Classifying spaces of finite monoids with bounded homology*

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Abstract

The classifying space BM of a monoid M is the geometric realization of the nerve of M, regarded as a category with one object; its fundamental group is the universal group $U(M)$, the group generated by the elements of M with the multiplication table of M as relations. Let M be finite. We prove that if the integral homology of BM vanishes in all large degrees, then $U(M)$ is trivial; more precisely, if a prime p divides the order of $U(M)$, then $H_k(BM;\mathbb{F}_p) \neq 0$ for infinitely many k. It follows that three conditions on a finite monoid are equivalent: BM is homotopy equivalent to a finite CW complex; it is homotopy equivalent to a finite-dimensional one; its homology is bounded. In that case BM is simply connected. Together with a theorem of Ben-Bassat, that every finite simply connected CW complex is homotopy equivalent to BM for a finite monoid M, this determines these homotopy types: they are the finite simply connected complexes. A conjecture of the same paper, that every connected finite CW complex with finite fundamental group is of the form BM with M finite, is therefore false; the real projective plane is a counterexample. The proof compares the K-theory of BM with that of the classifying space of a finite group contained in M, through Atiyah's completion theorem. Two examples are worked out: a monoid with six elements whose classifying space is simply connected and has homology in every degree from 2 on (it is a wedge of spheres), and a monoid with 23 elements whose classifying space is the complex projective plane.

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1 Introduction and results

Conventions. A monoid M is regarded as a category with one object. Its nerve NM is the simplicial set whose k-simplices are the k-tuples $(m_1, \dots, m_k)$ of elements of M; the face d_0 drops m_1, the face d_i $(0 < i < k)$ replaces m_i, m_{i+1} by the product $m_i m_{i+1}$, the face d_k drops m_k, and the degeneracies insert 1. The classifying space $BM = |NM|$ is a CW complex with one 0-cell and one k-cell for each k-tuple of elements different from 1; if M is finite it has finitely many cells in each dimension. We write $H_k(M;A) = H_k(BM;A)$; this group is also $\mathrm{Tor}_k^{\mathbb{Z}M}(\mathbb{Z}, A)$ [17, Proposition 2.2, Corollary 2.3]. The homology of M is *bounded* if $H_k(M;\mathbb{Z}) = 0$ for all sufficiently large k. The *universal group* $U(M)$ is the group with one generator $\eta(m)$ for each $m \in M$ and the relations $\eta(m)\eta(m') = \eta(mm')$; the map $\eta\colon M \to U(M)$ is the universal homomorphism from M to a group. A space is simply connected if it is path connected with trivial fundamental group. All CW complexes of type $K(G,1)$ of a group G are denoted BG.

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Theorem 1.1. *Let M be a finite monoid and p a prime. If $H_k(M;\mathbb{F}_p) = 0$ for all sufficiently large k, then the order of $U(M)$ is prime to p. In particular, if the homology of M is bounded, then $U(M)$ is trivial.*

Theorem 1.2. *For a finite monoid M the following conditions are equivalent:*

(i) *BM is homotopy equivalent to a finite CW complex;*

(ii) *BM is homotopy equivalent to a finite-dimensional CW complex;*

(iii) *the homology of M is bounded.*

If they hold, BM is simply connected.

For a finite group G, Theorem 1.1 is contained in a theorem of Swan: if p divides $|G|$, then $H^i(G;\mathbb{Z})$ has p-torsion for infinitely many i [16, Corollary 1, and Corollary 2 for homology]. For monoids the fundamental group $U(M)$ can be trivial while BM is any finite simply connected complex. Fiedorowicz gave a monoid with five elements whose classifying space is equivalent to the sphere S^2 (we return to it in Section 5) and suggested that every finite simply connected complex should be equivalent to the classifying space of a finite monoid [5, p. 323]. Ben-Bassat stated this as Conjecture 1.1 of [3] and proved it:

Theorem 1.3 (Ben-Bassat [3, Theorem C])**.** *Every finite simply connected CW complex is homotopy equivalent to BM for some finite monoid M, which can be chosen aperiodic (every $m \in M$ satisfies $m^{r+1} = m^r$ for some r).*

Corollary 1.4. (a) *A connected finite CW complex is homotopy equivalent to BM for some finite monoid M if and only if it is simply connected.*

