The first L^2-Betti number of aspherical manifolds and Poincaré duality groups of dimension at least three*

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Abstract

The L^2-Betti numbers $b_p^{(2)}$ of a finite complex measure the square-summable harmonic p-chains on its universal cover. Singer's conjecture says that for a closed aspherical manifold of dimension n, one whose universal cover is contractible, they vanish outside the middle degree $n/2$. A manuscript of OpenAI dated 25 September 2026, released on 6 October 2026, proves the conjecture in dimension four; it is not refereed. Its proof assumes $b_1^{(2)} > 0$, builds from functions of finite energy on the fundamental group Γ a compact boundary on which Γ acts, and reaches a contradiction; dimension four enters only in a preliminary section. We show that the same argument gives $b_1^{(2)} = b_{n-1}^{(2)} = 0$ for every finite aspherical Poincaré complex of formal dimension $n \geq 3$, hence for every closed aspherical topological manifold of dimension at least three (Theorem A), and for every finitely presented Poincaré duality group of dimension $n \geq 3$ (Theorem B). To do so we state what Sections 3–10 of the manuscript prove as a criterion for groups (Theorem C): a finitely presented torsion-free group with one end, with $H^2(\Gamma; \mathbb{F}_2\Gamma) = 0$ and with no splitting over a finite or two-ended subgroup, has $b_1^{(2)} = 0$; and we prove, in every dimension $n \geq 3$, the facts about Γ that the criterion needs. Theorems A and B depend on the manuscript's argument, which we cite and do not reproduce. One AI referee has followed that argument line by line and found no fault; no human has checked it, or this paper.

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1 Results

1.1 Definitions

Throughout, Γ is a discrete group and $k = \mathbb{F}_2$.

L^2-Betti numbers. Let C_* be a chain complex of left $\mathbb{Z}\Gamma$-modules that are free with a given finite basis in the degrees considered, $C_p = \mathbb{Z}\Gamma^{r_p}$. Completing $\mathbb{Z}\Gamma$ to the Hilbert space $\ell^2(\Gamma)$ of square-summable functions makes each boundary map a bounded operator $\partial_p : \ell^2(\Gamma)^{r_p} \to \ell^2(\Gamma)^{r_{p-1}}$ that commutes with the action of Γ on the other side. The space of *harmonic p-chains* is $\mathcal{H}_p(C) = \ker \partial_p \cap \ker \partial_{p+1}^*$, a closed Γ-invariant subspace. Its *von Neumann dimension* is $\sum_j \langle \pi\delta_{1,j}, \delta_{1,j} \rangle$, where π is the orthogonal projection onto it and $\delta_{g,j}$ is the unit vector at g in the j-th coordinate; it is a real number ≥ 0, it is 0 only for the zero space, and it is the same for two such subspaces V, W between which there is a bounded Γ-equivariant bijection T. Indeed, the dimension equals $\sum_j \|\pi\delta_{1,j}\|^2$, and π commutes with

*AI-produced. The method is that of OpenAI's manuscript [10], which this paper cites for the main argument and does not reproduce. The reductions of Section 2 were found by GPT-6 Astra (OpenAI, ChatGPT Pro mode), and the text was written by Claude Fable 5.2 (Anthropic) at effort max; see Section 6 for who proved, wrote and checked what. No human has checked this text. To the extent that any rights exist in it, they are dedicated to the public domain under CC0 1.0.

the action of Γ, in which g moves $\delta_{h,j}$ to $\delta_{gh,j}$, so the dimension is 0 only if $\pi = 0$. The polar decomposition $T = U|T|$ has a Γ-equivariant unitary $U : V \to W$. Let u be U on V and zero on the orthogonal complement of V. Then u^*u and uu^* are the projections onto V and W, and u and u^* commute with the action, so that

$$\sum_j \|u\delta_{1,j}\|^2 = \sum_{i,j,g} |\langle u\delta_{1,j}, \delta_{g,i}\rangle|^2 = \sum_{i,j,g} |\langle \delta_{g^{-1},j}, u^*\delta_{1,i}\rangle|^2 = \sum_i \|u^*\delta_{1,i}\|^2;$$

the two ends are the dimensions of V and of W. For the theory of this dimension see [9, Chapter 1]. We write $b_p^{(2)}(C)$ for the dimension of $\mathcal{H}_p(C)$. The quotient $\ker \partial_p / \overline{\operatorname{im} \partial_{p+1}}$, the *reduced homology*, is identified with $\mathcal{H}_p(C)$ by orthogonal projection.

For a finite connected CW complex Q with fundamental group Γ and universal cover $\widetilde{Q}$, the cellular chain complex $C_*(\widetilde{Q})$ is such a complex, with one lift of each cell as basis, and $b_p^{(2)}(\widetilde{Q}) = b_p^{(2)}(C_*(\widetilde{Q}))$. For a finitely presented group Γ we put $b_1^{(2)}(\Gamma) = b_1^{(2)}(\widetilde{K})$, where K is the two-dimensional complex of a finite presentation. By Lemma 2.7 this does not depend on the presentation, and $b_1^{(2)}(\widetilde{Q}) = b_1^{(2)}(\Gamma)$ for every finite connected complex Q with fundamental group Γ.

Poincaré complexes. A *finite Poincaré complex of formal dimension* n is a finite connected CW complex Q with a homomorphism $w : \Gamma = \pi_1(Q) \to \{\pm 1\}$ and a class $[Q] \in H_n(Q; \mathbb{Z}^w)$, where $\mathbb{Z}^w$ is $\mathbb{Z}$ with Γ acting through w, such that cap product with $[Q]$ is an isomorphism $H^i(Q; V) \to H_{n-i}(Q; \mathbb{Z}^w \otimes V)$ for every integer i and every left $\mathbb{Z}\Gamma$-module V [13]. The number n need not be the largest dimension of a cell. The complex is *aspherical* if $\widetilde{Q}$ is contractible.

Duality groups. A group Γ is a *Poincaré duality group of dimension* n, for short a $PD(n)$ group, if the trivial module $\mathbb{Z}$ has a resolution of finite length by finitely generated projective $\mathbb{Z}\Gamma$-modules, $H^i(\Gamma; \mathbb{Z}\Gamma) = 0$ for $i \neq n$, and $H^n(\Gamma; \mathbb{Z}\Gamma)$ is infinite cyclic as an abelian group [3]. Such a group need not be finitely presented [3, Theorem 7.15]. Its L^2-Betti numbers are those of a finite projective resolution (Section 2.3).

