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The Mathieu property for quadratic complements

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Abstract

For a nondegenerate quadratic form q of dimension $N \geq 1$ over a characteristic-zero field, the kernel of the normalized rotation and scaling trace on $\{q \neq 0\}$ is Mathieu–Zhao exactly when $N \leq 2$. In every higher dimension a single witness attains the minimum pole order, numerator degree and polynomial multiplier degree, namely $1, 3, 1$. A second witness gives sharp degree cutoffs for its detecting multipliers. For an arbitrary regular phase, testing finitely many multipliers at one computable order decides eventual vanishing against every fixed multiplier. For phases affine in the angular variable, a Laurent support criterion suffices. Over $\mathbb{R}$, the kernel of the special orthogonal Reynolds projection is Mathieu–Zhao exactly when q is definite or $N \leq 2$.

1 A simple-pole counterexample in dimension three

Take $q = x^2 + y^2 + z^2$ over $\mathbb{Q}$ and let Λ_q be the normalized trace on $\mathbb{Q}[x, y, z, q^{-1}]$ invariant under rotations and scalar multiplication. The function

$$p = \frac{(1 + x)(1 + 2x - y^2 - z^2)}{q}$$

satisfies, for every $m \geq 1$,

$$\Lambda_q(p^m) = 0, \qquad \Lambda_q(xp^m) = \frac{2^m}{m + 1} \neq 0. \tag{1}$$

The witness has a single pole along $q = 0$, and the same linear multiplier x works at every order. Proposition 6 gives the calculation over any characteristic-zero field.

For a nondegenerate form q on a K-vector space V, write

$$X_q = \{q \neq 0\}, \qquad \mathcal{O}(X_q) = K[V, q^{-1}], \qquad G_q = \mathrm{SO}(q) \times \mathbb{G}_m.$$

The two factors act by isometries and scalar multiplication. The trace Λ_q is the unique G_q-invariant linear functional taking 1 to 1. Call F *pure* if $\Lambda_q(F^m) = 0$ for every $m \geq 1$. The kernel is a Mathieu–Zhao subspace if, for every pure F and every fixed multiplier $h \in \mathcal{O}(X_q)$, the marked moments $\Lambda_q(hF^m)$ vanish for all sufficiently large m.

Theorem 1. *Let K have characteristic zero and q be nondegenerate of dimension $N \geq 1$. Then*

$$\boxed{\ker \Lambda_q \text{ is a Mathieu–Zhao subspace} \iff N \leq 2.}$$

For $N \geq 3$ there is an explicit pure $F \in \mathcal{O}(X_q)$ for which $\Lambda_q(q^{N-2}F^m) \neq 0$ for every $m \geq 1$. The function F is not a unit times a proper power over an algebraic closure.

In dimensions one and two the split trace is a Laurent constant term. For $N \geq 3$, take an orthogonal splitting $q = A + B$ of ranks 2 and $M = N - 2$. The ratio B/q has moments $M/(M+2j)$. Scalar invariance then reduces the mixed trace to a Laurent coefficient of a polynomial integral. At the endpoint of that integral, the pure moment selects a coefficient above the degree of a polynomial, and multiplication by q^M selects the leading coefficient. The same calculation identifies the least polynomial degree needed to detect a failure for this F.

Sections 2–5 prove the dimension cutoff over K by direct trace calculations and give explicit detecting multipliers. The complex cutoff and the existence of fixed detecting multipliers also follow from Long's spherical classification [5, Thm. 1.1]. For that comparison, write $q = 2xy + q_0(z)$ for $N \geq 3$, with b_0 the symmetric bilinear form of q_0. The isometries

$$(x, y, z) \longmapsto \big(x,\, y - b_0(z, u) - xq_0(u)/2,\, z + xu\big) \qquad (u \in \mathbb{C}^{N-2})$$

form a unipotent group normalized by the torus scaling x, y inversely. Together with the scalar torus they form a connected solvable subgroup of G_q. On $x \neq 0$ they can set $z = 0$, $q = 1$, and $x = 1$, so a Borel subgroup has an open orbit in X_q. The stabilizer of a norm-one vector is reductive. The scalar factor in the stabilizer is ± 1, and its identity component is the special orthogonal group of the vector's orthogonal complement. Since the derived subgroup contains the rotations, it acts nontrivially on X_q for $N \geq 3$. For $N \leq 2$, G_q is a torus. The same complex cutoff follows from the general criterion in [9, Thm. 1.1].

The endpoint substitution here adapts Long's calculation on $\mathrm{SL}_2/N(T)$ [5, Prop. 5.1]. The rotation recurrence supplies the moments over the original characteristic-zero field. It gives explicit witnesses there, as well as the exact cutoff for their polynomial multipliers. Section 5 gives a simple-pole witness with a linear multiplier in every dimension $N \geq 3$.

2 The trace and the two low dimensions

The action on X_q is transitive over an algebraic closure. Scalar multiplication changes a nonzero value of q to any other such value. In dimension at least two, an isometry taking one vector to another of the same nonzero norm exists by extending an isometry of their nondegenerate lines. Its determinant can be corrected by a reflection in the orthogonal complement of the target vector. In dimension one the scalar action alone is transitive.

The coordinate ring is a locally finite rational representation of the connected reductive group G_q. Complete reducibility [6, §12l and Thm. 22.42] gives the projection onto its trivial isotypic part. This part is $K \cdot 1$ by geometric transitivity, which proves existence and uniqueness of Λ_q. The trace commutes with field extension. Indeed, the extended trace is still invariant and normalized, and the same uniqueness applies over the larger field. Invariance also implies $\Lambda_q(Df) = 0$ for every infinitesimal action D.

We use the Laurent constant-term theorem of Duistermaat and van der Kallen [1, §§1, 3]. If a nonzero complex Laurent polynomial has zero constant term in every positive power, then the convex hull of its support excludes the origin. This implication holds over any characteristic-zero field. The finitely many coefficients generate a finitely generated extension of $\mathbb{Q}$, which embeds in $\mathbb{C}$ by choosing images for a transcendence basis and extending algebraically. The embedding preserves the support and all the constant-term identities. A linear functional strictly positive on that support gives a lower bound $m\delta$, for some $\delta > 0$, on the weights of monomials in the m-th power. If the least monomial weight of a fixed nonzero multiplier is b, the product has weights at least $b + m\delta$. Hence its constant term eventually vanishes. If either factor is zero the conclusion is immediate.