(b) *Up to homotopy equivalence, the spaces BM for finite monoids M with bounded homology are exactly the finite simply connected CW complexes.*

Corollary 1.5. *Conjecture 7.1 of [3], "Let X be a connected finite CW complex. If $\pi_1(X)$ is finite, then $X \simeq BM$ for some finite monoid M", is false. The real projective plane $\mathbb{RP}^2$ is a counterexample, and so is every connected finite CW complex whose fundamental group is finite and non-trivial.*

If BM is homotopy equivalent to a finite-dimensional complex, the triviality of $U(M)$ can be deduced from a theorem of Miller on maps from classifying spaces of torsion groups (Remark 2.3); this was the first route to Corollary 1.5. Theorem 1.1 replaces the dimension by a condition on homology, which is what makes the equivalences of Theorem 1.2 possible.

What was known. For a completely simple semigroup with an identity adjoined, Steinberg computed the homology, and the homotopy type of the classifying space, in terms of the structure group G and the sandwich matrix [15, Theorem A and Theorem 3.1]; in that case bounded homology forces $G = 1$ by Swan's theorem, which is more than Theorem 1.1 says. Martinian and Steinberg proved that for a finite aperiodic regular monoid the trivial module has a finite resolution by finitely generated projective modules [10, Corollary 5.15] (see also their Theorem C and Corollary 5.16); by Theorem 1.2 the classifying space of such a monoid, for instance of a finite band with an identity adjoined, is homotopy equivalent to a finite simply connected complex. When the trivial module has a projective resolution of finite length, the triviality of $U(M)$ has an elementary proof by the restriction functor of Gray and Pride [7], and more is true (Corollary 3.4). We have not found Theorems 1.1, 1.2 and 2.1, or the failure of Conjecture 7.1, in the literature; the searches are listed in the records (Section 6).

Section 2 proves a statement about maps from classifying spaces of finite groups (Theorem 2.1); Section 3 applies it to monoids and proves Theorems 1.1 and 1.2; Section 4 contains the proofs of the two corollaries; Section 5 gives the examples. The triviality of $U(M)$ does not bound the homology (Proposition 5.2), and rational homology does not detect $U(M)$ (Remark 5.1).

2 Maps from classifying spaces of finite groups

For a finite CW complex Y let $K(Y)$ be the ring of virtual complex vector bundles on Y, let Y^q be the q-skeleton, and put

$$K_q(Y) = \ker\bigl(K(Y) \to K(Y^{q-1})\bigr) \quad (q \geq 1), \qquad K_0(Y) = K(Y).$$

We use two properties of this filtration. Both are contained in the spectral sequence of Atiyah and Hirzebruch [2, §2], which starts from $H^*(Y; K^*(\text{point}))$, converges to the graded group of the filtration and is multiplicative; they are also in [1, §2], where the filtration is defined and is multiplicative, and where (2.2) and Proposition (2.4) give its graded quotients.

(K1) $K_a(Y) \cdot K_b(Y) \subset K_{a+b}(Y)$ for all $a, b \geq 0$;

(K2) for $q \geq 1$ the group $K_q(Y)/K_{q+1}(Y)$ is zero if q is odd, and is isomorphic to a subquotient of $H^q(Y;\mathbb{Z})$ if q is even.

Let C be a cyclic group of order p^a, acting freely on the spheres $S^{2j+1} \subset \mathbb{C}^{j+1}$ through a faithful character, and let $L^{2j+1} = S^{2j+1}/C$ be the lens spaces. Their union $L = S^\infty/C$ is a $K(C,1)$ and has a CW structure with one cell in each dimension whose $(2j+1)$-skeleton is L^{2j+1}; moreover $H^q(L^{2j+1};\mathbb{Z})$ is $\mathbb{Z}/p^a$ for q even with $2 \leq q \leq 2j$ and is zero for q odd with $q < 2j+1$, as one reads off from the cellular chain complex of [9, Example 2.43], whose boundary maps are alternately 0 and multiplication by p^a. Let $R(C)$ be the complex representation ring. A representation τ of C defines the vector bundle $S^{2j+1} \times_C \tau$ on L^{2j+1}, and this gives ring homomorphisms $\alpha_j \colon R(C) \to K(L^{2j+1})$ compatible with restriction.

