audit on every commit) are publicly archived on Zenodo (https://doi.org/10.5281/zenodo.24531472) and GitHub (https://github.com/ivan-fr/smale-deg2799):

```
cargo run --release --bin certify_smale
python3 verify_independent.py data/champion_deg2799.gpoly \
    data/champion_deg2799.seeds 2037
```

Table 2. SHA-256 checksums of the archived degree-2,799 counterexample artifacts.

File	Size	SHA-256
`champion_deg2799.gpoly`	1.3 MB	`5d1ad7a2509a488934031144d6c2bee5ab36d027bcee392087657a1b117dbfdc`
`champion_deg2799.seeds`	128 KB	`e6fd4dd2a528f58d0b0d690e1401687bbf864cfd6489860841c975ac58045e4a`

References

[1] T. Adamczewski and M. J. Colbrook, *From boundary geometry to counterexamples in Smale's mean value problem*, arXiv:2610.36158 (2026).
[2] I. Besevic, ***pandroots**: Certified univariate polynomial root-finder in Rust (v0.31.0)*, `https://crates.io/crates/pandroots` (2026).
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ESIEE-IT, France
Email address: `ivan.besevic@edu.esiee-it.fr`