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The Mathieu Property over Fields of Characteristic Zero:
Classification and Ground-Field Obstructions

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Abstract

Let K be a field of characteristic zero and X an affine geometrically homogeneous space of a group G with split reductive identity component. The normalized Reynolds kernel on $K[X]$ is Mathieu–Zhao if and only if $[G^\circ, G^\circ]$ acts trivially. A single active split SL_2 already gives a fixed marked counterexample for a connected reductive acting group. Both results allow $X(K) = \varnothing$. The proof transfers witnesses along principal orbits and specializes marked-moment recurrences over the ground field. We also classify connected reductive groups and compact adjoint semisimple torsors over real closed fields, and representative functions on compact Hausdorff homogeneous spaces. A long-root subgroup gives a further obstruction when a split central torus acts alongside an active compact factor of rank at least two.

1 Introduction

Let K be a field of characteristic zero. Let X be a smooth affine K-variety with a geometrically transitive action of a linear algebraic K-group G with reductive identity component. Thus $X_{\overline{K}}$ is a homogeneous quotient and its geometric stabilizers have reductive identity component by the affine Matsushima criterion [11]. If $x \in X(K)$, we may write $X = G/H$ with $H = G_x$. The normalized Reynolds functional $\Lambda_X : K[X] \to K$ projects onto the invariant constants. A function P is *pure* if $\Lambda_X(P^m) = 0$ for every $m \geq 1$. It is *safe* if for each fixed $Q \in K[X]$ one has $\Lambda_X(QP^m) = 0$ for all sufficiently large m. The kernel is Mathieu–Zhao precisely when every pure function is safe. We call (P, Q) a witness if P is pure and $\Lambda_X(QP^m) \neq 0$ at infinitely many orders m.

Theorem 1.1. *Suppose G° is split over K. Then*

$$\ker \Lambda_X \text{ is Mathieu–Zhao} \quad \Longleftrightarrow \quad [G^\circ, G^\circ] \text{ acts trivially on } X.$$

For $X = G/H$, the right-hand condition is $[G^\circ, G^\circ] \subseteq H$. For connected G, it also characterizes equality of the pure set and the geometric nullcone on $K[X]$.

Mathieu's first conjecture posed the purity-to-safety implication for representative functions on compact connected Lie groups [1]. His second asked whether pure functions on complex reductive spherical pairs are unstable. Long classified the compact connected group case and the complex spherical case [2, 3]. The classification on arbitrary complex reductive homogeneous spaces was proved in [4]. Over the ground field the answer can change: on the real quaternion norm-one group every nonzero real regular function has positive squared trace, while its complexification is SL_2 and admits a witness.

The negative direction of Theorem 1.1 uses only one active split SL_2 and works even when G is nonsplit. A general stabilizer either supplies an adjoint covariant or determines a normalizer

quotient of SL_2. In the second case an invariant denominator clears poles in the transferred witness. A recurrence and Hilbert irreducibility select parameters so a single ground-field multiplier detects infinitely many orders.

The absence of rational points also matters for compact groups. Over a real closed field, a torsor for a compact adjoint semisimple group has Mathieu–Zhao trace exactly when it has a rational point (Theorem 5.4). For instance, the isomorphism torsor from the quaternion division algebra to the split matrix algebra carries commuting compact and split adjoint actions with the same normalized trace.

We use the term *geometric nullcone* for the points whose orbit closure over an algebraic closure contains zero. For an individual regular function this can be tested in any finite-dimensional rational submodule containing it. For disconnected G, geometric instability is the same for G and G°, since the larger orbit closure is a finite union of translates. An unstable function is safe. Over an algebraic closure a contracting cocharacter gives strictly positive weights on that function, eventually dominating the finitely many weights of any fixed multiplier [9, 3]. A nonzero marked subsequence therefore excludes geometric instability.

2 Rank-one moment identities

Normalized Reynolds functionals commute with ground-field extension. An equivariant morphism between transitive spaces preserves their normalized functionals under pullback, by uniqueness. In particular, on a quotient G/H the trace is the restriction of the normalized left trace on $K[G]$. We use these facts also for central isogenies and products.

Lemma 2.1. *For split $S = \mathrm{SL}_2$, the quotients by the split diagonal torus normalizer and the two standard finite groups $2O, 2I$ admit witnesses over K.*

Proof. Use the binary forms

$$f_4 = X^2Y^2, \qquad f_8 = X^8 + 14X^4Y^4 + Y^8, \qquad f_{12} = XY(X^{10} + 11X^5Y^5 - Y^{10}).$$

For a form of degree $n = 2b$ write $a = b - 1$. For the rows u, v of a determinant-one matrix let $U = f(u)$, $F = f(v)$, and set

$$R = \frac{f(v + Fu)}{F}, \qquad P_f = U^a R^b, \qquad Q_f = F^a.$$

Taylor expansion in the added row shows R is polynomial. Every subgroup fixing f fixes the pair under right translation. For f_4, f_8, f_{12}, the groups are respectively the torus normalizer, $2O$ and $2I$. The pure moments vanish over $\mathbb{Q}$. The quartic has nonzero marked moments at all positive orders. For the two exceptional forms there are nonzero marked moments at orders m for which $28m - 3$, respectively $66m - 5$, is prime [4, Theorem 1.2 and Lemma 5.1]. In each progression the residue is coprime to the modulus, so Dirichlet's theorem gives infinitely many such orders. □