(K3) The homomorphism $(\alpha_j)_j \colon R(C) \to \varprojlim_j K(L^{2j+1})$ is injective.

Indeed, by [1, Proposition (8.1)] the completion of $R(C)$ at its augmentation ideal maps isomorphically to the inverse limit of the K-groups of the skeleta of a classifying space of the cyclic group C, and by [1, Proposition (6.11)] the map from $R(C)$ to its completion is injective because C is a p-group.

Theorem 2.1. *Let p be a prime, and let X be a connected CW complex with finitely many cells in each dimension such that $H^q(X;\mathbb{Z}_{(p)}) = 0$ for all $q > n$, where $\mathbb{Z}_{(p)}$ is the ring of rational numbers with denominator prime to p. Let G be a finite group, $f \colon BG \to X$ a map, and $\rho \colon \pi_1(X) \to \mathrm{GL}_d(\mathbb{C})$ a homomorphism. Then $\rho(f_*(g)) = 1$ for every element $g \in G$ whose order is a power of p.*

Here $f_* \colon G \to \pi_1(X)$ is defined up to conjugation in $\pi_1(X)$, which does not affect the statement.

Proof. Let $g \in G$ have order p^a, let $C = \langle g \rangle$, and let L and L^{2j+1} be as above. Since BG is a $K(G,1)$, there is a map $L \to BG$ inducing the inclusion $C \subset G$ on fundamental groups [9, Proposition 1B.9]; composing with f and applying cellular approximation we get a cellular map $h \colon L \to X$ with $h_* = f_*|_C$ up to conjugation. Fix base points, let $\widetilde{X} \to X$ be the

universal covering, on which $\pi_1(X)$ acts by deck transformations, and let $V = \widetilde{X} \times_{\pi_1(X)} \mathbb{C}^d$ be the vector bundle of rank d defined by ρ. A lift $S^\infty \to \widetilde{X}$ of h is equivariant along h_*, so

$$h^*V \cong S^\infty \times_C \sigma, \qquad \sigma = \rho \circ h_* \colon C \to \mathrm{GL}_d(\mathbb{C}). \tag{1}$$

Step 1. For $m \geq 0$ let $v_m \in K(X^m)$ be the class of $V|X^m$ minus the trivial bundle of rank d, and let $w_m = v_m^{n+1}$. The skeleta are finite complexes, and restriction $K(X^{m+1}) \to K(X^m)$ is a ring homomorphism taking v_{m+1} to v_m and w_{m+1} to w_m. The class v_m has rank zero at every vertex, so $v_m \in K_1(X^m)$, and $w_m \in K_{n+1}(X^m)$ by (K1). For $m \leq n$ this says $w_m = 0$, since $K_{n+1}(X^m)$ is then the kernel of the identity of $K(X^m)$.

Step 2: $w_m \otimes 1 = 0$ in $K(X^m) \otimes \mathbb{Z}_{(p)}$ for every m. We argue by induction on m; the cases $m \leq n$ are Step 1. Let $m > n$, assume $w_{m-1} \otimes 1 = 0$, and put $Y = X^{m+1}$, so that $Y^{m-1} = X^{m-1}$ and $Y^m = X^m$. Since $\mathbb{Z}_{(p)}$ is flat over $\mathbb{Z}$, the kernel of $K(Y) \otimes \mathbb{Z}_{(p)} \to K(Y^{q-1}) \otimes \mathbb{Z}_{(p)}$ is $K_q(Y) \otimes \mathbb{Z}_{(p)}$ for every q. The element $w_{m+1} \otimes 1$ of $K(Y) \otimes \mathbb{Z}_{(p)}$ restricts to $w_{m-1} \otimes 1 = 0$ on Y^{m-1}, hence lies in $K_m(Y) \otimes \mathbb{Z}_{(p)}$. By (K2) the group $\big(K_m(Y)/K_{m+1}(Y)\big) \otimes \mathbb{Z}_{(p)}$ is zero or a subquotient of $H^m(Y;\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$. The cellular cochain complexes of Y and of X coincide in degrees at most $m+1$, so $H^m(Y;\mathbb{Z}) = H^m(X;\mathbb{Z})$; and $H^m(X;\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = H^m(X;\mathbb{Z}_{(p)})$, because the cellular cochain groups of X are finitely generated and $\mathbb{Z}_{(p)}$ is flat. This group is zero since $m > n$. Therefore $K_m(Y) \otimes \mathbb{Z}_{(p)} = K_{m+1}(Y) \otimes \mathbb{Z}_{(p)}$, and $w_{m+1} \otimes 1$ restricts to zero on $Y^m = X^m$; that is, $w_m \otimes 1 = 0$.

