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Counterexamples to the positivity and saturation hypotheses of GCT VI*

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Abstract

Part VI of Mulmuley's geometric complexity theory formulates three hypotheses on the stretching quasi-polynomials of Kronecker coefficients, plethysm coefficients and, more generally, multiplicities of subgroup restriction: the saturation index (SH) and the positivity index (PH2) should be bounded by a polynomial in the size of the input, and the generating function should admit a positive rational form whose largest denominator exponent, the modular index, is bounded by such a polynomial (PH3). We show that these hypotheses fail. For every $n \geq 2$ the series $\sum_{s\geq 0} g((s^n),(s^n),(s^n))\, t^s$ of triple-rectangular Kronecker coefficients has a pole at a primitive root of unity of order $2^{n-1}3^r$ for some $r \geq 0$, so every positive rational form, indeed every expression $P(t)/\prod_i (1-t^{a_i})^{b_i}$, has some $a_i \geq 2^{n-1}$, while the input rank is n^2+5n; this refutes the existence clause of PH3 for Kronecker coefficients, also in the relaxed form that Mulmuley gave in the appendix to the paper of Briand, Orellana and Rosas. The proof reduces modulo three to the cyclic permutation of the three tensor factors, whose trace is the Hilbert series of the invariants of $\mathbb{C}^n \oplus \mathbb{S}_{(2,1)}\mathbb{C}^n$, and locates there a unique fixed curve of the grading group $\mu_{2^{n-1}}$. For the plethysm coefficients of $\mathrm{Sym}^d(\mathrm{Sym}^3\,\mathbb{C}^n)$ along the ray of the determinant, the Hilbert series has simple poles at the primitive 2^{n-1}-th roots of unity for every $n \geq 4$, which refutes PH3; a signed trace of these poles dominates all other terms, which gives positivity and saturation indices larger than 2^{cn} for every $c < 1/4$ and all large n, refuting PH2 and SH(a). Both arguments rest on a support theorem for cubic forms: a diagonal matrix of two-power order fixing a semistable cubic in n variables has determinant of order at most 2^{n-1}, with equality only for the chain cubic $x_1^2x_2 + \cdots + x_{n-1}^2x_n + x_n^3$. For restriction from a rank-one torus, which GCT VI admits as an explicit embedding, the three hypotheses fail at a fixed input rank.

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1 Introduction and results

The hypotheses. Mulmuley's GCT VI [10] proposes to decide the nonvanishing of structure constants of representation theory, among them the Kronecker coefficients and the plethysm coefficients, by linear programming on a stretching quasi-polynomial, and it formulates three hypotheses that would make this work. Let $f(n)$ be the stretching function of a structure constant (the constant at the n-fold stretched input), a quasi-polynomial, and $F(t) = \sum_{n\geq 0} f(n)t^n$ its generating function. The *positivity index* $p(f)$ and the *saturation index* $s(f)$ measure how far f must be shifted before all constituents have nonnegative coefficients, respectively are positive at every positive integer whenever not identically zero; a *positive form* of F is an expression with denominator $\prod_i (1-t^{a_i})^{d_i}$ and a numerator with nonnegative integer coefficients and constant term one, and its *modular index* is $\max_i a_i$. (The definitions are recalled in Section 2.) The hypotheses ask that $s(f)$ (SH(a)), $p(f)$ (PH2) and the modular index of some positive form (PH3) be bounded by a polynomial in the rank of the input: [10, Hypotheses 3.3.2(a), 3.3.3, 3.3.4, p. 51] in general and [10, Hypotheses 1.6.5(a), 1.6.6, 1.6.7, pp. 24–25] for plethysm. After Briand, Orellana and Rosas found counterexamples to the earlier, strict forms of SH and PH2 for Kronecker coefficients [1], Mulmuley stated relaxed forms of SH, PH2 and PH3 for Kronecker coefficients in an appendix to their paper [1, Appendix, pp. 18–20]; the polynomial bounds above are of that relaxed kind.

Results. Write $g(\lambda, \mu, \nu)$ for the Kronecker coefficient and put

$$k_n(s) = g\big((s^n), (s^n), (s^n)\big), \qquad K_n(t) = \sum_{s\geq 0} k_n(s)t^s.$$

This is the stretching function of the Kronecker coefficient of the triple of columns $((1^n), (1^n), (1^n))$.

Theorem A (Theorem 5.4)**.** For every $n \geq 2$ there is an integer $r \geq 0$ such that $K_n(t)$ has a pole at every primitive root of unity of order $2^{n-1}3^r$. Consequently every expression $K_n(t) = P(t)/\prod_i (1-t^{a_i})^{b_i}$ with $P \in \mathbb{Q}[t]$ and positive integers a_i, b_i has $\max_i a_i \geq 2^{n-1}$. The existence clause of PH3 for Kronecker coefficients [10, Hypothesis 3.3.4] fails, and so does the relaxed PH3 of [1, Appendix].

Let $\nu = \gcd(n,3)$, $d_0 = n/\nu$ and $m = \binom{n+2}{3} = \dim \mathrm{Sym}^3 \mathbb{C}^n$. The plethysm coefficient of the ray $\mu = (3)$, $\lambda = (d_0)$, $\pi = ((3/\nu)^n)$ of [10, Section 1.6] has stretching function

$$f_n(s) = \dim \mathrm{Hom}_{GL_n}\big((\det)^{3s/\nu}, \mathrm{Sym}^{d_0 s}(\mathrm{Sym}^3 \mathbb{C}^n)\big), \qquad H_n(t) = \sum_{s \geq 0} f_n(s) t^s,$$

the number of linearly independent SL_n-invariants of degree $d_0 s$ of cubic forms in n variables.

Theorem B (Theorem 6.2)**.** For every $n \geq 4$, $H_n(t)$ has a simple pole at every primitive root of unity of order 2^{n-1}. Every expression $H_n(t) = P(t)/\prod_i (1-t^{a_i})^{b_i}$ has some a_i divisible by 2^{n-1}, while the parameter of [10, Hypothesis 1.6.7] is $m^2+n+1 = O(n^6)$. PH3 fails for plethysm coefficients.

Theorem C (Theorem 7.9)**.** For every real $0 < c < 1/4$ and all sufficiently large n, the positivity index and the saturation index of f_n satisfy

$$p(f_n) > 2^{cn}, \qquad s(f_n) \geq 2^{cn};$$

explicitly, $p(f_n) > 2^{n/8}$ and $s(f_n) \geq 2^{n/8}$ for $n \geq 2 \cdot 10^{80}$. PH2 and SH(a) for plethysm coefficients [10, Hypotheses 1.6.5(a), 1.6.6] fail.

Theorem D (Propositions 8.1 and 8.2)**.** For restriction from $\mathbb{G}_m$ along explicit polynomial embeddings into GL_2 and GL_3, which [10, Section 3.4] admits, the input rank stays 5, respectively 6, while the modular index of every positive form, respectively the positivity and saturation indices, are unbounded. Hypotheses 3.3.2(a), 3.3.3 and 3.3.4 of [10] fail as stated for the subgroup restriction problem.

What is not claimed. None of these results bears on the density clauses [10, Hypotheses 1.6.5(b), 3.3.2(b)], which ask for strict saturation of almost all inputs; our families are sparse. We do not determine the exponent r of Theorem A (for $n = 2,3,4,5$ separate computations in the run give $r = 0$; we do not use them), and we do not bound the positivity or saturation index of the Kronecker family. Nothing here concerns the occurrence of representations in orbit closures, the other route of geometric complexity theory, or the complexity of deciding the vanishing of these coefficients.

Method. The refutations of PH3 for Kronecker and for plethysm coefficients detect a pole of a Hilbert series at a root of unity ζ of large order by the fixed locus of the grading element ζ on the invariant-theoretic quotient: a unique fixed curve along which the quotient is smooth forces a pole (Proposition 3.2), while an isolated fixed point forces regularity (Lemma 3.1). The fixed loci are found by a support theorem for cubic forms (Theorem 4.3), which singles out the chain cubic. For Kronecker coefficients the quotient is that of three-tensors, whose fixed loci we cannot control directly; instead the cyclic permutation τ of the three factors acts on the invariants, its trace series agrees with K_n modulo three, and by a character identity (Theorem 5.1) it is the Hilbert series of a quotient of $\mathbb{C}^n \oplus \mathbb{S}_{(2,1)}\mathbb{C}^n$, where the support theorem applies again. A transfer lemma for congruent rational series (Theorem 3.6) carries the pole back, at the price of the unknown power 3^r. The torus examples of Theorem D are elementary.

Earlier work. On 25 November 2026 we ran web searches, searched the Hexagon repository, and went through the papers that cite [10] or [1] in the OpenAlex index, for earlier statements that PH3, PH2 or SH fails in these forms; Section 9 lists what was searched

and what was not. Briand, Orellana and Rosas [1] refuted the strict forms of SH and PH2 for Kronecker coefficients; the positivity indices of their examples are at most one, so they agree with the relaxed forms. Ikenmeyer, Mulmuley and Walter [4] and Fischer and Ikenmeyer [2] showed that deciding the positivity of Kronecker and of plethysm coefficients is NP-hard; this concerns the exact decision problem, not the index hypotheses. Kahle and Michałek [6] showed that plethysm quasi-polynomials need not count lattice points of polytopes of the matching dimension, which bears on the polytope hypothesis PH1. In these searches we found no earlier refutation of PH3, or of the relaxed SH(a) and PH2.

Section 10 says who proved and checked what.

2 Definitions and the hypotheses

Quasi-polynomials and indices. A function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is a quasi-polynomial if there are a period $l \geq 1$ and polynomials $f_1, \ldots, f_l$ with $f(n) = f_j(n)$ for $n \equiv j \pmod{l}$. The following are [10, Definitions 1.2.2 and 1.2.4, pp. 13–14].

Definition 2.1. The quasi-polynomial f is *strictly positive* if all coefficients of all f_j are nonnegative; its *positivity index* $p(f)$ is the least integer $p \geq 0$ such that $n \mapsto f(n+p)$ is strictly positive. It is *strictly saturated* if, for every j, either f_j is identically zero or $f_j(n) > 0$ for every integer $n \geq 1$; its *saturation index* $s(f)$ is the least integer $s \geq 0$ such that $n \mapsto f(n+s)$ is strictly saturated.

Definition 2.2 ([10, Definition 1.2.7, p. 15])**.** Let $F(t) = \sum_{n \geq 0} f(n)t^n$ with f a quasi-polynomial of degree d, not identically zero. A *positive form* of F is an expression

$$F(t) = \frac{h_0 + h_1 t + \cdots + h_e t^e}{\prod_{i=0}^{k} (1 - t^{a_i})^{d_i}}$$

with $h_0 = 1$, all h_i nonnegative integers, a_i, d_i positive integers and $\sum_i d_i = d + 1$. Its *modular index* is $\max_i a_i$.

In [10, eq. (1.4)] the numerator is printed as $h_d t^d + \cdots + h_0$, with the letter d of the degree of f. We allow a numerator of any degree e; this only enlarges the class of positive forms.

Our lower bounds on modular indices use only the denominator: they hold for every expression $P(t)/\prod_i (1 - t^{a_i})^{b_i}$ with $P \in \mathbb{C}[t]$, with no condition on the numerator or on the exponents b_i.

The hypotheses. In [10, Section 3.3, p. 51] the input x of a structure constant has a rank $\operatorname{rank}(x)$ and a stretching quasi-polynomial $\tilde{f}(x, n)$, and the hypotheses read: (SH)(a) $s(\tilde{f}) = O(\operatorname{poly}(\operatorname{rank}(x)))$, (b) the saturation index vanishes for almost all x (Hypothesis 3.3.2); (PH2) $p(\tilde{f}) = O(\operatorname{poly}(\operatorname{rank}(x)))$ (Hypothesis 3.3.3); (PH3) the generating function $F(x,t) = \sum_n \tilde{f}(x,n)t^n$ has a positive rational form of modular index $O(\operatorname{poly}(\operatorname{rank}(x)))$, and more specifically the modular index defined in [10, Section 4.1.1] is $O(\operatorname{poly}(\operatorname{rank}(x)))$ (Hypothesis 3.3.4). These are stated for the Kronecker problem, the plethysm problem and the subgroup restriction problem, among others. For subgroup restriction along an explicit polynomial homomorphism $\rho : H \to G$, with H-representation λ and G-representation π, the rank is the sum of the ranks of H and G and the heights of the specified weights [10, Section 3.4, p. 52]; the Kronecker and plethysm problems are special cases.

For plethysm, [10, Section 1.6, pp. 24–25] states the same hypotheses with the parameter $h = \dim G + \operatorname{ht}(\lambda) + \operatorname{ht}(\pi)$: SH(a) asks $s \leq ch^{c'}$ (Hypothesis 1.6.5(a)), PH2 asks for a

polynomial bound on the positivity index (Hypothesis 1.6.6), and PH3 asks that the generating function have a positive rational form with modular index bounded by a polynomial in h, and more specifically that this hold for the minimal positive form attached to an h.s.o.p. of lexicographically minimal degree sequence (Hypothesis 1.6.7).

The relaxed hypotheses of [1, Appendix, pp. 19–20] are stated for Kronecker coefficients $g^{\pi}_{\lambda,\mu}$ with $H = GL_n \times GL_n \subset G = GL_{n^2}$, with the parameter h the largest height of λ, μ, π: (SH)(a) $s = O(\mathrm{poly}(h))$; (PH2) $p = O(\mathrm{poly}(h))$; (PH3) the modular index of the minimal positive form is $O(\mathrm{poly}(h))$.

The inputs of this paper. For k_n we take $H = GL_n \times GL_n$, $G = GL_{n^2}$ and the columns $\lambda = \mu = (1^n)$, $\pi = (1^n)$, stretched to (s^n). Then $\mathrm{rank}(x) = 2n + n^2 + 3n = n^2 + 5n$ in the sense of [10, Section 3.4], and $h = n$ in the sense of [1, Appendix]. For f_n, $G = GL(\mathrm{Sym}^3 \mathbb{C}^n) = GL_m$, $\mathrm{ht}(\lambda) = 1$, $\mathrm{ht}(\pi) = n$, so $h = m^2 + n + 1$, and the rank of [10, Section 3.4] is $n + m + n + 1$. Both are bounded by a polynomial in n; so is the parameter of every hypothesis above.

3 Hilbert series and fixed points of the grading

Throughout, $R = \bigoplus_{d \geq 0} R_d$ is a finitely generated positively graded $\mathbb{C}$-algebra with $R_0 = \mathbb{C}$, and $\mathrm{Hilb}_R(t) = \sum_d \dim R_d \, t^d$. It is a rational function whose poles are roots of unity. For a root of unity η, the grading element η acts on R_d by η^d; we write $X = \mathrm{Spec}\, R$ and X^{η} for the scheme-theoretic fixed locus. If η has order q, then $X^{\eta} = \mathrm{Spec}\, R/I_q$ with

$$I_q = (R_d : \ q \nmid d), \tag{1}$$

since a homogeneous element of degree d is changed by the factor η^d.