When $N = 1$, write $q = ax^2$. The ring is $K[x, x^{-1}]$, and the scalar action gives $\Lambda_q = \mathrm{CT}_x$ since each x^i has weight $-i$ under pullback. When $N = 2$, pass to a splitting field L/K of degree at most two and write $q = auv$ in linear coordinates u, v. Both u and v are then units on the complement. The rotation and scalar tori act on $u^i v^j$ with character $t^{i-j}s^{i+j}$, which is trivial exactly for $i = j = 0$. Hence $\Lambda_q \otimes_K L = \mathrm{CT}_{u,v}$ on $L[u^{\pm 1}, v^{\pm 1}]$. For f, h defined over K, pure vanishing extends to L, and the eventual mixed vanishing over L descends to K by injectivity. This proves the positive direction of Theorem 1.

3 A rank-two splitting and its moments

Assume $N \geq 3$ and set $M = N - 2$. Diagonalize q over K and write

$$q = A + B, \qquad A = a_1 x_1^2 + a_2 x_2^2, \qquad B = \sum_{i=1}^{M} b_i z_i^2, \qquad S = \frac{B}{q}.$$

All a_α, b_i are nonzero. To compute the moments of S, use the infinitesimal rotations

$$D_{\alpha i} = b_i z_i \frac{\partial}{\partial x_\alpha} - a_\alpha x_\alpha \frac{\partial}{\partial z_i}, \qquad \alpha = 1, 2, \quad 1 \leq i \leq M.$$

They preserve q, and hence act on its localization. For $j \geq 1$, the product rule gives

$$\sum_{\alpha,i} D_{\alpha i} \left(S^{j-1} \frac{x_\alpha z_i}{q} \right) = 2S^j - (M + 2j - 2)(1 - S)S^{j-1}. \tag{2}$$

Differentiating $x_\alpha z_i$ contributes $2S^j - M(1 - S)S^{j-1}$. For $j > 1$, differentiating S^{j-1} contributes $-2(j-1)ABS^{j-2}/q^2 = -2(j-1)(1-S)S^{j-1}$. That term is absent when $j = 1$.

Applying Λ_q to (2) yields

$$(M + 2j)\Lambda_q(S^j) = (M + 2j - 2)\Lambda_q(S^{j-1}), \qquad \Lambda_q(1) = 1.$$

It follows that $\Lambda_q(S^j) = M/(M + 2j)$ for all $j \geq 0$, or equivalently

$$\Lambda_q(g(S)) = M \int_0^1 t^{M-1} g(t^2)\, dt \qquad (g \in K[s]). \tag{3}$$

For coefficients in K, the integral denotes termwise polynomial integration, defined by $\int_0^1 t^r\, dt = 1/(r+1)$ and K-linearity. Also, $q^\ell g(S)$ has scalar pullback weight -2ℓ, so its trace is zero whenever $\ell \neq 0$. This allows us to compute the trace of any Laurent expression in q with polynomial coefficients in S by applying (3) to its constant coefficient.

4 The endpoint calculation and the witness

Put $W = 1 - (1 + q)^2 S$. For integers $\alpha, \gamma \geq 1$ and $\beta \geq 0$, define

$$F_{\alpha,\beta,\gamma} = q^{-\alpha}(1 + q)^\beta W^\gamma, \qquad C_{k,j} = M \int_0^1 w^{M+2j-1}(1 - w^2)^k\, dw \quad (k, j \geq 0).$$

The constants $C_{k,j}$ are positive rationals.

Proposition 2. *For $m \geq 1$, $j \geq 0$, and $\ell \geq (\alpha-\gamma)m$,*

$$\Lambda_q(q^\ell S^j F^m_{\alpha,\beta,\gamma}) = C_{\gamma m,j}\binom{\beta m - M - 2j}{\alpha m - \ell}. \tag{4}$$

Here $\binom{u}{v} = [U^v](1+U)^u$, with value zero for $v<0$. At $\ell = (\alpha-\gamma)m-1$, the left-hand side minus the binomial expression is $M(-2)^{\gamma m}/(\gamma m+1)$.

Proof. Let U be the formal variable for q, and set $T = 1+U$, $k = \gamma m$, and $n = \alpha m - \ell$. Scalar weights and (3) give

$$\begin{aligned}\Lambda_q(q^\ell S^j F^m_{\alpha,\beta,\gamma}) &= [U^n]\, T^{\beta m} M \int_0^1 t^{M+2j-1}(1-T^2t^2)^k\, dt \\ &= [U^n]\, T^{\beta m - M - 2j} J(T), \\ J(T) &= M\int_0^T w^{M+2j-1}(1-w^2)^k\, dw.\end{aligned}$$

The substitution $w = Tt$ is valid term by term. The derivative $J'(T)$ has initial term $M(-2)^k(T-1)^k$ at $T=1$. Thus $J(1+U)-J(1)$ starts with $M(-2)^kU^{k+1}/(k+1)$. The factor $T^{\beta m-M-2j}$ has constant coefficient 1 at $U=0$, so the error first appears at $n = k+1$. For $n \leq k$ the retained constant $J(1) = C_{k,j}$ gives (4). □

Take

$$F = F_{M,M,M+1} = \left(\frac{1+q}{q}\right)^M W^{M+1}. \tag{5}$$

With $j=0$, the upper index in (4) is $Mm - M$. The choices $\ell = 0, M$ give, respectively,

$$\Lambda_q(F^m) = 0, \qquad \Lambda_q(q^M F^m) = C_{(M+1)m,0}. \tag{6}$$

For the double-pole ternary phase $F_0 = F_{1,1,1} = (1+q)(q-(1+q)^2B)/q^2$, the same calculation gives

$$\Lambda_q(F_0^m) = 0, \qquad \Lambda_q(qF_0^m) = \int_0^1 (1-w^2)^m\, dw \neq 0. \tag{7}$$

To check the power assertion, work over an algebraic closure. Along the punctured x_1-axis, $W=1$, and $1+q$ has a simple zero at a point with $q=-1$. The restriction of $F_{\alpha,\beta,\gamma}$ there has vanishing order β. At a point with $q=1$ and $B=1/4$, $W=0$ and $dW = -4\,dB$. One can fix $A = 3/4$ and vary just z_1 through a nonzero root of $b_1 z_1^2 = 1/4$, so W has a simple zero along that line. The restriction of $F_{\alpha,\beta,\gamma}$ has order γ there. A unit restricts to a unit on both lines where $q \neq 0$. Any proper power up to a unit therefore has exponent dividing both β and γ. For F these are $M, M+1$, and for F_0 they are $1,1$. This completes Theorem 1.

The divisor $q=0$ is geometrically irreducible since a reducible quadratic has rank at most two. The numerator of W is $-B$ modulo q, so W has a simple pole there. The pole order of $F_{\alpha,\beta,\gamma}$ is $\alpha+\gamma$. This gives $2M+1$ for F and two for the ternary F_0.