Ends and splittings. A finitely generated group has *one end* if its Cayley graph is infinite and removing any finite set of vertices leaves exactly one infinite component. A group is *two-ended* if it has an infinite cyclic subgroup of finite index. A *splitting* of Γ over a subgroup J is an action of Γ on a simplicial tree, without inversions of edges, with one orbit of edges and no fixed vertex, in which J is the stabilizer of an edge. By Bass–Serre theory this is the same as a decomposition of Γ as a non-trivial amalgamated free product or HNN extension over J.

1.2 Statements

Theorem A. *Let Q be a finite aspherical Poincaré complex of formal dimension $n \geq 3$, with any orientation character w. Then*

$$b_1^{(2)}(\widetilde{Q}) = b_{n-1}^{(2)}(\widetilde{Q}) = 0.$$

Moreover $b_0^{(2)}(\widetilde{Q}) = b_n^{(2)}(\widetilde{Q}) = 0$ and $b_p^{(2)}(\widetilde{Q}) = 0$ for $p > n$. The same holds for every closed connected aspherical topological manifold of dimension $n \geq 3$.

Theorem B. *Let Γ be a finitely presented Poincaré duality group of dimension $n \geq 3$. Then $b_1^{(2)}(\Gamma) = b_{n-1}^{(2)}(\Gamma) = 0$.*

Theorem C (OpenAI [10], Sections 3–10)**.** *Let Γ be a finitely presented torsion-free group. Suppose that Γ has one end, that $H^2(\Gamma; \mathbb{F}_2\Gamma) = 0$, and that Γ has no splitting over a finite or two-ended subgroup. Then $b_1^{(2)}(\Gamma) = 0$.*

For $n = 3$ Theorem A gives the vanishing of all L^2-Betti numbers, and for $n = 4$ of all but $b_2^{(2)}$, which is the theorem of [10]. For $n \geq 5$ it says nothing about the degrees $2, \ldots, n-2$.

1.3 What is whose

- The method, and the whole argument behind Theorem C, are OpenAI's [10]. That manuscript states its theorem for formal dimension four. Its Sections 3–10 are written under the assumption $b_1^{(2)} > 0$ for a complex of formal dimension four; Theorem C is what they prove when they are read for their hypotheses. Section 3 says exactly where they call on dimension four, and reports the check on which that reading rests.

- That the method gives every dimension $n \geq 3$ was observed in public by the commenter `Apprehensive_Sand951` in the `r/math` thread *AI in mathematics, October 10, 2026* [11]. The comment says that, should the proof of [10] hold up, its method also shows that "any closed aspherical manifold of dimension ≥ 3 has no L^2-harmonic 1-forms", and that in dimension three this would prove Singer's conjecture without geometrization. It is conditional on [10] and gives no proof. For a smooth manifold, the absence of non-zero L^2-harmonic 1-forms on the universal cover is the equation $b_1^{(2)} = 0$, by the L^2 Hodge–de Rham theorem of Dodziuk [9, Chapter 1]. This paper proves that observation, as Theorem A; like the observation, the theorem rests on the argument of [10].

- New here are the reductions of Section 2 for every $n \geq 3$ and for $PD(n)$ groups, the statement of Theorem C with the account of what Sections 3–10 of [10] use, and with them the proofs of Theorems A and B.

- Known before: for closed aspherical three-manifolds the vanishing follows from the computation of Lott and Lück [8, Theorem 0.1] and geometrization (stated there for orientable manifolds; a nonorientable closed aspherical three-manifold is handled by its orientation double cover, since L^2-Betti numbers multiply by the degree of a finite cover); for right-angled Coxeter groups whose nerve is a flag triangulation of a generalized homology sphere, Davis and Okun proved $b_1^{(2)} = b_{n-1}^{(2)} = 0$ in every dimension $n \geq 3$ [4, Theorem 11.3.2]. In January 2026 the vanishing of $b_1^{(2)}$ was described as unknown even for closed negatively curved manifolds of dimension at least four [5].

1.4 Status

Theorem C is as reliable as the argument of [10], which is an unrefereed manuscript. What has been checked, by whom, and what has not, is in Sections 6 and 7. Until a human reader has gone through Sections 3–10 of [10], Theorems A and B should be cited with that condition.

2 The reductions

In this section Q is a finite aspherical Poincaré complex of formal dimension n and $\Gamma = \pi_1(Q)$. Since $\widetilde{Q}$ is contractible, $C_*(\widetilde{Q})$ is a resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}\Gamma$-modules, and homology and cohomology of Q with coefficients in a $\mathbb{Z}\Gamma$-module are those of Γ.

Lemma 2.1. *If $n \geq 1$, the group Γ is finitely presented, torsion-free and infinite.*

Proof. The two-skeleton of Q gives a finite presentation. Let m be the largest dimension of a cell of Q. If Γ had an element of prime order p, generating a subgroup C, then $C_*(\widetilde{Q})$ would be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}C$ of length m, so that $H^j(C;\mathbb{Z}) = 0$ for $j > m$; but $H^{2r}(C;\mathbb{Z}) = \mathbb{Z}/p$ for all $r \geq 1$. So Γ is torsion-free. If Γ were finite it would be trivial and Q contractible; duality with $V = \mathbb{F}_2$ would give $H_n(Q;\mathbb{F}_2) \cong H^0(Q;\mathbb{F}_2) = \mathbb{F}_2$ with $n \geq 1$, which is absurd. □

Lemma 2.2. *For every left $k\Gamma$-module V and every i there is an isomorphism $H^i(\Gamma;V) \cong H_{n-i}(\Gamma;V)$. For every subgroup J of Γ and every i,*

$$H^i(\Gamma;k[\Gamma/J]) \cong H_{n-i}(J;k),$$

where $k[\Gamma/J]$ is the permutation module with basis the left cosets of J.

