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On the Hernandez–Leclerc formula

Abstract. We prove the Hernandez–Leclerc geometric q-character formula for every real simple module in $\mathcal{C}^-$, in all finite Dynkin types and at every $q \in \mathbb{C}^\times$ that is not a root of unity. The coefficients are ordinary Euler characteristics of quiver Grassmannians of the prescribed generic injective kernel.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, with simple-root index set I, and let $q \in \mathbb{C}^\times$ be not a root of unity. We consider finite-dimensional type 1 modules over the untwisted quantum loop algebra $U_q(L\mathfrak{g})$. Drinfeld's current realization [6] supplies the currents defining highest ℓ-weights, and Chari and Pressley [3, Theorem 3.3] classify the simple modules by tuples of Drinfeld polynomials. Their spectral parameters are encoded by formal variables $Y_{i,a}$, where $i \in I$ and $a \in \mathbb{C}^\times$. A finite product of these variables with nonnegative exponents is a *dominant monomial*; we denote by $L(m)$ the simple module with highest monomial m.

The q-character introduced by Frenkel and Reshetikhin [11] records the dimensions of the simultaneous generalized ℓ-weight spaces as coefficients of Laurent monomials in the $Y_{i,a}$. It defines an injective ring homomorphism from the Grothendieck ring, whose multiplication comes from tensor product, to a Laurent polynomial ring [11, Theorem 3]. Thus it retains the spectral information in addition to the ordinary weight multiplicities. Frenkel and Mukhin [10] developed the combinatorics of these characters through simple-root monomials and screening operators. In particular, the lower monomials of $\chi_q L(m)$ are expressed as m times products of inverse simple-root monomials. Computing their coefficients is the character problem considered here.

In simply laced type, Nakajima [24] constructed finite-dimensional quantum affine representations from quiver varieties. The geometry describes standard modules and the intersection-cohomology multiplicities which determine their simple constituents. His t-analogs of q-characters refine these multiplicities and give a bar-invariant triangular basis whose specialization at $t = 1$ is the simple-character basis [25, Theorem 8.1]. Hernandez [17] developed an algebraic theory of (q, t)-characters for general finite type and formulated the corresponding canonical-specialization conjecture. This question concerns the identification of a deformed basis with the characters of actual simple modules.

A second geometric character construction arose from cluster algebras. Fomin and Zelevinsky [9] introduced these algebras by iterated exchange relations. Caldero and Chapoton [1] expressed cluster variables through Euler characteristics of quiver Grassmannians. For a representation M, these are varieties of subrepresentations with a fixed dimension vector. Derksen, Weyman and Zelevinsky [4, 5] extended representation mutation to quivers with potentials and established mutation formulas for the associated F-polynomials, degrees and E-invariants. These formulas provide the representation-theoretic counterpart of seed mutation that will be used below.

A simple module L is *real* if $L \otimes L$ is simple. The distinction between real and imaginary basis elements was developed by Leclerc [23] in the study of multiplication in the dual canonical basis. Hernandez and Leclerc [19] introduced finite spectral categories and proposed their monoidal categorification by cluster algebras. A cluster monomial is a product, with nonnegative exponents, of variables belonging to one seed. Their program relates these monomials to real

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simple modules. At level one, Hernandez and Leclerc proved the correspondence in types A_n and D_4, and Nakajima extended it to all ADE types [19, 26]. Hernandez's tensor-product theorem [18] also implies that every positive tensor power of a real simple module is simple.

In [20], Hernandez and Leclerc passed to the category $\mathcal{C}^-$. Its simple constituents have highest monomials in one fixed integral spectral component, with labels Y_{i,q^s} for $s \leq 0$. Their cluster algorithm computes the characters of Kirillov–Reshetikhin modules in every finite Dynkin type [20, Theorem 3.1]; their Theorem 4.8 gives a geometric formula for those modules. The proposed extension to all real simples is Conjecture 5.3 of that paper. To state it, let χ_q^- delete from a q-character every monomial containing a variable with positive spectral label. The exponents of m in the initial Kirillov–Reshetikhin monomials prescribe two injective modules; $K(m)$ is the kernel of a generic map from the negative part to the positive part. Let χ denote ordinary topological Euler characteristic and $\mathrm{Gr}_e K(m)$ the variety of subrepresentations with dimension vector e. With the simple-root monomials $A_{i,p}$ of (2.2), the conjecture is

$$\chi_q^-(L(m)) = m \sum_e \chi(\mathrm{Gr}_e K(m)) \prod_{(i,p)} A_{i,p}^{-e_{i,p}} \tag{1.1}$$

for every real simple $L(m) \in \mathcal{C}^-$. The vertices (i,p), the injective multiplicities and the boundary convention are specified in Section 2. The neighboring Conjecture 5.2 of [20] concerns the correspondence between real simple modules and cluster monomials.

Several families beyond the original Kirillov–Reshetikhin modules have been treated. Duan and Schiffler [7] proved the generic-kernel formula for prime snake modules of types A and B. Guo, Duan and Luo [16] obtained formulas for type-A Hernandez–Leclerc modules, and Tong, Duan and Luo [31] treated minimal affinizations of type G_2. More generally, Cao, Fujita and Murakami [2, Theorems 1.10 and 6.34] give an ordinary-Euler geometric q-character formula for every *reachable* simple, meaning one represented by a cluster monomial reached from the prescribed initial seed. Their Grassmannian module corresponds to the degree-zero cohomology of a rigid two-term injective complex under the identification with graded modules over a generalized preprojective algebra.

Finite spectral windows provide a setting for comparing these constructions. Duan and Schiffler [8] formulated real-simple and geometric-character conjectures for simply laced height categories; the sink–source, infinite-level specialization recovers (1.1). Qin [27] constructed common triangular bases compatible with mutation and proved monoidal categorification results for simply laced quantum affine categories. Fujita, Hernandez, Oh and Oya [12, 13] constructed isomorphisms among quantum Grothendieck rings and their cluster structures. Their finite-word construction retains the frozen variables, which record the ends of the spectral windows. The compatibility of these isomorphisms with canonical bases permits the transfer of complete degree labels between different types.

The canonical-specialization theorem is known in ADE from [25], in type B from [12], and in all classical types from Fujita and Qin [14, Theorem 2.13]. Tang [30, Theorem 1.1] proves it in every finite Dynkin type and at every non-root-of-unity $q \in \mathbb{C}^\times$. Consequently the canonical-basis isomorphisms of [13] specialize to isomorphisms preserving actual simple classes and their tensor product multiplicities. Appendix B supplies the corrected Laurent calculation for the G_2 generation argument in [30, Proposition 12.2].

The other immediate input is Juan's finite ADE realization theorem [22, Theorem 3.13, Proposition 6.6 and Corollary 6.7]. It assigns to every real simple in a finite ADE window a two-term presilting object, together with its decorated representation and its generic injective-kernel formula. Frozen tensor factors are included. The resulting self E-invariant vanishes, and its character coefficients are ordinary Euler characteristics. Section 3 states the precise input; Appendix A gives the coefficient specialization and the finite-kernel comparison.

Recent related results concern reachability and open orbits. Smith [29] proves reachability in finite ADE windows when each component of the high-defect subquiver is mutation-acyclic or has a dense cluster g-vector fan. He also proves reachability under a prehomogeneous separator condition on Poincaré–Birkhoff–Witt parameters. His open-orbit result [28] identifies specialized dual canonical and dual semicanonical elements on open-orbit components for symmetric Cartan data. The orbit considered in the following theorem lies in the Hom-space of the injective presentation fixed by the original highest monomial.

Theorem 1.1. *Let $\mathfrak{g}$ have any finite Dynkin type and let $q \in \mathbb{C}^\times$ be not a root of unity. For every real simple $L(m) \in \mathcal{C}^-$, the space of homomorphisms between the two injective modules prescribed by m has a unique dense open change-of-basis orbit. Its kernel $K(m)$ is finite-dimensional and satisfies* (1.1)*, with all the conventions of Hernandez–Leclerc* [20, Conjecture 5.3].

The proof transports a real simple to a finite ADE window, applies the realization theorem, and mutates the resulting decorated representation back while retaining every frozen degree coordinate. Full column rank of the extended exchange matrix then recovers the degree prescribed by m. The vanishing self E-invariant identifies the source and target injectives and gives an open orbit of differentials.

We then lower the finite-window boundary while keeping the upper boundary fixed. A bound on original path lengths stabilizes the relevant Hom and End spaces. The coefficient of the full dimension vector in a quiver-Grassmannian polynomial is one, so a fixed F-polynomial forces the increasing finite kernels to stabilize as submodules. This identifies their limit with the original $K(m)$. Sections 4–8 give these steps, and the parameter comparison in Section 3 extends the formula to every non-root-of-unity complex q.

2. The original quiver and the prescribed kernel

2.1. **Vertices and spectral labels.** Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $[b]_+ = \max(b, 0)$, and δ_{ij} is the Kronecker delta. Let $C = (c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{g}$. Choose positive integers d_i, with $\min_i d_i = 1$, such that $B = DC = (b_{ij})$ is symmetric, where $D = \operatorname{diag}(d_i)$. Write $r = \max_i d_i$; thus $r \in \{1, 2, 3\}$. We write $i \sim j$ when i, j are adjacent in the Dynkin diagram, and put $q_i = q^{d_i}$. All quantum-loop modules considered here are finite-dimensional and of type 1: the Cartan generators act semisimply with eigenvalues in $q_i^{\mathbb{Z}}$.

Let $Y_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$, be independent formal variables encoding the fundamental highest ℓ-weights. A monomial is *dominant* if its exponents are nonnegative. All monomials have finite support. The highest-ℓ-weight classification assigns a unique simple module $L(m)$ to each dominant monomial m [3, Theorem 3.3]. For a module M, write M_μ for its simultaneous generalized Cartan-current eigenspace with ℓ-weight monomial μ. Its q-character is

$$\chi_q(M) = \sum_\mu \dim_{\mathbb{C}}(M_\mu)\, \mu.$$

For a tensor subcategory $\mathcal{C}$ closed under subquotients, its Grothendieck ring $K_0(\mathcal{C})$ is generated by module classes with the relations from short exact sequences; multiplication is $[M][N] = [M \otimes N]$. Its additive basis consists of the simple classes. The q-character is an injective ring homomorphism [11, Theorem 3].

Consider the quiver on $I \times \mathbb{Z}$ with arrows

$$(i, p) \longrightarrow (j, p + b_{ij}) \qquad \text{whenever } b_{ij} \neq 0.$$

It has two isomorphic connected components [20, Lemma 2.2]. Fix one, with vertex set V, and put

$$W = \{(i, p + d_i) : (i, p) \in V\}, \qquad W^- = \{(i, s) \in W : s \le 0\},$$
$$V^- = \{(i, p) \in V : p + d_i \le 0\}. \tag{2.1}$$

The full subquiver on V^- is denoted by Γ^-. Its arrow set is Γ_1^-. We denote the length-zero path at a vertex v by e_v and compose paths from right to left: $\beta\alpha$ traverses α first. Its vertices are potential vertices. The corresponding spectral vertex is $(i, p + d_i)$. In particular, the upper boundary is $p + d_i \le 0$.

We write $Y_{i,s} = Y_{i,q^s}$. For $(i, p) \in V$, set

$$A_{i,p} = Y_{i,p-d_i} Y_{i,p+d_i} \prod_{j \ne i} \prod_{k=1}^{-c_{ji}} Y_{j,p-c_{ji}+1-2k}^{-1}. \tag{2.2}$$

An empty product is 1. This is precisely the root convention of [20, (6), (15)].

The root monomials are algebraically independent. To see this, note that $-c_{ji} \le d_i$ on every finite-type edge. Thus each neighboring label in $A_{i,p}$ is at most $p - c_{ji} - 1 < p + d_i$; its unique variable at the largest label is $Y_{i,p+d_i}$, with exponent one. In a nontrivial finite multiplicative relation $\prod A_{i,p}^{n_{i,p}} = 1$, choose the largest $H = p + d_i$ with $n_{i,p} \ne 0$. The exponent of $Y_{i,H}$ is exactly $n_{i,H-d_i}$, so one of these exponents is nonzero, a contradiction. Distinct polynomial monomials in the $A_{i,p}$ therefore remain distinct Laurent monomials in the Y-variables, which also proves algebraic independence.