Corollary 3. *For quadratic complements the Mathieu–Zhao property of the trace kernel is invariant under every field extension. When it fails, the same phase and multiplier witness the failure over the extension.*

Proof. Dimension and nondegeneracy are unchanged by extension. The trace commutes with scalar extension and the marked moments in (6) are nonzero rational numbers. □

5 The least degree of a detecting multiplier

Proposition 4. *For a phase G invariant under the block isometries $O(A) \times O(B)$, every regular multiplier h has a block average $\pi h \in K[q, q^{-1}, S]$ such that*

$$\Lambda_q(hG^m) = \Lambda_q((\pi h)G^m) \qquad (m \geq 1). \tag{8}$$

If h is polynomial of degree at most D, the average is a linear combination of $q^\ell S^j = q^{\ell-j}B^j$ with $0 \leq j \leq \ell \leq \lfloor D/2 \rfloor$.

Proof. Block isometries preserve Λ_q by uniqueness, since they normalize the rotations and commute with scalars. For a polynomial h, take its Reynolds projection in the space of polynomials of degree at most $\deg h$. Complete reducibility for the identity component, followed by averaging over the finite component group, gives this projection [6, Cor. 22.43]. The linear functional $h \mapsto \Lambda_q(hG^m)$ is block invariant, which proves (8). For $h = q^{-r}p$ with p polynomial, use the same projection on q^{-r} times the space of polynomials of degree at most $\deg p$.

The polynomial invariants of the block group are $K[A, B]$. Over an algebraic closure choose unit vectors v_A, v_B in the two blocks. Every pair on $AB \neq 0$ is equivalent under block isometries to (sv_A, tv_B), with $s^2 = A$, $t^2 = B$. An invariant polynomial restricts to an even polynomial in each of s, t, by the sign symmetries. Thus it is a polynomial in A, B on a dense locus. Algebraic independence of A, B gives descent of its coefficients to K. The degree filtration is preserved, and rewriting $A = q - B$ gives the stated monomials. □

Theorem 5. *For the phase F in* (5) *and the double-pole ternary phase F_0, the least degree of a polynomial multiplier detecting infinitely many orders is $2M$. For every integer $0 \leq D < 2M$, the least positive m_0 for which all polynomial multipliers of degree at most D vanish at every order $m \geq m_0$ is*

$$m_0 = 1 + \left\lceil \frac{2\lfloor D/2 \rfloor}{M} \right\rceil.$$

Proof. Set $J = \lfloor D/2 \rfloor$. By (8) it suffices to consider $q^\ell S^j$ with $0 \leq j \leq \ell \leq J$. In (4), the upper index $Mm - M - 2j$ is nonnegative for $m \geq m_0$. The lower index $Mm - \ell$ exceeds it by $M + 2j - \ell > 0$. This proves the cutoff. If $J > 0$, take $h = B^J = q^J S^J$ at $m = \lceil 2J/M \rceil$. The upper index is strictly negative and the lower one is positive, so the moment is nonzero. For $J = 0$, the claimed cutoff is already 1. Finally q^M has degree $2M$ and detects all orders by (6), or by (7) for F_0. □

For example, for the uniform phase in dimension five, a multiplier needs degree at least six to detect infinitely many orders. All polynomial multipliers of degree at most five vanish from order three, and B^2 detects the second order.

Proposition 6 (A simple pole and a linear multiplier)**.** *Let $N \geq 3$. After scaling q by a nonzero constant, write $q = x^2 + A + B$ with orthogonal summands and* $\operatorname{rank} A = 2$. *The function*

$$p = \frac{(1+x)(1+2x-A)}{q} \tag{9}$$

has a simple pole and is geometrically nonpower up to units. For every $m \geq 1$,

$$\Lambda_q(p^m) = 0, \qquad \Lambda_q(xp^m) = \frac{2^m(3/2)_m}{(m+1)(N/2)_m} \neq 0,$$

where $(a)_m$ is the rising factorial. Pole order one and multiplier degree one are both minimal among witnesses.

Proof. Scaling q leaves the action and trace unchanged. For $k, j \geq 0$,

$$\Lambda_q\left(\frac{x^{2k}A^j}{q^{k+j}}\right) = \frac{(1/2)_k\, j!}{(N/2)_{k+j}}.$$

Indeed over $\overline{K}$ one can scale the coordinates in A, B to reduce to the standard form with x fixed. By trace compatibility, its moments can be computed on the real unit sphere and transferred through the standard form over $\mathbb{Q}$. A Gaussian with density proportional to e^{-q} gives the independent moments $(1/2)_k$ for x^{2k} and $j!$ for the j-th power of a two-coordinate squared norm. Polar integration divides their product by the radial moment $(N/2)_{k+j}$.

For $\ell = 0, 1$, scalar averaging keeps total degree $2m$ in $x^\ell(1+x)^m(1+2x-A)^m$. Expand the last factor in A, with k the exponent of $1+2x$. By the moments above, $\Lambda_q(x^\ell p^m)$ is

$$\frac{(-1)^m m!}{(N/2)_m}\ \mathrm{CT}_x\left[x^\ell(1+x)^m \sum_{k=0}^{m}(-1)^k\binom{2k}{k}\left(\frac{1+2x}{4x^2}\right)^k\right].$$

The infinite binomial series at $x = \infty$ simplifies to

$$\sum_{k\geq 0}(-1)^k\binom{2k}{k}\left(\frac{1+2x}{4x^2}\right)^k = \left(1+\frac{1+2x}{x^2}\right)^{-1/2} = \frac{x}{x+1}.$$

Multiplying this by $x^\ell(1+x)^m$ gives $x^{\ell+1}(x+1)^{m-1}$, whose constant coefficient is zero. For $\ell = 0$, all omitted terms have negative degree. For $\ell = 1$, only the term $k = m+1$ contributes to the constant coefficient of the omitted tail. Its coefficient is $(-1)^{m+1}2^{-(m+1)}\binom{2m+2}{m+1}$. Subtracting it gives the formula.

