THE DUAL SMALE MEAN VALUE CONJECTURE FOR DEGREE EIGHT: BREAKING THE MINIMAL-MODULUS BARRIER VIA EXACT-INTEGER OUTWARD-ROUNDED VERIFICATION AND LEAN 4 FORMALIZATION

IVAN BESEVIC

ABSTRACT. Let $P(z) = z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C}[z]$ be a complex polynomial of degree $n \geq 2$. The Dual Smale Mean Value Conjecture (Dubinin–Sugawa and Ng, 2009) asserts that

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{n},$$

with equality if and only if $P(z) = \frac{1-(1-cz)^n}{nc}$ for some $c \in \mathbb{C} \setminus \{0\}$. While the primal Smale mean value conjecture was disproved in 2026, the dual conjecture remained open for all degrees $n \geq 8$; the previous record $n \leq 7$ (Hinkkanen–Kayumov–Khammatova, 2025) relied on bounding $|P(\zeta_1)/\zeta_1|$ at a single critical point ζ_1 of minimal modulus, a strategy that fails abruptly at $n = 8$ where $\min_{\zeta_1} 8|P(\zeta_1)/\zeta_1| \approx 0.6928 < 1$.

In this paper we break the minimal-modulus barrier for the first time and prove the Dual Smale Mean Value Conjecture unconditionally for all complex polynomials of degree $n = 8$, together with strict uniqueness of the extremal family. Our proof combines: (i) an exact-integer polynomial reformulation of $P(1)$, the single-variable and two-block Grace–Walsh–Szego (GWS) envelopes G_8 and H_8, and the secondary critical-point forms Q_j; (ii) a universal strict local minimum theorem around the extremal polynomial for all $n \geq 3$ with an explicit quantitative local ball $\max(|a-1|, |v-1|) \leq 1/100$ at $n = 8$; (iii) a Lean 4.36.0 formalization (1,904 lines, 24 theorems, zero `sorry`, standard axioms only) verifying Laguerre's polar derivative theorem, Grace–Walsh–Szego on closed disks, all exact integer identities, the quantitative local ball, the all-n real-critical-point case, unit-circle sum-of-squares certificates up to $n = 7$, an equal-modulus circle-clique theorem at $n = 8$, and the soundness of every interval-pruning and symmetric multiset-splitting rule; and (iv) a self-contained, outward-rounded IEEE-754 polar branch-and-bound certifier in Rust that exhausts the entire remaining 6-complex-variable domain in 72.75 seconds on 10 CPU cores.

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1. INTRODUCTION

In 1981, Smale [7] posed the celebrated mean value conjecture: for every complex polynomial P of degree $n \geq 2$ with $P(0) = 0$ and $P'(0) = 1$, does there exist a critical point ζ of P such that $|P(\zeta)/\zeta| \leq 1 - 1/n$ (or at least $\leq K$ for a universal constant)? In a remarkable development in 2026, the primal Smale conjecture was resolved in the negative with sharp global supremum $K^* = 4$ [1, 3].

In contrast, the *Dual Smale Mean Value Conjecture*, formulated independently by Dubinin–Sugawa [2] and Ng [6], asks for a sharp *lower* bound on the maximum over all critical points:

Theorem 1.1 (Main Theorem — Dual Smale Mean Value Conjecture for $n = 8$)**.** *Let $P(z) = z + a_2 z^2 + \cdots + a_8 z^8 \in \mathbb{C}[z]$ be any complex polynomial of degree 8 with $a_8 \neq 0$. Then*

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{8}. \tag{1}$$

Moreover, equality holds in (1) if and only if $P(z) = \frac{1-(1-cz)^8}{8c}$ for some $c \in \mathbb{C} \setminus \{0\}$.

Prior to this work, the Dual Smale Mean Value Conjecture was proved for $n \leq 7$ by Hinkkanen, Kayumov, and Khammatova [4] (building on earlier bounds for $n \leq 5$), for odd polynomials of

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arbitrary degree by Tang [8], and unconditionally up to the general lower bound $(n-1/2)^{1/n}/n^2$ by Jatar and Ng [5].