For use when a principal stabilizer is central, write a traceless matrix as

$$Z = \begin{pmatrix} a/2 & v \\ u & -a/2 \end{pmatrix}, \quad r = a^2 + 4uv, \qquad P_{\mathrm{ad}} = v^2(1 + au - u^3v)^4, \quad Q_{\mathrm{ad}} = u^2.$$

If R_S denotes the conjugation Reynolds projection, then

$$R_S(P_{\mathrm{ad}}^m) = 0, \qquad R_S(Q_{\mathrm{ad}} P_{\mathrm{ad}}^m) = \beta_m r^{3m-1}, \qquad \beta_m = \frac{2m}{3m+1} \frac{(2m)!(4m)!}{(6m)!}. \tag{1}$$

For $Z = ww^* - I/2$ with $w = (x, y)^{\mathsf{T}}$ a unit vector, $u = \bar{x}y$, $v = x\bar{y}$, $a = |x|^2 - |y|^2$, and

$$1 + au - u^3 v = (1 + |x|^2 u)(1 - |y|^2 u).$$

Under the uniform unit-sphere measure, $s = |x|^2$ is uniform on $[0, 1]$ and the angle of u is uniform. Put $n = 2m$. Angular averaging selects degree $k = n$ in u for the pure moment and $k = n - 2$ for the marked one. Expanding the two factors and integrating in s gives the common prefactor $(2n)!^2/(2n + k + 1)!$ times

$$\sum_{p+q=k} \frac{(-1)^q (n+p)!(n+q)!}{p!\, q!\, (2n-p)!\, (2n-q)!}.$$

For $k = n$, the factorial ratios cancel and the sum is zero. For $k = n - 2$, it equals

$$\frac{1}{(n-2)!} \int_0^1 \int_0^1 x^n y^n (1-x)(1-y)(x-y)^{n-2} \, \mathrm{d}y \, \mathrm{d}x.$$

Integrate on $y = xt$, $0 < t < 1$, and double by symmetry to obtain the displayed factor β_m.

At general radius, scaling Z by c multiplies r by c^2. A term of u-weight k in $(1+au-u^3v)^{4m}$ has total degree $2k$ in the matrix entries, with u, a, v of weights $1, 0, -1$. The selected pure and marked terms thus have total degrees $6m$ and $6m - 2$. This proves (1) on the entire traceless Hermitian real form. Zariski density gives the polynomial Reynolds identities over every characteristic-zero field.

3 Principal stabilizers

Suppose G is connected and let $S \to G$ be a nontrivial rank-one action with $S_{\overline{K}} \simeq \mathrm{SL}_2$. We establish the principal orbit statement for this geometric type, and then use splitness to descend a particular normalizer quotient.

Lemma 3.1. *There is a dense S-invariant open $U \subseteq X$ on which all geometric S-orbits are closed and their stabilizers are conjugate. The common stabilizer is proper and reductive.*

Proof. Work first over $\mathbb{C}$ using an embedding of an algebraic closure of a finitely generated field of definition. Choose a compact form C of $G_{\mathbb{C}}$ containing the image of a compact form of $S_{\mathbb{C}}$. After choosing a geometric base point, conjugate its stabilizer so that a Zariski-dense compact subgroup of it lies in C. These choices are possible by compact-form conjugacy and containment [10, Theorems 3.7–3.8]. The conjugated quotient is $G_{\mathbb{C}}$-equivariantly isomorphic to the original. Compact conjugation gives an antiholomorphic involution of the quotient and a Zariski-dense compact C-orbit of fixed points.

Embed the quotient into a finite-dimensional $S_{\mathbb{C}}$-module by taking a stable space of algebra generators closed under the involution. Average a Hermitian metric and choose it real on the fixed real form. For any fixed real vector v and any $A \in \mathrm{Lie}(\mathrm{SU}_2)$, the vector Av is real. Hence $\mathrm{Re}\langle iAv, v\rangle = 0$, which gives a closed orbit by the Kempf–Ness criterion [8]. The compact orbit meets the intersection of the maximal orbit dimension locus and the inverse image of the open

principal Luna stratum [7, Part III, §3]. The closed orbit in each fiber over that stratum has the same stabilizer type and thus maximal dimension. Any nonclosed orbit there would have that closed orbit in its boundary and would have strictly larger dimension. This proves the assertion on the inverse image of the principal stratum. The geometric principal open descends by uniqueness and base change of invariant algebras. The closed orbits have reductive stabilizers by Matsushima [11]. If their stabilizer were all of S, the action would be trivial on a dense open. □

Suppose now $S = \mathrm{SL}_{2,K}$ and let L represent the stabilizer class over $\overline{K}$. If L is noncentral, attach a group $M(L)$ as follows. When L° is a torus, use its normalizer. For a finite noncentral cyclic L, use the normalizer of its centralizer torus. For binary dihedral L of order $4r$ with $r \geq 3$, use the normalizer of the centralizer torus of the characteristic rotation subgroup generated by elements of order greater than four. For $Q_8, 2T, 2O, 2I$, use the full normalizer. It is $2O, 2O, 2O, 2I$, respectively. Indeed Q_8 and $2T$ are normal in $2O$, the centralizer of a finite noncyclic subgroup is the center, and the finite subgroup list rules out proper finite overgroups of $2O$ and $2I$ [6].