The category $\mathcal{C}^-$ consists of representations whose simple constituents have dominant highest monomials in the variables $Y_{i,s}$, $(i, s) \in W^-$. The truncation χ_q^- deletes from χ_q every monomial containing a variable $Y_{i,s}$ with $s > 0$ to a nonzero power. It is an injective ring homomorphism on $K_0(\mathcal{C}^-)$, by [20, Proposition 3.10].

2.2. **The potential.** For each adjacent ordered pair i, j, consider every cycle contained in Γ^- of the form

$$(i, p) \longrightarrow (j, p + b_{ij}) \longrightarrow (i, p + 2b_{ij}) \longrightarrow (i, p + 2b_{ij} + 2d_i) \longrightarrow \cdots \longrightarrow (i, p).$$

There are $-c_{ij}$ vertical arrows and two oblique arrows. Let S^- be the sum of these cycles, each taken once up to cyclic permutation. For a cycle $a_n \cdots a_1$, its cyclic derivative is

$$\partial_a(a_n \cdots a_1) = \sum_{k:\, a_k = a} a_{k-1} \cdots a_1 a_n \cdots a_{k+1}.$$

Every arrow occurs in only finitely many summands of S^-. Consequently all cyclic derivatives are finite sums of paths, and the ordinary path-algebra quotient

$$A^- = \mathbb{C}\Gamma^- / (\partial_a S^- : a \in \Gamma_1^-) \tag{2.3}$$

is defined. This is the algebra of [20, Definition 4.1]. We regard its modules as quiver representations; an arrow from u to v acts from the space at u to the space at v.

Let I_v be the injective representation with simple socle at v. In these conventions its space at u is the dual of the space of paths from u to v, modulo the relations. It satisfies

$$\mathrm{Hom}_{A^-}(N, I_v) \simeq \mathrm{D}\, N_v, \qquad \mathrm{D} = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C}). \tag{2.4}$$

The path estimates in Section 7 will in particular show that the fixed path spaces needed here are finite-dimensional.

2.3. **The degree and the two injective terms.** For $(i,s) \in W^-$, define

$$z_{i,s} = \prod_{\substack{k\geq 0 \\ s+2kd_i \leq 0}} Y_{i,s+2kd_i}. \tag{2.5}$$

Set $z_{i,t} = 1$ for $t > 0$. Then $Y_{i,s} = z_{i,s}/z_{i,s+2d_i}$. If

$$m = \prod_{(i,s)\in W^-} Y_{i,s}^{u_{i,s}}, \qquad u_{i,s} \in \mathbb{N},$$

with finite support, extend u by zero and put

$$g_{i,s} = u_{i,s} - u_{i,s-2d_i} \quad ((i,s) \in W^-), \qquad m = \prod_{(i,s)\in W^-} z_{i,s}^{g_{i,s}}. \tag{2.6}$$

Only coordinates in W^- occur in this expression. The prescribed injectives are

$$I(m)^- = \bigoplus_{(i,s)\in W^-} I_{i,s-d_i}^{[-g_{i,s}]_+}, \qquad I(m)^+ = \bigoplus_{(i,s)\in W^-} I_{i,s-d_i}^{[g_{i,s}]_+}. \tag{2.7}$$

The sign convention is that a generic map goes from $I(m)^-$ to $I(m)^+$.

For a finite-dimensional representation M, its ordinary F-polynomial is

$$F_M(y) = \sum_{e\in\mathbb{N}^{(V^-)}} \chi(\mathrm{Gr}_e\, M) y^e, \qquad y^e = \prod_{v\in V^-} y_v^{e_v}. \tag{2.8}$$

Here $\mathrm{Gr}_e\, M$ is the projective variety of subrepresentations of dimension vector e, and $\mathbb{N}^{(V^-)}$ denotes finitely supported nonnegative vectors. Only finitely many e occur. In particular, $F_M(0) = 1$. All Euler characteristics in this paper are the ordinary ones; for these projective varieties they also equal Euler characteristics with compact support.

Hernandez–Leclerc define $K(m)$ as the kernel of a generic map in $\mathrm{Hom}_{A^-}(I(m)^-, I(m)^+)$. Their genericity means a dense open set on which the finite kernel has constant F-polynomial. We prove the stronger statement that, for a real simple, this affine space has a unique dense open orbit. Thus its generic kernel has a well-defined isomorphism class.

Example 2.1. In type A_1, choose the branch with even spectral labels. For $m = Y_s$, $s \leq -2$, one has $I(m)^- = I_{s+1}$ and $I(m)^+ = I_{s-1}$. The vertical quiver has arrows $p \to p+2$; the generic map is the natural surjection, and its kernel is one-dimensional at $s+1$. Consequently

$$mF_{K(m)}(A^{-1}) = Y_s(1 + A_{s+1}^{-1}) = Y_s + Y_{s+2}^{-1}.$$

For $s = 0$, the negative degree at $s+2$ is outside W^-; then $I(m)^- = 0$, $K(m) = 0$, and the truncated character is Y_0. This also illustrates the vertex shift in (2.7).

3. Parameters and representation-theoretic inputs

3.1. **Formal and complex parameters.** The *integral spectral category* $\mathcal{C}_{\mathbb{Z}}(q)$ is the full Serre tensor category whose simple constituents have highest monomials in Y_{i,q^s}, $i \in I$, $s \in \mathbb{Z}$. The finite-seed results used below are formulated over $\overline{\mathbb{Q}(q)}$. We compare their integral spectral categories with those at complex parameters.

Lemma 3.1. *Choose $q_0 \in \mathbb{C}^\times$ transcendental over $\mathbb{Q}$. The embedding $\mathbb{Q}(q) \to \mathbb{C}$, $q \mapsto q_0$, extends to $k = \overline{\mathbb{Q}(q)}$. Extension of scalars identifies the simple bases, tensor-product composition multiplicities, and q-character coefficients of the integral spectral categories over k and at q_0. It preserves every finite spectral window and the property of being real.*

Proof. The extension of the field embedding exists because $\mathbb{C}$ is algebraically closed. Let M be a finite-dimensional simple module over k. Its action image $R \subseteq \mathrm{End}_k(M)$ is a finite-dimensional algebra acting faithfully and irreducibly. Its radical annihilates M, hence is zero. The Artin–Wedderburn decomposition of R, together with the faithfulness and simplicity of M, consists of a single full matrix algebra over k. Thus $R = \mathrm{End}_k(M)$. After extending scalars its image is still the full matrix algebra, so $\mathbb{C} \otimes_k M$ is simple.

The highest ℓ-weight classification, recalled in [20, §2.3], labels these simples by dominant monomials. Extension of scalars sends Y_{i,q^s} to Y_{i,q_0^s}. Every complex simple with one of these labels is the scalar extension of the formal simple with that label, by uniqueness in the classification. Distinct labels remain distinct. Extending a composition series is exact and keeps all its factors simple, so all composition multiplicities, including those of tensor products, are unchanged.

The Cartan currents on a fixed finite-dimensional module generate a finite-dimensional commutative action algebra. Since k is algebraically closed, its simultaneous generalized eigenspaces remain the same spaces after scalar extension. Their dimensions and distinct eigencharacters are preserved. This proves the assertion for q-characters. The window condition is a condition on the highest labels, and reality is the condition that the square of a simple class is a single simple class. □

Lemma 3.2. *Let $q, q' \in \mathbb{C}^\times$ be not roots of unity. There is an isomorphism of based Grothendieck rings*

$$K_0(\mathcal{C}_{\mathbb{Z}}(q)) \longrightarrow K_0(\mathcal{C}_{\mathbb{Z}}(q'))$$

which sends $[L(\prod Y_{i,q^s}^{u_{i,s}})]$ to $[L(\prod Y_{i,(q')^s}^{u_{i,s}})]$ and preserves every q-character coefficient. It preserves reality and the original truncation.

Proof. Write $q = \exp(\pi\sqrt{-1}\,\hbar)$. Since q is not a root of unity, $\hbar \notin \mathbb{Q}$. The set

$$\Pi_\hbar = \frac{\hbar}{2}\mathbb{Z}$$

satisfies $(\Pi_\hbar - \Pi_\hbar) \cap \mathbb{Z} = \{0\}$ and $\Pi_\hbar \pm \hbar/2 = \Pi_\hbar$. Its exponential image is $q^{\mathbb{Z}}$. The theorem of Gautam–Toledano Laredo in [15, §6.3] therefore gives the exact equivalence between the corresponding Yangian and quantum-loop spectral categories. Write $X_{i,a}$ for the additive highest-weight variable of a Yangian Drinfeld root a. The proposition in [15, §7.7] gives the q-character diagram, with

$$X_{i,a} \longmapsto Y^{\mathrm{raw}}_{i,\exp(2\pi\sqrt{-1}\,a)},$$

and explicitly identifies the induced map as an isomorphism of commutative Grothendieck rings.

Let $Y_\hbar(\mathfrak{g})$ denote the Yangian in the current presentation of [15, §2.2], with modes $x^\pm_{i,n}, \xi_{i,n}$, $n \in \mathbb{N}$, and degree-zero Cartan generators. Its Cartan current is

$$\xi_i^{(\hbar)}(u) = 1 + \hbar \sum_{n \geq 0} \xi_{i,n} u^{-n-1}.$$

There is an algebra isomorphism fixing the Cartan generators,

$$Y_\hbar(\mathfrak{g}) \longrightarrow Y_1(\mathfrak{g}), \qquad x^\pm_{i,n} \longmapsto \hbar^n x^\pm_{i,n}, \qquad \xi_{i,n} \longmapsto \hbar^n \xi_{i,n}.$$

For nonnegative integers a, b, the defining relation

$$[\xi_{i,a+1}, x^\pm_{j,b}] - [\xi_{i,a}, x^\pm_{j,b+1}] = \pm\,\frac{\hbar d_i c_{ij}}{2}(\xi_{i,a} x^\pm_{j,b} + x^\pm_{j,b}\xi_{i,a})$$

acquires the factor $\hbar^{a+b+1}$ on both sides. The root-current relation with $x^\pm_{i,a}$ in place of $\xi_{i,a}$ has the same factor. The relation

$$[x^+_{i,a}, x^-_{j,b}] = \delta_{ij}\xi_{i,a+b}$$

acquires $\hbar^{a+b}$. Each Serre relation has the common factor $\hbar^{a_1+\cdots+a_{1-c_{ij}}+b}$, and the Cartan relations are homogeneous with the Cartan generators in degree zero. These are all families of defining relations. Since $\hbar \neq 0$, the inverse scales the mode of index n by $\hbar^{-n}$. The displayed current is sent to $\xi_i^{(1)}(u/\hbar)$. Pullback thus sends a Yangian spectral root a to $\hbar a$. All the integral quantum-loop categories consequently have the same Yangian model, with roots in $\frac{1}{2}\mathbb{Z}$.

The superscript raw records the polynomial-root convention in [15, §7]. The factor

$$q_i^{-1}\frac{q_i^2 z - a}{z - a} = q_i \frac{z - a q_i^{-2}}{z - a}$$

has symmetric center aq_i^{-1}, where $q_i = q^{d_i}$. Thus the original variable Y_{i,q^s} corresponds to raw root q^{s+d_i}, and hence to the common Yangian label $(s + d_i)/2$. This shift is the same at q and q'. The resulting identification preserves exactly the integer labels s in Section 2.

Exactness preserves the simple bases. The q-character diagram preserves multiplication and coefficients. A tensor square has length one exactly when its Grothendieck class is one simple basis element, so reality is preserved. Deletion of monomials with labels $s > 0$ is also preserved. □

It is therefore enough to prove Theorem 1.1 at the transcendental parameter q_0. The right-hand side depends only on C, integer labels and the fixed potential. From now until the last step of the proof we work at q_0, using Lemma 3.1 when applying formal-parameter results.

3.2. **Seeds and representation-theoretic inputs.** An *ice seed* consists of algebraically independent variables $x = (x_u)_{u\in K}$, a subset $J \subseteq K$ of mutable vertices, and an extended exchange matrix $\widetilde{B} = (b_{uv})_{u\in K, v\in J}$. The vertices in $K \setminus J$ are frozen. Our sign convention is

$$b_{uv} = \#\{v \longrightarrow u\} - \#\{u \longrightarrow v\}.$$

Its principal quiver is the full quiver on J. For $a \in \mathbb{Z}^K$, set

$$x^a = \prod_{u\in K} x_u^{a_u}, \qquad \widehat{y}_v = \prod_{u\in K} x_u^{b_{uv}} \quad (v \in J).$$

Mutation at $k \in J$ replaces x_k and the extended matrix by

$$x_k' = x_k^{-1}\left(\prod_{i\in K} x_i^{[b_{ik}]_+} + \prod_{i\in K} x_i^{[-b_{ik}]_+}\right), \qquad x_i' = x_i \quad (i \neq k),$$

and, for $i \in K$, $j \in J$,

$$b_{ij}' = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & \text{otherwise.} \end{cases}$$

The Laurent polynomial representing a Grothendieck class in the specified seed is denoted by $\mathrm{char}(L)$.