1.1. **Why Degree $n=8$ Is a Fundamental Threshold.** Normalize P by scaling $z \mapsto \zeta_1 z$ so that a critical point of minimal modulus lies at $\zeta_1 = 1$. Writing the remaining $n-2$ critical points as $\zeta_k = 1/w_k$ $(k = 2, \dots, n-1)$, every w_k lies in the closed unit disk $\overline{\mathbb{D}} = \{w \in \mathbb{C} : |w| \le 1\}$, and

$$P'(z) = (1-z)\prod_{k=2}^{n-1}(1-w_k z).$$

Integrating from 0 to 1 yields the symmetric multiaffine form

$$P(1) = \int_0^1 (1-u)\prod_{k=2}^{n-1}(1-uw_k)\,du = \sum_{m=0}^{n-2}\frac{(-1)^m}{(m+1)(m+2)}\,e_m(w_2,\dots,w_{n-1}), \tag{2}$$

where e_m denotes the m-th elementary symmetric polynomial. By the Grace–Walsh–Szego (GWS) coincidence theorem on $\overline{\mathbb{D}}$, there exists $w \in \overline{\mathbb{D}}$ such that $P(1) = G_n(w)$, where

$$G_n(w) := \int_0^1 (1-u)(1-uw)^{n-2}\,du = \sum_{m=0}^{n-2}\frac{(-1)^m\binom{n-2}{m}}{(m+1)(m+2)}\,w^m. \tag{3}$$

For $2 \le n \le 7$, one has $n|G_n(w)| \ge 1$ everywhere on $\overline{\mathbb{D}}$, so the single minimal-modulus critical point $\zeta_1 = 1$ already satisfies $|P(\zeta_1)/\zeta_1| \ge 1/n$. At $n = 7$, the exact unit-circle identity (formally verified in Lean 4, Theorem 4.1)

$$\left|42\,G_7(e^{i\theta})\right|^2 - 36 = 14\,u\,(2u-1)^2(12u^2+4u+1), \qquad u = 1-\cos\theta, \tag{4}$$

reveals a double tangency to 1 at $\theta = \pm 60^\circ$ $(u = 1/2)$. Immediately at $n = 8$, this tangency breaks through 1: $8|G_8(w)|$ drops below 1 (down to ≈ 0.6928) on two disconnected lunes $L_8^{\pm} \subset \overline{\mathbb{D}}$ near $|w| \in [0.920, 1]$, $\arg(w) \in \pm[54.7^\circ, 80.3^\circ]$. Consequently, any proof for $n = 8$ must *necessarily* couple the minimal-modulus critical point $\zeta_1 = 1$ with the secondary critical-point values $P(\zeta_j)/\zeta_j$ $(j = 2, \dots, 7)$.

1.2. **Architecture of the Proof and Verification Boundary.** To maximize transparency and eliminate hidden numerical heuristics, we separate the proof into two cleanly audited layers:

(1) **Formal Lean 4.36.0 Kernel Proofs** (`lean/DualSmale/`, 1,904 lines, 24 theorems, zero `sorry`, zero `native_decide`, axioms strictly `[propext, Classical.choice, Quot.sound]`): Laguerre's polar derivative theorem and the Grace–Walsh–Szego coincidence theorem on closed disks; exact integer-coefficient polynomial identities for $P(1)$, G_8, H_8, and Q_j; the universal local minimum quadratic form for all $n \ge 3$ and the explicit quantitative local ball $\max(|a-1|, |v-1|) \le 1/100$ for H_8 (Stage 0); the all-n real-critical-point theorem, full $n \le 4$ proofs, and exact unit-circle sum-of-squares certificates for $n = 5, 6, 7$; the equal-modulus circle-clique impossibility theorem at $n = 8$; and the soundness of every directional projection, convex-combination, correlated-ratio Taylor remainder, and symmetric multiset binary splitting rule.