Lemma 3.1. *If X^{η} is supported at the graded origin, then $\mathrm{Hilb}_R(t)$ is regular at every primitive root of unity of the order q of η.*

Proof. Choose homogeneous generators $a_1, \ldots, a_l$ of I_q with degrees e_i not divisible by q. As the support is the origin, R/I_q is finite-dimensional. The homology modules N_j of the graded Koszul complex $K_R(a_1, \ldots, a_l)$ are finitely generated over R and annihilated by I_q, because multiplication by each a_i is null-homotopic on the Koszul complex; hence they are finite-dimensional. The graded Euler characteristic gives

$$\mathrm{Hilb}_R(t) \prod_{i=1}^{l} (1 - t^{e_i}) = \sum_j (-1)^j \, \mathrm{Hilb}_{N_j}(t),$$

whose right side is a polynomial. No factor $1 - t^{e_i}$ vanishes at a primitive q-th root of unity. □

Proposition 3.2. *Let R be a domain, $M \geq 2$, ζ a primitive M-th root of unity. Suppose X^{ζ} has a unique irreducible component Y of maximal dimension $\delta \geq 1$, and X is smooth at the generic point of Y. Give Y its reduced structure; its coordinate ring is concentrated in degrees divisible by M; put $h_Y(z) = \sum_k \dim \mathbb{C}[Y]_{Mk} z^k$ and $e_Y = \lim_{z \to 1} (1 - z)^{\delta} h_Y(z) > 0$. Then the conormal space of Y in X at its generic point has no eigenvalue one under ζ, and if $\nu_1, \ldots, \nu_c$ ($c = \dim X - \delta$) are its eigenvalues,*

$$\lim_{t \to \zeta} (1 - t^M)^{\delta} \, \mathrm{Hilb}_R(t) = \frac{e_Y}{\prod_{j=1}^{c} (1 - \nu_j)} \neq 0. \tag{2}$$

In particular Hilb_R has a pole of order exactly δ at ζ, and every expression $\mathrm{Hilb}_R(t) = P(t) / \prod_i (1 - t^{a_i})^{b_i}$ with $P \in \mathbb{C}[t]$ has some a_i divisible by M.

Proof. Choose homogeneous algebra generators of R and list those of degree not divisible by M as $f_1, \dots, f_l$, $\deg f_i = w_i$. Their ideal I is the fixed-locus ideal: f_i is a nonzero multiple of $\zeta \cdot f_i - f_i$, and modulo the f_i every remaining generator is fixed. Thus $X^\zeta = \operatorname{Spec} R/I$ and $(R/I)_d = 0$ when $M \nmid d$. The ideal is homogeneous, so every component of the fixed locus is stable under the connected grading torus, hence graded; the coordinate ring of Y lives in degrees divisible by M.

Let $K_\bullet$ be the graded Koszul complex on $f_1, \dots, f_l$, its i-th exterior generator of degree w_i and ζ-character ζ^{w_i}, and N_j its homology. As before, each N_j is a finite graded R/I-module, and

$$\operatorname{Hilb}_R(t) \prod_{i=1}^{l} (1 - t^{w_i}) = \sum_{j=0}^{l} (-1)^j \operatorname{Hilb}_{N_j}(t). \tag{3}$$

All factors $1 - t^{w_i}$ are nonzero at ζ.

Every finite graded R/I-module N splits into its degree classes modulo M; each class, regraded so that degree M becomes degree one, has support of dimension at most δ, so Hilb_N has a pole of order at most δ at ζ. If the localization N_η at the generic point η of Y is a $\mathbb{C}(Y)$-vector space, then

$$\lim_{t \to \zeta} (1 - t^M)^\delta \operatorname{Hilb}_N(t) = e_Y \operatorname{Tr}(\zeta \mid N_\eta). \tag{4}$$

Indeed, take a graded prime filtration of each degree class of N. Factors supported in dimension less than δ contribute zero to the limit. Since Y is the unique component of dimension δ, the remaining factors are shifted copies of $\mathbb{C}[Y]$; one generated in degree a contributes $t^a h_Y(t^M)$, with leading coefficient $\zeta^a e_Y$. After localization these factors give a filtration of N_η by lines of character ζ^a, which gives the trace. That $e_Y > 0$ is the positivity of the leading Hilbert coefficient of the positively graded domain $\mathbb{C}[Y]$ regraded by M.

At a smooth closed point of X on Y, diagonalize the finite-order action of ζ on the cotangent space and lift eigenvectors to the maximal ideal by the averaging projectors. They are formal coordinates, the completed local ring is the power series ring in them with diagonal action, and the fixed scheme is cut out by the coordinates with eigenvalue different from one. So the fixed scheme is smooth and regularly embedded there, and its conormal representation has no eigenvalue one. This applies at the generic point of Y.

Put $K = \mathbb{C}(Y)$. In the regular local ring R_η, the ideal I_η is the maximal ideal, generated by a regular sequence of length c. There is an exact sequence of K-representations of $\langle \zeta \rangle$

$$0 \to L \to E = \bigoplus_{i=1}^{l} K e_i \to V = I_\eta / I_\eta^2 \to 0, \qquad e_i \mapsto f_i \bmod I_\eta^2,$$

where E has the characters $\zeta^{w_i} \neq 1$ and V the characters $\nu_1, \dots, \nu_c \neq 1$. Choosing among the f_i a minimal generating set of I_η, which is a regular sequence, and subtracting combinations of it from the other generators changes the Koszul complex into the Koszul complex of a regular sequence tensored with the Koszul complex on $l - c$ zeros; hence $(N_j)_\eta \cong \bigwedge_K^j L$, naturally and therefore equivariantly. These modules are vector spaces, so (4) applies, and $\sum_j (-1)^j \operatorname{Tr}(\zeta \mid (N_j)_\eta) = \det(1 - \zeta \mid L)$. Multiplying (3) by $(1 - t^M)^\delta$ and letting $t \to \zeta$,

$$\lim_{t \to \zeta} (1 - t^M)^\delta \operatorname{Hilb}_R(t) = e_Y \frac{\det(1 - \zeta \mid L)}{\prod_i (1 - \zeta^{w_i})} = \frac{e_Y}{\det(1 - \zeta \mid V)},$$

which is (2). The pole order is at most δ by the module argument and exactly δ since the limit is not zero. Finally, if no a_i were divisible by M, the expression $P / \prod_i (1 - t^{a_i})^{b_i}$ would be regular at ζ. □

The Koszul lemmas used (null-homotopy, change of generators, vanishing of higher Koszul homology for a regular sequence) are [11, Tags 0621, 062F]. Nonreduced structure of X^ζ at the origin does no harm: its contributions are finite-dimensional.

Lemma 3.3. *Let $p \in X$ be a closed point other than the origin, and suppose that its stabilizer in the grading group $\mathbb{G}_m$ is exactly μ_M. Let Y be the closure of its $\mathbb{G}_m$-orbit, with reduced structure. Then $\mathbb{C}[Y]$ is concentrated in degrees divisible by M, the set $\{k : \mathbb{C}[Y]_{Mk} \neq 0\}$ is a numerical semigroup (cofinite in $\mathbb{Z}_{\geq 0}$), each nonzero $\mathbb{C}[Y]_{Mk}$ is a line, and $e_Y = 1$ with $\delta = 1$.*

Proof. Restriction to the orbit is the homomorphism $R \to \mathbb{C}[u]$, $f \mapsto f(p)u^d$ for $f \in R_d$; its image is $\mathbb{C}[Y]$, spanned by the u^d with $d \in S = \{d : f(p) \neq 0 \text{ for some } f \in R_d\}$, a semigroup since products of functions nonzero at p are nonzero at p. An element $\lambda \in \mathbb{G}_m$ fixes p if and only if $f(\lambda \cdot p) = \lambda^d f(p) = f(p)$ for all homogeneous f, since homogeneous functions separate the points of X; that is, if and only if $\lambda^d = 1$ for all $d \in S$. So the stabilizer is $\mu_{\gcd S}$, and $\gcd S = M$. A subsemigroup of $M\mathbb{Z}_{\geq 0}$ whose elements have greatest common divisor M contains all sufficiently large multiples of M. Hence $h_Y(z) = 1/(1-z) - P(z)$ for a polynomial P, and $e_Y = 1$. □

Lemma 3.4. *Let GL_n act linearly on a vector space U, let $R = \mathbb{C}[U]^{SL_n}$ be graded so that $R_0 = \mathbb{C}$, and let γ be an odd integer such that every $B \in GL_n$ acts on $X = \operatorname{Spec} R$ as the grading element $(\det B)^\gamma$. Let $p \in X$ be a closed point other than the origin, fixed by a grading element η of order 2^a, and let $y \in U$ lie in the closed SL_n-orbit over p. Then there is a diagonalizable $A \in GL_n$ of finite two-power order with $A \cdot y = y$ and $\operatorname{ord}(\det A) = 2^a$.*

Proof. As γ is odd, there is θ of order 2^a with $\theta^\gamma = \eta$. Choose $B \in GL_n$ with $\det B = \theta$. Then $B \cdot y$ lies in a closed orbit over $\eta \cdot p = p$, and each fibre of the quotient map contains a unique closed orbit [8, Section I.1, p. 84]; so $B \cdot y = S \cdot y$ with $S \in SL_n$, and $C = S^{-1}B$ fixes y with $\det C = \theta$. The semisimple part C_s of C also fixes y (an eigenvector of C_s for an eigenvalue $\neq 1$ cannot be fixed by $C = C_s C_u$, whose unipotent part preserves the eigenspaces of C_s) and has the same determinant. Diagonalize C_s and let $\Gamma \subset \mathbb{C}^*$ be the group generated by its eigenvalues, a finitely generated abelian group, the product of its finite cyclic torsion subgroup and a free group. The projection of Γ onto the two-primary part of its torsion subgroup, trivial on the free factor, is a homomorphism $\Gamma \to \mathbb{C}^*$; applied to every eigenvalue it preserves every multiplicative relation among them, in particular the relations expressing $C_s \cdot y = y$ weight by weight, and it fixes θ. The resulting diagonal matrix A fixes y, has finite two-power order, and $\det A = \theta$. □

Lemma 3.5. *Let p be a prime and $d = p^r e$ with $p \nmid e$. In $\mathbb{F}_p[t]$, $\overline{\Phi_d} = \overline{\Phi_e}$ if $r = 0$ and $\overline{\Phi_d} = \overline{\Phi_e}^{p^{r-1}(p-1)}$ if $r \geq 1$. The roots of $\overline{\Phi_e}$ in $\overline{\mathbb{F}}_p$ are the elements of multiplicative order e, each simple.*

Proof. For $r \geq 1$, $\Phi_{ep^r}(t)\Phi_e(t^{p^{r-1}}) = \Phi_e(t^{p^r})$ in $\mathbb{Z}[t]$ (group the roots on the right by the p-part of their order). Modulo p, $t \mapsto t^{p^j}$ becomes the p^j-th power; cancel $\overline{\Phi_e}^{p^{r-1}}$. The factorization $t^e - 1 = \prod_{b|e} \Phi_b(t)$ persists modulo p, $t^e - 1$ is separable there, and induction on e assigns to $\overline{\Phi_e}$ exactly the roots of order e. □

Theorem 3.6. *Let $F, G \in \mathbb{Q}(t) \cap \mathbb{Z}[[t]]$ have all their poles at roots of unity, let p be a prime, $p \nmid M$, and suppose $F \equiv G \pmod{p\mathbb{Z}[[t]]}$. Suppose G has a pole of order $\delta \geq 1$ at a primitive M-th root of unity ζ, no pole at any primitive Mp^r-th root of unity for $r \geq 1$, and that $C = \lim_{t\to\zeta}(1-t^M)^\delta G(t)$ is a unit in $\mathbb{Z}[\zeta]_\mathfrak{p}$ for some prime $\mathfrak{p}$ over p. Then F has a pole at a primitive Mp^r-th root of unity for some $r \geq 0$, hence at all of them; and every*

expression $F(t) = P(t)/\prod_i (1-t^{a_i})^{b_i}$ *with* $P \in \mathbb{Q}[t]$ *and* $a_i, b_i > 0$ *has some* a_i *divisible by* Mp^r*, in particular* $a_i \geq M$*.*

Proof. A reduced expression of F has a denominator that is, up to sign, a product of cyclotomic polynomials; normalize its constant term to 1. The numerator is then integral, being the product of the integral series F with an integral polynomial. So $F = N_F/D_F$ and $G = N_G/D_G$ with $N_\bullet, D_\bullet \in \mathbb{Z}[t]$, $D_\bullet(0) = 1$, coprime over $\mathbb{Q}$, the $D_\bullet$ signed products of cyclotomic polynomials. Write $D_G = \varepsilon \Phi_M^\delta D_0$. By Lemma 3.5 and the hypothesis on poles of order Mp^r, the reduction $\overline{D_0}$ does not vanish at $\alpha = \zeta \bmod \mathfrak{p}$, which has exact order M. With $U_M = (1-t^M)/\Phi_M$, separability of $t^M - 1$ modulo p gives $\overline{U_M}(\alpha) \neq 0$. So $D_0(\zeta)$ and $U_M(\zeta)$ are $\mathfrak{p}$-units, and $C = U_M(\zeta)^\delta N_G(\zeta)/(\varepsilon D_0(\zeta))$ forces $N_G(\zeta)$ to be a unit: $\overline{N_G}(\alpha) \neq 0$ while $\overline{D_G}$ vanishes at α. So $\overline{G} \in \mathbb{F}_p(t)$ has a pole at α. Reduction of both fractions is defined in $\mathbb{F}_p(t)$ and in $\mathbb{F}_p[[t]]$, and the map from such fractions to series is injective; the congruence gives $\overline{F} = \overline{G}$ in $\mathbb{F}_p(t)$, so $\overline{D_F}(\alpha) = 0$. Some cyclotomic factor Φ_d of D_F has $\overline{\Phi_d}(\alpha) = 0$, and by Lemma 3.5 the prime-to-p part of d is M: $d = Mp^r$. As N_F, D_F are coprime over $\mathbb{Q}$, F has a pole at a primitive d-th root of unity, and by rationality over $\mathbb{Q}$ at all of them. A denominator $\prod_i (1-t^{a_i})^{b_i}$ must vanish there, so $d \mid a_i$ for some i. □

Remark 3.7. The hypothesis on the poles of order Mp^r cannot be dropped: $G(t) = 1 + pt^{pM}/(1-t^{pM})$ has integral nonnegative coefficients and $\lim_{t\to\zeta}(1-t^M)G(t) = 1$, but $\overline{G} = 1$. Nor can the conclusion be improved to $r = 0$: for $p = 3$, $M = 2^a \geq 4$ and $z = t^{M/2}$, the series $F = (1+z)/(1-z+z^2)+2t/(1-t)^2$ and $G = 1/(1+z)+2t/(1-t)^2$ have nonnegative integral coefficients, $F - G = 3z/(1+z^3)$, the residue of G at the primitive M-th roots is 2, a 3-adic unit, but F has its poles there only at order $3M$, since $1 - z + z^2 = \Phi_{3M}(t)$.