In all cases $N_S(L) \subseteq M(L)$ and the assignment commutes with conjugation and Galois. The group $M(L)$ is conjugate to a self-normalizing K-subgroup M_0, namely the split diagonal torus normalizer or the stabilizer of f_8 or f_{12} in Lemma 2.1. The torus normalizer is self-normalizing by its identity component. The two finite groups are self-normalizing by maximality among finite groups. Conjugate L so $M(L) = M_0$. Over $\overline{K}$ the rule

$$S_x = sLs^{-1} \;\longmapsto\; sM_0$$

defines a rational equivariant map. Indeed, the image of $(s, y) \mapsto (sy, sM_0)$ from $S \times U^L$ is a constructible graph with one geometric point over each point of U. For $y \in U^L$, containment $L \subseteq S_y$ is equality by conjugacy, including equality of the numbers of components. The unique dominating component of the graph closure is birational to U by generic separability. For each Galois automorphism σ write $\sigma L = hLh^{-1}$. Since both attached groups are M_0, we have $h \in M_0$. Replacing L by σL therefore leaves the graph unchanged. This descends the map $\phi : X \dashrightarrow S/M_0$ over K without a rational point of X.

The two types of witness

If L is central, restriction to a closed orbit shows that the adjoint module occurs in $K[X]$. Restriction is surjective over the algebraic closure, the adjoint module occurs on S/L, and the intertwiner space commutes with base change. Hence there is a K-defined nonzero equivariant map to the adjoint space. There is also such a map over $\mathbb{C}$ taking Hermitian values on the compact orbit. In fact the antilinear involution $Z \mapsto \tau_{\mathrm{h}} \circ Z \circ \tau_X$ on the space of equivariant maps uses the compact conjugation τ_X of the quotient and $\tau_{\mathrm{h}}(M) = \bar{M}^{\mathsf{T}}$. Its real fixed space spans the complex space. Choose a nonzero fixed map. It has nonzero radius on an open subset of the compact orbit. By (1), therefore, all its positive marked moments are nonzero. It can be written in a finite span of maps defined over K. This gives a finite-parameter family of pure adjoint phases over K, whose marked sequences are generically nonzero.

If L is noncentral, pull back the witnesses P_0, Q_0 of Lemma 2.1 by $\phi : X \dashrightarrow S/M_0$. Denote the two rational functions by p, q. Their translates span a finite-dimensional rational submodule. Its common denominator ideal I in $K[X]$ is nonzero and S-stable. Over $\mathbb{C}$, choose a nonzero $a \in I$. Write $\tau_X^* a$ for compact conjugation on functions. Then $d = R_S(a\,\tau_X^* a)$ belongs to I^S. On the compact orbit it is nonnegative, and it is positive somewhere by density and the compact angular average. Hence $\Lambda_X(d^j) \neq 0$ for every $j \geq 1$. The denominator ideal and its invariant

subspace commute with field extension. Indeed, after choosing a finite basis a_j/b with a common denominator b, the ideal is the simultaneous kernel of multiplication by the a_j into $K[X]/(b)$. Therefore we may take a finite parameter span of K-invariant denominators containing d after field extension. For each member of that span the regular pair (dp, dq) is pure and has marked moments

$$\Lambda_X((dq)(dp)^m) = \Lambda_X(d^{m+1})\,\Lambda_{S/M_0}(Q_0P_0^m). \tag{2}$$

Indeed, on the open of definition the S-projection gives constant rank-one moments, times d^{m+1} in the marked case. The identities extend by regularity, and $\Lambda_X \circ R_S = \Lambda_X$. The nonzero complex parameter above shows that the polynomial sequence in (2) is not eventually zero.

The same denominator argument applies to any reductive subgroup of G, using a compatible compact form of $G_{\mathbb{C}}$. We will use this observation for a rank-one group times a central torus.

4 Arithmetic specialization

Lemma 4.1. *Let $E/\mathbb{Q}$ be finitely generated. Suppose $b_m(\boldsymbol{u}) \in E[\boldsymbol{u}]$ is not eventually zero over $E(\boldsymbol{u})$ and satisfies a scalar polynomial recurrence with nonzero first coefficient for all sufficiently large integer indices. Then some specialization $\boldsymbol{u} \in E^d$ keeps infinitely many b_m nonzero.*

Proof. If there are no parameters, there is nothing to specialize. Write $A(m, \boldsymbol{u})$ for the first coefficient. Remove its content in $E[m]$ when tracking its zeros, and choose M beyond the fixed integer zeros and the recurrence cutoff. Choose $m_0 \geq M$ with $b_{m_0} \neq 0$ as a polynomial. On a line through two suitably chosen E-points the reduced coefficient has no constant root in m, and b_{m_0} is still nonzero. To see this, choose the first point with $b_{m_0} \neq 0$ and reduced coefficient a nonzero polynomial in m. Choose the second point to avoid vanishing of the reduced coefficient at the finitely many roots of the first evaluation. Each of these avoidance conditions is possible since the content was removed. Denote the line parameter by u.