(2) **Outward-Rounded Exact-Integer Rust Certifier** (`verifier/src/`, 1,525 lines): using IEEE-754 `next_down`/`next_up` outward rounding, 4-ULP outward-rounded trigonometric bounds, exact integer coefficients $[420, -140, 70, -42, 28, -20, 15]/105$ (zero quadrature floats), and polar rotated-rectangle enclosures (`PBox`) with zero unit-disk overhang, the certifier exhausts Stages 1a, 1b, 1c, 2, and 3 in 72.75 seconds on 10 CPU cores, proving Theorem 1.1 unconditionally.

2. Grace–Walsh–Szego Coincidence and Exact Integer Identities

We formalize Laguerre's polar derivative theorem and the Grace–Walsh–Szego coincidence theorem directly from first principles in Lean 4 (`DualSmale/GraceWalshSzego.lean`).

Theorem 2.1 (Laguerre Polar Derivative and Grace–Walsh–Szego on Closed Disks)**.** *Let $m \geq 1$ and $F_m(c; z_1, \ldots, z_m) = \sum_{k=0}^{m} c_k e_k(z_1, \ldots, z_m)$ be a symmetric multiaffine polynomial with complex coefficients $c_k \in \mathbb{C}$. For any $r > 0$ and any $z_1, \ldots, z_m \in \mathbb{C}$ with $|z_i| \leq r$, there exists $\zeta \in \mathbb{C}$ with $|\zeta| \leq r$ such that*

$$F_m(c; z_1, \ldots, z_m) = F_m(c; \zeta, \ldots, \zeta).$$

Proof outline (formalized in `DualSmale/GraceWalshSzego.lean`). If $P \in \mathbb{C}[X]$ has $\deg P \leq m$ and no roots in $\overline{\mathbb{D}}$, then for any $w, z \in \overline{\mathbb{D}}$ with $w \neq z$ and every root a of P ($|a| > 1$), the algebraic identity

$$2\,\mathrm{Re}\left((1 - z\overline{w})\frac{a-w}{a-z}\right) = (1-|z|^2)\left|\frac{a-w}{a-z}\right|^2 + (1-|w|^2) + (|a|^2-1)\frac{|z-w|^2}{|a-z|^2} > 0$$

implies that the polar derivative $D_w P(z) = mP(z) + (w-z)P'(z)$ has no roots in $\overline{\mathbb{D}}$. Since D_w maps the diagonal polynomial of F_{m+1} to $(m+1)$ times the diagonal polynomial of the specialization $z_{m+1} = w$, induction on m yields zero-avoidance on $\overline{\mathbb{D}}$, and shifting $c_0 \mapsto c_0 - F_m(c; z)$ yields coincidence on $\overline{\mathbb{D}}$ (and on $|z| \leq r$ by scaling). □

Applying Theorem 2.1 to subsets of the 6 variables $w = (w_2, \ldots, w_7) \in \overline{\mathbb{D}}^6$ at degree $n = 8$ produces three exact integer-coefficient forms after multiplying by $105 \cdot 8 = 840$ (since $\mathrm{lcm}(2, 6, 12, 20, 30, 42, 56) = 840$):

Proposition 2.2 (Exact Integer-Coefficient Identities at Degree $n = 8$)**.** *For all $w = (w_2, \ldots, w_7) \in \mathbb{C}^6$, $a, v \in \mathbb{C}$, and any $j \in \{2, \ldots, 7\}$ with $w_j \neq 0$, letting $w_{-j} \in \mathbb{C}^5$ denote the remaining five coordinates:*

$$840\,P(1) = 420 - 140e_1(w) + 70e_2(w) - 42e_3(w) + 28e_4(w) - 20e_5(w) + 15e_6(w), \tag{5}$$

$$840\,G_8(v) = 420 - 840v + 1050v^2 - 840v^3 + 420v^4 - 120v^5 + 15v^6, \tag{6}$$

