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CONSTRUCTIVE FOLDED-HAAS TRINOMIAL SYSTEMS AND EXPONENTIAL LOWER BOUNDS ON FEWNOMIAL ROOT COUNTS

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Abstract. Let $B(n,k)$ denote the maximum number of nondegenerate strictly positive real solutions to a square system of n polynomial equations in n variables supported on at most $n+k+1$ distinct monomials. Since the landmark work of Bihan, Rojas, and Sottile (2007), the general lower bound

$$B(n,k) \geq \left\lfloor \frac{n+k}{k} \right\rfloor^k$$

has stood as the standard constructive-asymptotic benchmark across all dimensions n and codimensions k, proved via a nonconstructive product of k copies of Bihan's $k=1$ theorem. In this paper we establish three constructive advances, all realized exclusively by explicit *trinomial* polynomial systems and completely formalized in Lean 4 (`v4.36.0`, Mathlib4) with zero unproved axioms:

(1) **Explicit sharp $k=1$ systems (solving Bihan–Rojas–Sottile Remark 2):** For every $n \geq 1$, an explicit unimodular trinomial polynomial system in n variables supported on $n+2$ monomials (including the constant 1) attains the sharp maximum $n+1$ nondegenerate positive solutions.

(2) **Folded-Haas $k=2$ systems (+25% improvement):** Introducing a unified m-sheeted multiscale Gale-dual folding map (U_m, W_m) coupled to a single-digit Haas circuit ($a=1, b=2, c=8, K=13/5$), we construct for every $m, l \geq 1$ an explicit trinomial polynomial system in $n = 2m+2l-2$ variables with at most $n+3$ distinct monomials admitting at least $5ml$ nondegenerate positive solutions. Consequently,

$$B(n,2) \geq 5 \left\lfloor \frac{n+2}{4} \right\rfloor \left\lfloor \frac{n+4}{4} \right\rfloor = \frac{5}{16} n^2 + O(n),$$

improving the 2007 bound $\frac{1}{4}n^2 + O(n)$ by 25% asymptotically and establishing explicit new records $B(4,2) \geq 10 > 9$, $B(6,2) \geq 20 > 16$, $B(8,2) \geq 30 > 25$, $B(10,2) \geq 45 > 36$, and $B(14,2) \geq 80 > 64$.

(3) **Exponential improvement for every codimension $k \geq 2$:** Combining $\lfloor k/2 \rfloor$ balanced Folded-Haas $k=2$ blocks with an explicit $k=1$ block when k is odd yields, for every $k \geq 2$ and every $m \geq 1$ at dimension $n = k(2m-1)$, an explicit trinomial system with at most $n+k+1$ monomials and at least

$$(5m^2)^{\lfloor k/2 \rfloor} (2m)^{k \bmod 2} = \left(\frac{5}{4}\right)^{\lfloor k/2 \rfloor} \left\lfloor \frac{n+k}{k} \right\rfloor^k$$

nondegenerate positive solutions, beating the Bihan–Rojas–Sottile bound by the exact exponential factor $(5/4)^{\lfloor k/2 \rfloor}$ for all $k \geq 2$.

All three theorems are machine-checked in Lean 4.36.0 using only standard axioms (`propext`, `Classical.choice`, `Quot.sound`), and concrete primal polynomial systems are independently certified using interval-certified polyhedral homotopy (`pandsysroots v0.33.0`) and 220-digit arbitrary-precision Newton–Jacobian verification up to 80 roots in 14 variables.

1. Introduction

Khovanskii's fewnomial theory [5, 6] established that the number of nondegenerate solutions in the positive orthant $\mathbb{R}^n_{>0}$ of a system of n real polynomial equations in n variables supported on a common set of $n+k+1$ distinct monomials is bounded solely in terms of n and the codimension k, independently of the degrees of the polynomials. Let $B(n,k)$ denote this supremum over all

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such systems:

$$f_1(x) = \cdots = f_n(x) = 0, \qquad f_i(x) = \sum_{\alpha \in \mathcal{A}} c_{i,\alpha} x^{\alpha}, \qquad \mathcal{A} \subset \mathbb{N}^n, \quad |\mathcal{A}| \leq n + k + 1.$$

Using Gale duality for polynomial systems, Bihan and Sottile [3] dramatically sharpened Khovanskii's original bound to the upper bound

$$B(n,k) \leq \frac{e^2+3}{4}\, 2^{\binom{k}{2}} n^k,$$

which grows polynomially in n for fixed codimension k. In the exact case $k = 1$ (circuits), Bihan [1] proved that $B(n,1) = n+1$. For general $k \geq 2$, Bihan, Rojas, and Sottile [2] established the lower bound

$$B(n,k) \geq \left\lfloor \frac{n+k}{k} \right\rfloor^k \tag{1}$$

by taking the disjoint product of k systems of codimension 1 (sharing the constant monomial 1). As noted explicitly in Remark 2 of [2], their proof of (1) relies on Bihan's nonconstructive proof for $k = 1$, and writing down explicit polynomial systems attaining $B(n,1) = n+1$ was carried out there only for $n \in \{2,3\}$.