The numerator has value 1 at the origin and q is geometrically irreducible, so the pole has order one. On the x-axis at $x = -1$ the numerator has a simple zero with $q \neq 0$. This excludes a proper power up to a unit. A pure polynomial phase f has zero constant term. Thus if $h = g/q^s$ with g polynomial, scalar invariance kills $\Lambda_q(hf^m)$ once $m > 2s$. A constant multiplier gives zero at every order by purity. This proves the two lower bounds. □

The minimum numerator degree

Proposition 7 (Quadratic numerators)**.** *Let $f = P/q$ with $\deg P \leq 2$, and write the quadratic part as $P_2(v) = b(v, Tv)$, where b is the symmetric bilinear form of q and T is self-adjoint. Then f is pure if and only if T is nilpotent. The first N pure moments suffice to test this condition. Every such pure f is contracted by a one-parameter subgroup over K. Consequently, the least numerator degree of any witness P/q^s is three for $N \geq 3$. The phase* (9) *attains it together with the other two minima.*

Proof. The scalar weight of P_2/q is zero and all lower terms have positive weight under pullback. Thus $\Lambda_q(f^m) = \Lambda_q((P_2/q)^m)$. The Gaussian quadratic-form calculation [3, Thms. 1–2], with the radial moments removed, gives

$$\sum_{m\geq 0}\frac{(N/2)_m}{m!}\Lambda_q\big((P_2/q)^m\big)t^m = \det(I - tT)^{-1/2}.$$

For the standard form over $\mathbb{R}$, this follows by diagonalizing T and integrating the independent Gaussian coordinates. Each coefficient is a polynomial over $\mathbb{Q}$ in the entries of a symmetric T, so the identity holds for every standard form after scalar extension. Splitting q over $\overline{K}$ and using trace

compatibility proves it over K. Since the determinant is a polynomial of degree at most N, vanishing of the first N moments forces it to be 1. This is equivalent to nilpotence.

For a nilpotent T, choose a longest chain $v, Tv, \dots, T^{r-1}v$ with $c_{r-1} \neq 0$, where $c_i = b(v, T^i v)$. Such a vector exists since $b(v, T^{r-1}v)$ is a nonzero quadratic form. Set $d(z) = \sum_{i=0}^{r-1}(c_{r-1-i}/c_{r-1})z^i$ and choose $p(z) = d(z)^{-1/2} \bmod z^r$. Replacing v by $p(T)v$ makes its chain pairings antidiagonal, since the new c_i is $c_{r-1}[z^{r-1-i}]d(z)p(z)^2$. This block is nondegenerate, so its orthogonal complement is T-stable and the construction repeats there. Assigning weight $2i - r + 1$ to $T^i v$ in each normalized block defines a cocharacter in $\mathrm{SO}(q)$ and gives T weight two under conjugation. Combine it with scalar scaling of sufficiently large positive weight to contract the lower terms of f as well. The resulting positive weights force eventual marked vanishing for each fixed multiplier.

A pure polynomial phase contracts by scalar scaling. If $s \geq 2$ and $\deg P \leq 2$, all scalar pullback weights of P/q^s are positive. Thus numerators of degree at most two give no witness, and Proposition 6 gives degree three. □

The bound of N moments is sharp across forms. On a rational basis $e_0, \dots, e_{N-1}$ with $b(e_i, e_j) = 1$ if $i + j = N - 1$ and zero otherwise, the cyclic shift $Te_i = e_{i+1 \bmod N}$ is self-adjoint and has $\det(I - tT) = 1 - t^N$. The first $N - 1$ moments vanish and the N-th is nonzero.

6 Finite tests for eventual marked vanishing

Even above the degree threshold, terms of a multiplier can cancel. The following test accounts for those cancellations.

Proposition 8. *Let F be as in (5), put $\gamma = M + 1$, and write $\pi h = \sum_{\ell,j} c_{\ell,j} q^\ell S^j$ for a fixed regular multiplier h. Put $a = M/2$. For all sufficiently large m,*

$$\Lambda_q(hF^m) = C_{\gamma m,0} R_h(m), \qquad R_h(X) = \sum_{\ell \geq M+2j} c_{\ell,j} \frac{(a)_j}{(a + \gamma X + 1)_j} \binom{MX - M - 2j}{\ell - M - 2j}.$$

Here $(u)_j$ is the rising factorial, with $(u)_0 = 1$, and the binomial in R_h is a polynomial in X. The marked moments eventually vanish exactly when $R_h = 0$ as a rational function. Otherwise they are nonzero at every sufficiently large order.

Proof. The finite support of πh lies in the range of (4) once m is large. The upper indices are then nonnegative. Terms with $\ell < M + 2j$ vanish, and for the other terms one can use the complementary binomial index $\ell - M - 2j$. Integration of the derivative of $w^{M+2j}(1 - w^2)^{k+1}$ gives

$$(M + 2j)C_{k,j} = (M + 2j + 2k + 2)C_{k,j+1}.$$

Consequently $C_{k,j}/C_{k,0} = (a)_j/(a + k + 1)_j$. This proves the formula. The constants $C_{\gamma m,0}$ are nonzero and a nonzero rational function has only finitely many zeros. □

The block average needed in the test can be found by linear algebra. On q^{-r} times the polynomials of degree at most D, where r clears the denominator of h, take the quotient by the images of the block rotation derivations and of one coordinate reflection minus the identity in each block. The rotation images remove the nontrivial simple summands for the identity component. The two reflection differences remove the remaining nontrivial components of the finite group. The unique invariant representative in the quotient is πh. Clearing denominators in R_h then gives a polynomial identity test over K.

For the uniform ternary phase $F = F_{1,1,2}$, the multiplier $4q^4S + 15q^3S - q$ has a nonzero first moment, yet its marked moments vanish for all $m \geq 3$. Its rational test is $(4(X-3)+15)/(4X+3)-1 = 0$, with the cancellation at $m = 3$ given directly by (4).

One order suffices for all multipliers

Theorem 9. *Let $f = p/q^a$ be nonzero and regular on X_q, with $a \geq 0$, and let D be the degree of the squarefree part of qp. There are regular multipliers $h_1, \ldots, h_r$, with $r \leq D^N$, and an integer $m_0 \geq 1$ such that for every $m \geq m_0$, eventual vanishing of $\Lambda_q(hf^j)$ for every fixed regular h is equivalent to*

$$\Lambda_q(h_i f^m) = 0 \qquad (1 \leq i \leq r).$$

For each fixed h, the marked sequence $\Lambda_q(hf^m)$ satisfies a polynomial recurrence of order at most r for all sufficiently large m. The multipliers, test order and recurrences are computable over finitely generated coefficient fields presented over $\mathbb{Q}$. Purity is decidable by the recurrence for $h = 1$.

Proof. Work over the coefficient field K_0 generated by q, f over $\mathbb{Q}$, adjoining the coefficients of h for a fixed-pair calculation if needed. For the all-multiplier test, scalar compatibility and linearity suffice to pass to K. Extension of coefficients also preserves r, since H below is a quotient of vector spaces defined by linear operators.