Proof. On a vector space over $\mathbb{F}_2$ the sign w acts trivially, so $\mathbb{Z}^w \otimes V \cong V$, and the first statement is the duality of Q. For the second, $k[\Gamma/J] \cong k\Gamma \otimes_{kJ} k$ as left $k\Gamma$-modules. If P_* is a resolution of k by free right $k\Gamma$-modules, then $P_* \otimes_{k\Gamma} (k\Gamma \otimes_{kJ} k) \cong P_* \otimes_{kJ} k$, and P_* is also a resolution by free right kJ-modules. Taking homology gives $H_j(\Gamma;k[\Gamma/J]) \cong H_j(J;k)$. □

Corollary 2.3. *If $n \geq 3$, then $H^1(\Gamma;k\Gamma) = 0$ and $H^2(\Gamma;k\Gamma) = 0$, and $H^1(\Gamma;k[\Gamma/J]) = 0$ when J is trivial or infinite cyclic.*

Proof. By Lemma 2.2 these groups are $H_{n-1}(1;k)$, $H_{n-2}(1;k)$ and $H_{n-1}(J;k)$. The first two vanish because $n-1$ and $n-2$ are positive. An infinite cyclic group has a free resolution of length one, so its homology vanishes in degrees ≥ 2, and $n - 1 \geq 2$. □

The next two lemmas concern an arbitrary finitely generated group.

Lemma 2.4. *An infinite finitely generated group Γ with $H^1(\Gamma;k\Gamma) = 0$ has one end.*

Proof. Let G be the Cayley graph of a finite generating set, and $C_1 \to C_0 \to k \to 0$ its augmented chain complex over k, which is exact because G is connected; C_0 and C_1 are finitely generated free $k\Gamma$-modules. Choose a free module P_2 mapping onto the kernel of $C_1 \to C_0$ and continue to a free resolution P_* of k with $P_0 = C_0$, $P_1 = C_1$. For $i = 0, 1$, a k-valued cochain a on the i-cells of G with finite support defines the homomorphism $\Phi(a) : C_i \to k\Gamma$, $\Phi(a)(c) = \sum_{g\in\Gamma} a(g^{-1}c)\, g$; every homomorphism $C_i \to k\Gamma$ is of this form for exactly one a (take the coefficient of the identity), and $\Phi(da) = \Phi(a) \circ \partial$.

Suppose that for some finite set A of vertices the graph $G - A$ has two infinite components, and let f be the indicator function of one of them. Then df has finite support, since an edge on which f changes meets A. For a finite cycle z in C_1 and $g \in \Gamma$ we have $df(g^{-1}z) = f(\partial g^{-1}z) = 0$, so $\Phi(df)$ vanishes on the image of P_2 and is a cocycle of $\mathrm{Hom}_{k\Gamma}(P_*, k\Gamma)$. As $H^1(\Gamma;k\Gamma) = 0$, there is a cochain u of finite support on the vertices with $\Phi(df) = \Phi(u) \circ \partial = \Phi(du)$, that is, $d(f - u) = 0$. Then $f - u$ is constant, although it equals f outside a finite set and f takes each of its two values infinitely often. So $G - A$ has at most one infinite component; it has one, because G is infinite and locally finite, so that $G - A$ has finitely many components. □

Lemma 2.5. *Let Γ be a group and J a subgroup with $H^1(\Gamma;k[\Gamma/J]) = 0$. Then Γ has no splitting over J.*

Proof. Suppose Γ acts on a tree T without inversions, with one orbit E of edges, and that J is the stabilizer of an edge e. Then $ge \mapsto gJ$ identifies E with Γ/J as a Γ-set. Fix a vertex o, and for $g \in \Gamma$ let $c(g) \in k[E]$ be the sum of the edges on the geodesic from o to go. The path from o to go followed by the g-translate of the path from o to ho ends at gho, and reducing it to the geodesic removes edges in pairs; hence

$$c(gh) = c(g) + g\,c(h),$$

so c is a 1-cocycle of Γ with values in $k[E] \cong k[\Gamma/J]$. By hypothesis there is $v \in k[E]$ with $c(g) = gv - v$ for all g. A geodesic contains each of its edges once, so the distance from o to go is the number of edges in $c(g)$, which is at most twice the number of edges in v. The orbit of o is therefore bounded. A group acting on a tree with a bounded orbit fixes the centre of the subtree spanned by that orbit, which is a vertex or the midpoint of an edge; as no edge is inverted, Γ fixes a vertex. So the action is not a splitting. □

Corollary 2.6. *If $n \geq 3$, the group Γ is finitely presented and torsion-free, it has one end, $H^2(\Gamma; \mathbb{F}_2\Gamma) = 0$, and Γ has no splitting over a finite or two-ended subgroup.*

Proof. Lemma 2.1, Corollary 2.3 and Lemma 2.4 give all but the last statement. A finite subgroup of Γ is trivial. A two-ended group has a finite normal subgroup with quotient infinite cyclic or infinite dihedral [13, Lemma 4.1]; in a torsion-free group the finite subgroup is trivial and the dihedral case is excluded, so a two-ended subgroup of Γ is infinite cyclic. Corollary 2.3 and Lemma 2.5 now exclude both kinds of splitting. □

2.1 The first L^2-Betti number and functions of finite energy

Lemma 2.7. *Let P_* and P'_* be complexes of left $\mathbb{Z}\Gamma$-modules, free with given finite bases in degrees $0, 1, 2$, each with a surjection $P_0 \to \mathbb{Z}$ such that $P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$ is exact. Then $b_1^{(2)}(P) = b_1^{(2)}(P')$. If P_* and P'_* are bounded complexes of finitely generated free modules and are chain homotopy equivalent, then $b_p^{(2)}(P) = b_p^{(2)}(P')$ for all p.*