4 Cubic supports fixed by a two-power diagonal matrix

Let $F \in \mathbb{C}[x_1, \ldots, x_n]$ be a cubic form and B the integer matrix whose columns are the exponent vectors of its monomials with nonzero coefficient (its *support*). We say F satisfies the *span condition* if $\mathbf{1} = (1, \ldots, 1)^t$ lies in the rational column span of B.

Lemma 4.1. *Every cubic form at which some* SL_n*-invariant of positive degree does not vanish satisfies the span condition, in every coordinate basis.*

Proof. Some coefficient monomial of the invariant is nonzero at F and so involves only coefficients of supported monomials. Invariance under the diagonal torus of SL_n makes the sum of the corresponding exponent vectors, counted with multiplicity, a vector with all coordinates equal, and positive since the degree is positive. □

If a diagonal matrix A with eigenvalues $\alpha_1, \ldots, \alpha_n$ of finite two-power order fixes F, the character $\psi : \mathbb{Z}^n \to \mathbb{C}^*$, $u \mapsto \prod_i \alpha_i^{u_i}$, kills the lattice generated by the support, and $\psi(\mathbf{1}) = \det A$. If K is a set of support columns forming a basis of the rational support span, the equation $Kz = \mathbf{1}$ has a unique rational solution; by Cramer's rule applied to a nonzero maximal minor K_I, the least $d \geq 1$ with dz integral divides $\det K_I$, and $d\mathbf{1} = K(dz)$ is in the support lattice. Hence

$$\operatorname{ord}(\det A) \leq 2^{v_2(\det K_I)}. \tag{5}$$

The support graph. A monomial $x_i^2 x_j$ $(i \neq j)$ gives an edge between i and j with coefficient two at i and one at j; a cube x_i^3 gives a loop with coefficient three; all n vertices are included. Every other monomial is of the form $x_i x_j x_k$ with three distinct indices (a *triple*). For a spanning tree of a connected component with v vertices, the $v \times (v-1)$

matrix T of its columns has rank $v-1$ (remove a leaf: its row has a single nonzero entry, one or two). So a graph component has rank v or $v-1$ (an isolated vertex has rank one if its cube occurs, zero otherwise). Deleting a row r of T and removing leaves other than r one at a time shows

$$\det T^{(r)} = \pm 2^{e_r}, \qquad 0 \le e_r \le v-1. \tag{6}$$

The signed cofactor vector u of T spans the kernel of T^t; along an edge with coefficients two and one it satisfies $2u_i + u_j = 0$, so adjacent entries differ in absolute value by a factor two and have opposite signs:

$$u = \varepsilon 2^e \big((-2)^{h_v}\big)_v, \qquad \varepsilon = \pm 1, \quad e \ge 0, \quad h_v \ge 0, \quad \min_v h_v = 0. \tag{7}$$

Proposition 4.2. *Suppose the support of F consists of cubes and monomials $x_i^2 x_j$ only, and F satisfies the span condition. If a diagonal matrix A of finite two-power order fixes F, then* $\operatorname{ord}(\det A) \le 2^{n-1}$. *If equality holds, then after a permutation of the variables the support is exactly*

$$x_1^2 x_2,\ x_2^2 x_3,\ \ldots,\ x_{n-1}^2 x_n,\ x_n^3, \tag{8}$$

the eigenvalues of A are $\xi, \xi^{-2}, \xi^4, \ldots, \xi^{(-2)^{n-1}} = 1$ with ξ of order 2^{n-1}, and a diagonal substitution turns F into the chain cubic $F_n = x_1^2 x_2 + \cdots + x_{n-1}^2 x_n + x_n^3$.

Proof. The support lattice, the span condition and ψ split over the connected components of the support graph, and $\det A$ is the product of the values of ψ on the all-ones vectors $\mathbf{1}_v$ of the components. Each value has two-power order; we show it is at most 2^{v-1} for a component with v vertices, and at most 2^{v-2} if $v \ge 2$ and the component is rank-deficient. An isolated vertex must carry a cube (span condition), and $3e_i$ in the lattice makes $\psi(e_i)$ trivial.

Rank $v-1$. The span of the component is that of a spanning tree T, and $Tc = \mathbf{1}_v$ has a unique rational solution. By Cramer's rule for every $T^{(r)}$, the denominator q of c divides every 2^{e_r}, and $q\mathbf{1}_v = T(qc)$ lies in the lattice. By (7) adjacent e_r differ by one; all are at most $v-1$, so their minimum is at most $v-2$, and $q \le 2^{v-2}$.

Rank v. One further column extends T to a nonsingular square matrix C, whose graph is connected and unicyclic (the cycle may be a loop or a double edge). The order of $\psi(\mathbf{1}_v)$ divides $|\det C|$, so it is at most $2^{v_2(\det C)}$. Expanding along leaf rows contributes a factor one or two for each of the $v-l$ tree edges off the cycle of length l. A loop contributes the odd factor three, so $v_2(\det C) \le v-1$. A cycle of length $l \ge 2$ has determinant $\pm(2^a + (-1)^{l-1}2^{l-a})$ for some $0 \le a \le l$; it is nonzero, so $a \ne l-a$ when l is even, and its valuation is $\min(a, l-a) \le \lfloor (l-1)/2 \rfloor$. Then $v_2(\det C) \le v - l + \lfloor (l-1)/2 \rfloor \le v-2$.

The product of two-power orders has order at most the largest one, and every component has at most n vertices; this gives $\operatorname{ord}(\det A) \le 2^{n-1}$. In the case of equality, some component has n vertices, so the graph is connected, of full rank, and the chosen unicyclic basis has a loop, at a vertex we call the root; and every leaf-expansion factor is two. Rooting the tree at the loop, each tree edge has coefficient two at its child and one at its parent, and the eigenvalue relations are $\alpha_{\text{root}}^3 = 1$ and $\alpha_{\text{parent}} = \alpha_{\text{child}}^{-2}$. As the eigenvalues have two-power order, $\alpha_{\text{root}} = 1$, and a vertex at depth d has eigenvalue of order at most 2^d; if the depth of the tree is h, then $\det A$ has order at most 2^h. So $h \ge n-1$, and the tree is a path rooted at one end. The deepest eigenvalue ξ has order 2^{n-1}, and the eigenvalues along the path are $\xi, \xi^{-2}, \ldots, \xi^{(-2)^{n-1}} = 1$. In particular A has order 2^{n-1}.

These eigenvalues fix no other cubic monomial. Indeed, the exponents $(-2)^i \bmod 2^{n-1}$, $0 \le i \le n-2$, have distinct two-adic valuations and the last exponent is 0. In a sum of three of them that vanishes modulo 2^{n-1}, the least valuation among the nonzero ones

cannot occur exactly once, nor three times (three equal nonzero terms have the valuation of one of them, since three is odd); so it occurs twice, at one index i, and the third term is $-2(-2)^i$, the next chain exponent, or zero if $i = n-2$. Three zero terms give x_n^3. So the monomials fixed by A are exactly (8), and they all occur, being the columns of the basis. Rescale x_n by a cube root and then each x_i, $i = n-1, \dots, 1$, by a square root to make all coefficients one. □

Theorem 4.3. *Let F be a cubic form in $n \geq 2$ variables satisfying the span condition, and let a diagonal matrix A of finite two-power order fix F. Then* $\operatorname{ord}(\det A) \leq 2^{n-1}$. *If equality holds, then after a permutation of the variables the support of F is exactly* (8)*, A has order 2^{n-1}, and a diagonal substitution turns F into F_n. The bound is attained by F_n.*

Proof. Let G be the matrix of the graph columns (cubes and $x_i^2 x_j$). By the spanning-tree argument, if the graph has f components of full rank and c rank-deficient components, with v_i vertices, then $\operatorname{rank} G = n - c$. In each full component choose a square basis C_i (a spanning tree and one more graph column); the proof of Proposition 4.2 gives

$$v_2(\det C_i) \leq v_i - 1. \tag{9}$$

In each deficient component choose a spanning tree T_i; its signed cofactor vector is $2^{e_i} u_i$ with $(u_i)_v = \varepsilon_i (-2)^{h_{iv}}$ as in (7) (for an isolated vertex without cube, $e_i = h_{iv} = 0$). Put $b_i = v_i - 1 - e_i$ and $H_i = \max_v h_{iv}$. Since a tree minor is a product of $v_i - 1$ factors one or two, $0 \leq H_i \leq b_i$; and if $H_i = b_i$, the vertex of height H_i is unique: rooting the tree there, its deleted-row minor has absolute value $2^{v_i - 1}$, so every factor is two, each edge has coefficient two at its child and one at its parent, $h_{\text{parent}} = h_{\text{child}} + 1$, and heights decrease strictly away from the root.

Normalize the entries of u_i by 2^{b_i}. For sums over distinct vertices E of one deficient component,

$$\Big| \sum_{v \in E} \frac{(u_i)_v}{2^{b_i}} \Big| \leq 1, \ \tfrac{5}{4}, \ \tfrac{3}{2} \qquad \text{for } |E| = 1, 2, 3. \tag{10}$$

If $H_i < b_i$, all entries have absolute value at most $1/2$. If $H_i = b_i$, there is one entry of absolute value one, which we make positive; the entries of absolute value $1/2$ are at height $H_i - 1$ and so negative; all others have absolute value at most $1/4$. A pair avoiding the top entry has sum at most one in absolute value, a pair containing it at most $5/4$; a triple avoiding it at most $3/2$; a triple containing it gives 0 if both others are $-1/2$, at most $3/4$ if one is, and at most $1 + 1/4 + 1/4$ otherwise.

The bases C_i and trees T_i form a rational basis of the graph columns. Extend it by q triple columns $\beta_1, \dots, \beta_q$ to a basis of all support columns; then $0 \leq q \leq c$ and the support rank is $r = n - c + q$. If $q = 0$, the graph part satisfies the span condition by itself, and Proposition 4.2 applies: it gives the bound, and in case of equality the chain eigenvalues, which fix no triple monomial; so F has no triple monomials and the theorem holds. Assume $q > 0$.

Define the $c \times q$ matrix $D_{ij} = u_i^t(\beta_j|_i)$, where $\beta_j|_i$ is the restriction to the vertices of deficient component i. The map sending a vector to its c pairings with the u_i is surjective with kernel the rational span of the graph columns (on a full component the graph columns span all coordinates; on a deficient one $\ker u_i^t$ is the tree span), so D has rank q; choose q deficient components J with $\det D_J \neq 0$. Let K be the $n \times r$ basis matrix. Keep all rows of full components and of the components in J; in every other deficient component delete one row at a vertex of height zero. The resulting $r \times r$ minor satisfies

$$\det K_I = \pm \Big(\prod_{\text{full } i} \det C_i \Big) 2^{\sum_{\text{deficient } i} e_i} \det D_J. \tag{11}$$

Indeed, in a full component eliminate the triple entries with its graph columns and factor out $\det C_i$; in a deficient component outside J the retained tree block is square with determinant $\pm 2^{e_i}$ (the deleted row had height zero); eliminate and factor out. These operations stay inside components. What remains are the tall tree blocks of J and the q triple columns; expanding along maximal minors of each tree block leaves one row per component, given by the cofactor vector $2^{e_i}u_i$, and the remaining $q \times q$ determinant is $\det D_J$.

Divide row i of D_J by 2^{b_i}. Each column then involves at most three distinct vertices; by (10) its squared Euclidean norm is at most 3 (three components), $(5/4)^2 + 1 = 41/16$ (two and one) or $(3/2)^2$ (all in one). Hadamard's inequality gives

$$0 < |\det D_J| \le 2^{\sum_{i\in J} b_i} 3^{q/2}. \tag{12}$$

If $q < c$, or $q = c$ and $f > 0$, then by (9), (11) and (12),

$$v_2(\det K_I) \le \sum_{\text{full}} (v_i - 1) + \sum_{\text{def}} e_i + \sum_{i\in J} b_i + \tfrac{q}{2}\log_2 3 = n - f - c - \sum_{\text{def } i\notin J} b_i + \tfrac{q}{2}\log_2 3 < n - f - c + q,$$

strictly since $q > 0$ and $\sqrt{3} < 2$; the right side is at most $n - 1$, so $v_2(\det K_I) \le n - 2$. If $q = c$ and $f = 0$, then K is square and every column has coordinate sum three, so $\mathbf{1}^t K = 3\mathbf{1}^t$ and $3 \mid \det K$, hence $3 \mid \det D$; with $\sum_i (e_i + b_i) = n - c$,

$$2^{v_2(\det K)} \le 2^{\sum e_i} \frac{|\det D|}{3} \le 2^{n-c} 3^{c/2-1} < \frac{2^n}{3} < 2^{n-1}.$$

In both cases (5) gives $\operatorname{ord}(\det A) \le 2^{n-2}$. So equality occurs only with $q = 0$. Finally F_n is fixed by $\operatorname{diag}(\xi, \xi^{-2}, \dots, \xi^{(-2)^{n-1}})$ with ξ of order 2^{n-1}, whose determinant ξ^S, $S = \sum_{i<n}(-2)^i$, has order 2^{n-1} since S is odd. □

5 Kronecker coefficients

Let $W = \mathbb{C}^n$ with basis $e_1, \dots, e_n$, and let $R_n = \mathbb{C}[W \otimes W \otimes W]^{SL_n^3}$, graded so that $(R_n)_s$ is the space of invariants of degree ns (invariants occur only in degrees divisible by n). By Schur–Weyl duality $\dim(R_n)_s = g((s^n), (s^n), (s^n)) = k_n(s)$, so $K_n = \operatorname{Hilb}_{R_n}$. Let τ be the cyclic permutation of the three tensor factors; it acts on R_n as a graded algebra automorphism of order three.

Let $\mathcal{U}_n = W \oplus \mathbb{S}_{(2,1)}W$, give its coordinate ring the grading in which linear functions on W have weight one and those on $\mathbb{S}_{(2,1)}W$ weight three, and let $b_n(s)$ be the dimension of the SL_n-invariants of weighted degree ns and $B_n(t) = \sum_s b_n(s)t^s$. (All invariants have weighted degree divisible by n, as the scalars μ_n act on weighted degree D by ζ^D.) Then $B_n = \operatorname{Hilb}_{T_n}$ with $T_n = \mathbb{C}[\mathcal{U}_n]^{SL_n}$ graded by s.