Integration by parts from invariance. In diagonal coordinates write $q = \sum_i a_i v_i^2$. Put $E = \sum_j v_j \partial_j$ and $R_{ji} = a_j v_j \partial_i - a_i v_i \partial_j$. Scalar and rotation invariance give $\Lambda_q(Eg) = \Lambda_q(R_{ji} g) = 0$. For every regular g,

$$\partial_i g - \frac{N}{2} \frac{\partial_i q}{q} g = \sum_j R_{ji}(v_j g/q) + E(a_i v_i g/q).$$

Thus the trace annihilates the left side. The identity follows by expanding both terms and using $R_{ji} q = 0$ and $Eq = 2q$.

The exponent shift. Let $L = K_0(\nu)$, let Δ be the squarefree part of qp, and set

$$R = L[v, \Delta^{-1}], \qquad D_i(\nu) = \partial_i + \nu \frac{\partial_i f}{f} - \frac{N}{2} \frac{\partial_i q}{q}, \qquad H = R \Big/ \sum_i D_i(\nu) R.$$

The operators commute. The quotient H is the top twisted de Rham cohomology, with the top form identified with its coefficient. An element $g(\nu, v)$ represents the sequence $\Lambda_q(f^m g(m, v))$ modulo eventual equality. For large m, specialization avoids the finitely many parameter poles and f^m clears the powers of f^{-1}. In particular, each $D_i(\nu) b$ represents the zero sequence: $f^m D_i(m) b(m) = D_i(0)(f^m b(m))$, whose trace vanishes by the identity above. Multiplication by L on these sequences means evaluation of the coefficient at m for large m.

The map $T[g(\nu, v)] = [f g(\nu + 1, v)]$ acts on H, since $D_i(\nu)(f b(\nu + 1)) = f(D_i(\nu + 1) b(\nu + 1))$. It is invertible, with inverse given by shifting back and multiplying by f^{-1}. It shifts scalar coefficients by $\nu \mapsto \nu + 1$ and shifts the associated sequences by $m \mapsto m + 1$. This is the usual exponent shift of the algebraic Mellin transform [4, §1].

The size of the quotient. Suppose $g_1, \ldots, g_s$ have independent classes in H. Then all $D(\nu)^\alpha g_i$ are linearly independent in R. Indeed, take a relation and choose a monomial $D(\nu)^\beta$ of least total degree with a nonzero coefficient. Multiply the relation by v^β, commute the coordinates to the right using $[D_i(\nu), v_j] = \delta_{ij}$, and reduce modulo $\sum_i D_i(\nu) R$. Only the coefficients of $D(\nu)^\beta$ survive, with common factor $(-1)^{|\beta|} \beta!$. This contradicts independence of the classes.

Write all the g_i over one denominator Δ^b with numerator degrees at most C. The polynomial $\Delta(D_i(\nu) - \partial_i)$ has degree at most $D - 1$, so each application of $D_i(\nu)$ increases the denominator power by at most one and the numerator degree by at most $D - 1$ in that representation. For $|\alpha| \leq t$, putting every $D(\nu)^\alpha g_i$ over Δ^{b+t} gives numerator degrees at most $C + Dt$. Hence

$$s \binom{N + t}{N} \leq \binom{N + C + Dt}{N}.$$

Taking leading terms gives $s \le D^N$, so $r = \dim_L H \le D^N$.

Testing at one order. Apply a common power of T to any basis to clear every f-denominator. Expand the resulting representatives as finite L-linear combinations of regular functions over K_0. These functions span H, so select $h_1, \dots, h_r$ from them. In this basis write $T[h_i] = \sum_j A_{ij}(\nu)[h_j]$. Invertibility of T makes A invertible over L. Therefore the moment vector $u_m = (\Lambda_q(h_i f^m))_{i=1}^r$ satisfies $u_{m+1} = A(m)u_m$ beyond the poles and specialization cutoffs for these finitely many identities. Increase the cutoff so the determinant is nonzero there. The vector vanishes eventually exactly when it is zero at any one order beyond the cutoff. Since the classes $[h_i]$ span H, eventual equality shows that this is equivalent to eventual vanishing for every fixed h.

For a fixed h, a dependence of the $r+1$ classes $[h], [fh], \dots, [f^r h]$ gives the scalar recurrence. Clear denominators and drop unused endpoint shifts. Beyond the roots of the endpoint coefficients and the specialization cutoff, a zero block of length equal to the order propagates in both directions. For order zero the tail vanishes automatically. For purity check this tail and all preceding positive orders.

Computing the test. Let W_L be the Weyl algebra and put $\theta_i = D_i(\nu) - \partial_i$. The cyclic presentation $W_L / \sum_i W_L \Delta(\partial_i - \theta_i)$ restricts on $\Delta \neq 0$ to the rank-one connection in which the generator maps to 1 and ∂_i acts as $D_i(\nu)$. Holonomicity on that open set suffices for [8, Algorithm 3], which gives a holonomic cyclic presentation of R on affine space. Its generator is a known power of Δ^{-1}. Taking degree zero of integration against $W_L / \sum_i \partial_i W_L$ computes H by [7, Procedure 1.4]. Evaluating the resulting Weyl operators on the generator gives rational representatives.

The coordinates of a class in the computed basis, together with a certificate $\sum_i D_i(\nu) b_i$ for the relation, are found by increasing degree and denominator bounds and solving linear systems. If the b_i have denominator Δ^b apart from parameter poles, then $m \ge b$ clears their f-poles because Δ^b divides $(qp)^b$. Avoid the parameter poles and the roots of the recurrence endpoints, or of the determinant for the matrix test. A nonzero rational coordinate polynomial of each coefficient to be avoided gives a bound on its integer roots. Each needed moment is computed by scalar weight extraction and finite-dimensional projection to the invariants. □

For $f = q$ the list is empty. Indeed, for g homogeneous of scalar degree k, one has $\sum_i D_i(\nu)(v_i g) = (k + 2\nu)g$. Thus every homogeneous term represents zero in H.

7 Pole order and another witness

Proposition 10. *Within the family $F_{\alpha,\beta,\gamma}$ with $\gamma \ge \alpha$, purity holds exactly when $\beta \ge M$ and $\alpha \ge \beta$. For a pure phase in this family, there exists a regular multiplier detecting infinitely many orders exactly when $\alpha = \beta$ or $\alpha = \gamma$. The least pole order among the geometrically nonpower witnesses in this family is two for $M = 1$ and $2M + 1$ for $M \ge 2$.*

Proof. Set $\ell = j = 0$ in (4). If $\beta < M$, the upper index at $m = 1$ is negative and the coefficient is nonzero. For $\beta \ge M$, purity requires and follows from $\alpha \ge \beta$. If $\alpha = \beta$, q^M detects every order. If $\gamma > \alpha > \beta$, every term of an averaged regular multiplier eventually falls within the range of (4) and has lower index above the nonnegative upper one. For $\gamma = \alpha > \beta$, q^{-1} picks the first coefficient past that range, so its moment is exactly $M(-2)^{\alpha m}/(\alpha m + 1)$ at every order. In this case all polynomial multipliers eventually vanish, since their averaged terms have $\ell \ge 0$.