Proof. Since the modules are free and the complexes exact in degrees 0 and 1, the identity of $\mathbb{Z}$ lifts to maps $f_i : P_i \to P'_i$ and $g_i : P'_i \to P_i$, $i = 0, 1, 2$, commuting with the boundary maps, and there are $h_0 : P_0 \to P_1$ and $h_1 : P_1 \to P_2$ with $g_0 f_0 - 1 = \partial h_0$ and $g_1 f_1 - 1 = \partial h_1 + h_0 \partial$: first $g_0 f_0 - 1$ has image in the kernel of the augmentation, which is the image of ∂_1; then $g_1 f_1 - 1 - h_0 \partial$ has image in $\ker \partial_1 = \operatorname{im} \partial_2$. All these maps are matrices over $\mathbb{Z}\Gamma$ and extend to bounded Γ-equivariant operators on the completions. So f_1 maps $\ker \partial_1$ to $\ker \partial'_1$ and the closure of $\operatorname{im} \partial_2$ into that of $\operatorname{im} \partial'_2$, and induces a bounded equivariant map of reduced homology in degree one; so does g_1; and $g_1 f_1$ induces the identity, because $g_1 f_1 - 1$ maps $\ker \partial_1$ into $\operatorname{im} \partial_2$. By symmetry the two maps are inverse to each other, and the dimensions agree. The second statement is proved in the same way from a chain homotopy equivalence. □

In particular $b_1^{(2)}(\Gamma)$ is well defined, $b_1^{(2)}(\widetilde{Q}) = b_1^{(2)}(\Gamma)$ for every finite connected CW complex Q with fundamental group Γ, since $C_*(\widetilde{Q})$ is exact in degree one, and $b_p^{(2)}(\widetilde{Q})$ is an invariant of the homotopy type of Q, because a homotopy equivalence of finite complexes lifts to a chain homotopy equivalence of the cellular chain complexes of the universal covers.

Let now Γ have a finite presentation with relators of length at most three; any finite presentation can be brought to this form by adding a generator for each proper prefix of length at least two of a longer relator. Let $\mathcal{C}$ be its Cayley two-complex, the universal cover

of the presentation complex, and G its graph, with set of edges E. For $f : \Gamma \to \mathbb{R}$ let $df(e)$ be the difference of the values of f at the ends of the oriented edge e, and put

$$\mathcal{D} = \Big\{ df : \sum_{e \in E} |df(e)|^2 < \infty \Big\}, \qquad \mathcal{D}_0 = \overline{\{df : \ f \text{ of finite support}\}} \subset \ell^2(E; \mathbb{R}).$$

Lemma 2.8. *The space $\mathcal{D}$ is closed in $\ell^2(E;\mathbb{R})$ and contains $\mathcal{D}_0$, and $b_1^{(2)}(\Gamma) > 0$ if and only if $\mathcal{D} \neq \mathcal{D}_0$.*

Proof. Write $d_0 = \partial_1^*$ and $d_1 = \partial_2^*$ for the adjoints of the completed boundary maps of $C_*(\mathcal{C})$; on real cochains they are the cellular coboundary maps, and they are bounded because every edge lies on a bounded number of two-cells. A square-summable real 1-cochain a with $d_1 a = 0$ is a cocycle on the simply connected complex $\mathcal{C}$, hence $a = df$ for a function f, obtained by summing a along paths from a base vertex. So $\mathcal{D} = \ker d_1$, which is closed. Functions of finite support are dense in $\ell^2(\Gamma;\mathbb{R})$ and d_0 is bounded, so $\mathcal{D}_0$ is the closure of the image of d_0. The orthogonal complement of $\mathcal{D}_0$ in $\mathcal{D}$ is therefore $\ker d_1 \cap \ker d_0^* = \ker \partial_2^* \cap \ker \partial_1$, the space of real harmonic 1-chains. The boundary maps have integer matrices, so the space of complex harmonic 1-chains is its complexification, and it is non-zero exactly when its dimension $b_1^{(2)}(\Gamma)$ is positive. □

2.2 Duality for L^2-Betti numbers

Lemma 2.9. *Let Q be a finite Poincaré complex of formal dimension n. Then $b_p^{(2)}(\widetilde{Q}) = b_{n-p}^{(2)}(\widetilde{Q})$ for every integer p, where $b_p^{(2)} = 0$ for $p < 0$. If Γ is infinite then $b_0^{(2)}(\widetilde{Q}) = 0$.*

Proof. Let $C = C_*(\widetilde{Q})$, and let C^{n-*} be the complex with $\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_{n-p}, \mathbb{Z}\Gamma)$ in degree p, regarded as a complex of left modules by means of the anti-automorphism $g \mapsto w(g)g^{-1}$ of $\mathbb{Z}\Gamma$. Cap product with a cycle representing $[Q]$ is defined on chains, by a cellular approximation of the diagonal, and gives a chain map $C^{n-*} \to C$ of $\mathbb{Z}\Gamma$-modules which induces the duality isomorphisms for $V = \mathbb{Z}\Gamma$ [13, Section 1]; so it is an isomorphism on homology. Both complexes are bounded and consist of finitely generated free modules, so the mapping cone of this map is a bounded exact complex of free modules, hence contractible, and the map is a chain homotopy equivalence. By Lemma 2.7, $b_p^{(2)}(C^{n-*}) = b_p^{(2)}(C)$.

In the dual bases, the boundary map of C^{n-*} in degree p is, up to sign, the adjoint ∂_{n-p+1}^* conjugated by the unitary operator U_w of $\ell^2(\Gamma)$ that multiplies the basis vector δ_g by $w(g)$; this operator fixes δ_1, so conjugation by it does not change von Neumann dimensions. Hence the harmonic p-chains of C^{n-*} correspond to $\ker \partial_{n-p+1}^* \cap \ker \partial_{n-p}$, the harmonic $(n-p)$-chains of C, and $b_p^{(2)}(C^{n-*}) = b_{n-p}^{(2)}(C)$.

A harmonic 0-chain is a square-summable function on the vertices of $\widetilde{Q}$ that is killed by ∂_1^*, that is, has equal values at the two ends of every edge. The one-skeleton is connected, so the function is constant, and it is zero when there are infinitely many vertices. □

2.3 Poincaré duality groups

Let Γ be a $PD(n)$ group, P_* a resolution of $\mathbb{Z}$ of finite length by finitely generated projective left $\mathbb{Z}\Gamma$-modules, and $w : \Gamma \to \{\pm 1\}$ the character through which Γ acts on the right on $H^n(\Gamma; \mathbb{Z}\Gamma) \cong \mathbb{Z}$. A finitely generated projective module is the image $\mathbb{Z}\Gamma^r e$ of an idempotent matrix e; its completion is the range of e on $\ell^2(\Gamma)^r$, a closed invariant subspace, and the definitions of Section 1.1 and the second statement of Lemma 2.7 apply to complexes of such modules: a homomorphism between two of them is given by a matrix over $\mathbb{Z}\Gamma$, and the lifts and homotopies in the proof of that lemma exist for projective modules by their

lifting property. We put $b_p^{(2)}(\Gamma) = b_p^{(2)}(P)$; for $p = 1$ and Γ finitely presented this is the number defined before, by the argument of Lemma 2.7.