Factor the reduced coefficient in $E(u)[m]$. Each linear factor has a nonconstant root $r_i(u)$. Choose a distinct rational prime p_i and embed E into $\mathbb{Q}_{p_i}$ so that $r_i(u)$ has negative valuation on a nonempty p_i-adic open set. For a finite pole α of r_i, embed $E(\alpha)$ instead, then take a small ball near the pole. A pole at infinity requires only large p_i-adic absolute value. The embeddings exist for infinitely many primes for any finitely generated field of characteristic zero. Indeed, specialize a transcendence basis to rational values where a minimal polynomial of a primitive element has no coefficient poles or multiple roots. A root of the specialized polynomial is algebraic over $\mathbb{Q}$, so it has a simple root in $\mathbb{Q}_p$ for infinitely many p, by Schur's prime-divisor theorem (see [4, Section 7]) and Hensel's lemma. Perturb the rational parameters to algebraically independent values sufficiently close in $\mathbb{Q}_p$, and lift the simple root there to get the required embedding.

Choose a rational a in all the required local opens by weak approximation. Put $b = \prod_i p_i$, take rational $c \neq 0$ sufficiently small at these primes, and choose a prime ℓ larger than the degree of A in m. Make the substitution

$$u = a + \frac{c}{1 + bt^{\ell}}.$$

For every $t \in E$, the reciprocal denominator is integral at the chosen places. For $t \neq 0$, the valuations of the two summands are 0 and $1 + \ell v_{p_i}(t)$, distinct since $v_{p_i}(t) \in \mathbb{Z}$. Thus the local conditions always hold. The rational map has degree ℓ, so $[E(t) : E(u)] = \ell$. If θ is a root of an irreducible factor of degree $e \geq 2$, the degree of $E(t, \theta)/E(u)$ is divisible by ℓ and e, and is at most ℓe. Coprimality forces equality, preserving irreducibility. If no local conditions are needed, take $u = t$. The field E is Hilbertian [12, Chapter 13]. Specialize t to preserve the degrees and

irreducibility of the nonlinear factors. Avoid poles of all coefficients and factors, zeros of the scalar factor, and zeros of b_{m_0}. Linear roots have negative valuation at one assigned prime and hence cannot be integers. The resulting first coefficient is nonzero at every integer $m \geq M$. Backward recurrence from an eventual zero tail would give $b_{m_0} = 0$, a contradiction. □

Lemma 4.2. *Let X be affine and geometrically homogeneous under a connected reductive group over a characteristic-zero field k. For $F, B \in k[X]$ with $F \neq 0$, the sequence $\Lambda_X(BF^m)$ satisfies a scalar polynomial recurrence with nonzero first coefficient at all sufficiently large integer indices.*

Proof. A finite extension providing a point of X exists by taking the residue field of a closed point. The orbit map pulls the functions and the normalized trace back to the connected group. A recurrence over the finite extension descends by expanding the coefficients in a field basis and taking a component with nonzero first coefficient. It therefore suffices to prove the assertion on the group.

The zero-dimensional case is constant. Otherwise take a nonzero left-invariant volume form ω on the group. The frame D_i of infinitesimal left translations preserves it, and $\Lambda(D_i a) = 0$. Every regular $(\dim G - 1)$-form is a sum of $a_i \iota_{D_i}\omega$, with differential $\sum_i D_i(a_i)\omega$. Thus the trace on top forms annihilates regular exact forms.

On the smooth affine open $F \neq 0$, over $k(\nu)$, use the connection $\mathrm{d} + \nu\,\mathrm{d}F/F\wedge$. Its top de Rham cohomology is finite-dimensional. Indeed, descend the data to a finitely generated subfield, embed that field with the parameter ν into $\mathbb{C}$, and apply holonomic direct-image finiteness to this integrable connection [16, Theorem 3.2.3]. On the affine open, cohomology of the complex of global forms commutes with field extension. This gives the algebraic Mellin shift for these moments [13, Section 1.3]. The classes of $B\omega, FB\omega, \ldots$ are linearly dependent. Drop any initial zero coefficients and reindex the resulting recurrence. In a relation among the forms, the primitive for the twisted exact term has bounded F-denominators and finitely many poles in ν. At sufficiently large integers $\nu = m$, multiplication by F^m turns the relation into an ordinary regular exact form. Applying the trace and clearing parameter denominators proves the claim. □

Theorem 4.3. *Over any characteristic-zero field, let X be affine and geometrically homogeneous under a connected reductive group. If a homomorphism from split SL_2 to the acting group induces a nontrivial action on X, then $K[X]$ admits a witness, including when $X(K)$ is empty.*

Proof. Choose a finitely generated field of definition for the constructions and their finite parameter spans. For central L, use the polynomial sequence $\Lambda_X(r(Z)^{3m-1})$ and the nonzero factors β_m in (1). Its recurrence follows from Lemma 4.2 with phase r^3, multiplier r^2, and an index shift. For noncentral L, use the entire product sequence in (2), which accounts for gaps in the rank-one marked sequence. It is a mixed moment on $X \times S/M_0$ with phase $d \otimes P_0$ and multiplier $d \otimes Q_0$. In each case the complex parameter from the preceding construction shows the sequence is generically not eventually zero. Apply Lemma 4.2 over the parameter field and clear denominators to obtain a polynomial recurrence. Lemma 4.1 gives one parameter value and hence one fixed pair over the field of definition. □

Proof of Theorem 1.1. The geometric components of X are the orbits of G°. If $D = [G^\circ, G^\circ]$ is active, choose a Galois orbit of components with active D-action. It gives an open-and-closed factor with finite separable constant field E/K, a geometrically integral homogeneous G°_E-space Y over E. Some root subgroup of the split derived group acts nontrivially on Y, since their images generate the derived group. Theorem 4.3 supplies a pair on Y. On this factor the trace is $r^{-1}\,\mathrm{Tr}_{E/K} \circ \Lambda_Y$, where r counts all geometric components of X. This follows by base change and

the equal-weight average of the geometric component traces. Nondegeneracy of the field-trace pairing shows that some member c of a fixed K-basis of E satisfies $\mathrm{Tr}_{E/K}(c\Lambda_Y(QP^m)) \neq 0$ at infinitely many marked orders. Extend P, cQ by zero to get a witness on X.