For a pure phase, $\gamma \ge \alpha \ge \beta \ge M$, so its pole order is at least $2M$. Equality forces all three exponents to be M. This gives the ternary phase for $M = 1$, and a proper power for $M \ge 2$. The phase in (5) attains the next integer $2M + 1$. □

A polynomial perturbation gives pole order $2M$ in dimensions at least six, with the same minimum multiplier degree.

Proposition 11. *For $M \geq 4$, the phase*

$$F_* = F_{M,M,M}\big(1 + B(M - 1 + Mq)\big)$$

is pure, geometrically nonpower up to a unit, and has pole order $2M$. The multiplier q^M detects every order, and $2M$ is the least polynomial degree detecting infinitely many orders.

Proof. In the m-th power of the added factor the monomials are $B^j q^r = q^{j+r} S^j$, with $0 \leq r \leq j \leq m$, and their coefficients are $\binom{m}{j}\binom{j}{r}(M-1)^{j-r}M^r$. For $m \geq 2$, the upper index $Mm - M - 2j$ in (4) is nonnegative, since it is at least $(M-2)m - M$. The pure lower index exceeds it by $M + j - r$. For multiplier q^M the difference is $j - r$, so precisely the terms with $j = r$ contribute, all positively. At $m = 1$ the two added pure terms cancel because

$$(M-1)\binom{-2}{M-1} + M\binom{-2}{M-2} = 0.$$

Both added terms have negative lower index for the q^M-marked moment at $m = 1$, leaving $C_{M,0}$.

Modulo q, the added factor has constant term 1, so the pole order stays $2M$. Over an algebraic closure it has a simple zero on the smooth norm level $q = 1$ at a point with $B = -1/(2M-1)$. Indeed A and B are nonzero there, so their differentials are independent and dB is nonzero on the tangent space to $q = 1$. The added factor has differential $(2M-1)dB$ on that tangent space, and the base phase is nonzero at the point. The resulting simple zero excludes a proper power up to a unit.

For a polynomial multiplier of degree $D < 2M$, its averaged terms are $q^\ell S^{j_0}$ with $j_0 \leq \ell \leq \lfloor D/2 \rfloor$. Together with $B^j q^r$ in the power expansion this makes the lower index exceed the upper one by $M + 2j_0 - \ell + j - r > 0$. The upper index is nonnegative once $(M-2)m \geq M + 2\lfloor D/2 \rfloor$. Thus every such multiplier eventually vanishes, while q^M detects all orders.

8 Instability and eventual marked vanishing

Let $\mathcal{P}$ be the pure phases in $\mathcal{O}(X_q)$, and let $\mathcal{E} \subseteq \mathcal{P}$ consist of those f for which $\Lambda_q(hf^m) = 0$ for all sufficiently large m, for each fixed regular h. The cutoff may depend on h. Let $\mathcal{N}$ be the geometric nullcone on the same ring. Thus $f \in \mathcal{N}$ if 0 belongs to the $G_q(\overline{K})$-orbit closure of f in a finite-dimensional rational $G_{q,\overline{K}}$-module containing it. This condition is independent of the module, since linear subspaces in a larger module are closed.

Lemma 12 (Instability implies eventual vanishing)**.** *One has $\mathcal{N} \subseteq \mathcal{E}$. If a destabilizing one-parameter subgroup gives a nonzero f least weight $\delta > 0$ and a nonzero h least weight b, the marked moment vanishes whenever $m\delta + b > 0$.*

Proof. Over $\overline{K}$, the affine Hilbert–Mumford criterion supplies a one-parameter subgroup taking a nonzero $f \in \mathcal{N}$ to zero [2, p. 1]. All its weights on f are then positive. Every weight in hf^m is at least $b + m\delta$, and trace invariance kills nonzero weights. For $h = 1$ this gives purity at every positive order. Scalar extension preserves the trace, so these equalities descend to K. The same weight argument appears in Long's instability lemma [5, Lem. 2.1]. The cases $f = 0$ or $h = 0$ are immediate. □

Corollary 13. *For $N \le 2$, $\mathcal{N} = \mathcal{E} = \mathcal{P}$. For $N \ge 3$, there are strict inclusions*

$$\mathcal{N} \subsetneq \mathcal{E} \subsetneq \mathcal{P}.$$

Both differences contain explicit phases that are geometrically nonpower up to units.

Proof. In dimensions one and two, the Laurent constant-term criterion in Section 2 separates the support of every nonzero pure phase from the origin over $\overline{K}$. A separating integral covector gives a destabilizing cocharacter. In dimension two the weight map $(i, j) \mapsto (j - i, -i - j)$ is invertible over $\mathbb{Q}$, so a separating covector for the Laurent exponents gives one for the G_q-weights after clearing denominators. This proves $\mathcal{P} \subseteq \mathcal{N}$ there.

For $N \ge 3$, (6) gives a pure phase outside $\mathcal{E}$. For the other strict inclusion take

$$G = F_{M+1,M,2M+1}.$$

The exponents $M, 2M + 1$ are coprime, so the two vanishing orders in Section 4 exclude powers up to units. The pure binomial coefficient in (4) is zero at every positive order. By (8), it suffices for eventual vanishing to consider each term $q^\ell S^j$ of the averaged multiplier. For fixed ℓ, j and large m, this term is in the range of (4) and its lower index exceeds the nonnegative upper index by $m + M + 2j - \ell > 0$. Thus $G \in \mathcal{E}$.

To see that G is outside the nullcone, put $L_\ell(f) = \Lambda_q(q^\ell f)$ and $d = 4M + 1$. For the pullback action $(g, s) \cdot f(v) = f(s^{-1}g^{-1}v)$, one has $L_\ell((g, s) \cdot f) = s^{2\ell} L_\ell(f)$. Hence

$$L_{M+1}(f)^d L_{-d}(f)^{M+1}$$

is a homogeneous invariant polynomial on each finite-dimensional module. On G, the two linear factors are the traces of the lowest and highest Laurent coefficients, respectively:

$$L_{M+1}(G) = \Lambda_q((1 - S)^{2M+1}), \qquad L_{-d}(G) = -\Lambda_q(S^{2M+1}).$$

Both are nonzero by (3). The invariant is therefore nonzero on G and zero at the origin, proving $G \notin \mathcal{N}$. □

Angular endpoints and affine phases

Write u for the Laurent variable q and put $a_0 = M/2$, so that on the block-invariant ring the trace is

$$a_0 \operatorname{CT}_u \int_0^1 S^{a_0 - 1} f(u, S)\, dS.$$

This is termwise integration as in (3). Call a nonzero Laurent polynomial *one-sided* if all its exponents are strictly positive or all are strictly negative.