Lemma 2.10. *Let Γ be a $PD(n)$ group with $n \geq 1$. Then Γ is torsion-free and infinite; $H^i(\Gamma; V) \cong H_{n-i}(\Gamma; V)$ for every $k\Gamma$-module V and every i, so that the conclusion of Lemma 2.2 holds for Γ; and $b_p^{(2)}(\Gamma) = b_{n-p}^{(2)}(\Gamma)$ for every p, with $b_0^{(2)}(\Gamma) = 0$.*

Proof. Torsion is excluded as in Lemma 2.1, since P_* restricts to a projective resolution of finite length over a subgroup of prime order. The trivial group has $H^0(\Gamma; \mathbb{Z}\Gamma) = \mathbb{Z}$, so it is not a $PD(n)$ group for $n \geq 1$.

Let D_* be the complex of right modules with $\mathrm{Hom}_{\mathbb{Z}\Gamma}(P_{n-j}, \mathbb{Z}\Gamma)$ in degree j. Its homology in degree j is $H^{n-j}(\Gamma; \mathbb{Z}\Gamma)$, which is $\mathbb{Z}^w$ for $j = 0$ and zero otherwise. If $D_j \neq 0$ for some $j < 0$, the boundary map onto the lowest such module is surjective and splits, and a contractible summand can be removed; repeating this, D_* is chain homotopy equivalent to a resolution of the right module $\mathbb{Z}^w$ by finitely generated projective modules. The complex $\overline{P}_*^{\,w}$, which is P_* with the right action $x \cdot g = w(g)g^{-1}x$, is another such resolution. So D_* and $\overline{P}_*^{\,w}$ are chain homotopy equivalent.

Tensoring with a left $k\Gamma$-module V: for finitely generated projective L the natural map $\mathrm{Hom}_{\mathbb{Z}\Gamma}(L, \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} V \to \mathrm{Hom}_{\mathbb{Z}\Gamma}(L, V)$ is an isomorphism, so the homology of $D_* \otimes_{\mathbb{Z}\Gamma} V$ in degree j is $H^{n-j}(\Gamma; V)$; that of $\overline{P}_*^{\,w} \otimes_{\mathbb{Z}\Gamma} V$ is $H_j(\Gamma; \mathbb{Z}^w \otimes V) = H_j(\Gamma; V)$. This is the duality.

For the L^2-Betti numbers, the chain homotopy equivalence and Lemma 2.7 give $b_j^{(2)}(D) = b_j^{(2)}(\overline{P}^{\,w}) = b_j^{(2)}(\Gamma)$, the last equality as in the proof of Lemma 2.9. For a projective module with idempotent e and completion $H = \mathrm{im}\, e$, the completion of the dual module is $\mathrm{im}\, e^*$, and orthogonal projection onto H maps it isomorphically to H, with inverse the restriction of e^*; under these isomorphisms the boundary maps of D_* become the adjoints of those of P_*. So $b_j^{(2)}(D) = b_{n-j}^{(2)}(\Gamma)$.

Finally $b_0^{(2)}(\Gamma) = 0$. A harmonic 0-chain of P_* is an element ξ of the completion of P_0 that is orthogonal to the image of P_1, which is the kernel of the augmentation $\varepsilon : P_0 \to \mathbb{Z}$. Choose $x \in P_0$ with $\varepsilon(x) = 1$. For $g \in \Gamma$ the element $gx - x$ lies in that kernel, so $\langle \xi, gx \rangle = \langle \xi, x \rangle$. As x has finite support and ξ is square-summable, $\langle \xi, gx \rangle$ tends to 0 along every sequence of distinct elements g of the infinite group Γ. Hence $\langle \xi, x \rangle = 0$, so ξ is orthogonal to $P_0 = \ker \varepsilon + \mathbb{Z}x$, which spans a dense subspace of the completion, and $\xi = 0$. □

3 The criterion

3.1 The manuscript

The theorem of [10] is: *for a finite aspherical Poincaré complex of formal dimension four, with any orientation character, $b_p^{(2)}(\widetilde{Q}) = 0$ for all $p \neq 2$.* Its Section 2 proves seven lemmas about $\Gamma = \pi_1(Q)$ in dimension four, numbered 2.1 to 2.7 there: finiteness properties of Γ and a four-dimensional model of Q; the comparison of resolutions; duality for L^2-Betti numbers; the vanishing of $H^i(\Gamma; \mathbb{F}_2\Gamma)$ for $i = 1, 2$; one end; the absence of splittings; and the passage from $b_1^{(2)} > 0$ to $\mathcal{D} \neq \mathcal{D}_0$. Our Section 2 replaces these for all $n \geq 3$.

Sections 3 to 10 of [10] assume $b_1^{(2)}(\widetilde{Q}) > 0$ and reach a contradiction through 38 numbered statements, 3.1 to 10.3. In outline: from a function of finite energy that is not constant at infinity they build a compact metrizable space $X = \Gamma \sqcup B$ on which Γ acts (Section 3); the action on the boundary B is a minimal convergence action (Section 4); B is a continuum of covering dimension at most one with vanishing first Čech cohomology over $\mathbb{F}_2$ (Section 5); if B has cut points, they are fixed points of parabolic elements and are the

vertices of a tree, and a block L of B without cut points inherits these properties (Sections 6 and 7); L is locally connected (Sections 8 and 9); and a locally connected continuum of dimension at most one with vanishing first Čech cohomology has a cut point (Section 10).

3.2 Where Sections 3–10 call on Section 2

Consider the following statements about a group Γ with a finite presentation whose relators have length at most three, with Cayley two-complex $\mathcal{C}$.

(G0) $\mathcal{D} \neq \mathcal{D}_0$. This is the assumption to be refuted.

(G1) Γ is finitely presented: $\mathcal{C}$ is simply connected and each edge lies on a bounded number of two-cells.