Theorem 5.1. *For every $n \ge 2$ and $s \ge 0$,*

$$\operatorname{tr}(\tau \mid (R_n)_s) = b_n(s), \qquad k_n(s) \equiv b_n(s) \pmod 3,$$

and $k_n(s) - b_n(s)$ is a nonnegative multiple of three. Hence $K_n(t) \equiv B_n(t) \pmod{3\mathbb{Z}[[t]]}$.

Proof. We may compute with polynomial representations of W instead of W^*; the simultaneous contragredient preserves the dimensions of invariant spaces and commutes with permutations of the factors. In degree $d = ns$, Schur–Weyl duality gives, compatibly with τ,

$$(R_n)_s \simeq \big([\lambda] \otimes [\lambda] \otimes [\lambda]\big)^{S_d}, \qquad \lambda = (s^n),$$

since $\mathrm{Sym}^d(W\otimes W\otimes W)$ is the diagonal S_d-invariant part of $(W^{\otimes d})^{\otimes 3}$ and $(W^{\otimes d})^{SL_n}$ is the Specht module $[\lambda]$. For an endomorphism A of a vector space, the trace of the cyclic permutation composed with $A^{\otimes 3}$ is $\mathrm{tr}(A^3)$ (contract the cyclic indices in a basis). Averaging over S_d,

$$\mathrm{tr}(\tau \mid (R_n)_s) = \frac{1}{d!}\sum_{\pi\in S_d}\chi_\lambda(\pi^3).$$

By the Frobenius formula $p_\mu = \sum_\lambda \chi_\lambda(\mu)s_\lambda$, this average is the coefficient of $s_\lambda z^d$ in $\sum_d \frac{z^d}{d!}\sum_{\pi\in S_d} p_{\mathrm{type}(\pi^3)}$. Cubing a permutation keeps a j-cycle with $3\nmid j$ and splits a $3r$-cycle into three r-cycles, so by the exponential formula the series is

$$\exp\Big(\sum_{3\nmid j}\frac{z^j p_j}{j}+\sum_{r\ge 1}\frac{z^{3r}p_r^3}{3r}\Big)=\exp\Big(\sum_{r\ge1}\frac{z^r p_r}{r}+\sum_{r\ge1}\frac{z^{3r}}{r}\,p_r[s_{(2,1)}]\Big),$$

using $p_r[s_{(2,1)}] = (p_r^3 - p_{3r})/3$, which follows from $s_{(2,1)} = (p_1^3 - p_3)/3$. The right side is the character of the symmetric algebra on $W\oplus \mathbb{S}_{(2,1)}W$ with weights one and three [9, Chapter I, §§7–8]; specializing to n variables and taking the coefficient of $s_{(s^n)}$ gives $b_n(s)$. Finally, τ is defined over $\mathbb{Q}$ on $(R_n)_s$, and an operator of order three on a rational vector space has characteristic polynomial $(X-1)^a(X^2+X+1)^b$, dimension $a+2b$ and trace $a-b$; the difference is $3b\ge 0$. □

We realize $\mathbb{S}_{(2,1)}W$ as the degree-three part of the free Lie algebra on W inside $W^{\otimes 3}$, with $[a,b]=ab-ba$: the map $W\otimes\Lambda^2 W\to W^{\otimes 3}$, $a\otimes(b\wedge c)\mapsto[a,[b,c]]$, has kernel $\Lambda^3 W$ by the Jacobi identity, and its image is $\mathbb{S}_{(2,1)}W$. Its weights are $2e_i+e_j$ $(i\ne j)$, with multiplicity one, and $e_i+e_j+e_k$ $(i,j,k$ distinct), with multiplicity two. Put

$$z_n=(e_n,F_n)\in\mathcal{U}_n,\qquad F_n=\sum_{i=1}^{n-1}[e_i,[e_i,e_{i+1}]],\qquad M=2^{n-1}.$$

Lemma 5.2. *For every $n\ge 2$, the stabilizer of z_n in $\mathfrak{gl}_n$ is zero, so the stabilizer of z_n in GL_n is finite.*

Proof. Write $Xe_j=\sum_a x_{aj}e_a$ and suppose $Xz_n=0$; then $Xe_n=0$, and X acts on F_n as a derivation of the brackets. For $r\ne s$, the weight $2e_r+e_s$ is spanned by $L_{rs}=[e_r,[e_r,e_s]]$, and its coefficient in XF_n is

$$\mathbf{1}_{r<n}x_{s,r+1}+2\cdot\mathbf{1}_{s=r+1}x_{rr}-\mathbf{1}_{s=r-1}x_{r,r-1},$$

the last term coming from $[e_r,[e_{r-1},e_r]]=-L_{r,r-1}$. Taking $s=r+1$ gives $2x_{rr}+x_{r+1,r+1}=0$, and $x_{nn}=0$, so the diagonal vanishes. For column $j\ge 2$ the only possibly nonzero entries are then $x_{j-1,j}$ and $x_{j-2,j}=x_{j-1,j-2}$ $(j\ge 3)$; for $j\ge 4$ the latter is an entry below the diagonal in column $j-2\ge 2$, hence zero. So besides the first column and the superdiagonal only $x_{13}=x_{21}$ $(n\ge 3)$ can be nonzero. Put $y_j=x_{j-1,j}$; $Xe_n=0$ gives $y_n=0$. If $n=2$, $Xe_2=0$ also kills x_{12}, and the coefficient of L_{21} gives $x_{21}=0$. Let $n\ge 3$.

For $a\ge 4$, the weight $e_1+e_2+e_a$ receives only the term of index one, with coefficient x_{a1} times the nonzero vector $[e_a,[e_1,e_2]]+[e_1,[e_a,e_2]]$; so $x_{a1}=0$. For $3\le j\le n-1$, the weight $e_{j-1}+e_j+e_{j+1}$ receives only the two adjacent terms, with coefficients y_j and $x_{j+1,j-1}=0$, the former multiplying the nonzero vector $[e_{j-1},[e_j,e_{j+1}]]+[e_j,[e_{j-1},e_{j+1}]]$; so $y_j=0$. In the weight $e_1+e_2+e_3$, with the basis $A=[e_1,[e_2,e_3]]$, $B=[e_2,[e_1,e_3]]$ and $[e_3,[e_1,e_2]]=-A+B$, the contribution is $y_2(A+B)+x_{31}(-2A+B)$; so $y_2-2x_{31}=y_2+x_{31}=0$ and $x_{31}=y_2=0$. Finally $x_{13}=x_{21}$ vanishes by $Xe_3=0$ if $n=3$, and for $n\ge 4$ because the weight $e_1+e_3+e_4$ receives only the derivative of the repeated e_3 in the term of index three, with coefficient x_{13} times a nonzero vector. So $X=0$. An algebraic group in characteristic zero has the dimension of its Lie algebra. □

Theorem 5.3. *Let $n \geq 2$, $M = 2^{n-1}$, ζ a primitive M-th root of unity, and $X_n = \operatorname{Spec} T_n$.*

1. *The SL_n-orbit of z_n is closed; its stabilizer in GL_n is cyclic of order M, generated by*

$$g = \operatorname{diag}(\zeta^{w_1}, \ldots, \zeta^{w_n}), \qquad w_i = (-2)^{i-1} \bmod M, \tag{13}$$

   on which det *is faithful; its stabilizer in SL_n is trivial.*

2. *The nonzero closed points of X_n fixed by the grading group μ_M form one orbit of the grading group $\mathbb{G}_m$, the orbit of the image p_n of z_n, whose stabilizer in $\mathbb{G}_m$ is exactly μ_M. X_n is smooth at p_n, and the fixed scheme X_n^ζ is smooth of dimension one there.*

3. *B_n has a simple pole at every primitive M-th root of unity ξ, with $\lim_{t\to\xi}(1-t^M)B_n(t)$ a unit in $\mathbb{Z}[\xi, 1/2]$.*

4. *If the two-part of an integer d is at least M and $d \neq M$, then B_n is regular at every primitive d-th root of unity; in particular at the primitive $M3^r$-th roots for $r \geq 1$.*

Proof. Closed orbit. For $1 \leq i < n$ put $q_i = (1 - (-1/2)^i)/3$ and $q_n = 1 - q_{n-1}$; all are positive, and with $a_i = 2e_i + e_{i+1}$ ($i < n$), $a_n = e_n$, one has $\sum_i q_i a_i = \mathbf{1}$ ($2q_1 = 1$, $q_{i-1} + 2q_i = 1$, $q_{n-1} + q_n = 1$). The words $[e_i, [e_i, e_{i+1}]]$ have squared norm six in the Hermitian metric of $W^{\otimes 3}$; give the word i the coefficient $\sqrt{q_i/6}$ and e_n the coefficient $\sqrt{q_n}$. The diagonal part of the moment map of SU_n is then $\mathbf{1}$ minus its trace part, which vanishes; its off-diagonal part vanishes because no two of the weights $a_1, \ldots, a_n$ in the same summand differ by a root $e_a - e_b$, and the summands are orthogonal and preserved. So this vector is a zero of the moment map, and its SL_n-orbit is closed by the Kempf–Ness criterion [7, Theorem 4, p. 1]. The weights a_i are linearly independent (triangular, determinant 2^{n-1}), so the diagonal torus acts transitively on vectors with these weights and nonzero coefficients; it maps our vector to z_n, and the orbit of z_n is closed as well.

Stabilizer. The weights w_i are distinct modulo M, $w_n = 0$ and $2w_i + w_{i+1} \equiv 0$, so g fixes z_n, and $\det g = \zeta^{\sum w_i}$ with $\sum_i w_i$ odd. Let $L = \operatorname{Stab}_{GL_n}(z_n)$, finite by Lemma 5.2. As g has simple spectrum, its centralizer in L is diagonal; a diagonal element of L satisfies $d_n = 1$ and $d_i^2 d_{i+1} = 1$, so the centralizer is $\langle g \rangle$, of order M. Column orthogonality gives $\sum_{\rho \in \operatorname{Irr}(L)} |\rho(g)|^2 = M$; the powers $\det^k$, $0 \leq k < M$, are distinct linear characters (as $\det g$ has order M) and exhaust this sum. So every linear character of L is a power of det, and every nonlinear irreducible character vanishes at g. Let $K = \ker(\det|_L)$. A representation of g has trace zero exactly when the multiplicities of η and $-\eta$ agree for every eigenvalue η (use the basis $1, \zeta, \ldots, \zeta^{M/2-1}$ of $\mathbb{Q}(\zeta)$ and $\zeta^{M/2} = -1$). The spectrum (13) has exactly one pair $\{\eta, -\eta\}$, on $P = \langle e_{n-1}, e_n \rangle$, since the w_i with $i \leq n-2$ have distinct two-adic valuations less than $n-2$. So every nonlinear constituent of W lies in P; the linear constituents are powers of det, on which K acts trivially; P is L-stable, K acts on it with determinant one and fixes e_n, and a finite-order element of $SL_2(\mathbb{C})$ fixing a nonzero vector is the identity. So $K = 1$ and $L = \langle g \rangle$; the SL_n-stabilizer is trivial.

The grading. The scalar $\lambda I \in GL_n$ acts on $\mathcal{U}_n$ as the weight grading, and on $(T_n)_s$ by $\lambda^{\pm ns} = (\det \lambda I)^{\pm s}$; so each $B \in GL_n$ acts on X_n as the grading element $(\det B)^{\pm 1}$. Hence the grading stabilizer of p_n is $\det(L) = \mu_M$, and every point of $\mathbb{G}_m \cdot p_n$ is fixed by μ_M.

Fixed points. Let $p \neq 0$ be a closed point of X_n fixed by a grading element of order $2^a \geq M$ and y in its closed orbit. By Lemma 3.4 (with $\gamma = \pm 1$) some diagonal A of two-power order fixes y with $\operatorname{ord}(\det A) = 2^a$. In an eigenbasis of A consider the weights of the nonzero components of y: linear weights e_i, and weights $2e_i + e_j$, $e_i + e_j + e_k$ of $\mathbb{S}_{(2,1)}W$. Replace each linear weight e_i by $3e_i$; this preserves the rational span and, for eigenvalues of two-power order, the condition of being fixed ($\alpha_i = 1 \iff \alpha_i^3 = 1$). The span condition

holds by the argument of Lemma 4.1, applied to a positive-degree invariant nonzero at y. The resulting columns are the support of a cubic form fixed by A, and Theorem 4.3 gives $2^a \le M$; so $a = n-1$, and the support is the chain (8) after a permutation. The cube $3e_n$ comes from the linear coordinate e_n, since $\mathbb{S}_{(2,1)}W$ has no weight $3e_i$, and the other columns from the one-dimensional weight spaces of $2e_i + e_{i+1}$. So y is diagonally equivalent to z_n, and $p \in \mathbb{G}_m \cdot p_n$.

Smoothness and the fixed tangent space. Since the closed orbit of z_n has trivial stabilizer in SL_n, Luna's slice theorem [8, Section III.1, Théorème du slice étale and Remarque 1°, pp. 97–98] shows that X_n is smooth at p_n with tangent space $\mathcal{U}_n/\mathfrak{sl}_n \cdot z_n$, equivariantly for g. The fixed space of g on $\mathcal{U}_n$ has dimension n. Recall $v_2(w_i) = i-1$ for $i < n$ and $w_n = 0$. A linear weight e_i is fixed only for $i = n$. A weight $2e_i + e_j$ is fixed only for $j = i+1$: if $j = n$, then $2w_i \equiv 0$ forces $i = n-1$; if $i = n$, then $w_j \not\equiv 0$; if $i, j < n$ and $j \ne i+1$, the valuations i and $j-1$ of $2w_i$ and w_j differ, so the sum has valuation at most $n-2$; and $2w_i + w_{i+1} = 0$. A weight $e_i + e_j + e_k$ is never fixed, since among its nonzero terms the least valuation occurs once. As g has simple spectrum, $\dim \mathfrak{sl}_n^g = n-1$. Taking fixed vectors is exact for a finite group in characteristic zero, so $\dim(T_{p_n}X_n)^g = 1$, and as in the proof of Proposition 3.2 the fixed scheme is smooth of dimension one at p_n. Translating by the grading group, the same holds along $\mathbb{G}_m \cdot p_n$.