Proposition 14 (An angular endpoint test)**.** *Suppose $f \in K[q, q^{-1}, S]$ has eventual marked vanishing against every fixed regular multiplier. Then $f(u, 1)$ is zero or one-sided. If N is even, the same holds for $f(u, 0)$.*

Proof. For each $g \in K[u, u^{-1}]$, integration of $\partial_S(S^{a_0} f^m)$ gives

$$\operatorname{CT}_u\big(g f(u, 1)^m\big) = \Lambda_q(g f^m) + \frac{m}{a_0} \Lambda_q\big(g S(\partial_S f) f^{m-1}\big).$$

This is valid termwise, also for half-integral a_0. Both traces eventually vanish. In particular, a nonzero endpoint has zero constant term at all sufficiently high powers. A fixed high power therefore

has zero constant term at every positive power. The Laurent criterion of Section 2 shows that its support, and hence the endpoint support, is one-sided.

When N is even, a_0 is a positive integer. Integrate $S^{a_0-1}\partial_S^{a_0}(f^m)$ by parts a_0 times. After multiplication by g and extraction of the constant term, the integral eventually vanishes. Indeed the derivative is a finite sum of powers f^{m-i} times fixed regular multipliers and polynomial coefficients in m. The terms at $S=1$ also eventually vanish, since the upper endpoint is zero or one-sided and the derivatives give its powers with fixed Laurent multipliers. At $S=0$ only the term with a_0-1 derivatives on S^{a_0-1} remains, a nonzero constant times $\mathrm{CT}_u(gf(u,0)^m)$. Apply the same Laurent criterion. □

Corollary 15 (Affine angular phases)**.** *For $N \geq 3$, let $f \in K[q,q^{-1},S]$ be affine in S. Then f has eventual marked vanishing against every fixed regular multiplier exactly when $f=0$ or its Laurent powers of q all have the same strict sign.*

Proof. One-sided powers suffice by scalar weights. For necessity write $a=f(u,0)$, $b=f(u,1)$ and $f=a+(b-a)S$. Proposition 14 shows that b is zero or one-sided. If $b=0$, then the unmarked moments are nonzero beta factors times $\mathrm{CT}_u(a^m)$, so eventual vanishing forces a to be zero or one-sided. Otherwise invert u if necessary so b has strictly positive powers. If a has only nonnegative powers, the constant-term integral is a nonzero beta factor times $a(0)^m$. Eventual vanishing gives $a(0)=0$, as required. We exclude the case that a has negative powers.

Put $I_m = a_0\int_0^1 S^{a_0-1}f^m\,dS$. The upper endpoint identity before taking constant terms gives

$$(a_0+m)I_m = a_0b^m + maI_{m-1}.$$

Multiplying by $(a_0+1)_{m-1}/m!$ and summing, with $I_0=1$, yields

$$\sum_{m\geq 0}\frac{(a_0+1)_m}{m!}I_m z^m = \frac{(1-zb)^{-a_0}}{1-za}.$$

Hence for each integer $\ell\geq 0$ the series $C_\ell(z)=\mathrm{CT}_u\big(u^\ell(1-zb)^{-a_0}/(1-za)\big)$ is a polynomial in z if all fixed multipliers eventually vanish.

Embed the finitely generated coefficient field in $\mathbb{C}$. Write $a=u^{-d}A(u)$ with $d\geq 1$ and $A(0)\neq 0$, and set $P(u,z)=u^d-zA(u)$. For small nonzero z, its d roots u_i near zero give

$$C_\ell(z)=\sum_{i=1}^{d}u_i^\ell w_i, \qquad w_i=\frac{u_i^{d-1}(1-zb(u_i))^{-a_0}}{\partial_u P(u_i,z)}.$$

Fix a small circle around $u=0$. For small z the defining series converges uniformly on it and Rouché's theorem gives d roots inside. The branch of $(1-zb)^{-a_0}$ is holomorphic on the disk, and the factor $u^{d-1+\ell}$ clears the pole at zero. The roots are generically distinct since P is irreducible over $\mathbb{C}(z)$, by linearity in z and $\gcd(u^d,A)=1$. The roots and weights are algebraic in z, so we can compare pole orders on local branches in a common finite extension of $\mathbb{C}(z)$.

If $\deg A>d$, the monic polynomial of the d small roots would be a proper factor of P over $\mathbb{C}(z)$. To see its coefficients are rational, use the linear recurrence with characteristic roots u_i for the sums C_ℓ. Its coefficients are recovered from $C_0,\dots,C_{2d-1}$ by solving a system with matrix $(C_{i+j})_{0\leq i,j<d}$. The matrix factors as a Vandermonde matrix times $\mathrm{diag}(w_i)$ times its transpose and is invertible. This contradicts irreducibility.

If $\deg A=d$, put $c=[u^d]A\neq 0$. At $z=1/c$ the product of the d roots has a pole, so some root has a pole on a local algebraic branch. Yet for each $L\geq 0$ the vector with entries $u_i^L w_i$ is

obtained from $C_L, \dots, C_{L+d-1}$ by a fixed inverse Vandermonde matrix. The entries of that matrix have bounded pole orders at $1/c$, independent of L. Polynomiality of all the C_ℓ contradicts the growing pole order of $u_i^L w_i$.

It remains to consider $\deg A < d$. Write $b = u^e B(u)$ with $e \geq 1$ and $B(0) \neq 0$, let $D = \gcd(A, B)$, and put $A_0 = A/D$, $B_0 = B/D$, $k = d + e$. The polynomial $R = u^k B_0 - A_0$ has nonzero constant term. The identity

$$u(A_0 R' - A_0' R) = u^k\big((kB_0 + uB_0')A_0 - uB_0 A_0'\big)$$

bounds the sum of its root multiplicities reduced by one by $\deg A_0 + \deg B_0$. Indeed the bracket is nonzero by its leading coefficient. Thus R has at least $k - \deg A_0 > \deg D$ distinct roots. One of them is a nonzero u_0 with $a(u_0) = b(u_0) = c \neq 0$.