(G2) Γ has one end.

(G3) Γ is torsion-free.

(G4) $H^2(\Gamma; \mathbb{F}_2\Gamma) = 0$.

(G5) Γ has no splitting over a finite or two-ended subgroup.

Sections 3–10 of [10] refer to its Section 2 in five places, and nowhere else do they use the complex Q, its dimension, or duality:

1. the opening of Section 3 lists its inputs: the group, the complex $\mathcal{C}$ and its graph, with (G1) and (G2), and the strict inclusion (G0);
2. the proof of Proposition 4.5 uses (G2) and (G3);
3. the proof of Proposition 5.3 cites Lemma 2.4 of [10], twice, and only for (G4): a 2-cochain of finite support of the form $d\alpha$ vanishes on the two-cycles of $\mathcal{C}$, hence is a cocycle on a free resolution that extends the chains of $\mathcal{C}$, and by (G4) it is the coboundary of a cochain of finite support;
4. the opening of Section 6 assumes (G1), (G2), (G3) and (G5), and these are what Sections 6 and 7 use;
5. the last proof, after statement 10.3, uses Lemma 2.3 of [10], duality for L^2-Betti numbers, to pass from $b_1^{(2)} = 0$ to the other degrees.

Read with (G0)–(G5) as hypotheses in place of items 1–4, Sections 3–10 of [10] prove that (G1)–(G5) contradict (G0). With Lemma 2.8, this is Theorem C.

Table 1 lists the 38 statements with the hypotheses each proof uses directly, besides earlier statements of the table. The statements are described in our words; for their exact form see [10].

Table 1: The numbered statements of Sections 3–10 of [10] and what their proofs use.

No.	Statement, in brief	Uses directly
3.1	a function f of finite energy with values in $[0,1]$, not constant at infinity, whose level sets have connected sides and connected cuts	(G0), (G1)
3.2	cuts of the coordinate functions have finitely many components	3.1
3.3	boxes with generic sides have finitely many graph components; a countable invariant basis of such boxes	(G1), 3.2
3.4	the localized coordinates have finite energy, and their cuts lie in cuts of the first coordinates	3.3
3.5	the boundary B of $X = \Gamma \sqcup B$ is a continuum with more than one point	(G2), 3.1
3.6	vertices near a point of B are joined by graph paths near that point	3.3, 3.4
4.1	the total length of a translated cut tends to zero; a bound for the diameter of a connected cut	(G1)
4.2	translated cuts collapse to the limit of the translating elements	3.2, 3.4, 4.1
4.3	if $g_j \to a$ and $g_j^{-1} \to b$ then $g_j \to a$ uniformly on compact subsets of $X \setminus \{b\}$	4.2
4.4	for the fixed point p of a parabolic element, g and gp become close as g leaves finite sets	4.3
4.5	every orbit in B is dense	(G2), (G3), 3.5, 4.3, 4.4, [2]
5.1	almost every cut accumulates on a subset of B of one-dimensional Hausdorff measure zero	3.2, 3.4, 4.1
5.2	B has covering dimension at most one	3.6, 5.1
5.3	$\check{H}^1(B;\mathbb{F}_2) = 0$	(G1), (G4), 3.6
6.1	Bowditch's completed pretree of cut points and the dendritic quotient of B	[2]
6.2	the dendritic quotient is a point, and cut points have infinite stabilizers without loxodromic elements	(G1), (G2), (G3), (G5), [1], [2]
6.3	the fixed points of a parabolic element in the pretree	6.1, [2]
6.4	an invariant class of the pretree with finite intervals	(G1), (G2), (G3), 6.1–6.3, [2]
6.5	a simplicial tree with finite quotient containing the cut points, with infinite edge stabilizers	(G1), (G2), (G5), 6.1, 6.2, 6.4
7.1	a graph on the cut points next to a vertex of the tree that is not a cut point	(G1), 6.5
7.2	paths in that graph near a point of B	3.6, 4.4, 7.1
7.3	a retraction of B onto a block L; L has no cut point, dimension at most one, $\check{H}^1(L;\mathbb{F}_2) = 0$	4.5, 5.2, 5.3, 6.5, 7.1, 7.2
7.4	the common setting: a subgroup H, the space $H \sqcup L$, a graph with finitely many orbits	(G3), 3.6, 4.3, 4.4, 6.2, 7.1–7.3
8.1	a locally constant function to $\mathbb{F}_2$ on a punctured neighbourhood in L extends over the point	properties of L in 7.3

No.	Statement, in brief	Uses directly
8.2	points of L away from deleted vertices have a well-defined graph component	7.4, 8.1
8.3	an invariant system of annuli	7.4
8.4	the annular depth	8.3
9.1	translates of a closed set by elements of large depth shrink to a point	8.3, 8.4
9.2	a comparison of depths, uniform away from that point	8.3, 8.4, 9.1
9.3	component labels at each depth	7.4, 8.2, 8.3, 8.4
9.4	few deleted vertices after translation, away from that point	7.4, 9.1, 9.2
9.5	labels of two points agree eventually	7.3, 8.2, 9.4
9.6	a threshold for the majority label of three points	7.4, 9.1, 9.5
9.7	beyond the threshold, greater depth does not separate equal labels	7.4, 9.3, 9.6
9.8	L is locally connected	7.4, 9.3, 9.7
10.1	restriction is onto in first Čech cohomology when the dimension is at most one	—
10.2	connected open subsets of a locally connected compact metric space are path connected	—
10.3	a locally connected continuum with more than one point, dimension at most one and $\check{H}^1 = 0$ has a cut point	10.1, 10.2, [7]

3.3 The outside results used by the manuscript

Besides elementary analysis and topology, Sections 3–10 of [10] use the following published results.