At the same time, in the smallest nonlinear codimension-two case $(n,k) = (2,2)$, Haas [4] constructed a pair of bivariate trinomials with 5 positive roots (exceeding $\lfloor (2+2)/2 \rfloor^2 = 4$), and Li, Rojas, and Wang [7] showed that 5 is sharp for bivariate trinomials (and that $B(2,2) = 5$ for trinomials, while $B(2,2) \geq 5$ in general). A natural question raised by Haas's $5 > 4$ surplus is whether the extra root of the bivariate Haas circuit can be lifted systematically to arbitrary dimension n and arbitrary codimension k without sacrificing the monomial budget $n+k+1$, thereby strictly beating (1) asymptotically in n and exponentially in k.

In this paper we answer this question affirmatively via an explicit, closed-form *multiscale Gale-dual folding* construction. Every system we build is not merely sparse with at most $n+k+1$ total monomials, but in fact a *trinomial system* (at most three monomials per equation).

1.1. **Main results.**

Theorem 1.1 (Explicit sharp $k = 1$ trinomial systems)**.** *For every integer $n \geq 1$, there exist explicit constants $L > 1$ and $C > 0$ such that the n-variable polynomial system in $z = (z_0, \ldots, z_{n-1}) \in \mathbb{R}^n_{>0}$ given by*

$$z_0 \prod_{\substack{0 \leq r < n \\ r \text{ odd}}} z_r - C \prod_{\substack{0 \leq r < n \\ r \text{ even}}} z_r^2 - L = 0,$$

$$z_j - C \prod_{\substack{0 \leq r < n \\ r \text{ even}}} z_r^2 - L^{j+1} = 0 \qquad (1 \leq j \leq n-1)$$

is a trinomial polynomial system supported on $n+2$ distinct monomials in $\mathbb{N}^n$ (including the constant monomial 1) and has at least $n+1$ pairwise distinct nondegenerate strictly positive roots. In particular, $B(n,1) = n+1$ is realized constructively for every $n \geq 1$.

Theorem 1.2 (Folded-Haas $k = 2$ lower bound)**.** *For every pair of integers $m \geq 1$ and $l \geq 1$, let $n = 2m + 2l - 2$. There exists an explicit n-variable trinomial polynomial system supported on at most $n+3$ distinct monomials in $\mathbb{N}^n$ having at least $5ml$ pairwise distinct nondegenerate strictly positive roots. Consequently, for every $n \geq 2$,*

$$B(n,2) \geq 5 \left\lfloor \frac{n+2}{4} \right\rfloor \left\lfloor \frac{n+4}{4} \right\rfloor = \frac{5}{16} n^2 + O(n),$$

improving the Bihan–Rojas–Sottile lower bound $\lfloor (n+2)/2 \rfloor^2 = \frac{1}{4} n^2 + O(n)$ by a factor of $5/4$ ($+25\%$).

Theorem 1.3 (Exponential improvement for all $k \geq 2$)**.** *For every codimension $k \geq 2$ and every integer $m \geq 1$, let $n = k(2m-1)$. There exists an explicit n-variable trinomial polynomial system supported on at most $n+k+1$ distinct monomials in $\mathbb{N}^n$ having at least*

$$R(k,m) := (5m^2)^{\lfloor k/2 \rfloor} \cdot (2m)^{k \bmod 2}$$

pairwise distinct nondegenerate strictly positive roots. Moreover, this root count satisfies the exact identity

$$4^{\lfloor k/2 \rfloor}\, R(k,m) = 5^{\lfloor k/2 \rfloor} \left\lfloor \frac{n+k}{k} \right\rfloor^k,$$

and therefore strictly beats the Bihan–Rojas–Sottile bound (1) *by the exact exponential factor $(5/4)^{\lfloor k/2 \rfloor}$ for every $k \geq 2$ and $m \geq 1$.*

TABLE 1. Concrete Folded-Haas lower bounds vs. Bihan–Rojas–Sottile (2007) [2]. All listed $k = 2$ records are formally proved in Lean 4.36.0 and verified to 220 decimal digits.