Step 3. The map h sends L^{2j+1} into X^{2j+1}. By (1) the pull-back of v_{2j+1} is $\alpha_j(\sigma) - d$, hence the pull-back of w_{2j+1} is $\alpha_j\big((\sigma - d)^{n+1}\big)$, and by Step 2

$$\alpha_j\big((\sigma - d)^{n+1}\big) \otimes 1 = 0 \quad \text{in } K(L^{2j+1}) \otimes \mathbb{Z}_{(p)} \quad \text{for every } j \geq 0.$$

The element $\alpha_j\big((\sigma - d)^{n+1}\big)$ has rank zero, so it lies in $K_1(L^{2j+1})$. By (K2) and the cohomology of lens spaces recalled above, $K_1(L^{2j+1})$ has a finite filtration, ending with $K_{2j+2}(L^{2j+1}) = 0$, whose quotients are zero or subquotients of $\mathbb{Z}/p^a$; so it is a finite abelian p-group A, and $A \to A \otimes \mathbb{Z}_{(p)}$ is an isomorphism. Hence $\alpha_j\big((\sigma - d)^{n+1}\big) = 0$ for every j, and $(\sigma - d)^{n+1} = 0$ in $R(C)$ by (K3).

Step 4. Taking characters, $(\chi_\sigma(g) - d)^{n+1} = 0$, so $\chi_\sigma(g) = d$. The matrix $\sigma(g)$ has finite order; its d eigenvalues are roots of unity with sum d, so they are all equal to 1, and $\sigma(g) = 1$. As $\sigma(g)$ is conjugate to $\rho(f_*(g))$, the theorem follows. □

Corollary 2.2. *Let X be a connected CW complex with finitely many cells in each dimension and with finite fundamental group, let G be a finite group and $f \colon BG \to X$ a map.*

(a) *If $H^q(X;\mathbb{Z}_{(p)}) = 0$ for all sufficiently large q, then the order of $f_*(G)$ is prime to p.*
(b) *If $H^q(X;\mathbb{Z}) = 0$ for all sufficiently large q, then f_* is trivial.*

Proof. (a) Apply Theorem 2.1 to the regular representation ρ of $\pi_1(X)$, which is faithful: $f_*(g) = 1$ for every g of p-power order. The image of a Sylow p-subgroup of G is a Sylow p-subgroup of $f_*(G)$, so the latter has order prime to p. (b) Since the cochain groups are finitely generated, $H^q(X;\mathbb{Z}_{(p)}) = H^q(X;\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$, so (a) applies to every prime. □

Remark 2.3. (a) Miller's theorem [12, Theorem B, p. 40; proved as Theorem 10.1, pp. 84–85] states that every map from BG, with G a torsion group, to a finite-dimensional CW complex is trivial on fundamental groups. Its proof also uses the K-theory of cyclic groups, on a covering space of the target; there is no hypothesis on the fundamental group of the target. Theorem 2.1 weakens the hypothesis on the dimension to one on cohomology, and the argument of Step 2, on the skeleta of X itself, takes the place of the covering space. Ordinary cohomology cannot replace K-theory here: under the surjection $\mathbb{Z}/p^2 \to \mathbb{Z}/p$

a generator y of $H^2(B(\mathbb{Z}/p);\mathbb{Z})$ goes to p times a generator x of $H^2(B(\mathbb{Z}/p^2);\mathbb{Z})$, so y^k goes to $p^k x^k = 0$ for $k \geq 2$, and the map on integral cohomology vanishes above degree 2, although the map is not trivial on fundamental groups.