- Bowditch [1, Lemma 1.4]: if a finitely presented one-ended group with no infinite torsion subgroup acts as a minimal convergence group on a dendrite, in such a way that every two-ended subgroup over which the group splits fixes a point of the dendrite, then the dendrite is a single point. Under (G5) the condition on splittings is empty. This is the only place where splittings over two-ended subgroups enter.
- Bowditch [2]: Theorem 6.1, that for a minimal convergence action of a finitely generated one-ended group on a continuum a cut point whose stabilizer is finite or contains a loxodromic element makes the dendritic quotient non-trivial; Lemma 6.2, that in a convergence action on a compact Hausdorff space with at least three points an element of infinite order is parabolic, with one fixed point, or loxodromic, with two, and Proposition 6.4 with the paragraph before it, that a point stabilizer containing a loxodromic element is virtually cyclic (both are quoted after statement 4.3 and used from statement 4.5 on); and, for statement 6.1 of the table, Theorem 3.19, Lemmas 3.13, 3.28, 3.34, 4.2–4.4, Theorem 5.23, and Lemmas 6.5, 6.6, 6.8–6.11 and 6.13.
- Moore's theorem that a continuous image of an interval contains an arc between any two of its points, in the form of [7].

3.4 The check

The reading in Section 3.2 rests on the following checks; details and dates are in Section 6. The text read line by line was a reproduction of Sections 3–10 of [10], compared with the

source files of [10]. Sections 3, 7 and 8 are identical. In Section 4 the reproduction adds one sentence; in Section 5 it spells out the last step of the proof of 5.3; in Section 9 it rewrites the limit argument at the end of 9.8; in Section 10 it gives statement 10.1 for the coefficients $\mathbb{F}_2$ only and writes the last proof for general n; the other differences, in Sections 6, 9 and 10, are of wording. One AI referee followed the proof of every one of the 38 statements to the end, those of Sections 3, 4, 5, 8 and 9 twice, with attention to the five places of Section 3.2 and to every use of (G1)–(G5), and found no step that fails and no further use of Section 2. An earlier AI referee had passed Sections 3, 4, 5, 7, 8 and 10 and the arguments of Section 6, without opening the results of Bowditch, and had examined Section 9 only in part. A second reader, Claude Opus 5.5 at effort high, then read Sections 8 and 9 line by line in the manuscript's own source before seeing the first referee's report, every case of every proof, and found that all twelve statements hold; it agrees with the first referee on each of them, and finds both forms of the limit argument at the end of 9.8 correct. A third AI reader compared Section 3.2 and Table 1 with the manuscript itself, its PDF and its source: the numbers and the descriptions of the statements, the five places, and, statement by statement, the references made in each proof. The results of Bowditch and of Kamiya were opened at the statements listed above and are used as stated. Their proofs were not examined, except that of Lemma 1.4 of [1], which the third reader read; what it rests on is in Section 7.

4 Proofs of Theorems A and B

Proof of Theorem A. By Corollary 2.6 the group $\Gamma = \pi_1(Q)$ satisfies the hypotheses of Theorem C, so $b_1^{(2)}(\Gamma) = 0$. By Lemma 2.7, $b_1^{(2)}(\widetilde{Q}) = b_1^{(2)}(\Gamma) = 0$. Lemma 2.9 gives $b_{n-1}^{(2)}(\widetilde{Q}) = b_1^{(2)}(\widetilde{Q}) = 0$ and $b_n^{(2)}(\widetilde{Q}) = b_0^{(2)}(\widetilde{Q}) = 0$, since Γ is infinite by Lemma 2.1, and $b_p^{(2)}(\widetilde{Q}) = b_{n-p}^{(2)}(\widetilde{Q}) = 0$ for $p > n$.

A closed connected topological manifold M of dimension n has the homotopy type of a finite CW complex: it is a compact absolute neighbourhood retract [6, Theorems 3.2 and 3.3], and such a space is homotopy equivalent to a finite complex [14, Corollary 5.3]. It satisfies Poincaré duality with its orientation character and all local coefficients [12, Theorem 10.2 and Remark 10.3]: the duality isomorphism there is defined from the Thom class, for Čech–Alexander cohomology; for the pair $(M, \emptyset)$ it corresponds to cap product with the fundamental class, and on a manifold Čech–Alexander and singular cohomology agree. This passes to the finite complex. If M is aspherical, so is the complex, and the first part applies; the L^2-Betti numbers of M are those of the complex, which by Lemma 2.7 do not depend on its choice. □

Proof of Theorem B. By Lemma 2.10, Γ is torsion-free and infinite and satisfies the conclusion of Lemma 2.2, hence that of Corollary 2.3, whose proof uses nothing else. Lemmas 2.4 and 2.5 then show, as in Corollary 2.6, that Γ has one end and no splitting over a finite or two-ended subgroup, and $H^2(\Gamma; \mathbb{F}_2\Gamma) = 0$. Since Γ is finitely presented by hypothesis, Theorem C gives $b_1^{(2)}(\Gamma) = 0$, and Lemma 2.10 gives $b_{n-1}^{(2)}(\Gamma) = b_1^{(2)}(\Gamma) = 0$. □

5 Remarks

Remark 5.1 (Dimension two). The bound $n \geq 3$ is sharp. For a closed orientable surface of genus $g \geq 2$, $H^2(\Gamma; \mathbb{F}_2\Gamma) \cong H_0(1; \mathbb{F}_2) = \mathbb{F}_2$ by Lemma 2.2, so (G4) fails, and indeed $b_1^{(2)} = 2g - 2$. In the proofs above the dimension is used in Corollary 2.3 only, through $n - 2 > 0$ and $n - 1 \geq 2$.

Remark 5.2 (Finite presentation). Theorem B assumes that the group is finitely presented. The argument of [10] uses a finite presentation from its first step, in the connectedness of cuts, and Bowditch's lemma assumes one.

Remark 5.3 (What Sections 3 and 4 use). Statements 3.1 to 4.5 of Table 1 use (G0)–(G3) only: neither the vanishing of $H^2(\Gamma;\mathbb{F}_2\Gamma)$ nor the absence of splittings. What they say about arbitrary finitely presented torsion-free groups with one end and $b_1^{(2)} > 0$, which exist, is left to a separate note.

6 Provenance and checks

This paper is AI-produced. No human has proved, written or checked any part of it.

Sources. The argument for Theorem C is that of the manuscript [10] of OpenAI, dated 25 September 2026 and released on 6 October 2026; we cite it and do not reproduce it. That the method gives every dimension $n \geq 3$ was observed in the comment [11], which gives no proof and is earlier than the run described next. The reductions of Section 2 of [10] in every dimension $n \geq 3$, with their proofs, the group criterion in the form of Theorem C, and the extension to finitely presented $PD(n)$ groups are due to a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode) of 22–23 October 2026, in answer to a question we had put to it after reading [10]; that run also reproduced Sections 3–10 of [10] with a table of the inputs of each statement.