Let A be a finite-dimensional Jacobian algebra on a finite vertex set J, with indecomposable injectives I_v, $v \in J$. A *decorated representation* $\mathcal{M} = (M, V_0)$ consists of a finite-dimensional A-module M and a tuple $V_0 = ((V_0)_v)_{v\in J}$ of finite-dimensional vector spaces. Its F-polynomial is F_M. Take the minimal injective copresentation

$$0 \longrightarrow M \longrightarrow I_M^0 \xrightarrow{d} I_M^1.$$

Here I_M^0 is the injective envelope of M, I_M^1 is the injective envelope of I_M^0/M, and d is the quotient map followed by that envelope inclusion. Form the two-term complex, in degrees $0, 1$,

$$\Phi(\mathcal{M}) = (I_M^0 \xrightarrow{d} I_M^1) \oplus (0 \longrightarrow I_{V_0}), \qquad I_{V_0} = \bigoplus_{v\in J} I_v^{\dim(V_0)_v}. \tag{3.1}$$

Write inj$A$ for the category of finite-dimensional injective A-modules and $K^b(\mathrm{inj}A)$ for its bounded homotopy category: chain maps are identified modulo chain homotopy. We use the cohomological shift $X[1]^j = X^{j+1}$, with differential $-d_X^{j+1}$. Write $\Phi(\mathcal{M}) = (I^0 \to I^1)$ and identify injective classes with their multiplicity vectors on J. The degree and self E-invariant are

$$g(\mathcal{M}) = [I^1] - [I^0], \qquad E(\mathcal{M}) = \dim \mathrm{Hom}_{K^b(\mathrm{inj}A)}(\Phi(\mathcal{M}), \Phi(\mathcal{M})[1]). \tag{3.2}$$

These are [2, (4.3)–(4.4), Appendix A.19–A.20]. In particular, their E is the self E preserved in [5, Theorem 7.1]. The kernel of (3.1) is M. A two-term complex is *presilting* when its Hom to every positive shift of itself vanishes. For a complex in degrees $0, 1$, shifts by at least two have no overlapping nonzero terms, so this condition is equivalent to $E = 0$.

Theorem 3.3 (ADE realization)**.** *Let Δ be a simply laced finite Dynkin diagram, choose bipartite heights $\xi_i \in \{0, -1\}$ with $|\xi_i - \xi_j| = 1$ on each edge, and let $\ell \geq 1$ be an integer. Consider the finite window with spectral labels*

$$\xi_i - 2\ell,\ \xi_i - 2\ell + 2,\ \ldots,\ \xi_i.$$

In its standard Hernandez–Leclerc seed, let $L(m)$ be any real simple. There is a decorated representation $\mathcal{M} = (M, V_0)$ of the principal Jacobian algebra such that

$$E(\mathcal{M}) = 0, \qquad \mathrm{char}(L(m)) = x^{\widetilde{g}} F_M(\widehat{y}). \tag{3.3}$$

The mutable part of $\widetilde{g}$ is the injective degree of $\mathcal{M}$; all frozen coordinates are included. The polynomial F_M uses ordinary Euler characteristics. The conclusion holds with frozen tensor factors and decoration.

This is the combined output of [22, Theorem 3.13, Proposition 6.6, and Corollary 6.7]. Theorem 3.13 constructs a two-term presilting complex, so (3.2) gives self E-invariant zero. Proposition 6.6 identifies the original deletion truncation and prescribed finite kernel. Corollary 6.7 restores the frozen factors. The coordinate there denoted s for a mutable vertex corresponds to the potential coordinate $p = s - 1$ here. Appendix A details the Euler and boundary conventions.

Let $\mathcal{K}_t(\mathfrak{g})$ be the quantum Grothendieck ring, with canonical basis $L_t(m)$ indexed by dominant monomials. The parameter t here is the deformation parameter of this ring, and is distinct from the quantum-loop parameter q.

Theorem 3.4 (Actual simple specialization)**.** *For every finite Dynkin type and every dominant monomial m,*

$$L_t(m)\big|_{t=1} = \chi_q L(m).$$

We use [30, Theorem 1.1], with the Laurent generation calculation of Proposition 12.2 supplied in Appendix B. The statement concerns the ordinary finite-dimensional type 1 simple with highest monomial m. The classical cases are also covered by [14, Theorem 2.13].

Theorem 3.4 changes the canonical-basis isomorphisms of [13] into actual-simple based-ring isomorphisms at $t = 1$.

4. **Cofinal finite windows**

4.1. **The standard finite seed.** For integers $a \leq 0$, put

$$K_a = \{(i,p) \in V : a \leq p + d_i \leq 0\}, \tag{4.1}$$

$$J_a = \{(i,p) \in V : a \leq p - d_i < p + d_i \leq 0\}. \tag{4.2}$$

The rows of the finite standard seed are K_a, and its mutable columns are J_a. The frozen vertices are $K_a \setminus J_a$. Only the principal quiver on J_a is used to form its Jacobian algebra, denoted by

A_a. Initially the potential may be written with the orientation signs of [2]. Lemma 5.2 below identifies it with the restriction of S^-.

Write $\widetilde{B}_a = (b^a_{uv})_{u\in K_a, v\in J_a}$ for its extended exchange matrix, and

$$\widehat{y}_v = \prod_{u\in K_a} x_u^{b^a_{uv}} \quad (v \in J_a).$$

The standard monomial transformation is

$$x_{i,p} \longmapsto z_{i,p+d_i}, \qquad \widehat{y}_{i,p} \longmapsto A^{-1}_{i,p}. \tag{4.3}$$

All variables in the first product (2.5) lie within $[a, 0]$ when $(i, p) \in K_a$.

Let $\mathcal{C}_{[a,0]}$ be the Serre tensor subcategory whose highest monomials have labels in this window. The finite cluster algebra is isomorphic to $K_0(\mathcal{C}_{[a,0]})$. We use the convention that frozen variables are not inverted. For every simple $L(m)$ in this category, [2, Lemma 5.25] gives

$$\mathrm{char}_a(L(m)) = x^{g^a(m)} P_m(\widehat{y}), \qquad g^a(m)_{i,p} = u_{i,p+d_i} - u_{i,p-d_i}, \tag{4.4}$$

where

$$\chi_q^-(L(m)) = mP_m(A^{-1}), \qquad P_m(0) = 1. \tag{4.5}$$

The polynomial P_m is supported on J_a; the independence proved in Section 2 makes (4.5) unique. The cited lemma applies to every simple and uses the ring isomorphism of [20, Theorem 5.1].

It is useful to spell out the dependence on a. The definition of χ_q^- deletes only spectral labels above zero. For a fixed $L(m)$ it therefore does not change when a decreases. Since the q-character has finitely many monomials, the polynomial P_m in (4.5) has fixed finite support. All sufficiently deep windows contain that support and the support of (2.6). We regard the same P_m as a polynomial in larger sets of mutable variables by giving the new variables degree zero.

4.2. **Even prefixes and their boundaries.** We choose a cofinal sequence of the windows (4.1)–(4.2) to which the finite isomorphisms of [13] apply.

Proposition 4.1. *There is a strictly decreasing sequence of integers $(a_n)_{n\geq 1}$ tending to $-\infty$ such that finite mutation and relabeling give isomorphisms between the standard seed for $\mathcal{C}_{[a_n,0]}$ and bipartite ADE window seeds, with the following properties.*

1. *Every row, including every frozen row, is retained.*
2. *The nonlocalized finite cluster algebras are isomorphic.*
3. *At $t = 1$ these isomorphisms preserve the actual simple bases and real simples.*
4. *The extended exchange matrices have full column rank.*

Proof. We use the unfolding (Δ, σ) of [13, §4.3]. Here Δ is simply laced, σ is a diagram automorphism of order r, and the original vertex set is $I = \Delta_0/\langle\sigma\rangle$. Write $\imath \mapsto \bar{\imath}$ for the orbit projection; then d_i is the size of orbit i. The dual Coxeter number of $\mathfrak{g}$ is denoted by $h^\vee$. The nonsimply-laced data are

type of $\mathfrak{g}$	Δ	r	$h^\vee$
B_n	A_{2n-1}	2	$2n-1$
C_n	D_{n+1}	2	$n+1$
F_4	E_6	2	9
G_2	D_4	3	4

In type ADE, Δ is the original diagram and $\sigma = \mathrm{id}$. The automorphisms in the four rows are respectively reflection of A_{2n-1}, interchange of the two terminal vertices of the fork of D_{n+1}, the nontrivial automorphism of E_6, and triality of D_4. These are the conventions of [13, Table 1].

For precision, a Q-datum is a triple (Δ, σ, ξ) with integer heights satisfying the following conditions. If adjacent vertices $\imath, \jmath$ have equal orbit sizes, then $|\xi_\imath - \xi_\jmath| = d_{\bar{\imath}}$. If adjacent orbits $i = \{\imath\}$, j have sizes $1, r$, there is a unique $\jmath \in j$ such that

$$|\xi_\imath - \xi_\jmath| = 1, \qquad \xi_{\sigma^k \jmath} = \xi_\jmath - 2k \quad (1 \le k < r).$$

We require $\xi_\imath \equiv \varepsilon_{\bar{\imath}} \pmod 2$, where ε_i is the parity of the spectral labels of color i in the fixed component W.

Choose the height function of [13, Remark 5.7], so that

$$-2d_{\bar{\imath}} < \xi_\imath \le 0$$

for each unfolded vertex $\imath$. The associated infinite seed is exactly the original seed on W^-. Let w_0 be the longest Weyl group element of Δ, let N_Δ be its length, and define $\imath^*$ by $w_0(\alpha_\imath) = -\alpha_{\imath^*}$, where $\alpha_\imath$ is the corresponding simple root. An adapted word successively chooses a vertex whose height is larger than those of its neighbors and lowers that height by $2d_{\bar{\imath}}$. Take its first $2nN_\Delta$ positions, with $n \ge 1$. Denote the height change after one adapted longest-word block by $\mathscr{D}^{-1}$. Formula (4.7) and Proposition 5.1 of [13] give

$$(\mathscr{D}^{-1}\xi)_\imath = \xi_{\imath^*} - rh^\vee, \qquad \mathscr{D}^{-2n}\xi = \xi - H_n, \qquad H_n = 2nrh^\vee.$$

Two blocks remove the involution $\imath \mapsto \imath^*$.

The spectral labels in the prefix are precisely

$$\xi_\imath - H_n < s \le \xi_\imath, \qquad s \equiv \xi_\imath \pmod{2d_{\bar{\imath}}}.$$

Since $\xi_\imath$ is the largest nonpositive integer in its residue class and H_n is divisible by $2d_{\bar{\imath}}$, these are exactly the labels $-H_n < s \le 0$ in that class. The folding map is a bijection on the union of these classes. Thus, with

$$a_n = 1 - H_n, \tag{4.6}$$

the folded rows are exactly K_{a_n}. The last occurrence of each unfolded color is frozen. A label s is such a last occurrence precisely when $s - 2d_i < a_n$. Its complement is exactly J_{a_n}. This identifies the frozen boundary point by point.

On the ADE side choose the bipartite heights $0, -1$. The same $2n$ blocks have height span $2nh_\Delta$, where h_Δ is the Coxeter number, the order of a Coxeter element of Δ. In simply laced type this equals the dual Coxeter number. Each ADE column has nh_Δ positions. Consequently this is the window of Theorem 3.3 with $\ell + 1 = nh_\Delta \ge 2$.

Section 6.2 of [13] constructs the isomorphism for each finite prefix by commutations and braid moves. In a braid $iji \leftrightarrow jij$, the mutated first position has a later occurrence of its color within the same block, and hence is mutable. This remains true in the last block. The finite truncation conditions in [13, Lemmas 2.7–2.8 and Remark 2.10] are therefore satisfied. Sorting by heights or using the adapted-word order only changes a linear extension by commutations of nonadjacent colors. The complete matrices transform, so the frozen rows are retained.