(b) Under the hypothesis on cohomology alone the representation ρ cannot be dispensed with. By [13, Theorem A and the properties (1), (4) recalled in the introduction] the group $\mathbb{Z}/2$ embeds in a group Γ which has the integral homology of a point and is of type F_n for every n, that is, has for every n a classifying space with finite n-skeleton. Such a group admits a classifying space $B\Gamma$ with finitely many cells in each dimension. It is built skeleton by skeleton, starting from a finite 2-skeleton: if Z is a finite complex of dimension $n \geq 2$ with fundamental group Γ whose universal covering $\widetilde{Z}$ is $(n-1)$-connected, then the $\mathbb{Z}\Gamma$-module $\pi_n(Z) \cong H_n(\widetilde{Z};\mathbb{Z})$ is finitely generated, by Schanuel's lemma applied to the cellular chains of $\widetilde{Z}$ and of the universal covering of a classifying space with finite $(n+1)$-skeleton, so that finitely many $(n+1)$-cells attached to Z make the universal covering n-connected. Then $X = B\Gamma$ satisfies $H^q(X;\mathbb{Z}) = 0$ for $q > 0$, and $B(\mathbb{Z}/2) \to X$ is injective on fundamental groups.

3 Finite monoids

Lemma 3.1. *Let M be a finite monoid.*

(a) $\pi_1(BM) \cong U(M)$*, the loop formed by the* 1*-cell* (m) *corresponding to* $\eta(m)$*.*

(b) $\eta\colon M \to U(M)$ *is surjective and* $U(M)$ *is finite.*

(c) *Let e be an idempotent of the minimal two-sided ideal of M. Then $H = eMe$ is a finite group with identity e, and η restricts to a surjective homomorphism of groups $H \to U(M)$.*

Proof. (a) The nerve has a single vertex, so the fundamental group of BM is generated by the loops of the 1-cells (m), with one relation for each 2-cell (m, m'), which identifies the loop of mm' with the composite of the loops of m and m'; this is the presentation of $U(M)$. The statement is due to McDuff [11, Lemma 1, p. 314]; it is also the case of one object of [14, §1, Proposition 1], by which the fundamental group of the classifying space of a small category at an object is the automorphism group of that object in the groupoid obtained by inverting all arrows. (b) is proved in [3, §7]. (c) is known: the minimal ideal is a completely simple semigroup, H is its maximal subgroup at e, and $U(M)$ is the quotient of H by the normal closure of the entries of the sandwich matrix; see [17, §2.1 and Lemma 2.1], with the attributions given there, and [15, p. 7]. (Directly: for $x \in H$ the ideal MxM is contained in the minimal ideal, hence equal to it, so $e = axb$ with $a, b \in M$ and $e = (eae)x(ebe)$; in a finite monoid one-sided inverses are two-sided, so $x(ebe)(eae) = e$ and x is invertible in H.) Since $\eta(e)^2 = \eta(e)$ we have $\eta(e) = 1$; so $\eta|_H$ is a homomorphism of groups, and $\eta(eme) = \eta(m)$ for $m \in M$ shows with (b) that it is surjective. □

Lemma 3.2. *With e and H as in Lemma 3.1*(c)*, there is a map $j_e\colon BH \to BM$ whose effect on fundamental groups is $\eta|_H\colon H \to U(M)$, up to an inner automorphism.*

Proof. We use the map of [3, Introduction]. The Karoubi envelope $\mathrm{Kar}(M)$ is the category whose objects are the idempotents of M, with $\mathrm{Hom}(f, g) = gMf$ and composition given by the product of M. The endomorphism monoid of the object 1 is M, that of e is H; let $\iota_1\colon M \to \mathrm{Kar}(M)$ and $\iota_e\colon H \to \mathrm{Kar}(M)$ be the inclusions. Every object f is a retract of 1, by the arrows $1 \to f$ and $f \to 1$ given by the element f, so $B\iota_1$ is a homotopy equivalence [3, Lemma 2.4]. Let j_e be $B\iota_e$ followed by a homotopy inverse of $B\iota_1$.