$$\begin{aligned} 840\,H_8(a,v) &:= 840\,P(1)\big|_{(a,v,v,v,v,v)} \\ &= (420 - 700v + 700v^2 - 420v^3 + 140v^4 - 20v^5) \\ &\quad + a(-140 + 350v - 420v^2 + 280v^3 - 100v^4 + 15v^5), \end{aligned} \tag{7}$$

$$\begin{aligned} 840\,w_j^6\frac{P(\zeta_j)}{\zeta_j} &= (420w_j^6 - 140w_j^5) + e_1(w_{-j})(-140w_j^5 + 70w_j^4) + e_2(w_{-j})(70w_j^4 - 42w_j^3) \\ &\quad + e_3(w_{-j})(-42w_j^3 + 28w_j^2) + e_4(w_{-j})(28w_j^2 - 20w_j) + e_5(w_{-j})(-20w_j + 15). \end{aligned} \tag{8}$$

Proof. All four identities are verified algebraically in Lean 4 (`exact_p1_deg8_integer_identity`, `exact_G8_integer_identity`, `exact_H8_integer_identity`, and `exact_qj_deg8_integer_identity` in `DualSmale/PolynomialReduction.lean`). □

Notice that $8|P(1)| > 1 \iff |840\,P(1)| > 105$, and $8|P(\zeta_j)/\zeta_j| > 1 \iff |840\,w_j^6 P(\zeta_j)/\zeta_j|/|w_j|^6 > 105$. Every coefficient in (5)–(8) is a small exact integer representable without error in IEEE-754 binary64, completely eliminating numerical quadrature error.

3. Universal Local Minimum and Stage 0 Quantitative Ball

The extremal polynomial $P_n(z) = \frac{1-(1-z)^n}{n}$ corresponds to $w_2 = \cdots = w_{n-1} = 1$, where $nP_n(1) = 1$. Any global branch-and-bound proof must resolve an explicit neighborhood of this boundary point analytically.

Theorem 3.1 (Universal Second-Order Strict Positivity for All $n \geq 3$)**.** *Let $n \geq 3$ and $w_k = 1 + x_k + iy_k \in \overline{\mathbb{D}}$ ($k = 1, \ldots, n-2$), so that $2x_k + x_k^2 + y_k^2 \leq 0$. Then the exact second-order*

expansion of $|nP(1)|^2 - 1$ *satisfies*

$$\frac{2}{n-1}\sum_{k=1}^{n-2}(-x_k) - \frac{2}{(n-1)(n-2)}\Big(\Big(\sum y_k\Big)^2 - \sum y_k^2\Big) + \frac{1}{(n-1)^2}\Big(\sum y_k\Big)^2 \geq \frac{n}{(n-1)^2(n-2)}\sum_{k=1}^{n-2} y_k^2 + \frac{1}{n-1}\sum_{k=1}^{n-2} x_k^2.$$

Proof. Using $-x_k \geq (x_k^2+y_k^2)/2$, the difference between the left- and right-hand sides is bounded below by

$$\frac{n}{(n-1)^2(n-2)}\left((n-2)\sum_{k=1}^{n-2} y_k^2 - \Big(\sum_{k=1}^{n-2} y_k\Big)^2\right) \geq 0$$

by Cauchy–Schwarz (formalized in `universal_local_quadratic_lower_bound`). □

At $n = 8$, specializing to the single-variable GWS envelope $H_8(a, v)$ yields an explicit, strictly certified local ball around $(a, v) = (1, 1)$:

Theorem 3.2 (Stage 0 Quantitative Local Ball for H_8)**.** *For all* $(a, v) \in \overline{\mathbb{D}}^2 \setminus \{(1, 1)\}$ *satisfying* $|a - 1| \leq 1/100$ *and* $|v - 1| \leq 1/100$*, we have strictly*

$$8\,|H_8(a, v)| > 1.$$

Proof. Write $a = 1 + d_1$, $v = 1 + d_2$ with $d_1 = x_1 + iy_1$, $d_2 = x_2 + iy_2$, and expand $8H_8(1 + d_1, 1 + d_2) = Q + R$, where Q is the quadratic part $1 - \frac{1}{7}d_1 - \frac{5}{7}d_2 + \frac{5}{21}d_1d_2 + \frac{10}{21}d_2^2$ and every term of the higher-order remainder R carries a factor of d_2^2. Since $|d_1|, |d_2| \leq 1/100$, the triangle inequality gives $|R| \leq \frac{2}{100}|d_2|^2$. On the other hand, using $2x_i + x_i^2 + y_i^2 \leq 0$ and the exact quadratic form $\frac{1}{7}x_1^2 + \frac{5}{7}x_2^2 + \frac{4}{147}(y_1^2 + 5y_2^2 + 5(y_1 - y_2)^2)$, we obtain

$$|Q|^2 - \Big(1 + \frac{2}{100}|d_2|^2\Big)^2 \geq \frac{1}{7}x_1^2 + \Big(\frac{4}{147} - \frac{1}{500}\Big)y_1^2 + \Big(\frac{5}{7} - \frac{5}{100}\Big)x_2^2 + \Big(\frac{20}{147} - \frac{1}{25} - \frac{5}{100}\Big)y_2^2 > 0,$$

whence $|8H_8(a, v)| \geq |Q| - |R| > 1$ (formalized in `stage0_H8_local_ball` in `DualSmale/LocalMinimum.lean`). □

4. Analytic Results: Real Critical Points, $n \leq 7$, and Equal-Modulus $n = 8$

In addition to the $n = 8$ certification infrastructure, our Lean 4 library formalizes several standalone theorems illuminating the structure of the Dual Smale problem:

Theorem 4.1 (Real Critical Points, Low Degrees, and Exact $n = 7$ Factorization)**.** *The following hold unconditionally (formalized in* `DualSmale/DegreesUpToSeven.lean`*):*

1. *For every* $n \geq 2$ *and every real vector* $r \in [-1, 1]^{n-2}$*,* $\int_0^1 (1 - u)\prod_{k=1}^{n-2}(1 - ur_k)\,du \geq 1/n$.
2. *The Dual Smale Mean Value Conjecture holds for all complex polynomials of degrees* $n = 2, 3, 4$*, via the closed-disk sum-of-squares certificate* $|12G_4(w)|^2 - 9 = (t - 4u)^2 + 8u^2 + 8u + 2t \geq 0$ *(*$u = 1 - \operatorname{Re} w$*,* $t = 1 - |w|^2$*).*
3. *For every* $w = e^{i\theta}$ *on the unit circle (*$c = \cos\theta$*,* $u = 1 - c \in [0, 2]$*):*

$$|20G_5(w)|^2 - 16 = 10u(8(c-1)^2 + 1) \geq 0,$$
$$|30G_6(w)|^2 - 25 = 4u\big((60u + 20)(u - 1/3)^2 + \tfrac{20}{3}u + \tfrac{7}{9}\big) \geq 0,$$
$$|42G_7(w)|^2 - 36 = 14u(2u - 1)^2(12u^2 + 4u + 1) \geq 0.$$

At degree $n = 8$, we also obtain a purely combinatorial proof for the equal-modulus case ($|\zeta_1| = \cdots = |\zeta_7|$): if every critical point had $8|P(\zeta_j)/\zeta_j| \leq 1$, then Stage 1c forces every pairwise angular separation between the 7 unit-circle critical points into the interval $[30°, 146°]$ (or even $[31°, 145°]$ inside the interior of the allowed sector).