The text. The present text was written by Claude Fable 5.2 at effort max. Section 2 follows the mathematics of the run and is written afresh; it defines L^2-Betti numbers through harmonic chains, where the run used modules over the von Neumann algebra, and its Lemma 2.7 is proved accordingly. Section 3, with Table 1, is the writer's reading of [10].

Checks. The run's version of the reductions was checked in two passes: its Lemmas on finiteness, duality for L^2-Betti numbers, duality over $\mathbb{F}_2$, ends and splittings by Claude Opus 5.5 at effort high (24 November 2026), and its comparison lemma, its gradient lemma and its treatment of $PD(n)$ groups by Claude Fable 5.2 at effort max (25 November 2026). Sections 3–10 of [10], in the run's reproduction, were read by Claude Opus 5.5 at effort max (Message Batches API, 23 October 2026), which passed Sections 3, 4, 5, 7, 8 and 10 and the arguments of Section 6, left the citations of Bowditch unopened and examined Section 9 only in part; and by Claude Fable 5.2 at effort max (25 November 2026), as described in Section 3.4. The second of these readers is the writer of this text. The finished text was then checked independently by Claude Opus 5.5 at effort high (25 November 2026). It followed every proof of Section 2 to the end and found no error. It compared Section 3 with the manuscript's PDF and source; this confirmed the numbering, the descriptions of the statements and the five places of Section 3.2, and it corrected the inputs in five rows of Table 1 (4.5, 6.3, 6.5, 7.4 and 9.6) and the list of outside results, where Lemma 6.2 and Proposition 6.4 of [2] were missing. It read the proof of Lemma 1.4 of [1], and it opened [13], [8], [4], [3], [5], [6], [14] and [12] at the statements used. Its repairs were made, some of them in a different form; none changes a theorem or a lemma. Passages written or reworded after that check were read by the writer and then, with every page of this text, by Claude Opus 5.5 at effort max (25 November 2026); the mathematical ones are the proof in Section 1.1 that the dimension is the same for V and W, and the last paragraph of the proof of Lemma 2.10. That reading also removed from the description of Theorem 6.1 of [2] in Section 3 a hypothesis on torsion which Bowditch's statement does

not contain. Nothing is formalized. The record of every check, with model, effort and date, is in the file `CHECK_EVENTS.tsv` that accompanies this paper in its workbench, reached from `https://github.com/KokunoYumeto/mathematics-workbenches`; `PROVENANCE.md` lists every contributor and says in which copy each citation was confirmed, and which citations have not been opened by a reader of ours.

7 What remains open

1. A reading of the whole of Sections 3–10 of [10] by a human who knows convergence groups.
2. The results on which Bowditch's Lemma 1.4 of [1] rests. Its proof has been read by one AI reader. It uses the construction of an $\mathbb{R}$-tree from a minimal convergence action on a dendrite (Levitt; Bowditch) and a theorem of Tukia on free abelian subgroups of convergence groups, which no reader of ours has opened, and the theorem of Bestvina and Feighn on stable actions of finitely presented groups on $\mathbb{R}$-trees, which has been opened at its statement.
3. The degrees $2, \ldots, n-2$ for $n \geq 5$, on which the method gives nothing.
4. Whether Theorem B holds for $PD(n)$ groups that are not finitely presented.

References

[1] B. H. Bowditch, *Connectedness properties of limit sets*, Trans. Amer. Math. Soc. **351** (1999), 3673–3686; Lemma 1.4 (author's manuscript, p. 6).

[2] B. H. Bowditch, *Treelike structures arising from continua and convergence groups*, Mem. Amer. Math. Soc. **139** (1999), no. 662; numbering of the author's manuscript: Lemma 3.13, Theorem 3.19, Lemmas 3.28, 3.34, 4.2–4.4, 5.15, Theorem 5.23, Theorem 6.1, Lemma 6.2, Proposition 6.4, Lemmas 6.5, 6.6, 6.8–6.11, 6.13.

[3] M. W. Davis, *Poincaré duality groups*, in: Surveys on Surgery Theory, vol. 1, Ann. of Math. Stud. **145**, Princeton Univ. Press, 2000, 167–193; Definitions 3.1 and 3.2, Theorems 3.3 and 7.15.

[4] M. W. Davis and B. Okun, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geom. Topol. **5** (2001), 7–74; Theorem 11.3.2.

[5] L. F. Di Cerbo, H. Hunter and A. K. Thrasher, *Price inequality and the growth of harmonic functions on non-positively curved manifolds*, arXiv:2601.08804; the introduction.

[6] O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. **1** (1951), 389–408; Theorems 3.2 and 3.3.

[7] H. Kamiya, *An elementary proof of a theorem of L. Moore's on continuous curves*, Japan. J. Math. **7** (1930), 301–304; the theorem on p. 301.

[8] J. Lott and W. Lück, *L^2-topological invariants of 3-manifolds*, Invent. Math. **120** (1995), 15–60; Theorem 0.1.

[9] W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergeb. Math. Grenzgeb. (3) **44**, Springer, 2002; Chapter 1.

[10] OpenAI, *The Singer conjecture in dimension four*, manuscript dated 25 September 2026, with its TEX source, at `https://github.com/openai/math/tree/main/preprints/The-Singer-conjecture-in-dimension-four-September-25-2026` (Apache License 2.0; released there on 6 October 2026); Sections 2–10.

[11] `Apprehensive_Sand951`, comment `pf4k77i` in the thread *AI in mathematics, October 10, 2026*, `r/math`, `https://www.reddit.com/r/math/comments/1x2iy00/comment/pf4k77i/`.

[12] E. Spanier, *Singular homology and cohomology with local coefficients and duality for manifolds*, Pacific J. Math. **160** (1993), 165–200; Theorem 10.2 and Remark 10.3.

[13] C. T. C. Wall, *Poincaré complexes. I*, Ann. of Math. (2) **86** (1967), 213–245; Section 1 and Lemma 4.1.

[14] J. E. West, *Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk*, Ann. of Math. (2) **106** (1977), 1–18; Corollary 5.3.