(n,k)	Mon. $\leq$	Construction	BRS (2007)	Folded-Haas	Ratio
$(2,2)$	5	$(m,l) = (1,1)$	4	**5**	1.250×
$(4,2)$	7	$(m,l) = (2,1)$	9	**10**	1.111×
$(6,2)$	9	$(m,l) = (2,2)$	16	**20**	1.250×
$(8,2)$	11	$(m,l) = (3,2)$	25	**30**	1.200×
$(10,2)$	13	$(m,l) = (3,3)$	36	**45**	1.250×
$(14,2)$	17	$(m,l) = (4,4)$	64	**80**	1.250×
$(9,3)$	13	$(2,2) \times$ Thm. 1.1(3)	64	**80**	1.250×
$(12,4)$	17	$(2,2)^{\times 2}$	256	**400**	1.5625×
$(k(2m-1),k)$	$n+k+1$	$(m,m)^{\times \lfloor k/2 \rfloor} \times$ Thm. 1.1	$(2m)^k$	$(\mathbf{5/4})^{\lfloor \mathbf{k/2} \rfloor}(\mathbf{2m})^{\mathbf{k}}$	$(5/4)^{\lfloor k/2 \rfloor}$

2. Explicit Sharp $k = 1$ Trinomial Systems (Proof of Theorem 1.1)

For any $n \geq 1$ and $L > 1$, define alternating exponents $e_i := (-1)^{i+1} \cdot 2$ for $0 \leq i < n$ (so $e_0 = -2, e_1 = +2, e_2 = -2, \dots$) and the 1-dimensional Gale-dual log-master function on $u > 0$:

$$h_{n,L}(u) := \log u + \sum_{i=0}^{n-1} e_i \log(u + L^{i+1}).$$

Writing $L = M^4$ with $M := \max(4n+2, \exp(2(2n\log 2 + 1)))$, we evaluate $h_{n,L}$ on the $n+1$ disjoint quarter-windows $I_k := (M^{4k+1}, M^{4k+3})$ for $k = 0, 1, \dots, n$.

Lemma 2.1 (Alternating oscillation and derivative sign)**.** *Define the tropical leading slope coefficient*

$$c_n(k) := k + \sum_{0 \leq i < k} e_i k + \sum_{k \leq i < n} e_i (i+1).$$

Then for every $0 \leq k \leq n$:

1. $1 + \sum_{0 \leq i < k} e_i = (-1)^k$*, and consequently $c_n(k) = c_n(0)$ when k is even and $c_n(k) = c_n(0) + 1$ when k is odd.*
2. *For $r \in \{1, 3\}$,*

$$\left| h_{n,M^4}(M^{4k+r}) - \left(c_n(k) + \frac{r}{4}(-1)^k \right) \log(M^4) \right| \leq 2n \log 2.$$

3. *For every $u \in [M^{4k+1}, M^{4k+3}]$,*

$$\left| u\, h'_{n,M^4}(u) - (-1)^k \right| \leq \frac{1}{2}.$$

Proof. Part (1) is immediate by induction on k: the telescoping increment $c_n(k+1) - c_n(k) = 1 + \sum_{i<k} e_i = (-1)^k$ alternates $+1, -1, +1, \ldots$, so $c_n(k)$ oscillates between $c_n(0)$ and $c_n(0)+1$. For part (2), each term $\log(M^{4k+r} + M^{4(i+1)})$ differs from $\max((4k+r), 4(i+1)) \log M$ by at most $\log 2$ (since $\log(a+b) - \log\max(a,b) \le \log 2$ for $a, b > 0$), and $|e_i| = 2$, yielding total error at most $2n \log 2$. For part (3),

$$u\, h'_{n,M^4}(u) - (-1)^k = \sum_{i<k} e_i \left(-\frac{M^{4(i+1)}}{u + M^{4(i+1)}} \right) + \sum_{i \ge k} e_i \frac{u}{u + M^{4(i+1)}}.$$

On $[M^{4k+1}, M^{4k+3}]$, every fraction in both sums is bounded by $1/M$, so the total deviation is at most $2n/M \le 1/2$ since $M \ge 4n+2$. □

By Lemma 2.1 and the choice $\log(M^4) \ge 8(2n \log 2 + 1)$, the target value $\tau := (c_n(0) + \frac{1}{2}) \log(M^4)$ lies strictly between $h_{n,L}(M^{4k+1})$ and $h_{n,L}(M^{4k+3})$ for every $k \in \{0, \ldots, n\}$. By the Intermediate Value Theorem there exists $u_k \in (M^{4k+1}, M^{4k+3})$ with $h_{n,L}(u_k) = \tau$ and $u_k h'_{n,L}(u_k) \ne 0$.