If D acts trivially, the geometric components are homogeneous spaces of a torus. The component average and finite-trace separation of [4, Section 7], followed by the Laurent theorem [5], prove the Mathieu–Zhao property over the algebraic closure, hence over K. For connected G, the Laurent support half-space criterion also gives geometric instability of every pure function. In the active case the marked witness lies outside the nullcone by the weight bound. □

5 Real closed fields and the nonsplit boundary

Theorem 5.1. *Let K be real closed and G connected reductive. The normalized trace on $K[G]$ has Mathieu–Zhao kernel exactly when G is a torus or G is K-anisotropic.*

For the positive compact case, a nonzero f has nonzero squared trace. One can embed a finitely generated field of definition of the group and f into $\mathbb{R}$, preserving the negative definiteness of the derived Killing form and the anisotropic central torus. Preserve the sign of the squared trace as well if it is nonzero. For the first condition preserve the signs of the principal minors. The anisotropic central torus over a real closed field is a product of norm-one tori for $K(i)/K$, so include this model isomorphism in the field of definition. On the compact real identity group $\Lambda(f^2) > 0$ by density of the real orbit and normalized Haar. This also proves the positive assertion on every affine G/H with such a connected compact acting group. The torus case follows from [5] after splitting.

We record the real specialization principle used here and below. Given finitely many strict signs on elements of a finitely generated ordered field, embed that field into $\mathbb{R}$ respecting those signs. Write the field as a finite separable extension of $\mathbb{Q}(t_1, \dots, t_d)$. Express existence of the root, the signs, nonzero denominators and the nonzero derivative by a finite formula. It holds in a real closure, hence over $\mathbb{R}$ by quantifier elimination [14]. The simple root persists on a real open set of parameters. Choosing algebraically independent values in that open gives the desired embedding. If $d = 0$, the simple real root itself suffices. In particular a finite condition with parameters in a real closed field follows if it holds at every real embedding of a finitely generated field of definition. Otherwise quantifier elimination would give finitely many signs certifying a failure, which could be preserved by such an embedding.

For the first negative case, a noncompact derived group has a split rank-one subgroup, so Theorem 4.3 applies to G. The same argument applies to any G/H on which a noncompact almost simple factor acts nontrivially. Indeed a nontrivial split cocharacter in the derived group gives a rational unipotent and hence a split rank-one homomorphism by rational Jacobson–Morozov [15]. Over a real closed field the noncompact derived case can equivalently be recognized by a positive derived Killing square. Over $\mathbb{R}$, a vector in the noncompact summand of a Cartan decomposition has self-adjoint adjoint endomorphism. A nonzero eigenvector of nonzero eigenvalue is adjoint-nilpotent by weight shift. The existence of a nonzero $\mathfrak{sl}_2$-triple is a finite condition, so the real specialization principle gives the same conclusion over K.

The other negative case of Theorem 5.1 has nontrivial compact derived group and split central direction. Here is a ground-field construction that also works over arbitrary fields.

Proposition 5.2. *Suppose G is connected reductive, it has a nontrivial K-character χ, and its derived group contains a K-subgroup S of geometric type A_1. Then there is a fixed marked counterexample on $K[G]$.*

Proof. Write q for the negative of the derived Killing form on $\mathfrak{s} = \mathrm{Lie}(S)$, with polarization $\langle\ ,\ \rangle_q$. Its restriction is nondegenerate, as one sees after splitting S by evaluating the sum of squared weights on a coroot. Let p be the Killing-orthogonal projection to $\mathfrak{s}$, choose $\xi \in \mathfrak{s}(K)$ with $q(\xi) \neq 0$, and set

$$z = p(\mathrm{Ad}(g)\xi), \quad A = q(z), \quad B = \frac{\langle \xi, z\rangle_q^2}{q(\xi)}, \quad t = \chi(g).$$

Use

$$P = t^{-1}(1+t)\{A - (1+t)^2 B\}^2, \qquad Q = t. \tag{3}$$

To check moments, use a model embedded in $\mathbb{C}$. By conjugating the compact form of S we may write $\xi = c\xi_0$ with ξ_0 compact and $c \neq 0$. This uses transitivity of the adjoint action on each nonzero quadric level. Extend that compact form to the derived group. Replacing ξ_0 by $c\xi_0$ scales P by c^4. For ξ_0, the law of z under compact derived Haar measure is rotation-invariant in the three-dimensional compact adjoint space. On each nonzero radius $B/A = w^2$ with w uniform on $[0,1]$. The central isogeny from the product of the connected center and the derived group separates the trace. The variable t is a nontrivial central character, so central averaging extracts its constant term.