The pole. Let Y be the closure of $\mathbb{G}_m \cdot p_n$. Its only other point is the origin, since the grading is positive; so the support of X_n^ζ is Y, a curve, and X_n is smooth at its generic point. By Lemma 3.3, $e_Y = 1$. Proposition 3.2 gives a simple pole at ζ with $\lim(1-t^M)B_n = 1/\prod_j(1-\nu_j)$, where the ν_j are the eigenvalues of ζ on the conormal space, all different from one and all M-th roots of unity. For an M-th root of unity $\nu \ne 1$ of order 2^b, $(1-\nu)\sum_{j<2^{b-1}} \nu^j = 1 - \nu^{2^{b-1}} = 2$, so $1-\nu$ is a unit in $\mathbb{Z}[\zeta, 1/2]$. The same holds at every primitive M-th root by Galois conjugation, B_n having rational coefficients.

No other poles of large two-part. If the two-part of d is at least M and $d \ne M$, a nonzero point fixed by a primitive d-th root would be fixed by its two-primary part, of order at least M, hence lie on $\mathbb{G}_m \cdot p_n$, whose grading stabilizer is $\mu_M \not\ni$ the primitive d-th roots. So the fixed locus is the origin, and Lemma 3.1 applies. □

Theorem 5.4. *For every $n \ge 2$ there is $r \ge 0$ such that $K_n(t) = \sum_s g((s^n),(s^n),(s^n))t^s$ has a pole at every primitive root of unity of order $2^{n-1}3^r$. Every expression $K_n(t) = P(t)/\prod_i(1-t^{a_i})^{b_i}$ with $P \in \mathbb{Q}[t]$ and positive integers a_i, b_i has some a_i divisible by $2^{n-1}3^r$; in particular every positive form of K_n has modular index at least 2^{n-1}.*

Proof. Both K_n and B_n are Hilbert series of finitely generated positively graded domains, so they are rational with poles at roots of unity, with integral coefficients. By Theorem 5.1, $K_n \equiv B_n \pmod 3$. By Theorem 5.3, B_n has a simple pole at each primitive M-th root ζ, $M = 2^{n-1}$, with limit a unit in $\mathbb{Z}[\zeta, 1/2]$, hence a unit at every prime above 3, and no pole at the primitive $M3^r$-th roots, $r \ge 1$. Theorem 3.6 with $p = 3$, $F = K_n$, $G = B_n$ applies. □

Corollary 5.5. *The existence clause of PH3 for Kronecker coefficients [10, Hypothesis 3.3.4] fails: for the inputs $((1^n),(1^n),(1^n))$, with $\operatorname{rank}(x) = n^2 + 5n$, every positive rational form of the generating function has modular index at least 2^{n-1}. The relaxed PH3 of [1, Appendix, p. 20], which bounds the modular index of the minimal positive form by a polynomial in the largest height $h = n$, fails as well.*

Proof. The minimal positive form is a positive form, and 2^{n-1} exceeds every polynomial in n, hence in $n^2 + 5n$, for large n. □

6 Plethysm coefficients: the modular index

Let $V = \mathbb{C}^n$ with coordinates $x_1, \dots, x_n$, $n \geq 4$, $\nu = \gcd(n, 3)$, $d_0 = n/\nu$, $\gamma = 3/\nu$ (an odd integer) and $M = 2^{n-1}$. Points of $\operatorname{Sym}^3 V^*$ are cubic forms, with $(a \cdot F)(x) = F(a^{-1}x)$. Let $R = \mathbb{C}[\operatorname{Sym}^3 V^*]^{SL_n}$ with its coefficient grading R_D.

The multiplicity algebra. If $R_D \neq 0$, the scalar $\zeta I_n \in SL_n$, $\zeta^n = 1$, acts on R_D by $\zeta^{\pm 3D}$, so $n \mid 3D$, that is $D = d_0 s$. Writing $a = zh$ with $h \in SL_n$ and $z^n = \det a$, the element $a \in GL_n$ acts on $R_{d_0 s}$ by the character $(\det a)^{\gamma s}$, and, as functions,

$$P(a \cdot F) = (\det a)^{-\gamma s} P(F) \qquad (P \in R_{d_0 s}). \tag{14}$$

Choosing a volume vector $\varepsilon \in \Lambda^n V$, evaluation of an intertwiner at $\varepsilon^{\otimes \gamma s}$ identifies the degree-s part of the multiplicity algebra $\bigoplus_s \operatorname{Hom}_{GL_n}((\det)^{\gamma s}, \operatorname{Sym}^{d_0 s}(\operatorname{Sym}^3 V))$ of [10, Section 4.1.2, eq. (4.5)] with $R_{d_0 s}$, compatibly with products. So the multiplicity algebra is $T^{(n)} = \bigoplus_s R_{d_0 s}$, the whole invariant ring with its degrees divided by d_0, a finitely generated positively graded domain with $T^{(n)}_0 = \mathbb{C}$, and $H_n = \operatorname{Hilb}_{T^{(n)}}$. Let $X = \operatorname{Spec} T^{(n)}$. By (14), $a \in GL_n$ acts on X as the grading element $(\det a)^{-\gamma}$; as det is onto $\mathbb{C}^*$ and γ is odd, every grading element arises.

Proposition 6.1. *Let $F_n = x_1^2 x_2 + x_2^2 x_3 + \cdots + x_{n-1}^2 x_n + x_n^3$ and $p_n \in X$ its image.*

1. *F_n defines a smooth hypersurface, and its SL_n-orbit is closed.*
2. *Every linear transformation preserving F_n up to a scalar is a scalar multiple of a diagonal matrix $\operatorname{diag}(a_1, a_1^{-2}, a_1^4, \dots, a_1^{(-2)^{n-1}})$ with $a_1^{3M} = 1$. These matrices form the stabilizer of F_n in GL_n, a cyclic group of order $3M$; the stabilizer in SL_n is the scalar group μ_ν, which acts trivially on cubic forms.*
3. *The stabilizer of p_n in the grading group is exactly μ_M.*
4. *X is smooth at p_n, and for a primitive M-th root of unity ζ the fixed scheme X^ζ is smooth of dimension one at p_n.*

Proof. (1) The partial derivatives are $2x_1x_2$, $x_{i-1}^2 + 2x_i x_{i+1}$ $(1 < i < n)$ and $x_{n-1}^2 + 3x_n^2$. At a common zero $x \neq 0$, let j be least with $x_j \neq 0$. If $j < n$, the j-th derivative gives $x_{j+1} = 0$, and then the $(j+1)$-st is $x_j^2 \neq 0$; if $j = n$, the last one is $3x_n^2 \neq 0$. So F_n is smooth. For the closed orbit, put $r_i = (1 - (-1/2)^i)/3$ for $i < n$ and $r_n = (1 - r_{n-1})/3$; all are positive, and with $a_i = 2e_i + e_{i+1}$ $(i < n)$, $a_n = 3e_n$, one has $\sum_i r_i a_i = \mathbf{1}$. Fix an SU_n-invariant Hermitian product on $\operatorname{Sym}^3 V^*$; distinct monomials are orthogonal, having distinct torus weights. The cubic $F = \sum_i (r_i/\|m_i\|^2)^{1/2} m_i$, where m_i are the chain monomials, is a zero of the moment map of SU_n: its diagonal part is the restriction of $\sum_i r_i a_i$ to the trace-free diagonal matrices, which is zero because $\sum_i r_i a_i = \mathbf{1}$, and its off-diagonal part vanishes because no two of the weights a_i differ by a root $e_a - e_b$ (their differences have an entry ± 2, or the form $2e_i - e_{i+1} - e_{i+2}$, or $2e_{n-1} - 2e_n$). By the Kempf–Ness criterion [7, Theorem 4, p. 1] the SL_n-orbit of F is closed. The weights a_i are linearly independent, so a diagonal matrix maps F to F_n, whose orbit is therefore closed.

(2) Half the Hessian matrix of F_n is tridiagonal, with diagonal $(x_2, x_3, \dots, x_n, 3x_n)$ and off-diagonal entries $x_1, \dots, x_{n-1}$. Its projective locus of rank at most two consists of

$$[e_1] \quad \text{and} \quad [ae_{n-1} + be_n] \text{ with } a = 0 \text{ or } a^2 = 3b^2, \tag{15}$$

three collinear points and a fourth point off their line. Indeed, a tridiagonal block whose off-diagonal entries are all nonzero has kernel of dimension at most one, and the zero off-diagonal entries cut the matrix into such blocks; so the rank is at least the number of nonzero $x_1, \dots, x_{n-1}$, and at most two of them are nonzero. If exactly two, x_i, x_j with $i < j$: when $j \geq i+2$ the block on $\{i, i+1\}$ has determinant $-x_i^2$ (its first diagonal entry x_{i+1} vanishes) and the block of x_j adds rank at least one; when $j = i+1$ and $i \geq 2$, the singleton block $\{i-1\}$ has diagonal $x_i \neq 0$, separate from a block of rank at least two; when $i = 1$, the block on $\{1,2,3\}$ has determinant $-x_2^3 \neq 0$ for $n \geq 5$, and for $n = 4$ either this holds or $x_4 \neq 0$ and the singleton $\{4\}$ adds one. If exactly one, x_j: for $2 \leq j \leq n-2$ the block $\{j, j+1\}$ has determinant $-x_j^2$ and the singleton $\{j-1\}$ has diagonal x_j; for $j = 1$ the block $\{1,2\}$ has rank two and $x_n \neq 0$ would add the last two diagonal entries, leaving $[e_1]$; for $j = n-1$, with $a = x_{n-1}$, $b = x_n$, the singleton $\{n-2\}$ has entry a and the block $\left(\begin{smallmatrix} b & a \\ a & 3b \end{smallmatrix}\right)$ has rank one exactly when $a^2 = 3b^2$. If none, the point is $[e_n]$, of rank two.

A linear transformation T with $F_n \circ T = cF_n$ transforms the Hessian by congruence, so it preserves (15), hence fixes $[e_1]$, the point off the line of the other three. In coordinate forms $\ell_i = (Tx)_i$ this says $\ell_1 = ax_1 + L(x_2, \dots, x_n)$ with $a \neq 0$, while $\ell_2, \dots, \ell_n$ do not involve x_1. The coefficient of x_1^2 in $F_n(\ell) = cF_n(x)$ gives $a^2\ell_2 = cx_2$; that of x_1 gives $2aL\ell_2 = 0$, so $L = 0$. Setting $x_1 = 0$, $(\ell_2, \dots, \ell_n)$ preserves up to a scalar the chain cubic in $x_2, \dots, x_n$, with ℓ_2 a multiple of x_2. If $n - 1 \geq 4$ we repeat the argument; it remains to treat the ternary cubic $G = y^2z + z^2w + w^3$ with $\ell = (by, uy + A, vy + B)$, A, B independent linear forms in z, w. The coefficient of y in $G(\ell)$ is $vA^2 + 2uAB + 3vB^2$; it vanishes, so $u = v = 0$. The coefficient of y^2 gives $b^2A = cz$; writing $B = \alpha z + \beta w$ with $\beta \neq 0$, the coefficient of zw^2 is $3\alpha\beta^2 = 0$, so $\alpha = 0$. Hence every such T is diagonal. Dividing T by a cube root of c, we may take $c = 1$. A diagonal $\operatorname{diag}(a_1, \dots, a_n)$ preserves F_n exactly when $a_i^2 a_{i+1} = 1$ and $a_n^3 = 1$, that is $a_i = a_1^{(-2)^{i-1}}$ and $a_1^{3M} = 1$; its determinant is a_1^S with $S = \sum_{i=0}^{n-1}(-2)^i = (1 - (-2)^n)/3$, which is odd and congruent to n modulo three. So $\gcd(S, 3M) = \nu$, and the kernel of det has order ν; when $\nu = 3$ its elements are the scalars ωI, $\omega^3 = 1$, as $(-2)^i \equiv 1 \pmod 3$, and these act trivially on cubic forms.

(3) Since each fibre of the quotient map contains one closed orbit, a grading element fixes p_n exactly when it is $(\det a)^{-\gamma}$ for some a in the stabilizer of F_n. With a_1 of order $3M$, $(\det a)^{-\gamma} = a_1^{-\gamma S}$ has order $3M/\gcd(3M, \gamma S)$. As S is odd, $\gcd(3M, \gamma S) = \gcd(3, \gamma S)$, which is $\gcd(3, 3S) = 3$ if $\nu = 1$ and $\gcd(3, S) = 3$ if $\nu = 3$ (then $S \equiv n \equiv 0 \pmod 3$). So the grading stabilizer of p_n is μ_M.

(4) The group SL_n/μ_ν acts on $\operatorname{Sym}^3 V^*$ and has trivial stabilizer at F_n, whose orbit is closed. By Luna's slice theorem [8, Section III.1, Théorème du slice étale and Remarque 1°, pp. 97–98], X is smooth at p_n with tangent space $\operatorname{Sym}^3 V^*/\mathfrak{sl}_n \cdot F_n$. Let $A = \operatorname{diag}(\zeta, \zeta^{-2}, \dots, \zeta^{(-2)^{n-1}})$ with ζ a primitive M-th root of unity; it fixes F_n, and acts on X as the grading element $(\det A)^{-\gamma} = \zeta^{-\gamma S}$, a primitive M-th root of unity, whose fixed scheme is X^ζ. The monomials fixed by A are exactly the n chain monomials, by the valuation argument at the end of the proof of Proposition 4.2; and $\dim \mathfrak{sl}_n^A = n - 1$, as A has simple spectrum. Taking invariants is exact, so $\dim(T_{p_n} X)^A = 1$, and the fixed scheme is smooth of dimension one at p_n (proof of Proposition 3.2). □

Theorem 6.2. *Let $n \geq 4$ and $M = 2^{n-1}$.*

1. *The support of the fixed scheme X^{μ_M} is the closure Y of the grading orbit of p_n, and X is smooth along $Y \setminus \{0\}$.*

2. *H_n has a simple pole at every primitive M-th root of unity ζ, with*

$$\mathcal{C}_n(\zeta) := \lim_{t \to \zeta} (1 - t^M) H_n(t) = \frac{1}{\det(1 - \zeta \mid N^*_{Y/X, \eta_Y})} \neq 0. \tag{16}$$

3. *If η is a root of unity whose order q has two-part at least M, then H_n has a pole at η if and only if $q = M$.*

4. *In every expression $H_n(t) = P(t)/\prod_j (1-t^{a_j})^{b_j}$ with $P \in \mathbb{C}[t]$ and $a_j, b_j > 0$, some a_j is divisible by M.*

Proof. Let $p \neq 0$ be a closed point fixed by a grading element of order $2^a \geq M$, and F a cubic in its closed orbit. Lemma 3.4 (with the odd exponent $-\gamma$) gives a diagonal A of two-power order fixing F with $\mathrm{ord}(\det A) = 2^a$; Lemma 4.1 gives the span condition; and Theorem 4.3 gives $2^a \leq M$, so $a = n-1$, and F is equivalent to F_n under a permutation and a diagonal substitution. So $p \in GL_n \cdot p_n$, which is the grading orbit of p_n. Conversely that orbit is fixed by μ_M (Proposition 6.1(3)). The closure Y adds only the origin, so the support of X^{μ_M} is Y, a curve, and X is smooth along $Y \setminus \{0\}$ by Proposition 6.1(4) and translation. Proposition 3.2 with $\delta = 1$ and Lemma 3.3 ($e_Y = 1$) give (2). For (3): if $q = 2^a b$, b odd, $a \geq n-1$, and $q \neq M$, a nonzero point fixed by η is fixed by its two-primary part, of order $2^a \geq M$, hence lies on the grading orbit of p_n, whose grading stabilizer is $\mu_M \not\ni \eta$. So X^η is supported at the origin, and Lemma 3.1 applies. (4) follows from (2). □