The algebra convention of [13, Definition A.1] uses nonnegative powers in cluster monomials and does not invert the frozen variables. Hence the finite isomorphism identifies the actual finite Grothendieck subrings. The global isomorphism preserves the canonical bases by [13, Theorems 6.1 and 6.3]; Theorem 3.4 identifies their specializations with actual simple bases. If a simple class from the source window has image $[L']$, that image lies in the finite target subring. A Serre subcategory has Grothendieck group freely spanned by its simple classes, a subset of the global simple basis. Linear independence therefore forces L' itself to lie in the target window. If $[L]^2$ is one simple class, so is $[L']^2$; thus L' is real.

Finally, [13, Corollary 1.3] gives a compatible pair for every finite prefix. The product of the transpose of its extended exchange matrix with the compatible skew form has a nonzero

diagonal mutable block. The exchange matrix therefore has full column rank. This property is preserved by the finite seed transformations. □

5. **Transporting arbitrary decorated representations**

5.1. **The principal potentials.** The standard principal potentials in the preceding windows are rigid and Jacobi-finite by [2, Proposition 6.1]. Here Jacobi-finite means that the completed Jacobian algebra is finite-dimensional. Rigidity means that every positive-length cyclic path is, modulo cyclic equivalence, in the Jacobian ideal. The ordinary and completed Jacobian algebras agree for the standard potentials. In a finite-dimensional completed Jacobian algebra the arrow ideal is nilpotent, so its modules are the nilpotent representations used in the mutation formulas. We need to identify the potential obtained after mutation with the standard one at the endpoint.

Lemma 5.1. *On a finite loop-free quiver, two rigid Jacobi-finite potentials are formally right equivalent. The resulting equivalence preserves the ordinary quiver Grassmannians of finite-dimensional nilpotent representations, their F-polynomials, their injective degrees and their self E-invariants.*

Proof. Let $\mathfrak{m}$ be the arrow ideal in the completed path algebra. Fix a sufficiently high jet order N. The positive-length potentials modulo cyclic equivalence and $\mathfrak{m}^{N+1}$ form a finite-dimensional affine space. The group of algebra automorphisms of the truncated path algebra fixing the vertex idempotents and preserving the arrow ideal is an algebraic group.

The tangent image of its orbit at a potential S consists of cyclic classes of sums

$$\sum_a f_a\, \partial_a S,$$

where f_a is a path combination with the same endpoints as a, of positive length. By rigidity every cyclic class is a sum of classes of $u(\partial_a S)v$. Cyclic rotation replaces such a class by that of $vu(\partial_a S)$. Since a is not a loop, an endpoint-compatible coefficient vu cannot have length zero. It is therefore an allowed infinitesimal arrow substitution. The tangent image is the entire space of jets.

Over $\mathbb{C}$ an algebraic group orbit is smooth and locally closed. A full-dimensional orbit in this affine space is open. Both rigid potentials thus have nonempty open jet orbits in the same irreducible space, and these orbits intersect. Their jets are right equivalent. A jet automorphism lifts to a continuous automorphism of the completed path algebra: its invertible linear part allows the inverse to be solved successively in the arrow filtration.

Jacobi-finiteness gives $\mathfrak{m}^s$ contained in the Jacobian ideal for some s. The finite determinacy theorem [21, Theorem 3.16] says that the potential is then $(s+1)$-determined. Taking a jet order sufficient for both potentials turns the jet equivalence into formal right equivalence.

On a finite-dimensional nilpotent representation all sufficiently long paths act by zero. Every formal arrow substitution therefore acts by a finite sum. A family of vertex subspaces stable under the old arrows is stable under all the new arrows; the inverse substitution gives the converse. Thus, inside the same product of ordinary Grassmannians, the two submodule varieties agree. This argument also holds for families, so it identifies their scheme structures. The equivalence fixes the vertex idempotents and is exact. It preserves injective classes, copresentations and their homotopy spaces, and hence the degrees and E. □

Lemma 5.2. *The signed standard potential and the unsigned potential S^-, restricted to any finite principal window, are identified by an arrow rescaling which fixes every vertex. These rescalings may be chosen simultaneously on Γ^-.*

Proof. Because the diagram is a tree, choose recursively nonzero complex numbers t_i satisfying

$$t_i^{-c_{ij}} = -t_j^{-c_{ji}}$$

on every edge. At each step a nonzero root exists in $\mathbb{C}$, and the tree gives no compatibility condition around a cycle. Multiply every vertical arrow of color i by t_i. For the pair of oblique arrows across an edge i, j, choose their product of scaling factors to be $\omega_{ij}/t_i^{-c_{ij}}$, where $\omega_{ji} = -\omega_{ij}$ are the signs in the signed potential. The cycle using vertical arrows of color i then receives coefficient ω_{ij}, and the cycle using color j receives coefficient $-\omega_{ij}$. The same choice at every height gives the asserted rescaling. It restricts to every window and does not change any submodule condition. □

Under reduced mutation, Jacobi-finiteness and rigidity are preserved by [4, Corollaries 6.6 and 6.11]; rigidity also gives nondegeneracy by [4, Corollary 8.2]. Consequently, along the finite sequences of Proposition 4.1, all reduced principal potentials remain rigid and Jacobi-finite. At the endpoint Lemma 5.1 identifies the resulting principal potential with the standard one.

5.2. **Every row of the character.** For $\mathcal{M} = (M, V_0)$ and a mutable vertex k, let

$$\beta_k : M_k \longrightarrow \bigoplus_{a:k\to j} M_j, \qquad x \longmapsto (M_a x)_a, \qquad h_k(\mathcal{M}) = -\dim\ker\beta_k.$$

Lemma 5.3. *Suppose an ice seed has extended matrix $\widetilde{B}$. Let $\mathcal{M}$ be a decorated representation of its principal quiver with potential, and let $\widetilde{g}$ be an integral vector on all rows whose mutable part is its representation degree. Write*

$$X = x^{\widetilde{g}} F_M(\widehat{y}), \qquad \widehat{y}_j = \prod_i x_i^{b_{ij}}.$$

After mutation at a mutable vertex k, X has the same form for the mutated representation and seed. The new degree on every row is

$$\widetilde{g}'_k = -\widetilde{g}_k, \qquad \widetilde{g}'_i = \widetilde{g}_i + [b_{ik}]_+\widetilde{g}_k - b_{ik}h \quad (i \neq k), \tag{5.1}$$

where $h = h_k(\mathcal{M})$. In particular frozen rows obey the same formula. The self E-invariant is unchanged.

Proof. The identities of [5, Lemma 5.2] apply to arbitrary decorated representations. Write $\mu_k\mathcal{M} = (M', V_0')$ and put $h' = h_k(\mu_k\mathcal{M})$. The coefficient variables mutate by

$$y'_k = y_k^{-1}, \qquad y'_j = y_j y_k^{[b_{kj}]_+}(1+y_k)^{-b_{kj}} \quad (j \neq k).$$

The cited identities give

$$\widetilde{g}_k = h - h', \qquad (1+y_k)^h F_M(y) = (1+y_k^{-1})^{h'} F_{M'}(y').$$

These identities use ordinary Euler characteristics.

Set

$$P_+ = \prod_i x_i^{[b_{ik}]_+}, \qquad P_- = \prod_i x_i^{[-b_{ik}]_+}.$$

The exchange relation is $x_k x'_k = P_+ + P_-$, and $\widehat{y}_k = P_+/P_-$. The matrix mutation formula gives $\widehat{y}'_k = \widehat{y}_k^{-1}$ and, for $j \neq k$,

$$\begin{aligned}\widehat{y}'_j &= \widehat{y}_j(P_+ + P_-)^{-b_{kj}} P_+^{[b_{kj}]_+} P_-^{-[-b_{kj}]_+} \\ &= \widehat{y}_j \widehat{y}_k^{[b_{kj}]_+}(1+\widehat{y}_k)^{-b_{kj}}.\end{aligned}$$

Indeed, after writing $P_+ = P_-\widehat{y}_k$, the exponent of P_- is $-b_{kj} + [b_{kj}]_+ - [-b_{kj}]_+ = 0$. Thus substituting $y_j = \widehat{y}_j$ sends y'_j to $\widehat{y}'_j$. Substitution into the F-identity yields

$$\begin{aligned} x^{\widetilde{g}} F_M(\widehat{y}) &= x^{\widetilde{g}} \widehat{y}_k^{\widetilde{g}_k - h}(1+\widehat{y}_k)^{-\widetilde{g}_k} F_{M'}(\widehat{y}') \\ &= (x'_k)^{-\widetilde{g}_k} \prod_{i \neq k} x_i^{\widetilde{g}_i} P_+^{\widetilde{g}_k - h} P_-^h F_{M'}(\widehat{y}'). \end{aligned}$$

The exponent contributed to row $i \neq k$ is

$$[b_{ik}]_+(\widetilde{g}_k - h) + [-b_{ik}]_+ h = [b_{ik}]_+ \widetilde{g}_k - b_{ik} h.$$

This proves (5.1) for every row. On mutable rows it is the degree mutation formula of [5]. The self E-invariant is preserved by [5, Theorem 7.1]. □

5.3. **Recovering the original degree.**

Lemma 5.4. *Let $\widetilde{B} : \mathbb{Z}^J \to \mathbb{Z}^K$ be injective and let $g, g_0 \in \mathbb{Z}^K$. Suppose $F, F_0 \in \mathbb{Z}[y_j : j \in J]$, both with constant term 1, and*

$$x^g F(x^{\widetilde{B}}) = x^{g_0} F_0(x^{\widetilde{B}}).$$

Then $g = g_0$ and $F = F_0$.

Proof. Injectivity ensures that distinct monomials on either side remain distinct after substitution. In particular, the constant term on the left contributes x^g with coefficient 1. It occurs on the right, so $g = g_0 + \widetilde{B}e$ for some $e \in \mathbb{N}^J$. Likewise $g_0 = g + \widetilde{B}f$ for $f \in \mathbb{N}^J$. Thus $\widetilde{B}(e+f) = 0$, and injectivity gives $e + f = 0$. Since both are nonnegative, $e = f = 0$. After canceling x^g, injectivity of the monomial substitution gives $F = F_0$. □

Proposition 5.5. *Fix a real simple $L(m) \in \mathcal{C}^-$. In each sufficiently deep window a_n of Proposition 4.1, there is a decorated A_{a_n}-representation $\mathcal{M}_n = (M_n, V_n)$ such that*

$$E(\mathcal{M}_n) = 0, \qquad g(\mathcal{M}_n) = g^{a_n}(m)|_{J_{a_n}}, \qquad F_{M_n} = P_m.$$

The character with its complete degree is $x^{g^{a_n}(m)} P_m(\widehat{y})$.

Proof. Map $[L(m)]$ to the ADE window by Proposition 4.1. Its image is a real simple in that finite category. Apply Theorem 3.3, keeping its frozen degree and decoration. Mutate its representation along the inverse finite seed sequence. Lemma 5.3 preserves the complete character and $E = 0$. At the endpoint use Lemmas 5.1 and 5.2 to regard the result as a representation of A_{a_n}. These equivalences preserve its ordinary F-polynomial.

The resulting expression $x^{\widetilde{g}} F_{M_n}(\widehat{y})$ is the same algebra element as $[L(m)]$. On the other hand, its expression in the original seed is (4.4). Both F-polynomials have constant term 1, and $\widetilde{B}_{a_n}$ has full column rank. Lemma 5.4 therefore gives $\widetilde{g} = g^{a_n}(m)$ on all rows and $F_{M_n} = P_m$. This constructs the representation separately in each sufficiently deep window. Section 8 compares their kernels. □

6. **The prescribed injective presentation**

We apply the injective complexes (3.1)–(3.2) to identify the prescribed kernel.

Lemma 6.1. *Let $f : I^0 \to I^1$ be a minimal two-term complex of injective A-modules and suppose $\mathrm{Hom}_{K^b(\mathrm{inj}A)}(f, f[1]) = 0$. If $g = [I^1] - [I^0]$, then*

$$I^0 = \bigoplus_{v \in J} I_v^{[-g_v]_+}, \qquad I^1 = \bigoplus_{v \in J} I_v^{[g_v]_+}. \tag{6.1}$$

The orbit of f under $\mathrm{Aut}_A(I^1) \times \mathrm{Aut}_A(I^0)$ is the unique dense open orbit of $\mathrm{Hom}_A(I^0, I^1)$.