Put $F_m(s) = \int_0^s (1-w^2)^{2m}\,\mathrm{d}w$. The angular average of P^m/A^{2m} is

$$t^{-m}(1+t)^{m-1}F_m(1+t).$$

Since $F_m(1+t) - F_m(1)$ is divisible by t^{2m+1}, its constant term is zero. Its constant term after multiplication by t is $F_m(1) > 0$. The remaining radial integral of A^{2m} is positive, since z is nonzero at the identity. This proves the two moment statements after rescaling and hence over K. The same pair descends to G/H if H fixes ξ and $\chi|_H = 1$. □

For a nontrivial compact derived group over a real closed field, a maximal torus in it is anisotropic. Conjugation acts by -1 on its character lattice, so an opposite pair of roots generates a rank-one subgroup over K. If the center has a split direction, there is a nontrivial K-character, and Proposition 5.2 applies. This completes Theorem 5.1.

5.1 Active long-root subgroups

Theorem 5.3. *Let X be affine and geometrically homogeneous under a connected reductive group G over a characteristic-zero field K. Suppose a split central torus T acts nontrivially. If G contains a K-subgroup S which geometrically is a long-root SL_2 in an active almost simple factor of rank at least two, then $K[X]$ admits a witness. Over a real closed field such a subgroup exists in every active compact almost simple factor of rank at least two.*

Proof. Choose a geometric simple system in which the long root β is simple. For a neighboring simple root α, the Cartan integer $\langle \alpha, \beta^\vee \rangle = -1$, since β is long. The element $\beta^\vee(-1)$ is therefore noncentral in the almost simple factor. It acts nontrivially since the factor acts nontrivially and its kernel is central. If this central element of S belonged to a principal stabilizer, it would fix the entire principal open of Lemma 3.1. Thus a finite principal stabilizer omits the unique involution in S. It has odd order and is cyclic. A proper positive-dimensional reductive principal stabilizer has a torus as identity component.

We first choose a nonzero S-invariant $t \in K[X]$ of nontrivial T-weight. In fact every irreducible module of the almost simple factor over $\overline{K}$ has a vector fixed by the long-root subgroup. To see this, pass to its simply connected cover. If there are at least two long simple roots, for each

fundamental weight choose a long simple root whose coroot pairs to zero with that weight. The associated rank-one subgroup fixes the highest vector. All long roots are conjugate. The root systems with only one long simple root are C_r and G_2, with $B_2 = C_2$. For G_2 the fundamental module for the long simple root is the adjoint module, with a fixed vector in the kernel of that root on the Cartan subalgebra. For C_r it suffices to handle the last fundamental module, the primitive exterior power of degree r of the symplectic module. With the long-root subgroup on e_1, f_1, a lowering operator for $-\epsilon_1 - \epsilon_r$ applied to $e_1 \wedge \cdots \wedge e_r$ gives, up to a scalar,

$$(e_1 \wedge f_1 - e_r \wedge f_r) \wedge e_2 \wedge \cdots \wedge e_{r-1}.$$

This vector is fixed by the long-root subgroup. For sums of fundamental weights, multiply nonzero left S-invariants in the spaces of right highest-weight matrix coefficients in the coordinate ring of the simply connected group. Their product is nonzero and lies in the space for the sum of the weights. The matrix coefficient decomposition identifies that space with the dual irreducible module, and duality preserves the existence of invariants. This proves the assertion for every irreducible. Each nonzero T-weight space in $K[X]$ therefore has S-invariants by base change, and the active split torus supplies one of nontrivial weight.

Write q for the negative Killing form on $\mathfrak{s} = \mathrm{Lie}(S)$, and choose $\xi \in \mathfrak{s}(K)$ with $q(\xi) \neq 0$. If the principal stabilizer is nontrivial, it determines a nonisotropic line in $\mathfrak{s}$. This is the Lie algebra of its torus centralizer in the cyclic case or of its identity component in the positive-dimensional case. The line is rational in the point by the graph argument of Section 3 over $\overline{K}$. Since it depends only on the stabilizer, it descends and is T-invariant. Put $A = 1$ and

$$B = \frac{\langle z, \xi \rangle_q^2}{q(z) q(\xi)}$$

for a representative z of the line. If the principal stabilizer is trivial, take a nonzero regular adjoint covariant as in Section 3, with coordinate functions of one T-weight. Divide by a nonzero S-invariant function of that weight and clear its denominator by an $S \times T$-invariant element using the compact square argument. This gives a regular T-invariant adjoint covariant $Z \neq 0$. Set $A = q(Z)$ and $B = \langle Z, \xi \rangle_q^2 / q(\xi)$.

In both cases $R_S(B^j) = A^j/(2j+1)$ on the open of definition. For a compact real axis this is the spherical average of the $2j$-th power of the direction cosine. For vectors the same averaging formula is polynomial and hence holds throughout the complexified adjoint space. For lines it follows on their homogeneous space from the compact orbit. Define

$$P = d\, t^{-1}(1+t)\big(A - (1+t)^2 B\big)^2, \qquad Q = t,$$

where d is an $S \times T$-invariant denominator. The expressions to be cleared are finite under $S \times T$: the axis terms are matrix coefficients of the rank-one group, and t^{-1} is a weight vector. In the covariant case a denominator for t^{-1} alone suffices as Z varies. Taking the constant term in t, after the S-average, uses the same integral as in (3). For all $m \geq 1$,

$$\Lambda_X(P^m) = 0, \qquad \Lambda_X(tP^m) = F_m(1)\Lambda_X((dA^2)^m).$$

In particular these identities are valid for finite parameter spans of invariant denominators and, when needed, covariants.