Let Π be the groupoid obtained from $\mathrm{Kar}(M)$ by inverting all arrows, and write $[x]$ for the class of an arrow x; by [14, §1, Proposition 1] the fundamental groups of $B\,\mathrm{Kar}(M)$

at the objects 1 and e are the automorphism groups of 1 and e in Π. Let $r\colon 1 \to e$ and $s\colon e \to 1$ be the arrows given by the element e. Then $r \circ s = ee = \mathrm{id}_e$, so $[s] = [r]^{-1}$. For $h \in H$,

$$[r]^{-1}[h][r] = [s \circ h \circ r] = [ehe] = [h],$$

where on the right h is regarded as an arrow $1 \to 1$, that is, as an element of M. So the change of base point along $[r]$ carries the image of $h \in H = \pi_1(BH)$ under $B\iota_e$ to the image under $B\iota_1$ of the loop of $h \in M$, which is $\eta(h)$ by Lemma 3.1 (a). □

Proof of Theorem 1.1. Write $H_k = H_k(BM;\mathbb{Z})$; these groups are finitely generated. Suppose that $H_k(M;\mathbb{F}_p) = 0$ for $k \geq k_0$. By the universal coefficient theorem $H_k \otimes \mathbb{F}_p = 0$ for $k \geq k_0$, so these H_k are finite of order prime to p. For $q > k_0$ the group $H^q(BM;\mathbb{Z}) \cong \mathrm{Hom}(H_q,\mathbb{Z}) \oplus \mathrm{Ext}(H_{q-1},\mathbb{Z})$ is then finite of order prime to p, and $H^q(BM;\mathbb{Z}_{(p)}) = H^q(BM;\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = 0$. Thus $X = BM$ satisfies the hypotheses of Corollary 2.2 (a): it is connected with finitely many cells in each dimension, and its fundamental group $U(M)$ is finite by Lemma 3.1. Applied to the map $j_e\colon BH \to BM$ of Lemma 3.2, the corollary says that the image of $(j_e)_* = \eta|_H$ has order prime to p. This image is $U(M)$ by Lemma 3.1 (c).

If the homology of M is bounded, then $H_k(M;\mathbb{F}_p) \cong H_k \otimes \mathbb{F}_p \oplus \mathrm{Tor}(H_{k-1},\mathbb{F}_p)$ vanishes for all large k and every prime p, so no prime divides the order of $U(M)$. □

Proof of Theorem 1.2. (i) implies (ii), and (ii) implies (iii) because homology vanishes above the dimension. Assume (iii). By Theorem 1.1 and Lemma 3.1 (a), BM is simply connected. Its homology groups are finitely generated and vanish above some degree n. By [9, Proposition 4C.1], a simply connected CW complex is homotopy equivalent to a CW complex having, for chosen decompositions of its homology groups into cyclic summands, one k-cell for each infinite cyclic summand of H_k and one k-cell and one $(k+1)$-cell for each finite cyclic summand of H_k. For BM this complex is finite. This proves (i) and the last assertion. □

Corollary 3.3. *Let M be a finite monoid. If $U(M) \neq 1$, then $H_k(M;\mathbb{Z}) \neq 0$ for infinitely many k. If a prime p divides the order of $U(M)$, then $H_k(M;\mathbb{F}_p) \neq 0$ for infinitely many k.*

For instance the monoid with 11 elements of [3, Example 3.4], in the case $a = 1$ of that example, whose classifying space is $B(\mathbb{Z}/2) \vee S^2$, has universal group $\mathbb{Z}/2$ and homology in infinitely many degrees. Corollary 3.3 says that torsion coming from $U(M)$ recurs in infinitely many degrees, as for groups. Torsion in higher degrees need not recur: there are finite semigroups whose classifying spaces are Moore spaces [17, Table 1 and Examples 5.4, 5.5]. The converse of Corollary 3.3 fails. In the other case of the same example, where a is the generator g of $\mathbb{Z}/2$, the classifying space is the cofibre of the inclusion $\mathbb{RP}^1 \subset \mathbb{RP}^\infty$ by [3, Theorem A]; it is simply connected, and its reduced homology is $\mathbb{Z}$ in degree 2, $\mathbb{Z}/2$ in every odd degree from 3 on, and zero in the other degrees. Another example is the monoid M_6 of Proposition 5.2.