Proof. A chain map from f to $f[1]$ is determined by a map $I^0 \to I^1$. Writing the null-homotopies gives

$$\mathrm{Hom}_{K^b(\mathrm{inj}A)}(f, f[1]) \simeq \frac{\mathrm{Hom}_A(I^0, I^1)}{\{af - fb : a \in \mathrm{End}_A(I^1),\ b \in \mathrm{End}_A(I^0)\}}. \tag{6.2}$$

Minimality says that the differential lies in the radical of the category of finite direct sums of indecomposable injectives. That radical is a two-sided ideal under composition. If I_v occurred in both terms, the map whose corresponding component is id_{I_v} and whose other components vanish would be nonzero modulo this radical. Every $af - fb$ is radical, so this map would give a nonzero class in (6.2). This contradicts the hypothesis. The two terms therefore have disjoint indecomposable supports, which proves (6.1).

The group acts by $(u, v) \cdot f = ufv^{-1}$. Over the dual numbers $\mathbb{C}[\epsilon]/(\epsilon^2)$,

$$(1 + \epsilon a)f(1 - \epsilon b) = f + \epsilon(af - fb).$$

Its differential at the identity is therefore $(a, b) \mapsto af - fb$. Equation (6.2) says that this differential surjects onto the tangent space of the affine space $\mathrm{Hom}_A(I^0, I^1)$. The orbit is locally closed and has the full dimension of this space, hence is open. The affine space is irreducible; two nonempty open orbits would intersect and therefore be the same orbit. The open orbit is consequently unique and dense. □

Corollary 6.2. *In each sufficiently deep cofinal window a_n, the prescribed finite injective terms for the mutable degree $g^{a_n}(m)|_{J_{a_n}}$ admit a unique dense open orbit of differentials. Its kernel $K_n(m)$ satisfies*

$$F_{K_n(m)} = P_m.$$

Proof. The complex (3.1) of the representation in Proposition 5.5 is minimal, has kernel M_n, and its Hom to its shift by one vanishes by (3.2). Lemma 6.1 gives exactly the positive and negative injective terms of the prescribed mutable degree. Their generic kernel is the module part of that representation, whose F-polynomial is P_m. □

7. **Stability with a fixed upper boundary**

We now work with the original unsigned algebra A^-. By Lemma 5.2, the algebras A_a may be identified with the restrictions of its potential to J_a. We will prove the fixed-boundary stability needed to pass from Corollary 6.2 to the original kernel.

7.1. **A bound for the original paths.**

Lemma 7.1. *Fix vertices $u = (i, p)$ and $v = (j, p')$ of Γ^-. There is a bound on the lengths of all paths from u to v which are nonzero in A^-. One possible bound is*

$$\mathrm{length}(\pi) \le \frac{p' - p + (r+2)M(p')}{2}, \qquad M(p') = (r+1)\big(-p' + 2(r-1)\big). \tag{7.1}$$

Proof. A vertical arrow of color i raises the height by $2d_i$; an oblique arrow across i, j lowers it by $|b_{ij}|$. The derivative of the potential with respect to an interior oblique arrow is a two-term relation. It moves an oblique arrow $j \to i$, followed by $-c_{ij}$ vertical arrows of color i, to $-c_{ji}$ vertical arrows of color j, followed by the same oblique arrow, with a minus sign. At the upper boundary the missing term gives a zero relation. These are exactly the relations used in the proof of [20, Proposition 4.17].

For a nonzero path, apply these relations to move vertical arrows to the front as far as possible. Termination can be seen by weighting each vertical arrow by its height rise: moving a vertical block to the left of an oblique arrow strictly decreases the sum, over all oblique arrows, of the

vertical rise lying to their right. Each relation preserves the total vertical rise, the ordered oblique colors, and the number of oblique arrows.

After removing the initial vertical segment, every maximal vertical run following an oblique arrow $j \to i$ has length strictly less than $-c_{ij}$. A nonempty run can therefore occur only at a short vertex $d_i = 1$ next to a multiple edge. If another vertical run follows after just one oblique arrow, that next run can again be moved to the left and merged with the first. Thus the remaining nonempty runs may be arranged with at least two oblique arrows between them, as in [20, Proposition 4.17].

Except for a final vertical tail, group each run with the oblique arrow just before and just after it. These blocks are disjoint. In lacing 2, a run has one vertical arrow; the entering arrow drops height 2, and the leaving arrow drops at least 1. The whole block has strictly negative height change. In lacing 3, the diagram is G_2; both oblique arrows drop height 3, and the intervening run has at most two vertical arrows, so the block drops height by at least 2. Each such block has length at most $r + 1$. Other oblique arrows may be treated as single descending blocks. In the simply laced case there are only these single-arrow blocks. The final ungrouped vertical tail has at most $r - 1$ short vertical arrows and raises height by at most $2(r - 1)$.

The endpoint of the initial vertical segment has height at most zero. Since the final endpoint has height p', the total descent of the blocks is at most $-p' + 2(r - 1)$. Each block descends by at least one. If O is the number of oblique arrows, this gives

$$O \le (r + 1)(-p' + 2(r - 1)) = M(p').$$

The moves preserve O, so this bound also holds for the number of oblique arrows in the original path.

Let V_π be the number of vertical arrows in the original path. Its original height balance is

$$\sum_{\text{vertical arrows}} 2d_i = p' - p + \sum_{\text{oblique arrows}} |b_{ij}|.$$

In finite type, $|b_{ij}| \le r$ for $i \ne j$. Therefore

$$2V_\pi \le p' - p + rO, \qquad V_\pi + O \le \frac{p' - p + (r + 2)M(p')}{2}.$$

If a boundary relation occurred during the rearrangement, the original path would already be zero. For a nonzero path all the preceding estimates apply, proving (7.1). □

7.2. **Corners and injective maps.** For a finite principal set J_a, let $\mathfrak{a}_a$ be the ideal of A^- generated by the vertex idempotents outside J_a. The natural finite quotient is

$$A^-/\mathfrak{a}_a \simeq A_a. \tag{7.2}$$

Indeed, for a retained arrow, a cycle not contained in J_a contributes a derivative path through a deleted vertex and hence becomes zero. A derivative with respect to a deleted arrow has a deleted endpoint. The surviving relations are exactly the derivatives of the restricted potential.

Lemma 7.2. *For any fixed finite set $T \subset V^-$, all the maps*

$$e_u A^- e_v \longrightarrow e_u A_a e_v \qquad (u, v \in T)$$

are isomorphisms for sufficiently negative a, and they respect multiplication. These spaces are finite-dimensional.

Proof. Lemma 7.1 bounds the lengths of all nonzero paths between each fixed pair of endpoints. The quiver is locally finite, so only finitely many paths of bounded length can start at a fixed vertex. In particular each fixed corner is finite-dimensional.

For nonempty T, let L_T be a common nonnegative integer bound for these path lengths and let p_T be the least height of a vertex in T. Choose

$$a < p_T - r - 2rL_T.$$

A deleted vertex $(i,p) \in V^- \setminus J_a$ satisfies $p - d_i < a$, hence $p < a + d_i \le a + r$. Each arrow changes height by at most $2r$. A path of length at most L_T starting in T therefore has every height at least $p_T - 2rL_T > a + r$, and cannot meet a deleted vertex. Every path between vertices of T through a deleted vertex is consequently zero in A^-. Such paths span the corner of $\mathfrak{a}_a$, so that ideal has zero intersection with each of these corners. For empty T the assertion is vacuous. Surjectivity follows from (7.2). There are only finitely many pairs in T, so one bound suffices for all of them. The maps are restrictions of algebra quotient maps and preserve composition. □

Define the left exact truncation functor

$$T_a N = \mathrm{Hom}_{A^-}(A_a, N) = \{x \in N : \mathfrak{a}_a x = 0\}. \tag{7.3}$$

The second description is understood in the representation category with its vertex idempotents. It identifies $T_a N$ with the largest subrepresentation supported on J_a. For $v \in J_a$, adjunction and (2.4) show that

$$T_a I_v = I_v^a,$$

the injective with socle v over A_a.

Corollary 7.3. *Fix finite direct sums I^0, I^1 of the injectives I_v. For all sufficiently negative a, restriction identifies, compatibly with composition,*

$$\begin{gathered} \mathrm{End}_{A^-}(I^0) \simeq \mathrm{End}_{A_a}(T_a I^0), \qquad \mathrm{End}_{A^-}(I^1) \simeq \mathrm{End}_{A_a}(T_a I^1), \\ \mathrm{Hom}_{A^-}(I^0, I^1) \simeq \mathrm{Hom}_{A_a}(T_a I^0, T_a I^1). \end{gathered} \tag{7.4}$$

All these spaces are finite-dimensional. The identifications preserve the change-of-basis actions and their tangent maps.

Proof. For indecomposable terms, the adjunction (2.4) gives

$$\mathrm{Hom}_{A^-}(I_u, I_v) \simeq \mathrm{D}(I_u)_v \simeq e_u A^- e_v.$$

The last equality uses finite-dimensionality of the corner. The same formula holds in A_a, and restriction corresponds to the corner quotient. Apply Lemma 7.2 to the finite set of socle vertices of $I^0 \oplus I^1$ and then take the required matrix spaces. Composition compatibility identifies the endomorphism algebras, their groups of units, and the maps $af - fb$. □

Lemma 7.4. *Every element of a finite direct sum of the I_v generates a finite-dimensional subrepresentation. Consequently, for every subrepresentation N of such a sum and every decreasing cofinal sequence $a_n \to -\infty$, with $a_{n+1} \le a_n$,*

$$N = \bigcup_n T_{a_n} N.$$

Proof. First take $x \in (I_v)_u$. If a path $\alpha : u \to w$ acts nontrivially on x, there is a path $\beta : w \to v$ such that $x(\beta\alpha) \neq 0$. The concatenated path is nonzero in A^-. Lemma 7.1, applied to the fixed endpoints u, v, therefore bounds the length of α. Only finitely many paths of this length start at u; they span the subrepresentation generated by x. It is finite-dimensional and has finite support. An arbitrary element has finitely many vertex and direct-sum components, so the same conclusion holds for it.

Choose n such that the support of this generated subrepresentation is in J_{a_n}. Every deleted idempotent annihilates it, and hence so does the ideal $\mathfrak{a}_{a_n}$. The element is in $T_{a_n} N$. As n increases,

the sets J_{a_n} increase and the ideals $\mathfrak{a}_{a_n}$ decrease. Thus these are increasing subrepresentations whose union is N. □

7.3. **The top Grassmannian term.**

Lemma 7.5. *For every finite-dimensional quiver representation X and every vertex v,*

$$\deg_{y_v} F_X = \dim X_v. \tag{7.5}$$

Proof. Every subrepresentation has dimension at most $\dim X_v$ at vertex v, so the left-hand side is no larger than the right-hand side. The Grassmannian of subrepresentations of the full dimension vector $\dim X$ consists only of X itself. It is a point. Hence the coefficient of $y^{\dim X}$ in F_X is 1, proving the opposite inequality. □

8. **Proof of the formula**

Proof of Theorem 1.1. First take the parameter q_0 of Lemma 3.1, and fix a real simple $L(m) \in \mathcal{C}^-$. Let $I^- = I(m)^-$ and $I^+ = I(m)^+$ be exactly the injectives in (2.7). Choose n_0 so large that for every $n \geq n_0$ all the following conditions hold:

(1) $L(m) \in \mathcal{C}_{[a_n,0]}$;
(2) every nonzero coordinate of the original degree, after the shift $p = s - d_i$, lies in J_{a_n};
(3) the fixed polynomial P_m in (4.5) is supported on J_{a_n};
(4) the three isomorphisms (7.4) hold for I^-, I^+.

Each condition holds eventually: the first three concern finite sets, and the fourth is Corollary 7.3.

By Corollary 6.2, for each such n there is a differential

$$f_n : T_{a_n} I^- \longrightarrow T_{a_n} I^+$$

with zero tangent cokernel and with $F_{\ker f_n} = P_m$. Its terms are the indicated ones: condition (2) makes the finite mutable degree precisely the restriction of the original degree, with no omitted nonzero boundary coordinate.

Fix $n = n_0$, and lift f_{n_0} by the Hom isomorphism to a map

$$f : I^- \longrightarrow I^+.$$

The two End isomorphisms and their composition compatibility identify its tangent quotient with the zero quotient for f_{n_0}. They also show that, for every $n \geq n_0$, the restricted map $T_{a_n} f$ has zero tangent quotient. Its orbit and the orbit of f_n are nonempty open orbits in the same affine space $\operatorname{Hom}_{A_{a_n}}(T_{a_n} I^-, T_{a_n} I^+)$. They coincide by irreducibility. Thus

$$F_{\ker(T_{a_n} f)} = P_m \qquad (n \geq n_0). \tag{8.1}$$

This establishes compatibility of the independently constructed finite objects using the unique open orbit.