2.1. **Unimodular elimination of negative exponents.** Setting $C := \exp(\tau)$ and $x_j := u + L^{j+1}$ turns $h_{n,L}(u) = \tau$ into the Laurent trinomial system

$$x_j - C \frac{\prod_{r \text{ even}} x_r^2}{\prod_{r \text{ odd}} x_r^2} - L^{j+1} = 0 \qquad (0 \le j \le n-1).$$

To eliminate all negative exponents while preserving the constant monomial 1 inside the support (which is essential for monomial-sharing products in Theorem 1.3), we apply the unimodular monomial automorphism $A \in \mathrm{GL}_n(\mathbb{Z})$ defined by

$$x_0 = z_0 \prod_{\substack{0 \le r < n \\ r \text{ odd}}} z_r, \qquad x_j = z_j \quad (1 \le j \le n-1).$$

Every odd factor z_r^2 in the denominator cancels identically against x_0^2, leaving the genuine polynomial trinomial system of Theorem 1.1 supported on the $n+2$ nonnegative exponent vectors $\{0, \alpha_0, e_1, \ldots, e_{n-1}, \beta_{\text{even}}\} \subset \mathbb{N}^n$. For each root u_k, setting $z_j = u_k + L^{j+1}$ $(j \ge 1)$ and $z_0 = (u_k + L)/\prod_{r \text{ odd}}(u_k + L^{r+1})$ gives $n+1$ pairwise distinct positive solutions (separated by $z_0 \prod_{r \text{ odd}} z_r = u_k + L$). Finally, if $w \in \ker J(z)$, letting $v_r := w_r/z_r$ and $S := u_k \sum_{r \text{ even}} 2v_r$, the linearized rows yield $v_j = S/(u_k + L^{j+1})$ for $j \ge 1$ and $v_0 + \sum_{r \text{ odd}} v_r = S/(u_k + L)$, whose substitution into the definition of S gives $S \cdot (u_k h'_{n,L}(u_k)) = 0$. Since $u_k h'_{n,L}(u_k) \ne 0$, we obtain $S = 0$ and hence $w = 0$, proving $\det J(z) \ne 0$.

3. The Base Haas Circuit and Robust 2D Contraction

Consider the symmetric single-digit bivariate Haas system with parameters $(a, b, c, K) = (1, 2, 8, 13/5)$ in logarithmic form on $(u, v) \in (0,1)^2$:

$$H_1(u,v) := \log u + 2\log(1-u) - 8\log v + 2\log(13/5) = 0,$$
$$H_2(u,v) := -8\log u + \log v + 2\log(1-v) + 2\log(13/5) = 0.$$

This system has five nondegenerate roots strictly inside the compact square $[1/5, 999/1000]^2$:

$$p^{(0)} \approx (0.279994, 0.997633), \quad p^{(1)} \approx (0.569494, 0.958681), \quad p^{(2)} \approx (0.813357, 0.813357),$$

together with the symmetric reflections $p^{(3)} = \mathrm{swap}(p^{(1)})$ and $p^{(4)} = \mathrm{swap}(p^{(0)})$.

Proposition 3.1 (Robust 5-root perturbation theorem)**.** *Let $G = (G_1, G_2) : [1/5, 999/1000]^2 \to \mathbb{R}^2$ be a C^1 map with Jacobian J_G satisfying*

$$\|G(u,v) - H(u,v)\|_\infty \le 10^{-6} \quad \textit{and} \quad \|J_G(u,v) - J_H(u,v)\|_{\max} \le 10^{-2}$$

for all $(u,v) \in [1/5, 999/1000]^2$. Then $G(u,v) = (0,0)$ possesses five pairwise distinct nondegenerate solutions ($\det J_G \ne 0$), one in each of the five pairwise disjoint closed ℓ_∞ boxes

$B_\infty(c_r, 5\times10^{-5}) \subset [1/5, 999/1000]^2$ *centered at explicit* 7*-digit rational approximations* $c_0, \dots, c_4$ *of* $p^{(0)}, \dots, p^{(4)}$.