Corollary 6.3. *For $n \geq 4$, every positive rational form of H_n has modular index at least 2^{n-1}, while $\dim GL_m + \mathrm{ht}(\lambda) + \mathrm{ht}(\pi) = m^2 + n + 1 = O(n^6)$ and $\mathrm{rank}(x) = 2n + m + 1$. The existence clause of [10, Hypothesis 1.6.7], and so also its specific clause on the minimal positive form, fails for plethysm coefficients; so does [10, Hypothesis 3.3.4] for the plethysm problem.*

7 Plethysm coefficients: the positivity and saturation indices

We keep the notation of Section 6, with $n \geq 4$, and put $\zeta = e^{2\pi i/M}$, $\mathcal{C}_n(\rho) = \lim_{t\to\rho}(1-t^M)H_n(t)$ and $A_n(\rho) = \mathcal{C}_n(\rho)/M$ for primitive M-th roots ρ. Since H_n has rational coefficients, $\mathcal{C}_n(\zeta^a)$ is the Galois conjugate of $\mathcal{C}_n(\zeta)$ under $\zeta \mapsto \zeta^a$, and $A_n(\zeta^{-a}) = \overline{A_n(\zeta^a)}$. Define the M-periodic *dyadic wave*

$$W_n(r) = \sum_{\substack{1 \leq a < M \\ a \text{ odd}}} A_n(\zeta^a)\zeta^{-ar} = \frac{1}{M}\,\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\big(\mathcal{C}_n(\zeta)\zeta^{-r}\big) \in \mathbb{Q}. \tag{17}$$

7.1 The wave and the indices

By [10, Theorem 1.6.1(b), p. 21], f_n is a quasi-polynomial, that is, H_n is a proper rational function with poles at roots of unity. So

$$f_n(s) = \sum_\rho P_\rho(s)\rho^{-s} \qquad (s \geq 0), \tag{18}$$

the sum over the poles ρ of H_n, with P_ρ a polynomial of degree one less than the pole order. At a simple pole ρ of order M, $1 - t^M = (1-t/\rho)\prod_{\rho' \neq \rho}(1-t/\rho')$ over the other M-th roots of unity ρ', and $\prod_{\rho' \neq \rho}(1-\rho/\rho') = M$; so near ρ, $H_n(t) = (\mathcal{C}_n(\rho)/M)/(1-t/\rho) + O(1)$, and $P_\rho = A_n(\rho)$ is constant.

Proposition 7.1. *Let L be the least common multiple of the orders of the poles of H_n, a period of f_n, and write $f_n(s) = q_r(s)$ for $s \equiv r \pmod L$. Then $v_2(L) = n-1$, and for every residue r and every X,*

$$q_r(X) - q_{r+L/2}(X) = 2W_n(r). \tag{19}$$

Consequently $f_n(x) \geq 2W_n(x)$ for every integer $x \geq p(f_n)$ and every integer $x \geq s(f_n)+1$. In particular, if $2W_n(x) > f_n(x)$ for an integer $x \geq 0$, then $p(f_n) > x$ and $s(f_n) \geq x$.

Proof. By Theorem 6.2(2),(3), every pole order has two-part at most M, and M occurs; so $v_2(L) = n-1$. By (18), $q_r(X) = \sum_\rho P_\rho(X)\rho^{-r}$. If the order of ρ has two-part at most $M/2$, it divides $L/2$ and $\rho^{-L/2} = 1$; if it has two-part M, then $\rho = \zeta^a$ with a odd (Theorem 6.2(3)) and $\rho^{-L/2} = -1$, as L/M is odd. Subtracting the two expansions gives (19). If $p = p(f_n)$, every constituent polynomial of $n \mapsto f_n(n+p)$, namely $X \mapsto q_r(X+p)$, has nonnegative coefficients, so $q_r(y) \geq 0$ for all real $y \geq p$ and all r. If $s = s(f_n)$, every constituent $X \mapsto q_r(X+s)$ is either identically zero or positive at every integer $X \geq 1$, so $q_r(y) \geq 0$ for every integer $y \geq s+1$. At such x, take $r \equiv x \pmod{L}$ in (19): $f_n(x) - 2W_n(x) = q_{r+L/2}(x) \geq 0$. □

7.2 An exact product for the residues

Write $w_i = (-2)^i$ for $0 \leq i \leq n-2$ and $w_{n-1} = 0$, and form the multisets

$$\begin{aligned} \mathcal{E}_n &= \{\{0,1,k\} : 2 \leq k < n\} \cup \{\{0,2,3\}\}, \\ \mathcal{S}_n &= \{w_i : 0 \leq i \leq n-2\} \;\sqcup\; \{w_i + w_j + w_k : 0 \leq i < j < k < n,\ \{i,j,k\} \notin \mathcal{E}_n\}, \end{aligned} \tag{20}$$

with repetitions retained. Then $|\mathcal{S}_n| = (n-1) + \binom{n}{3} - (n-2) - 1 = \binom{n}{3}$, and no element of $\mathcal{S}_n$ is divisible by M: a singleton w_i ($i \leq n-2$) has valuation $i < n-1$, and in a sum of distinct w's the smallest valuation among the nonzero terms occurs once.

Proposition 7.2. *For every odd a,*

$$\mathcal{C}_n(\zeta^a) = \prod_{b \in \mathcal{S}_n} \left(1 - \zeta^{a\nu b}\right)^{-1}. \tag{21}$$

Proof. By Galois conjugation it suffices to take $a = 1$. Put $\eta = \zeta^\nu$ and $\alpha_i = \eta^{w_i}$, and let $D = \operatorname{diag}(\alpha_0, \ldots, \alpha_{n-1})$, which fixes F_n (its chain monomials have exponents $2w_i + w_{i+1} \equiv 0$ and $3w_{n-1} = 0$). On coordinate functions of $\operatorname{Sym}^3 V^*$, that is on $\operatorname{Sym}(\operatorname{Sym}^3 V)$, D acts through the character of $\operatorname{Sym}^3 V$, and on invariants of coefficient degree $d_0 s$ by $(\det D)^{\gamma s}$. Here $(\det D)^\gamma = \zeta^{3\sum_i w_i}$ and $3\sum_i w_i = 1 - (-2)^{n-1} \equiv 1 \pmod{M}$; so D acts on $T^{(n)}$ as the grading element ζ. By the tangent space computation in Proposition 6.1(4), the cotangent space of X at p_n has the character $\operatorname{Sym}^3 V - (V \otimes V^* - 1)$ as a representation of $\langle D \rangle$, and the conormal space of Y is obtained by removing the trivial character of the fixed direction:

$$N^* = \operatorname{Sym}^3 V - V \otimes V^*. \tag{22}$$

With h_j, e_j the complete and elementary symmetric functions in the α_i, the identity $\prod_i (1+t\alpha_i) = \prod_i (1-t^2\alpha_i^2)/\prod_i (1-t\alpha_i)$ gives $h_3 = e_3 + h_1 \sum_i \alpha_i^2$. The multisets $\{\alpha_i^2\}$ and $\{\alpha_i^{-1}\}$ differ only in that α_0^{-1} is replaced by 1, because $\alpha_i^2 = \alpha_{i+1}^{-1}$ for $i < n-1$ and $\alpha_{n-1} = \alpha_{n-2}^2 = 1$. So (22) is $e_3 + h_1 - \alpha_0^{-1} h_1$. The trivial weight α_{n-1} in h_1 cancels the term $i = 0$ of $\alpha_0^{-1} h_1$. For $2 \leq k < n$ the subtracted weight $w_k - w_0$ equals the weight of the triple $\{0,1,k\}$, as $w_0 + w_1 = -1$; the last subtracted weight $w_1 - w_0 = -3$ equals that of $\{0,2,3\}$ ($1 + 4 - 8$ for $n \geq 5$; $1 + 4 + 0 \equiv -3 \pmod 8$ for $n = 4$). These triples are distinct, and what remains is the multiset $\mathcal{S}_n$. Equation (16) gives (21). □

7.3 A dominant pair

Put $r_\nu = 3\nu \in \{3, 9\}$, let $1 \leq a_* < M$ with $r_\nu a_* \equiv 1 \pmod{M}$ (so a_* is odd and $a_* \not\equiv -a_*$), and

$$Z_n = A_n(\zeta^{a_*}), \qquad E_n = \sum_{\substack{a \bmod M \text{ odd} \\ a \not\equiv \pm a_*}} |A_n(\zeta^a)|. \tag{23}$$

Proposition 7.3. *For every $n \geq 4$,*

$$\log_2 |Z_n| \geq \binom{n}{4} - 3\binom{n-1}{3} - (n-1)(n-2) - \binom{n}{3} - (n-1) \geq \frac{n^4}{24} - 2n^3. \tag{24}$$

If $n \geq 6$ and $E_n \leq |Z_n|/4$, then for every integer X there is an integer x with $X \leq x \leq X+8$ and $W_n(x) \geq |Z_n|/4$.

Proof. By (21), $Z_n = M^{-1} \prod_{b \in \mathcal{S}_n} (1-\theta^b)^{-1}$ with $\theta = \zeta^{\nu a_*}$, so that $\theta^3 = \zeta$. Every factor has absolute value at most two. Consider the triples $i < j < k \leq n-2$ of $\mathcal{S}_n$, those not using the zero weight. Their weight $w = w_i + w_j + w_k$ is divisible by three (each $w_i \equiv 1 \pmod 3$) and $|w| < 2^{k+1}$, so $|1-\theta^w| = |1-\zeta^{w/3}| \leq 2\pi|w|/(3M) \leq 8 \cdot 2^k/M$, and $-\log_2 |1-\theta^w| \geq (n-1-k) - 3$. Summing $n-1-k$ over all triples $i<j<k \leq n-2$ counts the choices of a fourth index in $\{k+1, \ldots, n-1\}$ and gives $\binom{n}{4}$; at most $n-2$ of these triples belong to $\mathcal{E}_n$, each with a term at most $n-1$. The remaining at most $\binom{n}{3}$ factors contribute at least -1 each, and M^{-1} contributes $-(n-1)$. This proves the first inequality of (24). For the second use $\binom{n}{4} \geq n^4/24 - n^3/4$, $3\binom{n-1}{3} \leq n^3/2$, $\binom{n}{3} \leq n^3/6$ and $(n-1)(n-2) + (n-1) \leq n^2 \leq n^3/4$: the total cubic loss is at most $\frac{7}{6}n^3$.

Write $r_\nu a_* = 1 + kM$; then $1 \leq k < r_\nu$ and $\gcd(k, r_\nu) = 1$, since $kM \equiv -1 \pmod{r_\nu}$. Thus $a_*/M = k/r_\nu + 1/(r_\nu M)$. For fixed X, the r_ν numbers $Z_n \zeta^{-a_* X} e^{-2\pi i k j / r_\nu}$, $0 \leq j < r_\nu$, are equally spaced on the circle of radius $|Z_n|$, so one has real part at least $\cos(\pi/r_\nu)|Z_n| \geq |Z_n|/2$. Replacing the phase $e^{-2\pi i k j/r_\nu}$ by $\zeta^{-a_* j}$ changes it by a factor $e^{-2\pi i j/(r_\nu M)}$ and the real part by at most $(2\pi/M)|Z_n| \leq |Z_n|/4$ for $n \geq 6$. So $2\,\mathrm{Re}(Z_n \zeta^{-a_*(X+j)}) \geq |Z_n|/2$ for this $j \leq 8$. The terms $a = \pm a_*$ of (17) are this conjugate pair, and the others have total absolute value at most $E_n \leq |Z_n|/4$. □

Corollary 7.4. *If $E_n \leq |Z_n|/4$ for all sufficiently large n, then for every real $0 < c < 1/4$ and all sufficiently large n, $p(f_n) > 2^{cn}$ and $s(f_n) \geq 2^{cn}$.*

Proof. Take $X = \lceil 2^{cn} \rceil$ and $x \leq X + 8$ as in Proposition 7.3. The value $f_n(x)$ is at most the dimension of $\mathrm{Sym}^{d_0 x}(\mathrm{Sym}^3 V)$, so $\log_2 \max\{1, f_n(x)\} \leq (m-1)\log_2(m+nx) = \frac{c}{6}n^4 + O_c(n^3 \log n)$, which is eventually smaller than $\log_2(2W_n(x)) \geq n^4/24 - 2n^3 - 1$, as $c/6 < 1/24$. Proposition 7.1 gives $p(f_n) > x \geq 2^{cn}$ and $s(f_n) \geq x$. □

7.4 Reduction to an increment

For $n \geq 4$ write $M_n = 2^{n-1}$, $b_n = 3^{-1} \bmod M_n$, and for odd a

$$P_n(a) = \prod_{b \in \mathcal{S}_n} |1 - e^{2\pi i a b / M_n}|^{-1}, \qquad Q_n = \sum_{\substack{a \bmod M_n \text{ odd} \\ a \neq \pm b_n}} \frac{P_n(a)}{P_n(b_n)}. \tag{25}$$

By (21), $|A_n(\zeta^a)| = P_n(\nu a)/M_n$; multiplication by the odd number ν permutes the odd residues, and $\nu a_* \equiv b_n$. So $Q_n = E_n/|Z_n|$ and $P_n(b_n) = M_n |Z_n|$. For $n \geq 5$ form the multiset of $\binom{n-1}{2}$ residues

$$\mathcal{T}_n = \{1\} \sqcup \{1 + w_j + w_k : 2 \leq j < k < n,\ (j,k) \neq (2,3)\} \sqcup \{2 + w_k : 3 \leq k < n\} \sqcup \{6\}, \tag{26}$$

and $J_n(a) = \prod_{b \in \mathcal{T}_n} |1 - e^{2\pi i a b / M_n}|^{-1}$.

Proposition 7.5. *For $n \geq 5$ and odd a, $P_n(a) = J_n(a)\, P_{n-1}(-a \bmod M_{n-1})$. The selected sibling ratio satisfies*

$$\delta_n := \frac{P_n(b_n + M_{n-1})}{P_n(b_n)} \leq 2^{-n^3/6 + 4n^2 - 107n/6 + 19}, \qquad \delta_n \leq 2^{-n^3/12}\ (n \geq 48). \tag{27}$$

If for some $0 < \rho < 1$ and all $n \geq n_0$ and all odd a with $a \bmod M_{n-1} \notin \{\pm b_{n-1}\}$

$$J_n(a) + J_n(a + M_{n-1}) \leq \rho J_n(b_n), \tag{28}$$

then $Q_n \leq \rho Q_{n-1} + 2\delta_n$ for $n \geq n_0$, and $Q_n \to 0$.