Put $K = \ker f$ and $X_n = T_{a_n} K$. Left exactness of T_{a_n} gives

$$X_n = \ker(T_{a_n} f).$$

These are finite-dimensional, since they are submodules of injectives over the finite-dimensional algebra A_{a_n}. By Lemma 7.4,

$$X_n \subseteq X_{n+1}, \qquad K = \bigcup_n X_n.$$

Regard their dimension vectors and F-polynomials as vectors and polynomials on all of V^-, with zero coordinates outside J_{a_n}. Equations (8.1) and (7.5) imply

$$\dim(X_n)_v = \deg_{y_v} P_m \qquad (v \in V^-,\ n \geq n_0).$$

The increasing submodules X_n therefore have exactly the same finite dimension vector. Each inclusion is an equality. Their union is the finite-dimensional module $K = X_{n_0}$, and $F_K = P_m$.

The space $\mathrm{Hom}_{A^-}(I^-, I^+)$ and both endomorphism algebras are finite-dimensional by Corollary 7.3. The zero tangent quotient for f makes its change-of-basis orbit open in this affine space, by the same argument as in Lemma 6.1. It is the unique dense open orbit. Its kernel is therefore the original generic kernel $K(m)$. Using (4.5) gives

$$\chi_q^-(L(m)) = mP_m(A^{-1}) = mF_{K(m)}(A^{-1}),$$

which is (1.1).

Finally let $q \in \mathbb{C}^\times$ be not a root of unity. Lemma 3.2 sends a real simple with highest label m at q to the real simple with the same integer label at q_0, and preserves its truncated character coefficient by coefficient. The quiver, the potential, the degree, and the two injective terms in (2.7) depend only on these integer labels and the Cartan matrix. The same is true of their generic kernel and its ordinary F-polynomial. The identity at q_0 therefore proves the identity at q, with precisely the original labels and cutoff. □

Appendix A. Ordinary Euler characteristics in the ADE input

We give the coefficient and support comparison in the ADE input [22, Theorem 3.13 and Corollary 6.7].

A.1. **The coefficient specialization.** Let the Dynkin diagram be simply laced, with vertex set I, and choose integers ξ_i such that $|\xi_i - \xi_j| = 1$ when $i \sim j$. Write $Y_{i,r} = Y_{i,q^r}$ and

$$A_{i,r} = Y_{i,r-1}Y_{i,r+1}\prod_{j\sim i} Y_{j,r}^{-1}.$$

Fix a dominant monomial $m = \prod_{i,r} Y_{i,r}^{w_{i,r}}$ in a finite window such that $L(m)$ is real. Choose an orientation Q of the Dynkin diagram and let $\overline{Q}$ be its double, obtained by adjoining a reverse arrow a^* to each $a \in Q_1$. The preprojective algebra is

$$\Pi = \mathbb{C}\overline{Q}/\left(e_i \sum_{a\in Q_1} (aa^* - a^*a)e_i : i \in I\right).$$

Give every arrow degree -1 and put $B = \Pi[u]$, where u is central of degree 2, as in [22, Section 4]. The symbols Hom_B^0 and $\mathrm{Ext}_B^{1,0}$ refer to maps and extensions in the category of graded B-modules with degree-preserving maps. Set

$$J_i = \mathrm{Hom}_{\mathbb{C}}(e_i\Pi, \mathbb{C}), \qquad P_w = \bigoplus_{i,r} J_i\langle r+2\rangle^{w_{i,r}}.$$

The socle of J_i has degree zero, and $\langle a\rangle$ raises degrees by a. The full kernel supplied by the realization is a rigid graded module $K = (P_w, U)$, where U is the action of u and $\mathrm{Ext}_B^{1,0}(K,K) = 0$.

For a finitely supported dimension vector $e = (e_{i,s})$, $\mathrm{Gr}_B(e,K)$ denotes the variety of graded B-submodules of K with that dimension vector. Juan constructs a smooth graded quiver variety X_e, a regular function f_e on it, and an identification

$$Z_e = \mathrm{Gr}_B(e,K)$$

with a proper invariant union of critical components of f_e. These assertions, including isolation of this union, are [22, Lemma 5.1 and Proposition 5.3]. Moreover, the cut grading and the

inverse framing cocharacter give an algebraic $\mathbb{C}^*$-action satisfying $f_e(z \cdot x) = z f_e(x)$; see [22, Lemma 5.4 and equation (5.11)]. For nonempty X_e, put $D_e = \dim_{\mathbb{C}} X_e$ and

$$H_e = R\Gamma(Z_e, \phi_{f_e}[-1]\mathbb{Q}_{X_e}[D_e]), \qquad P_e(t) = \sum_j t^{-j} \dim_{\mathbb{Q}} H^j(H_e). \tag{A.1}$$

Empty varieties contribute zero. Here $\phi[-1]$ is the perverse-normalized vanishing-cycle functor, as in [22, equation (5.12)].

Theorem 6.3 of [22] identifies, before either specialization,

$$P_e(t) = a_{e,0}(t), \tag{A.2}$$

where $a_{e,0}(t)$ is the zero-stratum multiplicity in Nakajima's proper-map decomposition. By [26, equation (3.18)], $a_{e,0}(1)$ is the coefficient of $m \prod_{i,s} A_{i,s-1}^{-e_{i,s}}$ in the character of the actual simple $L(m)$. The cohomological variable t is independent of the loop parameter q. Juan writes $t = v^2$ in the quantum character; specializing $v = 1$ gives $t = 1$.

Proposition A.1. *With these conventions,*

$$a_{e,0}(1) = \chi\big(\mathrm{Gr}_B(e, K)\big). \tag{A.3}$$

Consequently,

$$\chi_q(L(m)) = m \sum_e \chi(\mathrm{Gr}_B(e, K)) \prod_{i,s} A_{i,s-1}^{-e_{i,s}}. \tag{A.4}$$

Proof. The precise vanishing-cycle input is [22, Lemma 6.4]: if X is smooth of pure dimension D, $f(z \cdot x) = z f(x)$, and $Z \subset f^{-1}(0)$ is a proper invariant union of critical components, then

$$\chi(Z, \phi_f[-1]\mathbb{Q}_X[D]) = (-1)^D \chi(Z).$$

We apply this cited lemma to (X_e, f_e, Z_e). Since $(-1)^{-j} = (-1)^j$, definition (A.1) gives

$$P_e(-1) = (-1)^{D_e} \chi(Z_e). \tag{A.5}$$

On the other hand, the parity statement for the zero-stratum coefficient is

$$a_{e,0}(t) \in t^{D_e} \mathbb{Z}_{\geq 0}[t^{-2}] \qquad \text{[26, equation (3.18) and Remark 3.20].} \tag{A.6}$$

Every exponent in this polynomial is congruent to D_e modulo two. Thus (A.2) implies $P_e(-1) = (-1)^{D_e} P_e(1)$. Comparing with (A.5) cancels the sign and proves (A.3). Distinct finite dimension vectors give distinct monomials: at the largest spectral index where their difference is nonzero, the exponent of $Y_{i,s}$ in their ratio is the negative of that nonzero entry. The coefficient dictionary therefore gives (A.4). □

This is the specialization in [22, Theorem 6.5].

A.2. **The prescribed finite kernel.** Fix $\ell \geq 1$ and write the highest-monomial window and the mutable window as

$$\mathcal{Y}_{\ell,\xi} = \{(i, r) : \xi_i - 2\ell \leq r \leq \xi_i,\ r \equiv \xi_i \pmod 2\},$$
$$W_{\ell,\xi} = \{(i, s) : \xi_i - 2\ell + 2 \leq s \leq \xi_i,\ s \equiv \xi_i \pmod 2\}.$$

Assume that w is supported on $\mathcal{Y}_{\ell,\xi}$. Let $\chi_q^{\leq \xi}$ delete each monomial containing a variable $Y_{i,s}$ with $s > \xi_i$ to a nonzero power. The lower-support calculation below identifies this with deletion outside the full highest-monomial window $\mathcal{Y}_{\ell,\xi}$. Extend w by zero and put $g_{i,s}^{\mathrm{full}} = w_{i,s} - w_{i,s-2}$. The full reduced injective presentation in [22, Proposition 4.3] has the terms

$$I_- = \bigoplus_{i,s} \mathbb{I}(i, s)^{[-g_{i,s}^{\mathrm{full}}]_+}, \qquad I_+ = \bigoplus_{i,s} \mathbb{I}(i, s)^{[g_{i,s}^{\mathrm{full}}]_+}, \tag{A.7}$$

where $\mathbb{I}(i,s)$ is the graded injective with socle at (i,s). The module K is the kernel of a map in the open orbit of $\mathrm{Hom}_B^0(I_-,I_+)$.

The graded cover of B has vertices $(i,s)\in I\times\mathbb{Z}$; its morphisms from (i,s) to (j,t) are $e_jB_{t-s}e_i$. Graded B-modules are representations of this category. For a finite set $S\subset I\times\mathbb{Z}$, let B_S be its quotient by the vertex idempotents outside S, and let $\Gamma_S N$ be the largest submodule of N supported on S. For $(i,s)\in S$, write $\mathbb{I}_S(i,s)$ for the indecomposable injective B_S-module with socle at (i,s). Lemma 4.4 of [22] gives

$$\Gamma_S\mathbb{I}(i,s)=\begin{cases}\mathbb{I}_S(i,s), & (i,s)\in S,\\ 0, & (i,s)\notin S,\end{cases}$$

and identifies the restricted generic kernel with $\Gamma_S K$. Indeed, Γ_S is right adjoint to the exact inclusion of B_S-modules and hence is left exact. The map

$$\mathrm{Hom}_B^0(I_-,I_+)\longrightarrow\mathrm{Hom}_{B_S}(\Gamma_S I_-,\Gamma_S I_+)$$

is surjective: compose a map on the right with $\Gamma_S I_+\hookrightarrow I_+$ and extend it from $\Gamma_S I_-\subset I_-$ by injectivity. This surjective linear map is open. The image of the ambient open orbit is contained in one finite change-of-basis orbit, so that orbit is open. Left exactness identifies its kernel with $\Gamma_S K$. In particular,

$$K_{\ell,\xi}(m)\simeq\Gamma_{W_{\ell,\xi}}K, \tag{A.8}$$

with precisely the positive and negative injective multiplicities prescribed by $g^{\mathrm{full}}|_{W_{\ell,\xi}}$.

We also record why the original monomial deletion has exactly this effect, as in [22, Proposition 6.6]. A nonzero degree-t component of J_j at color i requires $t\geq\mathrm{dist}(i,j)$, where $\mathrm{dist}(i,j)$ is the distance in the Dynkin diagram. A component of $J_j\langle r+2\rangle$ occurring in P_w therefore has degree

$$s\geq\xi_j-2\ell+2+\mathrm{dist}(i,j)\geq\xi_i-2\ell+2.$$

Thus every coordinate (i,s) in the support of K has $s\geq\xi_i-2\ell+2$. Its root factor $A_{i,s-1}$ has same-color indices $s-2,s$ and neighboring indices $s-1$, all at or above their respective lower highest-monomial bounds. Every term in (A.4) therefore satisfies the lower cutoff. Deletion above ξ is exactly deletion outside $\mathcal{Y}_{\ell,\xi}$. If a submodule dimension vector e has $e_{i,s}>0$ with $s>\xi_i$, choose such a coordinate with maximal $s-\xi_i$, and then with maximal ξ_i. For $j\sim i$, a nonzero $e_{j,s+1}$ would have either larger excess, when $\xi_j=\xi_i-1$, or the same excess and larger height, when $\xi_j=\xi_i+1$. Hence all these neighboring coordinates vanish. The exponent of the out-of-window variable $Y_{i,s}$ in $m\prod_{j,t}A_{j,t-1}^{-e_{j,t}}$ is consequently

$$-e_{i,s}-e_{i,s+2}+\sum_{j\sim i}e_{j,s+1}<0.$$

The monomial is deleted. Conversely, if e is supported on $W_{\ell,\xi}$, every factor $A_{i,s-1}$ involved has its same-color indices $s-2,s$ and its neighboring indices $s-1$ in $\mathcal{Y}_{\ell,\xi}$. The monomial is retained. Submodules of K supported on $W_{\ell,\xi}$ are exactly the submodules of $\Gamma_{W_{\ell,\xi}}K$, with the same Grassmannians. Applying this criterion to (A.4) yields

$$\chi_q^{\leq\xi}(L(m))=mF_{K_{\ell,\xi}(m)}(y_{i,s}=A_{i,s-1}^{-1}),\qquad F_{K_{\ell,\xi}(m)}(y)=\sum_e\chi(\mathrm{Gr}_e\,K_{\ell,\xi}(m))y^e.$$

A.3. **Finite scope and frozen factors.** The finite enlargement used in [22, Theorem 6.3] can be chosen once for all $K^{\oplus r}$ with r a positive integer. Their supports and injective socle vertices coincide with those of K; increasing r changes only multiplicities. The root factors for $e\leq r\dim K$ likewise introduce no new spectral positions. One therefore chooses a single strip containing these positions and places its new frozen row below the support of w.