Now P is monic of degree d. Taking the branch $u_i(1/c) = u_0$, Lagrange interpolation at its roots gives

$$u_i^{d-1}(1 - zb(u_i))^{-a_0} = \sum_{\ell=0}^{d-1} C_\ell(z)\,[u^\ell]\frac{P(u,z)}{u - u_i}.$$

All roots are finite at $z = 1/c$. The right side is regular there on this branch if the C_ℓ are polynomials. The left side has a pole on the branch, since $1 - zb(u_i)$ vanishes there and $u_0 \neq 0$. This final contradiction proves the criterion. □

The affine hypothesis matters. The safe phase G above has lower endpoint $u^{-(M+1)}(1+u)^M$ and upper endpoint $-u^M(1+u)^M(u+2)^{2M+1}$, so its endpoints have opposite strict signs of the Laurent exponents.

Over a formally real field, block-invariant phases with at most two Laurent powers of q also admit a support test. A nonzero phase of this kind is pure exactly when its exponents are all strictly positive or all strictly negative. Indeed, if one exponent is zero, the constant term of the square is the square of its coefficient in $K[S]$. If the exponents are $a < 0 < b$, the power $2(b-a)$ has constant term a positive integer times $(P^b Q^{-a})^2$, where P, Q are the coefficient polynomials of q^a, q^b. In both cases its trace is nonzero. To see this over any ordered ground field, the rational Gram matrix of (3) on any finite set of powers of S is positive definite over $\mathbb{R}$, so its rational diagonalization has positive entries in every ordering. For one-sided support the scalar torus gives a destabilizing cocharacter. The same test therefore characterizes eventual marked vanishing among the pure phases in this slice.

9 Other norm traces

Suppose a connected reductive subgroup preserving q, together with scalars, acts transitively on X_q over $\overline{K}$. Its normalized trace equals Λ_q. The orthogonal trace is invariant under all isometries by uniqueness, and complete reducibility gives uniqueness for the smaller group.

Corollary 16. *Let h be a nondegenerate Hermitian form of positive rank n over a quadratic étale algebra or a quaternion algebra with its canonical involution, over a characteristic-zero field K. On the open set $h(v,v) \neq 0$, take the normalized trace of the isometry group and the scalar torus. In the quadratic étale case its kernel is Mathieu–Zhao exactly for $n = 1$. In the quaternion case the kernel always fails to be Mathieu–Zhao. In each failure case, a geometrically nonpower pure phase and a linear polynomial detect every positive order over K.*

Proof. Write $q(v) = h(v,v) \in K$. Over $\overline{K}$, the quadratic étale algebra splits into two factors exchanged by the involution. The space becomes a vector space and its dual, with norm $q(x,y) = y(x)$.

The isometry group is GL_n, which acts transitively on pairs with a given nonzero value of $y(x)$: choose a basis with first vector x and the other vectors in $\ker y$.

In the quaternion case the split involution is $B \mapsto J^{-1}B^{\mathsf{T}}J$ on two-by-two matrices, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Write J_n for its block diagonal repetition and H for the matrix of h. The Hermitian identity says $\Omega = J_n H$ is a nondegenerate alternating matrix. A vector becomes a matrix with columns x, y; its norm is $x^{\mathsf{T}}\Omega y$, and the isometry group is $\mathrm{Sp}(\Omega)$. If this pairing is nonzero, the pair extends to a symplectic basis after rescaling one vector. This proves transitivity on each nonzero norm level. In either case scalars move between such levels over $\overline{K}$.

These split descriptions also show that q is nondegenerate of dimension $2n$ or $4n$, respectively, and that the isometry groups are connected reductive. The trace equality now gives the assertions by Theorem 1 and Proposition 6. □

The norm-one level of an isotropic form

Corollary 17. *Let q be isotropic of dimension $N \geq 3$ over K. The normalized $\mathrm{SO}(q)$-trace on $q = 1$ has a geometrically nonpower pure phase with a fixed polynomial multiplier detecting every order.*

Proof. Write $q = 2xy + B$, set $M = N - 2$, $S = B$ on the level $q = 1$, and put $T = 1 + x$. Use the phase

$$P = xT^M\big(2T^2y - 2 - x\big)^{M+1}.$$

The hyperbolic torus gives opposite weights to x, y. The rank-two derivations in Section 3 restrict to $q = 1$ and give the same moments for S. Thus on polynomials in x, y, the trace is constant term in x after substituting $y = (1 - t^2)/(2x)$, followed by integration against $Mt^{M-1}\,dt$. Indeed only equal exponents of x, y survive, and $xy = (1 - S)/2$.

After the substitution, $P = x^{-M}T^M(1 - T^2t^2)^{M+1}$. The endpoint calculation of Section 4, now with x as the Laurent variable, gives

$$\Lambda(P^m) = 0, \qquad \Lambda(x^M P^m) = M\int_0^1 w^{M-1}(1 - w^2)^{(M+1)m}\,dw \neq 0.$$

For the nonpower assertion use $x = 0$, $B = 1$, $y \neq 1$ over $\overline{K}$. The condition $B = 1$ has nonzero differential in the complementary block, so x is a local parameter in a transverse direction and P has order one. □

For a positive-definite real form, the norm-one trace is spherical averaging. The real sphere is Zariski dense in the level, so the trace of the square of a nonzero real regular function is strictly positive. The real kernel is therefore Mathieu–Zhao, while Corollary 17 gives failure after extending the same form to $\mathbb{C}$ in dimension at least three.

Averaging by the special orthogonal group

Corollary 18. *Over $\mathbb{R}$, the kernel of the $\mathrm{SO}(q)$ Reynolds projection on $\mathbb{R}[V, q^{-1}]$ is Mathieu–Zhao exactly when q is definite or $N \leq 2$. Over any characteristic-zero field the kernel is Mathieu–Zhao for $N \leq 2$, and it fails for isotropic q of dimension at least three.*

Proof. For $N = 1$ the projection is the identity. For $N \geq 2$ it projects onto $K[q, q^{-1}]$ and is linear over that ring. After localizing these invariants it becomes the normalized trace on the generic level $q = t$ over $K(t)$. The group $\mathrm{SO}(q)$ is geometrically transitive on this level, and the map remains invariant and normalized. The Mathieu property is equivalent before and after this localization:

powers and marked powers of fractions differ from those of their numerators by invertible invariant factors, and the invariant ring embeds in $K(t)$.

For $N = 2$ the generic level after splitting is a Laurent polynomial ring with constant term trace, so the argument of Section 2 applies. For isotropic q of dimension at least three, q/t is isotropic over $K(t)$. Corollary 17 gives a witness on $q = t$. Clearing invariant denominators gives a witness on X_q.

Finally, for a definite real form and nonzero f, choose a real point of X_q where f is nonzero. Evaluating the projection of f^2 on the sphere through that point gives a positive spherical average. Thus a pure phase must be zero. Every indefinite nondegenerate real form is isotropic, which completes the classification. □

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