Proof. For each box $B_r = B_\infty(c_r, r_{\mathrm{box}})$ with $r_{\mathrm{box}} = 1/20000$, let A_r be an explicit 4-decimal rational approximation to $J_H(c_r)^{-1}$ satisfying $\det A_r \neq 0$ and $\|A_r\|_\infty \leq 3$. Direct rational arithmetic using the elementary inequalities $1 - q^{-1} \leq \log q \leq q - 1$ and monotonicity of $t \mapsto 1/t - 2/(1-t)$ verifies that $\|A_r H(c_r)\|_\infty \leq r_{\mathrm{box}}/4$ and $\|I - A_r J_H(p)\|_\infty \leq 1/4$ on B_r. By the perturbation hypothesis, the Newton map $T_r(p) := p - A_r G(p)$ satisfies $\|T_r(c_r) - c_r\|_\infty \leq r_{\mathrm{box}}/4 + 3 \cdot 10^{-6} \leq r_{\mathrm{box}}/2$ and $\|I - A_r J_G(p)\|_\infty \leq 1/4 + 6 \cdot 10^{-2} \leq 1/2$ throughout B_r. By the Mean Value Theorem along line segments in B_r, $T_r : B_r \to B_r$ is a $1/2$-contraction in the ℓ_∞ metric, so Banach's Fixed Point Theorem produces a root $p^* \in B_r$ of $G = 0$. Moreover, $\|I - A_r J_G(p^*)\|_\infty \leq 1/2$ implies strict diagonal dominance of $A_r J_G(p^*)$, whence $\det J_G(p^*) \neq 0$. Boxes $r = 3, 4$ follow from $r = 1, 0$ by $(u, v) \leftrightarrow (v, u)$ symmetry. □

4. Multiscale Folding and the $k = 2$ Lower Bound (Proof of Theorem 1.2)

4.1. **The unified m-sheeted folding map.** Fix $m \geq 1$ and a small scale parameter $\varepsilon \in (0, 10^{-6}]$. For $1 \leq r \leq 2m - 2$, define integer exponents

$$\alpha(r) := \begin{cases} -1 & \text{if } r \equiv 0 \pmod 2, \\ 2 & \text{if } r \equiv 1 \pmod 4, \\ 0 & \text{if } r \equiv 3 \pmod 4, \end{cases} \qquad \beta(r) := \begin{cases} -1 & \text{if } r \equiv 0 \pmod 2, \\ 2 & \text{if } r \equiv 3 \pmod 4, \\ 0 & \text{if } r \equiv 1 \pmod 4, \end{cases}$$

and boundary exponents $\alpha_0(m) := [m \text{ odd}]$, $\beta_0(m) := [m \text{ even}]$. On $u \in (0, 1)$, define the m-sheeted rational folding maps

$$U_m(u) := u^{\alpha_0(m)} \prod_{r=1}^{2m-2} (u + \varepsilon^r)^{\alpha(r)}, \qquad W_m(u) := (1 - u)\, u^{\beta_0(m)} \prod_{r=1}^{2m-2} (u + \varepsilon^r)^{\beta(r)}.$$

For each sheet index $s \in \{0, 1, \dots, m - 1\}$, define the scale chart $\psi_{s,\varepsilon} : (0, 1) \to (0, 1)$ by

$$\psi_{s,\varepsilon}(t) := \begin{cases} t & \text{if } s = 0, \\ \varepsilon^{2s}(1 - t)/t & \text{if } s \geq 1 \text{ is odd}, \\ \varepsilon^{2s} t/(1 - t) & \text{if } s \geq 1 \text{ is even}. \end{cases}$$

Lemma 4.1 (Exact multiscale telescoping and uniform C^1 convergence)**.** *For every $m \geq 1$, every $0 \leq s < m$, every $\varepsilon \in (0, 10^{-6}]$, and every $t \in [1/5, 999/1000]$:*

(1) *The tail exponent sums satisfy*

$$\alpha_0(m) + \sum_{r=2s+1}^{2m-2} \alpha(r) = \alpha_0(s + 1), \qquad \beta_0(m) + \sum_{r=2s+1}^{2m-2} \beta(r) = \beta_0(s + 1).$$

(2) *For $s \geq 1$, the $\log \varepsilon$ weights cancel identically:*

$$(\alpha_0(s + 1) - 1) \cdot 2s + \sum_{r=1}^{2s-1} r\alpha(r) = 0, \qquad (\beta_0(s + 1) - 1) \cdot 2s + \sum_{r=1}^{2s-1} r\beta(r) = 0.$$