Finally, the normalization by frozen factors is part of [22, Proposition 4.2 and Corollary 6.7]. In color i its multiplicity is $\delta_i = \min_{0\le k\le \ell} w_{i,\xi_i-2k}$. Removing δ_i full columns gives the unique dominant vector with the same mutable differences and column minimum zero. Restoring these columns adds projective-injective full intervals to the rigid operator and the corresponding frozen tensor factors to $L(m)$. A full column has zero difference on $W_{\ell,\xi}$, so the prescribed finite kernel remains $K_{\ell,\xi}(m)$; the highest-monomial prefactor is the original m. Thus the ordinary Euler formula applies with arbitrary frozen factors. The realization theorem includes every rigid object, including its decoration, as stated in [22, Corollary 6.7].

Appendix B. A Laurent normalization in type G_2

We prove the Laurent unit-ideal identity required in [30, Proposition 12.2] and [32, Lemma 4.2.4], using explicit coordinates. The identity is pulled back to the original variables before it is applied to the current homomorphism.

Throughout this appendix, $q \in \mathbb{C}^\times$ is not a root of unity. We number the long vertex by 1 and the short vertex by 2, and use

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad (d_1, d_2) = (3, 1), \qquad 2 \longrightarrow 1.$$

Fix nonnegative gauge ranks a_1, a_2, write $\Omega = \{(i, r) : 1 \le r \le a_i\}$, and put

$$R_\Omega = \mathbb{C}[w_{i,r}^{\pm 1} : (i, r) \in \Omega].$$

This is a Laurent polynomial ring in finitely many independent variables. A magnetic charge is a tuple $\gamma = (\gamma_{i,r}) \in \mathbb{Z}^\Omega$.

B.1. **The one-step coefficients.** Let e_α denote the coordinate vector at $\alpha \in \Omega$. In the abelian multiplication formula of [32, (3.2.20), (3.2.25)], denote the coefficient of the magnetic class r_γ in

$$r_{\varepsilon e_\alpha}\, r_{\gamma-\varepsilon e_\alpha} = c_\alpha^\varepsilon(\gamma) r_\gamma$$

by $c_\alpha^\varepsilon(\gamma)$, where $\varepsilon \in \{+1, -1\}$ and $\varepsilon\gamma_\alpha > 0$. The framing contribution is 1: at the selected coordinate the two charges ε and $\gamma_\alpha - \varepsilon$ have the same weak sign, and at every other coordinate the first charge is zero. Thus these coefficients have no flavor factor.

Put $\Delta_{rs} = \gamma_{2,r} - 3\gamma_{1,s}$. For the edge $2 \to 1$, the factor in [32, (3.2.25)] is

$$1 - q^{-3-2b_0} w_{2,r} w_{1,s}^{-1}.$$

Put $\eta = \varepsilon e_\alpha$. The three integers b_0, b_1, b_2 in that formula satisfy

$$3\eta_{1,s} - \eta_{2,r} = b_0 - b_1,$$
$$3(\gamma_{1,s} - \eta_{1,s}) - (\gamma_{2,r} - \eta_{2,r}) = b_1 - b_2,$$

and one of the two conditions

$$b_0, b_2 \ge 0 > b_1, \qquad \text{or} \qquad b_1 \ge 0 > b_0, b_2.$$

Adding the two equations gives $b_2 = b_0 + \Delta_{rs}$. For $+e_{2,r}$, one has $b_1 = b_0 + 1$. Only the second sign condition is possible, and it forces $b_0 = -1$ and $\Delta_{rs} \le 0$. For $+e_{1,s}$, one has $b_1 = b_0 - 3$. Only the first condition is possible, giving $0 \le b_0 \le 2$ and $\Delta_{rs} \ge -b_0$. For $-e_{2,r}$, one has $b_1 = b_0 - 1$. The first condition gives $b_0 = 0$ and $\Delta_{rs} \ge 0$; the second is impossible. For $-e_{1,s}$, one has $b_1 = b_0 + 3$. Only the second condition is possible, and it gives $-3 \le b_0 \le -1$. Writing $b_0 = -1 - t$ gives $0 \le t \le 2$ and $\Delta_{rs} \le t$. For the long-vertex steps we reverse the quotient using

$$1 - q^k w_{2,r} w_{1,s}^{-1} = -q^k w_{2,r} w_{1,s}^{-1} (1 - q^{-k} w_{1,s} w_{2,r}^{-1}).$$

Substitution of the four ranges into these factors gives the exponents -1, $3+2t$, -3, and $1-2t$, respectively. Up to multiplication by Laurent units, the four coefficients are the following products:

$$P^+_{2,r} = \prod_{s:\,\Delta_{rs}\le 0} (1 - q^{-1} w_{2,r} w_{1,s}^{-1}), \tag{B.1}$$

$$P^+_{1,s} = \prod_{\substack{r,\ 0\le t\le 2 \\ \Delta_{rs}\ge -t}} (1 - q^{3+2t} w_{1,s} w_{2,r}^{-1}), \tag{B.2}$$

$$P^-_{2,r} = \prod_{s:\,\Delta_{rs}\ge 0} (1 - q^{-3} w_{2,r} w_{1,s}^{-1}), \tag{B.3}$$

$$P^-_{1,s} = \prod_{\substack{r,\ 0\le t\le 2 \\ \Delta_{rs}\le t}} (1 - q^{1-2t} w_{1,s} w_{2,r}^{-1}). \tag{B.4}$$

Here t is integral and an empty product is 1.

For either sign define

$$\Gamma_\varepsilon = \{\alpha \in \Omega : \varepsilon\gamma_\alpha > 0\}.$$

Every variable in P^ε_α, for $\alpha \in \Gamma_\varepsilon$, is indexed by Γ_ε. Indeed, for a positive short coordinate, $\Delta_{rs} \le 0$ implies $\gamma_{1,s} > 0$; for a positive long coordinate, $\Delta_{rs} \ge -t$ implies $\gamma_{2,r} \ge 3\gamma_{1,s} - 2 \ge 1$. For a negative short coordinate, $\Delta_{rs} \ge 0$ implies $\gamma_{1,s} < 0$; for a negative long coordinate, $\Delta_{rs} \le t$ implies $\gamma_{2,r} \le 3\gamma_{1,s} + 2 \le -1$. These inequalities supply the closure needed below, including for charges with both positive and negative coordinates.

In a factor $1 - q^k w_\alpha w_\beta^{-1}$, the selected coordinate is α, and the zero relation is

$$w_\beta = q^k w_\alpha, \qquad \text{equivalently} \qquad w_\alpha = q^{-k} w_\beta. \tag{B.5}$$

Thus an exponent defined by the latter, current-to-neighbor ratio is the negative of the factor exponent k. Introduce auxiliary variables only for this Laurent calculation:

$$w_{i,r} = q^{s_i} v_{i,r}, \qquad s_1 = 0, \quad s_2 = 2. \tag{B.6}$$

Under this substitution the factor exponent becomes $k' = k + s_i - s_j$, when $\alpha = (i,r)$ and $\beta = (j,s)$. All possibilities are recorded in Table 1; the inequalities in (B.1)–(B.4) may select only a subset of them.

TABLE 1. Factor exponents in the G_2 one-step products.

Step	Selected vertex / neighbor	k	k'
Positive	Short / long	-1	1
Positive	Long / short	$3, 5, 7$	$1, 3, 5$
Negative	Short / long	-3	-1
Negative	Long / short	$-3, -1, 1$	$-5, -3, -1$

B.2. **The Laurent unit ideal.**

Lemma B.1. *For $\varepsilon \in \{+1, -1\}$ with $\Gamma_\varepsilon \neq \varnothing$, the polynomials P^ε_α, $\alpha \in \Gamma_\varepsilon$, generate the unit ideal of*

$$R_{\Gamma_\varepsilon} = \mathbb{C}[w_\alpha^{\pm 1} : \alpha \in \Gamma_\varepsilon].$$

This assertion remains valid after tensoring with any commutative $\mathbb{C}$-algebra.

Proof. The substitution (B.6) is an isomorphism from R_{Γ_ε} to the corresponding Laurent ring in the v_α. Suppose that all the transformed polynomials vanish at a point of $(\mathbb{C}^\times)^{\Gamma_\varepsilon}$. For each α choose a vanishing factor and draw the arrow $\alpha \longrightarrow \beta$ to its other coordinate. The preceding inequalities ensure that $\beta \in \Gamma_\varepsilon$. If a product is empty it cannot vanish, so the choice is possible at every vertex under the stated supposition.

A finite directed graph in which every vertex has an outgoing arrow contains a directed cycle. Write such a cycle as $\alpha_0, \alpha_1, \dots, \alpha_m = \alpha_0$, with $m > 0$, and write $\alpha_j = (i_j, r_j)$. The selected zero relations give

$$v_{\alpha_{j+1}} = q^{k'_j} v_{\alpha_j}, \qquad q^E = 1, \qquad E = \sum_{j=0}^{m-1} k'_j.$$

By Table 1, every k'_j is strictly positive when $\varepsilon = +1$ and strictly negative when $\varepsilon = -1$. Hence $E \neq 0$, a contradiction. Notice also that the change in the cycle sum is zero:

$$\sum_{j=0}^{m-1} (s_{i_j} - s_{i_{j+1}}) = 0.$$

The substitution therefore preserves the cycle obstruction.

There is no common zero on the algebraic torus. The weak Nullstellensatz for the finitely generated complex algebra R_{Γ_ε} implies that the ideal is the whole ring. Pulling back along (B.6) gives Laurent polynomials b_α in the original variables such that

$$\sum_{\alpha \in \Gamma_\varepsilon} b_\alpha P_\alpha^\varepsilon = 1. \tag{B.7}$$

Tensoring this identity with any commutative $\mathbb{C}$-algebra preserves the displayed equality. □

The omitted Laurent units can be absorbed in the coefficients b_α. Thus (B.7) applies to the original one-step products in the full gauge Laurent ring, tensored with the flavor ring. If $\gamma \neq 0$, use its positive coordinates when any exist, and otherwise use its negative coordinates. Equation (B.7), with the Laurent units absorbed, expresses r_γ as a sum of gauge-Laurent multiples of $r_{\varepsilon e_\alpha} r_{\gamma - \varepsilon e_\alpha}$. Each residual charge has $\sum_\alpha |\gamma_\alpha|$ smaller by one. Starting from $r_0 = 1$, induction gives the abelian generation assertion of [32, Lemma 4.2.4]. The flavor-equivariant nil-Hecke and spherical constructions of [32, Lemmas 4.2.3 and 4.2.5] then use the same identity over the flavor ring and after every base change of that ring, as required in [30, §12.1]. After the identity has been pulled back, all currents, gauge coordinates, highest monomials, and spectral labels retain their original definitions.