Theorem 4.2 (Circle-Clique Impossibility at $n = 8$)**.** *There do not exist* 6 *angles* $0 \le \theta_0 < \cdots < \theta_5 < 360^\circ$ *whose pairwise circular distances all lie in* $[31^\circ, 145^\circ]$*. In particular, no* 7*-point equal-modulus configuration can satisfy the single-variable exclusion condition at all* 7 *critical points simultaneously.*

Proof. Formalized in `circle_clique_no_six` and `circle_clique_no_seven` in `DualSmale/EqualModulusEight.lean` by cyclic-shift reduction to the maximal adjacent gap. □

5. Outward-Rounded Polar Certification for Degree $n = 8$

To complete the proof of Theorem 1.1 for arbitrary $w = (w_2, \ldots, w_7) \in \overline{\mathbb{D}}^6$, we execute a five-stage outward-rounded interval branch-and-bound certification in Rust (`verifier/src/`).

5.1. **Rigorous Arithmetic and Polar Enclosures.** All floating-point interval bounds use IEEE-754 directed rounding via `f64::next_down` and `f64::next_up` after every elementary operation (`interval.rs`), with conservative 4-ULP outward enclosures for sin and cos. A key algorithmic obstacle in certifying unit-disk polynomials in Cartesian boxes is corner overhang past $|w| = 1$, which causes spurious branch-and-bound blowup near the boundary. We eliminate this completely by parameterizing boxes in polar coordinates $B = [r_0, r_1] \times [\theta_0, \theta_1] \subset \overline{\mathbb{D}}$ (`PBox`), bounded rigorously by an outward-rounded rotated rectangle aligned with the midpoint angle $\theta_m = (\theta_0 + \theta_1)/2$. Every point $re^{i\theta} \in B$ satisfies $r \le 1$ identically, with zero leakage outside $\overline{\mathbb{D}}$.

5.2. **Soundness of All Pruning and Splitting Rules.** For a 6-tuple of polar boxes $B_2 \times \cdots \times B_7 \subset \overline{\mathbb{D}}^6$, a node is pruned whenever at least one of the following certified criteria holds (each proved sound in `DualSmale/IntervalSoundness.lean`):

(1) **Two-Block GWS Modulus Pruning**: if $k \in \{1, \ldots, 6\}$ boxes lie inside $|w| \le R_k$ for the Stage 1b certified thresholds

$$(R_1, R_2, R_3, R_4, R_5, R_6) = (0.420,\ 0.725,\ 0.820,\ 0.865,\ 0.892,\ 0.912),$$

then applying Theorem 2.1 separately to the k inner and $6 - k$ outer coordinates yields $8|P(1)| > 1$.

(2) **GWS Safe-Disk Pruning**: if all 6 boxes lie inside a disk $D(c, \rho)$ certified in Stage 1a to satisfy $8|G_8(w)| > 1$, then Theorem 2.1 implies $8|P(1)| > 1$.

(3) **First-Order Affine Enclosures for $P(1)$ and Q_j**: writing $w_k = c_k + \delta_k$ around the rotated-rectangle center c_k, we separate the exact center value, the linear gradient term in δ_k, and the quadratic-and-higher Taylor remainder. For $Q_j(w) = 840 w_j^6 P(\zeta_j)/\zeta_j$, we use the correlated-ratio decomposition

$$\frac{Q_j}{w_j^6} = \gamma + \frac{\Delta_1}{c_j^6} + \frac{\Delta_0 + \Delta_2}{w_j^6} + \Delta_1 \frac{c_j^6 - w_j^6}{c_j^6 w_j^6},$$

proved sound in `correlated_ratio_lower_bound`, which cancels the dominant w_j^6 sensitivity before taking the directional projection $\mathrm{Re}(d \cdot z) \le |z|$ (`re_dir_mul_le_norm`).

(4) **Convex-Combination and Multi-Point Average Pruning**: when $8|P(1)|$ and $8|P(\zeta_j)/\zeta_j|$ have opposing gradients across a box, we test dyadic convex combinations $(1 - \alpha)\,\mathrm{Re}(d_1 \cdot 8P(1)) + \alpha\,\mathrm{Re}(d_j \cdot 8P(\zeta_j)/\zeta_j) \le \max(8|P(1)|, 8|P(\zeta_j)/\zeta_j|)$ (`convex_combo_re_le_max`) and equal-weight averages over active critical points (`avg_re_le_bound`).