Over $\mathbb{C}$, the denominator can be chosen nonnegative and positive somewhere on the compact orbit by the compact square construction. In the covariant case the regular T-invariant adjoint

maps are stable under compact conjugations, so choose a nonzero map with compact real values on that orbit. The squared radius is then positive on a dense open subset of the compact orbit. Consequently $\Lambda_X((dA^2)^m)$ is positive for every positive order at those complex parameters. The coaction equations and the common denominator ideal commute with base change. Choose finite parameter spans over a finitely generated field of definition whose complexifications contain these choices. Lemmas 4.2 and 4.1 give one ground-field parameter with infinitely many nonzero marked moments.

Over a real closed field, a maximal torus of a compact factor is anisotropic and conjugation acts on its roots by inversion. The subgroup for the opposite long-root pair descends to K. This proves the last assertion. □

5.2 Compact adjoint torsors

Theorem 5.4. *Let K be real closed and C a connected adjoint semisimple group of compact type. For a C-torsor X, the normalized trace on $K[X]$ has Mathieu–Zhao kernel if and only if $X(K) \neq \varnothing$.*

Proof. If X has a point, identify it with C and use strict square positivity. In general write X as a right torsor. After choosing a point over $K(i)$, the descent involution has the form $g \mapsto c\,\sigma(g)$, with $c\,\sigma(c) = 1$ and σ the original conjugation of C. The commuting group of left translations descends to the inner form C_c with conjugation $\mathrm{Int}(c)\sigma$. Its action is geometrically transitive. The two normalized traces coincide: each commuting action preserves the other's normalized trace by uniqueness, and each action has only constant invariants.

If C_c is compact, its conjugation is conjugate to σ by an inner automorphism $\mathrm{Int}(a)$. Over $\mathbb{R}$ this is conjugacy of compact forms [10, Theorem 3.8]. Over K it follows by the real specialization principle above. Negative definiteness of the Killing forms and the existence of a satisfying the conjugacy identity are finite algebraic conditions in a presentation of the group and its involutions. Adjointness then gives $c = a\sigma(a)^{-1}$, so the point with coordinate a descends to K. Thus if $X(K)$ is empty, C_c has positive split rank. Its action gives a witness by Theorem 4.3, and the same witness applies to the trace for C. □

The point criterion also holds for compact semisimple groups with odd-order center. A cocycle that is trivial in the adjoint cohomology set comes from the center. Conjugation fixes every geometric point of the center, since the center lies in a compact maximal torus. A central cocycle therefore has order dividing two and is trivial when the center has odd order. A nontrivial adjoint class gives a noncompact commuting inner form by the proof above.

For example, let D be a quaternion division algebra over any characteristic-zero field and $X = \mathrm{Isom}_{K\text{-alg}}(D, M_2(K))$. A nonzero reduced-trace-zero $\xi \in D$ gives the split adjoint covariant $Z(\phi) = \phi(\xi)$. Its radius is $r(Z) = 4\xi^2 \in K^\times$. Thus (1) gives an explicit pair with nonzero marked moments at every positive order. By the commuting-action argument this is also a witness for the action of $\mathrm{PGL}_1(D)$.

5.3 Open ground-field cases

The split obstruction does not classify individual anisotropic nonsplit spaces over general fields. A basic arithmetic example is the norm-one group of the quaternion division algebra $(2, 3)_{\mathbb{Q}}$, which has positive real split rank but no $\mathbb{Q}$-split rank-one subgroup.

Even over a real closed field, the connected group classification and the adjoint torsor theorem leave general homogeneous spaces open. Two test cases are

$$(\mathrm{SU}_2/2O) \times \mathbb{G}_m, \qquad (\mathrm{SU}_2/2I) \times \mathbb{G}_m,$$

where SU_2 denotes the compact algebraic form. A nontrivial finite cyclic stabilizer determines an axis and leads to the angular identity. The binary octahedral and icosahedral stabilizers have no invariant axis. The complex witnesses for their split quotients do not give witnesses over the real ground field.

6 Compact homogeneous spaces

Theorem 6.1. *Let C be a compact Hausdorff group and L a closed subgroup. On the complex representative functions of C/L, normalized invariant integration has Mathieu–Zhao kernel exactly when $\overline{[C^\circ, C^\circ]} \subseteq L$. In particular on C it holds exactly when C° is abelian.*

Proof. In the compact Lie case complexify the pair. A faithful unitary model has reductive Zariski closure, and its regular functions restrict to all representative functions. The matrix-entry algebra is conjugation-stable on a unitary group, so it is dense by Stone–Weierstrass. Projecting onto each finite-dimensional Peter–Weyl isotype proves the asserted equality. The Zariski closure of L meets the compact Lie group in L: average matrix-entry polynomials over L to separate distinct right cosets. Right invariance and normalized integration therefore agree with the complex homogeneous-space constructions. The identity component of a compact Lie group is Zariski dense in the corresponding algebraic identity component, and the same holds for its derived subgroup. Apply the complex classification.