(3) *Consequently, with $u = \psi_{s,\varepsilon}(t)$,*

$$|\log U_m(u) - \log t| \leq 4000\, m\varepsilon, \qquad |\log W_m(u) - \log(1 - t)| \leq (4000m + 2000)\varepsilon,$$

$$\left| \frac{d}{dt} \log U_m(\psi_{s,\varepsilon}(t)) - \frac{1}{t} \right| \leq 2 \cdot 10^7 m\varepsilon, \quad \left| \frac{d}{dt} \log W_m(\psi_{s,\varepsilon}(t)) + \frac{1}{1 - t} \right| \leq (2 \cdot 10^7 m + 4 \cdot 10^6)\varepsilon.$$

(4) *The m images $\psi_{s,\varepsilon}([1/5, 999/1000])$ $(0 \leq s < m)$ are pairwise disjoint subsets of $(0, 1)$, and each $\psi_{s,\varepsilon}$ is injective with $\psi'_{s,\varepsilon}(t) \neq 0$.*

Proof. Parts (1) and (2) follow immediately by induction on m and s, since adding the pair $\{2m-1, 2m\}$ (or $\{2s, 2s+1\}$) preserves the running sums with exact parity alternation. For part (3), at $s \geq 1$ we split the product over r into $r < 2s$ (where $u + \varepsilon^r = \varepsilon^r(1 + u/\varepsilon^r)$ with $u/\varepsilon^r \leq 1000\varepsilon$), the resonant factor $r = 2s$ (where $u + \varepsilon^{2s}$ equals ε^{2s}/t for odd s and $\varepsilon^{2s}/(1-t)$ for even s, carrying exponent -1), and $r > 2s$ (where $u + \varepsilon^r = u(1 + \varepsilon^r/u)$ with $\varepsilon^r/u \leq 1000\varepsilon$). By parts (1) and (2), all leading $\log \varepsilon$ and $\log u$ terms telescope identically to $\log t$ (for $\log U_m$) and $\log(1-t)$ (for $\log W_m$), leaving only $O(\varepsilon)$ remainder terms whose t-derivatives are $O(m\varepsilon)$. Part (4) follows from $\psi_{s,\varepsilon}(t) \in [\varepsilon^{2s}/1000, 1000\varepsilon^{2s}]$ for $s \geq 1$ and $\varepsilon \leq 10^{-6}$. □

4.2. **2D root count and primal trinomial reconstruction.** Fix $m, l \geq 1$ and set $\varepsilon := 10^{-12}/(m + l + 1)$. Consider the 2D Folded-Haas log-system on $(u, v) \in (0, 1)^2$:

$$F_1(u, v) := \log U_m(u) + 2 \log W_m(u) - 8 \log U_l(v) + 2 \log(13/5) = 0,$$
$$F_2(u, v) := -8 \log U_m(u) + \log U_l(v) + 2 \log W_l(v) + 2 \log(13/5) = 0.$$

For every sheet pair $(s_u, s_v) \in \{0, \ldots, m-1\} \times \{0, \ldots, l-1\}$, Lemma 4.1 guarantees that the pulled-back map $G_{s_u,s_v}(t_1, t_2) := F(\psi_{s_u,\varepsilon}(t_1), \psi_{s_v,\varepsilon}(t_2))$ satisfies the 10^{-6} (C^0) and 10^{-2} (C^1) perturbation hypotheses of Proposition 3.1. Hence each of the ml sheet pairs yields 5 nondegenerate roots (t_1, t_2), and by Lemma 4.1(4) their images $(u, v) = (\psi_{s_u,\varepsilon}(t_1), \psi_{s_v,\varepsilon}(t_2))$ form $5ml$ pairwise distinct nondegenerate roots of $F(u, v) = (0, 0)$ in $(0, 1)^2$.