(5) **Symmetric Multiset Binary Splitting**: since the objective $\max(|P(1)|, \max_j |P(\zeta_j)/\zeta_j|)$ is invariant under all $6! = 720$ permutations of $(w_2, \ldots, w_7)$, whenever a group of m identical polar boxes B is bisected into $B = L \cup R$, it suffices to branch into the $m + 1$ subproblems placing $n_L \in \{0, \ldots, m\}$ points in L and $m - n_L$ points in R (`sym_multiset_binary_split_cover`).

5.3. **Complete Certification Results.** Define the conjugate polar sectors

$$S_+ := \{re^{i\theta} : r \in [0.420, 1],\ \theta \in [30^\circ, 146^\circ]\}, \qquad S_- := \overline{S_+}.$$

By complex-conjugation symmetry $P(z) \mapsto \overline{P(\overline{z})}$, any configuration in $(S_+ \cup S_-)^6$ has $(n_+, n_-) \in \{(6,0),(5,1),(4,2),(3,3)\}$ points in (S_+, S_-) up to conjugation. Table 1 reports the exact execution log of `cargo run --release` (10 threads, Apple Silicon, zero warnings, 2/2 unit tests passed).

TABLE 1. Complete outward-rounded Rust certification summary for degree $n = 8$.

Stage	Certified Domain	Nodes / Leaves	Time	Method
0	$\max(\lvert a-1\rvert, \lvert v-1\rvert) \le 0.01$	—	formal	Lean 4 (`stage0_H8_local_ball`)
1a	5 GWS safe disks for G_8	391	2.14 ms	2D outward-rounded B&B
1b	6 two-block GWS bounds R_k	1,175,626	1.82 s	4D polar outward-rounded B&B
1c	$\overline{\mathbb{D}} \setminus (S_+ \cup S_-)$ via H_8	453,538	0.34 s	4D polar B&B outside Stage 0
2	Mixed splits $(5,1),(4,2),(3,3)$	63,002	0.70 s	$8\lvert P(1)\rvert > 1$ polar B&B
3	All 6 points in S_+ (S_+^6)	5,160,052	69.88 s	10-thread sym. B&B + 4D H_8
Total	**All of $\overline{\mathbb{D}}^6 \setminus \{(1,\dots,1)\}$**	**6,852,609**	**72.75 s**	**100% outward-rounded PASS**

In Stage 3, the 5,160,052 certified leaves break down by pruning rule as: modulus threshold (18,051), GWS safe disk (14), $P(1)$ alone (1,941,040), single secondary critical point Q_j (2,161,959), and convex-combination / average gradient cancellation (1,038,988), with 3,965,772 symmetric splits, maximum recursion depth 33, and 15 memoized 4D H_8-exclusions on sub-boxes of S_+ (77,424 additional 4D nodes).

5.4. **Proof of Theorem 1.1 (Including Strict Uniqueness).**

Proof of Theorem 1.1. Normalize P so that a critical point of minimal modulus is $\zeta_1 = 1$, giving $w = (w_2, \dots, w_7) \in \overline{\mathbb{D}}^6$. If $w = (1,1,1,1,1,1)$, then $P'(z) = (1-z)^7$ and $P(z) = \frac{1-(1-z)^8}{8}$, giving equality $\lvert P(1)/1\rvert = 1/8$.

Suppose $w \neq (1,\dots,1)$. First, assume that some coordinate satisfies $w_k \notin S_+ \cup S_-$. Applying Theorem 2.1 to the remaining five coordinates $w_{-k} \in \overline{\mathbb{D}}^5$ yields $v \in \overline{\mathbb{D}}$ with $P(1) = H_8(w_k, v)$. If $(w_k, v) \neq (1,1)$, then either $\max(\lvert w_k - 1\rvert, \lvert v-1\rvert) \le 1/100$, in which case Theorem 3.2 gives $8\lvert P(1)\rvert = 8\lvert H_8(w_k, v)\rvert > 1$, or $\max(\lvert w_k - 1\rvert, \lvert v-1\rvert) > 1/100$, in which case Stage 1c certifies $8\lvert H_8(w_k, v)\rvert > 1$.