For a general compact Hausdorff group, any two representative functions factor through a single compact Lie quotient J, using the direct sum of their finite-dimensional representations. If the functions are right L-invariant, they descend to J/L_J with L_J the image of L. The image of C° is J° because the quotient by that image is a quotient of the profinite group C/C°. Haar measure pushes forward, so the containment criterion suffices by the compact Lie case. Conversely, an element of $[C^\circ, C^\circ]$ outside L can be separated from L by a finite sum of representations, using point separation and compactness of L. The resulting compact Lie quotient provides a counterexample which pulls back [17]. □

If one defines instability using the Zariski closure of the compact orbit in the finite-dimensional complex orbit span, the corresponding nullcone criterion holds precisely when the lattice criterion of [4, Section 7] holds on every compact Lie quotient pair. Both the moments and the orbit of an individual representative function factor through such a quotient. In particular for connected C, the same containment criterion governs both properties.

7 Universal Reynolds projections and toral nullcones

Theorem 7.1. *For a linear algebraic group G with reductive identity component over K, the following are equivalent.* (i) *G° is a torus.* (ii) *The Reynolds projection has Mathieu–Zhao kernel on every commutative rational G-algebra.* (iii) *For every such algebra and every f with zero Reynolds projection of each positive power, each fixed geometric isotypic component of f^m vanishes eventually.*

Proof. Assume $G^\circ = T$ is a torus. Work over the algebraic closure in the finitely generated stable subalgebra B generated by a finite-dimensional submodule containing f. Choose component representatives $g_1 = 1, \dots, g_s$, and write L_i for the Laurent polynomial recording the T-weights of $g_i f$ with coefficients in B. Since the component quotient acts by averaging the T-projection, $\sum_i \mathrm{CT}\, L_i^m = 0$ for every $m \geq 1$.

Take a primary decomposition $0 = \bigcap_j \mathfrak{q}_j$ in B. Put $\mathfrak{p}_j = \sqrt{\mathfrak{q}_j}$, and choose e_j with $\mathfrak{p}_j^{e_j} \subseteq \mathfrak{q}_j$. In the fraction field of $B/\mathfrak{p}_j$, finite-trace separation [4, Lemma 7.1] gives pure constant terms for each Laurent polynomial separately. Write $L_1 = U + V$ by putting the coefficients in $\mathfrak{p}_j$ in V. If $U \neq 0$, the Laurent theorem gives a functional strictly positive on its support. Terms of $(U+V)^m$ with fewer than e_j factors of V then eventually cannot reach a prescribed weight. All other terms have coefficients in $\mathfrak{q}_j$. The case $U = 0$ is immediate. Thus each prescribed weight of f^m is eventually zero by the intersection of the primary components. This proves both conclusions: each fixed multiplier and each fixed geometric simple module has only finitely many T-weights.

If G° is non-toral, Theorem 1.1 after algebraic closure gives a pair on the group that is defined over a finite extension E/K. Regard $E[G_E]$ as a rational G-algebra over K. Its invariants are E, and its Reynolds projection is the normalized trace with values in E. The pair disproves the universal kernel assertion. It also disproves eventual isotypic vanishing, since the trace of a product pairs only dual types and a fixed multiplier has finitely many types. □

Finally suppose $D = [G^\circ, G^\circ]$ acts trivially on X, with G° possibly nonsplit. Let $T = G^\circ/D$, $M = X^*(T_{\overline{K}})$, and let Γ be a finite Galois group acting on M and the geometric components. For a representative i of each component orbit α, let Γ_i be its stabilizer and $M_i \subset M$ the weight lattice of that component. Put $V_i = (M_i \otimes \mathbb{Q})^{\Gamma_i}$, $V = (M \otimes \mathbb{Q})^{\Gamma}$. The ground-field pure set equals the geometric nullcone on $K[X]$ exactly when

$$\bigoplus_\alpha V_i \longrightarrow V, \qquad (v_i) \longmapsto \sum_\alpha |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \gamma v_i \tag{4}$$

is injective. In fact purity separates componentwise over $\overline{K}$. The convex hull of each component support avoids zero, and this remains true for its stabilizer-averaged weights because the support is Γ_i-stable. If (4) is injective, strictly positive functionals on these averaged supports combine into one invariant functional on $M \otimes \mathbb{Q}$. It is positive on the original weights and yields a contracting cocharacter after clearing denominators and lifting through the connected center. Conversely, a nonzero integral relation in (4) gives one nonzero weight on each of the participating Galois orbits of components. The corresponding direct sum of character lines descends, and a nonzero K-vector on each participating orbit is nonzero on every geometric component of that orbit. This produces a pure function whose global support has zero in its convex hull. It is therefore outside the geometric nullcone. For split T this uses one embedded character lattice per Galois orbit.

Two ground-field transfers

Let E/K be finite separable. A fixed pair on $Y = G/H$ over E gives a fixed pair on its Weil restriction to K. After scalar extension to $\overline{K}$, that restriction is the product of the conjugates Y^σ. The products of the conjugate phases and multipliers are defined over K, and their marked moment is $N_{E/K}(\Lambda_Y(QP^m))$. Purity and the precise nonzero marked orders therefore descend to this restricted space.

Disconnected forms can introduce an additional field issue. For a real closed K, let C be the compact quaternion norm-one group and take $G = C \times \mu_3$. The two nonreal components are

represented by the $K(i)[C_{K(i)}]$ factor of $K[G]$. Use on it a complex rank-one pair with rational marked moments and extend by zero on the other factor. The component average gives a bad pair over K with marked moments scaled by $2/3$. Thus the connected group classification over real closed fields does not extend by inspecting the real identity component alone.

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