B.3. **Numerical inversion of the quantum Cartan matrix.** We give the coefficient calculation in [30, §12.1] for both G_2 and F_4. For F_4, number the vertices along the chain with symmetrizer $(2, 2, 1, 1)$, $c_{23} = -1$ and $c_{32} = -2$. Let $a_i \in \mathbb{N}$, $i \in I$, be the gauge ranks, let $\ell_i = \sum_j c_{ji} a_j \geq 0$ be the framing ranks, and let $z_{i,t}$, $1 \leq t \leq \ell_i$, be the invertible framing coordinates. Set

$$W_j(u) = \prod_{t=1}^{a_j} (1 - w_{j,t}/u), \qquad Z_i(u) = \prod_{t=1}^{\ell_i} (1 - q_i^2 z_{i,t}/u), \qquad \delta_j = \prod_{t=1}^{a_j} w_{j,t}.$$

On the determinant cover introduce units D_j with $D_j^2 = \delta_j$. The geometric Cartan current is

$$\Psi_i(u) = \left(\prod_j D_j^{c_{ji}} \right) Z_i(u) \frac{\displaystyle\prod_{j \neq i} \prod_{k=1}^{-c_{ji}} W_j(q_j^{-c_{ji}-2k} u)}{W_i(u) W_i(q_i^{-2} u)}. \tag{B.8}$$

For the paired framings of [30, §§12.1–12.2], the ordinary current map uses invertible framing coefficients κ_i satisfying

$$\kappa_i^2 q_i^{2\ell_i-2a_i}\prod_{j\neq i} q_j^{-a_jc_{ji}}\prod_{t=1}^{\ell_i} z_{i,t}=1.$$

For a single paired factor, let ν be its framing color and $b\in\mathbb{C}^\times$ its spectral multiplier. Its framing ranks are $\ell_i=2\delta_{i\nu}$, and the two zeros of Z_ν are bq^{s-d_ν} and $bq^{s+d_\nu+H}$. Here $H=12$ for G_2 and $H=18$ for F_4; $s=1$ for color 1 of G_2 and color 3 of F_4, and $s=0$ in the other four cases. Each gauge rank in these six factors is even [30, §12.2]. Since the zeros of Z_i are $q_i^2 z_{i,t}$,

$$\prod_{t=1}^{\ell_i} z_{i,t}=b^{2\delta_{i\nu}}q^{\delta_{i\nu}(2s+H-4d_i)}.$$

Thus the normalization units are explicitly

$$\kappa_i(b)=b^{-\delta_{i\nu}}q^{-\frac{1}{2}\left(d_i(2\ell_i-2a_i)-\sum_{j\neq i}d_ja_jc_{ji}+\delta_{i\nu}(2s+H-4d_i)\right)}.$$

The exponent of q is integral. Squaring this formula and multiplying by the preceding framing product verifies the normalization identity. For products of paired factors, the gauge and framing ranks add and the units κ_i multiply; the same identity follows by multiplication. These units are Laurent functions of the spectral multipliers.

The image of the ordinary Cartan current is $\phi_i(u)=\kappa_i\Psi_i(u)$, and its leading coefficient is

$$k_i=\phi_i(\infty)=\kappa_i\prod_j D_j^{c_{ji}}.$$

Both k_i and k_i^{-1} are images of the ordinary Cartan generators.

For each integer $n>0$ define

$$h_{i,n}=n[u^{-n}]\log\frac{\phi_i(u)}{k_iZ_i(u)},\qquad p_{j,n}=\sum_{t=1}^{a_j}w_{j,t}^n.$$

Here $[u^{-n}]$ denotes coefficient extraction in a formal series at infinity. The argument of the logarithm has constant term one. The factor κ_i cancels, and

$$\log W_j(cu)=-\sum_{n>0}\frac{c^{-n}p_{j,n}}{n}u^{-n}\qquad(c\in\mathbb{C}^\times).$$

Applying this identity to (B.8) gives

$$h_{i,n}=(1+q_i^{2n})p_{i,n}-\sum_{j\neq i}\sum_{k=1}^{-c_{ji}}q_j^{n(c_{ji}+2k)}p_{j,n}. \tag{B.9}$$

Thus the column vectors $h_n=(h_{i,n})_i$ and $p_n=(p_{i,n})_i$ satisfy $h_n=B(q^n)p_n$, where

$$B_{ii}(x)=1+x^{2d_i},\qquad B_{ij}(x)=-\sum_{k=1}^{-c_{ji}}x^{d_j(c_{ji}+2k)}\quad(i\neq j). \tag{B.10}$$

For G_2 in the numbering above,

$$B_{G_2}(x)=\begin{pmatrix}1+x^6 & -x^3-x-x^{-1}\\ -x^3 & 1+x^2\end{pmatrix}.$$

Its determinant is

$$(1+x^6)(1+x^2)-x^3(x^3+x+x^{-1})=1-x^4+x^8=\Phi_{24}(x).$$

For F_4 in this numbering,

$$B_{F_4}(x) = \begin{pmatrix} 1+x^4 & -x^2 & 0 & 0 \\ -x^2 & 1+x^4 & -1-x^2 & 0 \\ 0 & -x^2 & 1+x^2 & -x \\ 0 & 0 & -x & 1+x^2 \end{pmatrix}.$$

Let $\mathfrak{d}_j$ be its leading principal determinant of order j, with $\mathfrak{d}_0 = 1$. Expansion in the last row of each tridiagonal principal submatrix yields

$$\begin{aligned} \mathfrak{d}_1 &= 1 + x^4, \\ \mathfrak{d}_2 &= (1+x^4)\mathfrak{d}_1 - x^4\mathfrak{d}_0 = 1 + x^4 + x^8, \\ \mathfrak{d}_3 &= (1+x^2)\mathfrak{d}_2 - x^2(1+x^2)\mathfrak{d}_1 = 1 + x^{10}, \\ \mathfrak{d}_4 &= (1+x^2)\mathfrak{d}_3 - x^2\mathfrak{d}_2 = 1 - x^6 + x^{12} = \Phi_{36}(x). \end{aligned}$$

Here Φ_m is the mth cyclotomic polynomial. At $x = q^n$ these determinants are nonzero, since q is not a root of unity. Their inverses are complex scalars, so (B.9) recovers all power sums $p_{j,n}$ from Cartan-current coefficients.

Let $e_{j,k}$ be the elementary symmetric polynomial of degree k in $w_{j,1}, \dots, w_{j,a_j}$. Newton's identities give the explicit recursion

$$e_{j,0} = 1, \qquad k e_{j,k} = \sum_{n=1}^{k} (-1)^{n-1} e_{j,k-n} p_{j,n} \quad (1 \le k \le a_j).$$

Consequently all coefficients of

$$W_j(u) = \sum_{k=0}^{a_j} (-1)^k e_{j,k} u^{-k}$$

are recovered. Its top elementary symmetric coefficient is $e_{j,a_j} = \delta_j$.

The inverse determinant coefficients are recovered in the ordinary gauge Laurent ring from the leading units. Put

$$\tau_i = \kappa_i^{-2} k_i^2 = \prod_j \delta_j^{c_{ji}}.$$

The ordinary Cartan matrices have determinant one, and

$$(C_{G_2}^{\mathsf{T}})^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \qquad (C_{F_4}^{\mathsf{T}})^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}.$$

Thus

$$\delta_i = \prod_j \tau_j^{((C^{\mathsf{T}})^{-1})_{ij}}. \tag{B.11}$$

The same formula with all exponents negated recovers δ_i^{-1}. The elementary symmetric coefficients together with δ_i^{-1} generate the symmetric Laurent polynomials in the gauge variables. On the determinant cover the unsquared leading units also give

$$D_i = \prod_j (\kappa_j^{-1} k_j)^{((C^{\mathsf{T}})^{-1})_{ij}},$$

and the negative exponents recover D_i^{-1}. This proves the coefficient-generation statement there as well.

The formal inverse of $Z_i(u)$ has coefficients polynomial in the framing coordinates, because $Z_i(\infty) = 1$. All other divisions above are by nonzero complex scalars, integers, or the displayed units. The calculation therefore takes place over the prescribed flavor ring and persists after each of its base changes, together with the identity of Lemma B.1.

References

[1] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), 595–616, doi:10.4171/CMH/65.

[2] P. Cao, R. Fujita, and K. Murakami, *Partial F-invariants and cluster categorifications*, preprint (2026), arXiv:2609.08781v1.

[3] V. Chari and A. Pressley, *Quantum affine algebras and their representations*, in *Representations of Groups*, CMS Conf. Proc. **16**, Amer. Math. Soc., 1995, 59–78, arXiv:hep-th/9411145.

[4] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, Selecta Math. (N.S.) **14** (2008), 59–119. Version used: arXiv:0704.0649v4.

[5] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations II: Applications to cluster algebras*, J. Amer. Math. Soc. **23** (2010), 749–790. Version used: arXiv:0904.0676v3.

[6] V. G. Drinfeld, *A new realization of Yangians and of quantum affine algebras*, Dokl. Akad. Nauk SSSR **296** (1987), 13–17; English translation, Soviet Math. Dokl. **36** (1988), 212–216.

[7] B. Duan and R. Schiffler, *A geometric q-character formula for snake modules*, J. London Math. Soc. (2) **102** (2020), 846–878, doi:10.1112/jlms.12341.

[8] B. Duan and R. Schiffler, *Real simple modules over simply-laced quantum affine algebras and categorifications of cluster algebras*, J. Algebra **685** (2026), 608–672, doi:10.1016/j.jalgebra.2025.07.043. Preprint version: arXiv:2305.08715v2.

[9] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529, doi:10.1090/S0894-0347-01-00385-X.

[10] E. Frenkel and E. Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, Comm. Math. Phys. **216** (2001), 23–57, arXiv:math/9911112.

[11] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of W-algebras*, in *Recent Developments in Quantum Affine Algebras and Related Topics*, Contemp. Math. **248**, Amer. Math. Soc., 1999, 163–205, arXiv:math/9810055.

[12] R. Fujita, D. Hernandez, S.-j. Oh, and H. Oya, *Isomorphisms among quantum Grothendieck rings and propagation of positivity*, J. Reine Angew. Math. **785** (2022), 117–185. Version used: arXiv:2101.07489v3.

[13] R. Fujita, D. Hernandez, S.-j. Oh, and H. Oya, *Isomorphisms among quantum Grothendieck rings and cluster algebras*, preprint (2023), arXiv:2304.02562v2.

[14] R. Fujita and F. Qin, *Freezing operators in representation theory of quantum loop algebras*, preprint (2026), arXiv:2601.00687v3.

[15] S. Gautam and V. Toledano Laredo, *Yangians, quantum loop algebras and abelian difference equations*, J. Amer. Math. Soc. **29** (2016), 775–824. Version used: arXiv:1310.7318v7.

[16] J. Guo, B. Duan, and Y.-F. Luo, *Combinatorics of the q-characters of Hernandez–Leclerc modules*, J. Algebra **605** (2022), 253–295, doi:10.1016/j.jalgebra.2022.04.024.

[17] D. Hernandez, *Algebraic approach to q,t-characters*, Adv. Math. **187** (2004), 1–52, arXiv:math/0212257v5.

[18] D. Hernandez, *Simple tensor products*, Invent. Math. **181** (2010), 649–675, arXiv:0907.3002v2.

[19] D. Hernandez and B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), 265–341, doi:10.1215/00127094-2010-040.

[20] D. Hernandez and B. Leclerc, *A cluster algebra approach to q-characters of Kirillov–Reshetikhin modules*, J. Eur. Math. Soc. **18** (2016), 1113–1159, doi:10.4171/JEMS/609.

[21] Z. Hua and G.-S. Zhou, *Noncommutative Mather–Yau theorem and its applications to Calabi–Yau algebras*, preprint (2018), revised 2019, arXiv:1803.06128v5.

[22] Hernandez Juan, *Quantum affine representations and cluster characters*, preprint (2026), Hexagon:2611.00712v1.

[23] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$*, Transform. Groups **8** (2003), 95–104, arXiv:math/0202148v2.

[24] H. Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), 145–238, doi:10.1090/S0894-0347-00-00353-2.

[25] H. Nakajima, *Quiver varieties and t-analogs of q-characters of quantum affine algebras*, Ann. of Math. (2) **160** (2004), 1057–1097, arXiv:math/0105173v2.

[26] H. Nakajima, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51** (2011), 71–126. Version used: arXiv:0905.0002v5.
[27] F. Qin, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*, Duke Math. J. **166** (2017), 2337–2442, doi:10.1215/00127094-2017-0006.
[28] B. Smith, *A proof of the open orbit conjecture*, preprint (2026), Hexagon:2611.00923v1.
[29] B. Smith, *On reachability in Hernandez–Leclerc categories*, preprint (2026), Hexagon:2611.00975v1.
[30] Q. Tang, *The quantum Kazhdan–Lusztig conjecture for quantum loop algebras*, preprint (2026), Hexagon:2611.00322v2.
[31] J. Tong, B. Duan, and Y.-F. Luo, *The q-characters of minimal affinizations of type G_2 arising from cluster algebras*, Comm. Algebra **51** (2023), 565–585, doi:10.1080/00927872.2022.2106373.
[32] M. Varagnolo and E. Vasserot, *Representations of shifted affine quantum groups and Coulomb branches*, preprint (2025), arXiv:2503.06262v3.