It remains to verify that $(w_k, v) = (1,1)$ cannot occur when $w \neq (1,\dots,1)$. Indeed, if $w_k = 1$, consider the degree-5 symmetric multiaffine form $F_5(w_{-k}) := 840P(1) - 105$, whose diagonal polynomial is

$$Q(z) := 840H_8(1,z) - 105 = -5(z-1)\big(z^4 - 7z^3 + 21z^2 - 35z + 35\big).$$

By Theorem 3.2 and Stage 1c, $8\lvert H_8(1,z)\rvert > 1$ for every $z \in \overline{\mathbb{D}} \setminus \{1\}$, so Q has a simple root at $z = 1$ ($Q'(1) = -75 \neq 0$) and its four remaining roots $a_1, \dots, a_4$ all satisfy $\lvert a_i\rvert > 1$. Since $w \neq (1,\dots,1)$, there exists a coordinate $w_\ell \in \overline{\mathbb{D}} \setminus \{1\}$ among w_{-k}. Its polar derivative $D_{w_\ell}Q(z) = 5Q(z) + (w_\ell - z)Q'(z)$ satisfies $D_{w_\ell}Q(1) = -75(w_\ell - 1) \neq 0$. For any $z \in \overline{\mathbb{D}} \setminus \{1\}$, either $z = w_\ell$ (giving $D_{w_\ell}Q(w_\ell) = 5Q(w_\ell) \neq 0$) or $z \neq w_\ell$, in which case the Laguerre identity of Theorem 2.1 gives

$$2\,\mathrm{Re}\left((1 - z\overline{w_\ell})\frac{a - w_\ell}{a - z}\right) = (1 - \lvert z\rvert^2)\left|\frac{a - w_\ell}{a - z}\right|^2 + (1 - \lvert w_\ell\rvert^2) + (\lvert a\rvert^2 - 1)\frac{\lvert z - w_\ell\rvert^2}{\lvert a - z\rvert^2},$$

which is ≥ 0 for $a = 1$ and strictly > 0 for each $a = a_i$ ($\lvert a_i\rvert > 1$). Hence $D_{w_\ell}Q$ is zero-free on all of $\overline{\mathbb{D}}$. Applying Theorem 2.1 to the remaining four coordinates of w_{-k} implies $F_5(w_{-k}) \neq 0$, i.e., $P(1) \neq 1/8$. Consequently $v \neq 1$, so $(w_k, v) \neq (1,1)$ and we conclude $8\lvert P(1)\rvert > 1$ strictly.

Finally, if all six coordinates lie in $S_+ \cup S_-$, then by conjugation symmetry we fall into either Stage 2 (mixed half-plane splits $(5,1)$, $(4,2)$, $(3,3)$, where $8\lvert P(1)\rvert > 1$ strictly) or Stage 3

($w \in S_+^6$, where $\max\big(8|P(1)|, \max_{2\le j\le 7} 8|P(\zeta_j)/\zeta_j|\big) > 1$ strictly). This establishes both (1) and strict uniqueness of the extremal family. □

6. Reproducibility and Software Artifacts

All Lean 4 proofs and the Rust certifier are publicly available at `https://github.com/ivan-fr/dual-smale-n8` and verified automatically by GitHub Actions CI:

```
# 1. Verify all 24 Lean 4.36.0 theorems and audit axioms
cd lean && lake build DualSmale && lake env lean check_axioms.lean

# 2. Run Rust unit tests and full outward-rounded degree-8 certification
cd verifier && cargo test --release && cargo run --release
```

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ESIEE-IT, France

Email address: `ivan.besevic@edu.esiee-it.fr`