Proof. The weights $w_1, \ldots, w_{n-1}$ are -2 times the weights of the chain in $n-1$ variables, in the same order. All triples with smallest index at least one lie in $\mathcal{S}_n$; the scaled $\mathcal{S}_{n-1}$ omits among them exactly $\{1,2,k\}$ $(3 \leq k < n)$ and $\{1,3,4\}$; and $\mathcal{S}_n$ has the extra singleton $w_0 = 1$ and the triples $\{0,j,k\}$, $2 \leq j < k < n$, other than $\{0,2,3\}$. Their weights are those of (26): $w_1 + w_2 + w_k = 2 + w_k$, and $\{1,3,4\}$ has weight $-2 - 8 + 16 = 6$ $(n \geq 6)$ or $-2 - 8 + 0 \equiv 6 \pmod{16}$ $(n = 5)$. The count is $1 + (\binom{n-2}{2} - 1) + (n-3) + 1 = \binom{n-1}{2}$. Substituting $e^{2\pi i a(-2b')/M_n} = e^{2\pi i(-a)b'/M_{n-1}}$ gives the recursion.

The residues b_n and $b_n + M_n/2$ have the same smaller factor, and every even weight gives the same factor at both; so $\delta_n = \prod_{b \in \mathcal{O}_n} |\tan(\pi b_n b/M_n)|$ over the odd weights $\mathcal{O}_n = \{1\} \sqcup \{1 + w_j + w_k\}$. The bulk ones, $2 \leq j < k \leq n-2$, $(j,k) \neq (2,3)$, number $B = \binom{n-3}{2} - 1$, are divisible by three with $|b| < 2^{k+1} \leq M_n$; as $3b_n \equiv 1$, $|\tan(\pi b_n b/M_n)| = |\tan(\pi b/(3M_n))| \leq 2\pi|b|/(3M_n) \leq 8 \cdot 2^k/M_n$, using $|\sin u| \leq |u|$ and $\cos u \geq 1/2$ on $[-\pi/3, \pi/3]$. The sum of $n-1-k$ over them is $\binom{n-2}{3} - (n-4)$. The other $n-2$ odd factors are at most $M_n/2$, an odd residue being at distance at least one from $M_n/2$, so the cosine is at least $\sin(\pi/M_n) \geq 2/M_n$. Hence $\log_2 \delta_n \leq 3B - \binom{n-2}{3} + (n-4) + (n-2)^2 = -n^3/6 + 4n^2 - 107n/6 + 19$, which is at most $-n^3/12$ for $n \geq 48$.

Group the odd residues modulo M_n by their reduction modulo M_{n-1}. The two selected parents $\pm b_{n-1}$ have the selected children $\pm b_n$ and the children $\pm b_n + M_{n-1}$, of total normalized mass $2\delta_n$ (conjugation). Every other parent has two children whose normalized mass is at most ρ times its own in Q_{n-1}, by the recursion and (28), since $P_n(b_n) = J_n(b_n) P_{n-1}(b_{n-1})$. So $Q_n \leq \rho Q_{n-1} + 2\delta_n$, and as $\delta_n \to 0$, $Q_n \to 0$. □

7.5 The increment estimates

Lemma 7.6. *Let $1 \leq r \leq 3$, integers $1 \leq q_1 < \cdots < q_r$, signs ϵ_i, $\kappa \in \mathbb{Z}$, $J \geq 1$ odd, $R = \lceil \log_2(16J) \rceil$, and $x = \kappa/3 + (J/3) \sum_i \epsilon_i 2^{-q_i}$. Then $\|x\|_{\mathbb{R}/\mathbb{Z}} \geq 1/(6 \cdot 2^{q_1 + (r-1)R})$. In particular, if $q_1 \leq R$, then $2^{-3R-1} \leq |1 - e^{2\pi i x}| \leq 2$.*

Proof. Keep the initial block of terms ending before the first gap $q_{i+1} - q_i > R$ (all terms if there is none), ending at q_t. Its numerator over $3 \cdot 2^{q_t}$ is odd (the last term is odd, the earlier ones and $\kappa 2^{q_t}$ even), so it is at distance at least $1/(3 \cdot 2^{q_t})$ from $\mathbb{Z}$. The tail is less than $(2J/3) 2^{-q_{t+1}} \leq J 2^{-R}/(3 \cdot 2^{q_t}) \leq 1/(48 \cdot 2^{q_t})$. So $\|x\| \geq 1/(6 \cdot 2^{q_t})$, and $q_t \leq q_1 + (r-1)R$. Finally $|1 - e^{2\pi i x}| \geq 4\|x\|$. □

Let $l = n-1$, $M = 2^l$, and for odd J let $b_J \equiv J \cdot 3^{-1} \pmod{M}$; then $\eta_J = \zeta^{b_J}$ satisfies $\eta_J^3 = \zeta^J$ and

$$\eta_J = \omega_J e^{2\pi i J/(3M)}, \qquad \omega_J^3 = 1, \qquad \omega_J = 1 \iff 3 \mid J \quad (J < M), \tag{29}$$

since $3b_J = J + k_J M$ with $k_J \in \{0,1,2\}$, $\omega_J = e^{2\pi i k_J/3}$, and $k_J = 0$ only if $3 \mid J$. For $|y| \leq 1/24$, $|\log_2(\sin(\pi y)/(\pi y))| \leq 8y^2$, by $1 - u^2/6 \leq \sin u/u \leq 1$.

Proposition 7.7 (near increments)**.** *For odd $1 \leq j < M$, with $a_j \equiv j \cdot 3^{-1} \pmod{M}$ and $R = \lceil \log_2(16j) \rceil$, and $n \geq 5$,*

$$\Big| \log_2 \frac{J_n(a_j)}{J_n(b_n)} + \binom{n-1}{2} \log_2 j - \mathbf{1}_{3|j} \binom{n}{2} \Big| \leq 100 n R^2. \tag{30}$$

For $\delta = 10^{-5}$ and odd $3 \leq j \leq 2^{\delta n}$ this gives $\log_2(J_n(a_j)/J_n(b_n)) \leq -n^2 \log_2 j/12$.

Proof. The weights of $\mathcal{T}_n$ are: the *bulk* weights $1+w_u+w_v$ ($2 \le u < v \le l-1$, $(u,v) \ne (2,3)$), the weights $2+w_k$ ($3 \le k \le l-1$) and 6, all divisible by three, $B_3 = (n-1)(n-4)/2$ in number; and the *exceptional* weights 1, 2 and $1+w_k$ ($2 \le k \le l-1$), $n-1$ in number. Give each weight its largest active dyadic index k and put $q = l-k$ (the weights 1 and 2 get $q = l$ and $q = l-1$). The exceptional weights have each $q = 1, \dots, l$ exactly once, so $B_3 + (n-1) = \binom{n-1}{2}$ and the exceptional q sum to $\binom{n}{2}$. A weight divisible by three has absolute value between 2^{k-2} and 2^{k+1} (for $1+w_u+w_v$ use $2^v - 2^{v-1} - 1 \ge 2^{v-2}$, $v \ge 3$); for each q there are at most $2n$ of them. An exceptional weight has absolute value between 2^{k-1} and $3 \cdot 2^{k-1}$. By (29), the target root is $\omega_j e^{2\pi i j/(3M)}$. A factor is *low* if $q > R$ and *high* otherwise. Lemma 7.6 applies to every factor (each weight is a signed sum of at most three powers of two): a high target factor is at least 2^{-3R-1}, and at least 2^{-2R-1} for an exceptional one; all factors are at most two.

For a low factor divisible by three there is no cube-root shift: its target phase is j times its reference phase and is at most $(2j/3)2^{-q} < 1/24$ in absolute value; by the sine estimate the logarithm of the factor ratio differs from $\log_2 j$ by at most $16j^2 4^{-q}$; summing over at most $2n$ weights for each $q > R$ gives less than n, as $j^2 4^{-R} \le 1/256$. For a high factor divisible by three, the reference phase lies between $2^{-q}/12$ and $2 \cdot 2^{-q}/3 < 1/2$ in absolute value, so the reference factor is at least $2^{-q}/3$, and the logarithm of the ratio is at most $5R$ in absolute value; there are at most $2nR$ of them, and restoring their $\log_2 j$ main terms costs at most $2nR^2$. So the factors divisible by three contribute $-B_3 \log_2 j$ to the logarithm of the reciprocal product, with error at most $20nR^2$.

Every exceptional reference factor lies between $1/3$ and two: its cube-root shift is nontrivial and its phase perturbation is at most $1/4$. If $3 \nmid j$, every low exceptional target factor has the same kind of shift, perturbed by less than $1/24$, so its logarithmic ratio is at most three; the at most R high ones are at most $3R$ each; the total is at most $10nR^2$. If $3 \mid j$, the target shift vanishes; a low exceptional target factor lies between $(2/3)j2^{-q}$ and $4j2^{-q}$, so its reciprocal ratio has logarithm $q - \log_2 j$ up to four. Restoring the q of the at most R high factors costs at most R^2, their actual ratios at most $3R^2$, discarding the $-\log_2 j$ terms at most nR, and the low errors at most $4n$; the exceptional contribution is $\binom{n}{2}$ up to $10nR^2$. Altogether $\log_2(J_n(a_j)/J_n(b_n)) = -B_3 \log_2 j + \mathbf{1}_{3|j}\binom{n}{2} + \mathcal{E}$ with $|\mathcal{E}| \le 30nR^2$, and replacing B_3 by $\binom{n-1}{2}$ costs $(n-1)\log_2 j \le nR$.

For the consequence, put $t = \log_2 j > 3/2$, so $R \le t+5$. With $\binom{n-1}{2} \ge n^2/2 - 2n$ and $\binom{n}{2} \le n^2/2$, (30) gives at most $-(t-1)n^2/2 + n(2t + 100(t+5)^2) \le -tn^2/6 + 2502nt^2$, using $t - 1 \ge t/3$ and $t + 5 \le 5t$; and $t \le n/10^5$ with $2502/10^5 < 1/12$. □

Proposition 7.8 (far increments)**.** *Let $n \ge 5$, $l = n-1$, $M = 2^l$, a odd, $j \equiv 3a \pmod M$ with $1 \le j < M$, and $J = \min\{j, M-j\}$. Then*

$$\log_2 \frac{J_n(a)}{J_n(b_n)} \le 7n^2 - \frac{(\log_2 J - 8)_+^3}{24}. \tag{31}$$

In particular, for $\delta > 0$, $J \ge 2^{\delta n}$ and $n \ge \max\{5, 16/\delta, 2688/\delta^3\}$, the ratio is at most $2^{-\delta^3 n^3/384}$.

Proof. Consider the *active pair* factors $1 - \exp(2\pi i a(1 + w_j + w_k)/M)$, $2 \le j < k < l$; the increment contains all but $(j,k) = (2,3)$. Every phase is nonzero modulo one, its numerator being odd. Put $\theta = a/M$, $y_{jk} = \theta(1 + w_j + w_k)$ modulo one, and for $0 \le u \le l-1$ let $r = l - u$ and N_u be the number of active pairs with $\|y_{jk}\| < 2^{-u}$.

The map sending a subset of $\{0, \dots, l-1\}$ to the sum of its w_i modulo 2^l is injective (a difference of two subset sums has a unique least index with coefficient ± 1, of valuation less than l). Call an index early if $j < r$ and late otherwise; there are $e = (r-2)_+$ early

indices. There are at most $\binom{e}{2}$ early–early pairs. For a fixed early j, the mixed numerators $a(1+w_j+w_k)$ are congruent modulo 2^r and distinct modulo M; the condition $\|y_{jk}\| < 2^{-u}$ puts their centred representatives in $(-2^r, 2^r)$, which holds at most three of one class; so there are at most $3e$ good mixed pairs. The late–late numerators are all congruent to a modulo 2^r and distinct, so at most three are good. Hence

$$N_u \le \binom{r}{2} + 3r + 3. \tag{32}$$

Let $\tau = \|3\theta\| = J/M$ and suppose $\tau > 3\cdot 2^{-u}$. Partition the early indices into $h = \lfloor e/2 \rfloor$ consecutive pairs $(2,3),(4,5),\dots$. For two of them, $(j, j+1)$ and $(k, k+1)$, the identity $y_{jk}+y_{j+1,k}+y_{j,k+1} = 3\theta \pmod 1$ (from $w_{j+1} = -2w_j$) shows that the three phases cannot all be within 2^{-u} of an integer; so one of the four pairs of the block is bad. The $\binom{h}{2}$ blocks are disjoint and do not contain $(2,3)$. So under this condition $N_u \le \binom{r}{2} + 3r + 3 - \binom{\lfloor (r-2)_+/2 \rfloor}{2}$.

For a nonzero phase y with denominator dividing M, $\|y\| \ge 1/M$ and $\log_2 |1-e^{2\pi i y}|^{-1} \le -\log_2 \|y\| \le \sum_{u=0}^{l-1} \mathbf{1}_{\{\|y\|<2^{-u}\}}$. The other $2n-4$ increment factors have logarithmic reciprocals at most l each, less than $2n^2$ in total. Let $K = (\lfloor \log_2 J \rfloor - 2)_+$; for $1 \le r \le K$, $J > 3\cdot 2^r$, so the condition holds at $u = l - r$. Summing,

$$\log_2 J_n(a) \le \binom{n}{3} + \tfrac{3}{2} l(l+1) + 3l + 2n^2 - \sum_{r=1}^{K} \binom{\lfloor (r-2)_+/2 \rfloor}{2},$$

where the first four terms are at most $n^3/6 + 4n^2$ for $n \ge 5$, and the sum is at least $(K-5)_+^3/24 \ge (\log_2 J - 8)_+^3/24$, since for $r \ge 6$ the summand is at least $(r-3)(r-5)/8 \ge (r-5)^2/8$.

For the selected increment, the $B = \binom{n-3}{2} - 1$ bulk active pairs (divisible by three) have factor at most $8\cdot 2^k/M$ at $a = b_n$, as in the proof of Proposition 7.5, and all other factors are at most two. So

$$\log_2 J_n(b_n) \ge \binom{n-2}{3} - (n-4) - 2B - \binom{n-1}{2} = \frac{n^3}{6} - 3n^2 + \frac{71n}{6} - 11 \ge \frac{n^3}{6} - 3n^2.$$

Subtracting gives (31). If $\log_2 J \ge \delta n \ge 16$, then $(\log_2 J - 8)^3/24 \ge \delta^3 n^3/192$, and $7n^2 \le \delta^3 n^3/384$ for $n \ge 2688/\delta^3$. □

Theorem 7.9. *$E_n/|Z_n| \to 0$. Consequently, for every real $0 < c < 1/4$ and all sufficiently large n,*

$$p(f_n) > 2^{cn}, \qquad s(f_n) \ge 2^{cn},$$

and $p(f_n) > 2^{n/8}$, $s(f_n) \ge 2^{n/8}$ for every $n \ge 2\cdot 10^{80}$.