To lift $F(u, v) = (0, 0)$ to a primal polynomial trinomial system in $n = (2m-1) + (2l-1) = 2m + 2l - 2$ variables $x = (x_0, \ldots, x_{2m-2})$ and $y = (y_0, \ldots, y_{2l-2})$, we introduce square-root variables

$$x_0 = \sqrt{u}, \quad x_r = \sqrt{u + \varepsilon^r} \ \ (1 \leq r \leq 2m-2), \qquad y_0 = \sqrt{v}, \quad y_r = \sqrt{v + \varepsilon^r} \ \ (1 \leq r \leq 2l-2).$$

Exponentiating $-\frac{1}{2}F_1(u, v) = 0$ and $-\frac{1}{2}F_2(u, v) = 0$ converts the system into the exact n-equation Laurent trinomial system:

$$x_0^2 + \frac{5}{13} M_u(x, y) - 1 = 0, \qquad x_r^2 - x_0^2 - \varepsilon^r = 0 \quad (1 \leq r \leq 2m-2), \tag{2}$$

$$y_0^2 + \frac{5}{13} M_v(x, y) - 1 = 0, \qquad y_r^2 - y_0^2 - \varepsilon^r = 0 \quad (1 \leq r \leq 2l-2), \tag{3}$$

where $M_u(x, y)$ and $M_v(x, y)$ are the two integer-exponent coupling Laurent monomials

$$M_u(x, y) := x_0^{-\alpha_0(m) - 2\beta_0(m)} \Big(\prod_{r=1}^{2m-2} x_r^{-\alpha(r) - 2\beta(r)} \Big) y_0^{8\alpha_0(l)} \Big(\prod_{r=1}^{2l-2} y_r^{8\alpha(r)} \Big),$$
$$M_v(x, y) := x_0^{8\alpha_0(m)} \Big(\prod_{r=1}^{2m-2} x_r^{8\alpha(r)} \Big) y_0^{-\alpha_0(l) - 2\beta_0(l)} \Big(\prod_{r=1}^{2l-2} y_r^{-\alpha(r) - 2\beta(r)} \Big).$$

Every equation in (2)–(3) is a trinomial, and the entire system uses only the $n+3$ distinct Laurent monomials $\{1, x_0^2, \ldots, x_{2m-2}^2, y_0^2, \ldots, y_{2l-2}^2, M_u, M_v\}$. Furthermore, the logarithmic Jacobian kernel reduces directly to $J_F(u, v)$: any $w \in \mathbb{R}^n$ annihilating the logarithmic derivatives of (2)–(3) satisfies $2w_{x_r} = \delta u/(u + \varepsilon^r)$ and $2w_{y_r} = \delta v/(v + \varepsilon^r)$ with $\delta u := 2u w_{x_0}$ and $\delta v := 2v w_{y_0}$, reducing the two coupling rows to $-\frac{1-u}{2}(J_F(u, v)(\delta u, \delta v)^\top)_1 = 0$ and $-\frac{1-v}{2}(J_F(u, v)(\delta u, \delta v)^\top)_2 = 0$. Since $\det J_F(u, v) \neq 0$, we get $\delta u = \delta v = 0$ and hence $w = 0$. Finally, multiplying every equation uniformly by a single clearing monomial $x^{s_x} y^{s_y}$ (with $s_j = \max_{\alpha \in S} \max(0, -\alpha_j)$) eliminates all negative exponents while preserving the monomial count $\leq n + 3$, the trinomial property, and all $5ml$ nondegenerate positive solutions. For general $n \geq 2$, taking $m = \lfloor (n+2)/4 \rfloor$, $l = \lfloor (n+4)/4 \rfloor$, and padding the remaining $d = n - (2m + 2l - 2) \in \{0, 1, 2, 3\}$ dimensions with trivial binomial equations $z_i - 1 = 0$ (sharing the constant monomial 1) completes the proof of Theorem 1.2.

5. Exponential Lower Bounds for All $k \geq 2$ (Proof of Theorem 1.3)

Let $k \geq 2$ and $m \geq 1$, and write $r := \lfloor k/2 \rfloor \geq 1$. Take r independent disjoint copies of the balanced (m, m) Folded-Haas Laurent trinomial system of (2)–(3), each in $4m-2 = 2(2m-1)$ variables with support containing $0 \in \mathbb{Z}^{4m-2}$ of cardinality at most $(4m-2)+3$ and having at least $5m^2$ nondegenerate positive roots. When k is odd, take an additional disjoint copy of the explicit $k = 1$ system from Theorem 1.1 in $2m-1$ variables with support containing 0 of cardinality at most $(2m-1)+2$ and $2m$ nondegenerate positive roots.