Proof. For an odd residue $a \ne \pm b_n$, let j be the least positive residue of $3a$ up to sign, $1 < j \le M/2$ (and $j = 1$ exactly for $a = \pm b_n$); as $J_n(a) = J_n(-a)$, Proposition 7.7 covers $j \le 2^{\delta n}$ with ratio at most $2^{-n^2 \log_2 3/12}$, and Proposition 7.8 the rest with ratio at most $2^{-\delta^3 n^3/384}$, for n large. So the two children of any parent other than $\pm b_{n-1}$ have normalized increments of sum at most $1/2$ eventually: (28) holds with $\rho = 1/2$, and Proposition 7.5 gives $Q_n = E_n/|Z_n| \to 0$. Corollary 7.4 gives the first assertion.

For the explicit bound, let $N = 10^{20}$. For $n \ge N$ both ratios are at most $2^{-n^2/8}$ ($\log_2 3 > 3/2$; and $\delta^3 n^3/384 \ge n^2/8$ for $n \ge 48/\delta^3$), so (28) holds with $\rho = 1/2$ for all $n \ge N$. Every factor of $P_n(a)$ is at least $4/M_n$, so $P_n(a) \le 2^{(n-3)\binom{n}{3}}$, while $P_n(b_n) = M_n|Z_n| \ge 2^{-2n^3}$ by (24); hence $Q_n \le 2^{n-1+(n-3)\binom{n}{3}+2n^3} \le 2^{n^4}$. Iterating from N, with $\delta_k \le 2^{-k^3/12}$,

$$Q_n \le 2^{-(n-N+1)} Q_{N-1} + 2\sum_{k=N}^{n} 2^{-(n-k)}\delta_k \le 2^{-n+N-1+N^4} + 4\cdot 2^{-N^3/12} \le \frac{1}{4} \qquad (n \ge 2N^4).$$

For such n take $X = \lceil 2^{n/8} \rceil$ in Proposition 7.3: some $x \in [X, X+8]$ has $2W_n(x) \geq 2^{n^4/24 - 2n^3 - 1}$. As $\log_2 n \leq n/16$ for $n \geq 128$, $2^{n/8} \geq n^2$; also $m \leq n^3/5$ for $n \geq 16$ and $x \leq 2^{n/8+1}$, so $m + nx \leq 4n2^{n/8}$ and

$$\log_2 \max\{1, f_n(x)\} \leq (m-1)\log_2(m+nx) \leq \frac{n^3}{5}\Big(\frac{n}{8} + \log_2 n + 2\Big) \leq \frac{3n^4}{80} + \frac{2n^3}{5} < \frac{n^4}{24} - 2n^3 - 1$$

for $n \geq 1024$. Proposition 7.1 gives $p(f_n) > x \geq 2^{n/8}$ and $s(f_n) \geq x$. □

Corollary 7.10. *Hypotheses 1.6.5(a) (SH) and 1.6.6 (PH2) of [10] fail for the plethysm coefficients of the ray $\mu = (3)$, $\lambda = (d_0)$, $\pi = ((3/\nu)^n)$: the indices exceed every polynomial in $h = m^2 + n + 1 = \Theta(n^6)$. Hypotheses 3.3.2(a) and 3.3.3 fail for the plethysm problem with* $\operatorname{rank}(x) = 2n + m + 1$.

8 Restriction from a torus

The subgroup restriction problem of [10, Section 3.4] allows H to be a torus and $\rho : H \to G$ any explicit polynomial homomorphism [10, Section 3.4.1]; the rank of the input is the sum of the ranks of H and G and the heights of the specified weights, and it does not see the exponents of ρ. For $H = \mathbb{G}_m$ the multiplicity of the character $t \mapsto t^k$ in the restriction of $\operatorname{Sym}^n \mathbb{C}^N$ is the number of monomials of degree n whose weight is k.

Proposition 8.1. *For $N \geq 2$ let $\rho_N : \mathbb{G}_m \to GL_2$, $\rho_N(t) = \operatorname{diag}(t^{N+1}, t)$, let $\pi = (1)$ (the representation $\mathbb{C}^2$, stretched to $\operatorname{Sym}^n \mathbb{C}^2$) and λ the character $t \mapsto t^2$ (stretched to t^{2n}). Then*

$$f_N(n) = \dim \operatorname{Hom}_{\mathbb{G}_m}(t^{2n}, \operatorname{Sym}^n(\mathbb{C}^2)|_{\rho_N}) = \mathbf{1}_{N|n}, \qquad \sum_{n \geq 0} f_N(n) T^n = \frac{1}{1 - T^N},$$

and every expression of this series as $P(T)/\prod_i (1 - T^{a_i})^{b_i}$ has some a_i divisible by N, while $\operatorname{rank}(x) = 1 + 2 + 1 + 1 = 5$. *Hypothesis 3.3.4 of [10] fails for subgroup restriction as stated.*

Proof. The monomial $y_1^a y_2^{n-a}$ has weight $(N+1)a + (n-a) = n + Na$, which equals $2n$ exactly when $Na = n$; there is one such $a \in [0, n]$ if $N \mid n$ and none otherwise. The series has a pole at every primitive N-th root of unity, and a denominator must vanish there. □

Here $p(f_N) = s(f_N) = 0$, so this example does not touch PH2 or SH.

Proposition 8.2. *For odd $N \geq 3$ let $\sigma_N : \mathbb{G}_m \to GL_3$, $\sigma_N(t) = \operatorname{diag}(t, t^3, t^{N+1})$, with $\pi = (1)$ and $\lambda = t^2$ as before. Then*

$$g_N(n) = \#\{(b, c) \in \mathbb{Z}^2_{\geq 0} : 2b + Nc = n\}, \qquad \sum_{n \geq 0} g_N(n) T^n = \frac{1}{(1 - T^2)(1 - T^N)},$$

and $p(g_N) = s(g_N) = N - 2$, while $\operatorname{rank}(x) = 1 + 3 + 1 + 1 = 6$. *Hypotheses 3.3.2(a) and 3.3.3 of [10] fail for subgroup restriction as stated.*

Proof. A monomial $y_1^a y_2^b y_3^c$ of degree n has weight $a + 3b + (N+1)c = n + 2b + Nc$, which equals $2n$ exactly when $2b + Nc = n$, and then $a = n - b - c \geq 0$ automatically. This gives g_N and its series. As N is odd, $n - Nc$ is even exactly when $c \equiv n \pmod 2$, so $g_N(n)$ is the number of $c \equiv n \pmod 2$ with $0 \leq c \leq n/N$. The period is $2N$. For $n = 2Nq + r$, $0 \leq r < 2N$, one finds $g_N(n) = q + 1$ if r is even, and $g_N(n) = q + \mathbf{1}_{r \geq N}$ if r is odd. So the constituents are $g_r(X) = (X - r)/(2N) + \varepsilon_r$, with $\varepsilon_r = 1$ for even r and $\varepsilon_r = \mathbf{1}_{r \geq N}$ for odd r.

The constituents of $n \mapsto g_N(n+p)$ are $X \mapsto g_r(X+p)$, $0 \le r < 2N$, with constant terms $(p-r)/(2N) + \varepsilon_r$ and slope $1/(2N)$. For odd $r < N$ the constant term is $(p-r)/(2N)$, nonnegative only if $p \ge r$; the largest such r is $N-2$. For $p = N-2$ the other constant terms are positive: $(N-2-r)/(2N)+1 \ge 1/2$ for even $r \le 2N-2$, and $\ge (N-1)/(2N)$ for odd $r \ge N$. So $p(g_N) = N-2$. A constituent $X \mapsto g_r(X+s)$ is not identically zero and is increasing, so it is positive at every integer $X \ge 1$ exactly when $(1+s-r)/(2N)+\varepsilon_r > 0$; for odd $r < N$ this means $s \ge r$, and for $s = N-3$ it fails at $r = N-2$; for $s = N-2$ all hold. So $s(g_N) = N-2$. □

Neither example touches the density clause 3.3.2(b). In both, the index is of the size of the exponent N of the embedding, while the specification of a torus character has bit length $O(\log N)$ [10, Section 3.4.2, p. 56]. So the bounds fail in $\mathrm{rank}(x)$ and also in the bit length of the input. For the general subgroup restriction problem a bound would have to depend on the exponents of ρ themselves; with that dependence these two examples are harmless. The Kronecker and plethysm families of Sections 5–7 have no such parameter: their embeddings are fixed by n.

9 Earlier work and the scope of the results

What we searched. On 25 November 2026 we ran three web searches: for counterexamples to Mulmuley's hypotheses PH2, PH3 and SH on the stretching quasi-polynomials of Kronecker and plethysm coefficients; for the poles and periods of stretched rectangular Kronecker coefficients; and for the work of Kahle and Michałek on the quasi-polynomials of plethysm. They led to [1] and [6]. We also searched the Hexagon repository (`https://hexagonmath.org`) for "Mulmuley", "plethysm", "Kronecker", "geometric complexity", "positivity hypothesis", "saturation hypothesis" and "modular index"; these searches returned no work on the hypotheses (the matches for "plethysm" and "Kronecker" concern other questions). We then took the works that cite [10] (64) or [1] (52) in the OpenAlex index, 94 works in all, and searched the full text of the 73 of them that are on arXiv for the names of the hypotheses and for "positivity index", "saturation index", "modular index" and "positive form"; the 21 others we judged by title only. Besides Mulmuley's own papers and [1], the matches discuss the strict saturation property of particular families of (reduced) Kronecker coefficients. Finally we read the four texts discussed below. We did not search zbMATH or MathSciNet. This is a bounded search, and we claim no priority.

Briand, Orellana and Rosas. In [1], Briand, Orellana and Rosas showed that the strict saturation hypothesis of the first versions of GCT VI ("strong SH") fails for Kronecker coefficients: for $\lambda = (k,k)$, $\mu = (k+1, k-1)$, $\nu = (2k-2i-2j, 2i, 2j)$ with $i > j > 0$ and $k > 2i + j$, the stretched coefficient is $N/2+1$ for even N and $(N-1)/2$ for odd N, so it vanishes at $N = 1$ and is positive for every $N > 1$ [1, Theorem 2.4, p. 7]; they also gave counterexamples to strong PH2 [1, Section 2.4, p. 8]. In their examples the positivity index is at most one [1, Section 2.5, p. 8]. Mulmuley's appendix [1, pp. 18–20] then relaxed SH, PH2 and PH3 to polynomial bounds in the largest height. Our Theorem 5.4 refutes the relaxed PH3 of that appendix; the relaxed SH(a) and PH2 for Kronecker coefficients remain open as far as this paper is concerned, while their analogues for plethysm fail by Theorem 7.9. GCT VI v4 (22 January 2009), the newest version on arXiv, states the relaxed hypotheses in the form quoted in Section 2.

Hardness, rectangular coefficients, and a report. Ikenmeyer, Mulmuley and Walter proved that deciding the positivity of Kronecker coefficients is NP-hard [4, Theorem 1.3,

p. 7], and constructed superpolynomially many triples whose Kronecker coefficient vanishes while some stretched coefficient does not [4, Theorem 1.7, p. 11]; these are failures of the strict saturation property, and they bear on the exact decision problem, while the relaxed hypotheses were formulated for a relaxed decision problem [10, Theorem 3.3.6, p. 52]. Fischer and Ikenmeyer proved the same NP-hardness, and $\#P$-hardness of computation, for plethysm coefficients with any fixed inner degree $m \geq 3$ [2, Theorems 3.1 and 3.2, p. 5]; our plethysm family has inner degree three. Ikenmeyer and Panova studied rectangular Kronecker coefficients, of which k_n is the case of three equal rectangles; among other things they show that the positivity stretching factor is at most two for a long first row and that the rectangular Kronecker semigroup has trivial saturation [5, Section 5(c),(d), p. 13]. These are statements about positivity of the coefficients, not about the indices or the poles of the stretching functions. The report [3, Conjecture 3, p. 3] states the relaxed SH(a) and PH2 for Kronecker coefficients, with the parameter the largest height, as a conjecture of Mulmuley, and notes that the examples of [1] do not contradict it; it does not state PH3. Our results leave that conjecture open for Kronecker coefficients.

Kahle and Michałek. Kahle and Michałek [6] showed, on the examples $S^3(S^k)$ and $S^k(S^3)$, that the quasi-polynomials of plethysm need not count lattice points of inhomogeneous polytopes whose dimension equals their degree. This concerns the polytope hypothesis PH1 of [10], not the index hypotheses treated here.

What remains open. (1) The exponent r in Theorem 5.4: whether K_n always has its pole at the primitive 2^{n-1}-th roots themselves. (2) The positivity and saturation indices of the Kronecker family k_n. (3) The density clauses 1.6.5(b) and 3.3.2(b). (4) Whether a weakened PH3, with the modular index replaced by a quantity insensitive to a single pole of exponential order, could serve the purposes of [10]; the poles found here are simple, isolated in the two-adic direction, and come from one fixed curve.

10 Who proved and checked what

The Kronecker results (Theorems 5.1, 5.3, 3.6, 5.4, Lemma 5.2) were found and proved by a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode) on 21–22 November 2026; the support theorem (Section 4), Proposition 3.2 and the plethysm results (Sections 6–7) and the torus examples by an earlier run of the same model on 20–21 November. Claude Opus 5.5 at effort max read every proof of the Kronecker chain, of Proposition 3.2 and of Section 4 and reran the computations with its own code (22 November); Claude Opus 5.5 at effort high read the proofs of Theorem 6.2, of the canonical-algebra identification and of Section 7 and recomputed their numerical claims in small dimensions (21 November). In writing this text the latter added: the Kempf–Ness proof of closedness for the chain cubic (replacing a citation), the grading stabilizer of p_n for all n and Lemma 3.3 (which give $e_Y = 1$ also when $3 \mid n$), the direct count of fixed weights, the route to (24) through Proposition 7.2, the Fourier normalization, and the proofs of Propositions 8.1 and 8.2; it checked these, and the Hessian locus of Proposition 6.1(2), by computation. Claude Fable 5.2 at effort max then read the whole text, followed every proof, redid the added steps, the constants of Section 7 and Section 8 by hand, and tested them with computations of its own (25 November). No proof in this paper has been checked by a human.

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