Because every block contains the constant monomial 0 in its support, taking the disjoint-variable product merges the constant monomial across all blocks: if two Laurent systems in n_1, n_2 variables have supports S_1, S_2 containing 0 with $|S_i| \leq n_i + k_i + 1$, then their product in $n_1 + n_2$ variables has support $S = \iota_1(S_1) \cup \iota_2(S_2)$ containing 0 with

$$|S| \leq |S_1| + |S_2| - 1 \leq (n_1 + n_2) + (k_1 + k_2) + 1,$$

preserves the trinomial property of every equation, has block-diagonal Jacobian, and multiplies the nondegenerate positive root counts. Thus the combined system has dimension

$$n = r \cdot 2(2m-1) + (k \bmod 2)(2m-1) = k(2m-1),$$

total Laurent monomial count at most $n + k + 1$, and at least $R(k, m) = (5m^2)^r (2m)^{k \bmod 2}$ nondegenerate positive roots. Applying a single uniform clearing monomial x^s yields a genuine trinomial polynomial system in $\mathbb{N}^n$ with at most $n+k+1$ monomials and $R(k, m)$ nondegenerate positive roots. Since $\lfloor (n+k)/k \rfloor = \lfloor (k(2m-1)+k)/k \rfloor = 2m$, we obtain

$$4^r R(k, m) = (20m^2)^r (2m)^{k \bmod 2} = 5^r (4m^2)^r (2m)^{k \bmod 2} = 5^r (2m)^k = 5^{\lfloor k/2 \rfloor} \left\lfloor \frac{n+k}{k} \right\rfloor^k,$$

proving Theorem 1.3.

6. Formal Verification in Lean 4 and Independent Numerical Certification

Every proof in this paper is backed by an open-source, three-layer verification suite hosted at `https://github.com/ivan-fr/fewnomial-folded-haas`.

6.1. **Lean 4.36.0 / Mathlib4 formalization.** The Lean 4 project (`lean/FoldedHaas/`) formally defines genuine multivariate polynomial systems `SparseSystem n` with nonnegative integer exponents $\alpha : \mathrm{Fin}\, n \to \mathbb{N}$, exact calculus partial derivatives, the $n \times n$ Jacobian matrix `jacobian sys x`, the total monomial support `totalMonomials sys`, the trinomial predicate `IsTrinomialSystem sys`, and nondegenerate positive root counts `HasAtLeastRoots sys R`. It is organized into four modules:

- `PrimalSystem.lean`: Multivariate polynomial evaluation, exact partial derivative verification, Laurent-to-polynomial uniform clearing, disjoint products, and trivial padding.
- `ExplicitK1.lean`: Complete proof of Theorem 1.1 (`explicit_k1_trinomial_system`) via the unimodular change of variables $A \in \mathrm{GL}_n(\mathbb{Z})$.
- `BaseHaas.lean`: Complete 2D Newton–Banach contraction proof of Proposition 3.1 (`base_haas_five_perturbed_roots`).
- `FoldedTheorem.lean`: Multiscale folding identities (Lemma 4.1), Theorem 1.2 (`folded_haas_k2_bound`, `folded_haas_k2_all_n`), Theorem 1.3 (`fewnomial_lower_bound_all_k`), and explicit record theorems `B_2_2_ge_5` through `B_14_2_ge_80`.

Running `#print axioms` on all main theorems confirms zero `sorryAx`, zero `native_decide`, and dependence strictly on the standard Lean axioms `[propext, Classical.choice, Quot.sound]`.

6.2. **Independent numerical certifications.** To provide concrete numerical realizations alongside the symbolic Lean proofs, we include two independent verifiers:

(1) **Certified polyhedral homotopy (`pandsysroots v0.33.0`):** The Rust binary `src/main.rs` certifies six explicit primal trinomial systems via polyhedral homotopy and Krawczyk interval verification: Theorem 1.1 for $B(1,1) \geq 2$, $B(2,1) \geq 3$, $B(3,1) \geq 4$, $B(4,1) \geq 5$, and Theorem 1.2 for $B(2,2) \geq 5$ and $B(4,2) \geq 10$ (clearing all 10/10 positive roots out of 501 certified complex paths in 2.50 s with maximum interval radius 3.67×10^{-11} and maximum primal residual 8.88×10^{-16}).

(2) **220-digit arbitrary-precision verification:** The pure-Python script `verify_220digits.py` verifies primal polynomial residuals and exact Jacobian determinants at 240-digit working precision for Theorem 1.1 ($n = 1, \ldots, 6$, residuals $\leq 1.41 \times 10^{-238}$) and Theorem 1.2 up to $(m, l) = (4, 4)$, certifying all 80 nondegenerate positive roots of the 14-variable, 17-monomial trinomial system with maximum residual $\leq 2.52 \times 10^{-231}$ and minimum pairwise separation ≥ 0.162.

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