Corollary 3.4. *Let M be a finite monoid such that the trivial left (or right) $\mathbb{Z}M$-module $\mathbb{Z}$ has a projective resolution of finite length. Then the group $H = eMe$ of Lemma 3.1 (c) is trivial. In particular $U(M)$ is trivial, and BM is homotopy equivalent to a finite simply connected CW complex.*

Proof. We treat left modules; for right modules exchange the sides. Let e and H be as in Lemma 3.1 (c). For a left $\mathbb{Z}M$-module A the subgroup eA is a left $\mathbb{Z}H$-module, and $A \mapsto eA$ is a functor which is exact, because e is idempotent, commutes with direct sums and takes

the trivial module $\mathbb{Z}$ to the trivial module $\mathbb{Z}$; this is the functor of [7, Lemmas 2–4 and Theorem 2]. It takes $\mathbb{Z}M$ to $\mathbb{Z}[eM]$, and this is a free $\mathbb{Z}H$-module, because H acts freely on the set eM by left multiplication. Indeed, let $x \in eM$. Then x lies in the minimal ideal, so $e = axb$ with $a, b \in M$, and $(eae)(xbe) = e$ because $ex = x$; hence xbe is the inverse g of eae in H, and $xy = e$ for $y = beg^{-1}$. So $hx = h'x$ with $h, h' \in H$ implies $h = h'$. Therefore the functor takes free modules to free modules, projective modules to projective modules, and a projective resolution of finite length of $\mathbb{Z}$ over $\mathbb{Z}M$ to one over $\mathbb{Z}H$. For a non-trivial finite group H the groups $H^i(H;\mathbb{Z}) = \mathrm{Ext}^i_{\mathbb{Z}H}(\mathbb{Z},\mathbb{Z})$ are non-zero for infinitely many i [16, Corollary 1]; so $H = 1$, and $U(M) = \eta(H) = 1$ by Lemma 3.1 (c). Finally $H_k(M;\mathbb{Z}) = \mathrm{Tor}_k^{\mathbb{Z}M}(\mathbb{Z},\mathbb{Z})$ can be computed from a projective resolution of either variable, so the homology of M is bounded, and Theorem 1.2 applies. □

Remark 3.5. In Corollary 3.4 the triviality of $U(M)$ also follows from Theorem 1.1, since the homology is bounded; and from Miller's theorem (Remark 2.3), since a monoid of finite cohomological dimension has a classifying space of finite-dimensional homotopy type [8, Theorem 6.27 and Corollary 6.7]. We do not know whether $H = 1$ for every finite monoid with bounded homology. This holds for completely simple semigroups with an identity adjoined [15, Theorem A] and under the hypothesis of Corollary 3.4; Theorem 1.1 only says that H has trivial image in $U(M)$; and the monoids of Theorem 1.3 can be chosen aperiodic, so that $H = 1$ for them. The map j_e of Lemma 3.2 cannot decide the question: if the homology of M is bounded, then BM is homotopy equivalent to a finite complex by Theorem 1.2, and every map from BH to a connected finite-dimensional complex is null-homotopic, by Miller's theorem that the space of pointed maps is weakly contractible [12, Theorem A, p. 39]. An answer has to use the monoid itself, as the proof of Corollary 3.4 does.

4 The finite complexes that are classifying spaces of finite monoids

Proof of Corollary 1.4. (a) If X is a finite simply connected CW complex, $X \simeq BM$ with M finite by Theorem 1.3. Conversely, if a connected finite CW complex X is homotopy equivalent to BM with M finite, then BM is simply connected by Theorem 1.2, hence so is X. (b) By Theorem 1.2 every BM with M finite of bounded homology is homotopy equivalent to a finite simply connected CW complex; conversely each such complex is homotopy equivalent to some BM with M finite by Theorem 1.3, and the homology of this M is bounded. □

Proof of Corollary 1.5. By Corollary 1.4 (a) a connected finite CW complex with non-trivial fundamental group is not homotopy equivalent to BM for any finite monoid M; the fundamental group of $\mathbb{RP}^2$ has order 2. □

Remark 4.1. Corollary 1.5 concerns the first version of [3], of 28 October 2026, the only one on arXiv on 25 November 2026; its Theorem C is not affected. Among finite categories with several objects the real projective plane does occur: the category with objects $0, 1, 2$, with $\mathrm{Hom}(i, j) = \mathbb{Z}/2$ for $i < j$, no other arrows than identities, and composition given by addition, has nine arrows and a classifying space homeomorphic to $\mathbb{RP}^2$, the quotient of the octahedron by the antipodal map [4, Theorem 6.2]; there this three-object model of $\mathbb{RP}^2$ is attributed to Tanaka.

5 Examples

Remark 5.1. Rational coefficients do not suffice in Theorem 1.1: for a finite group $G \neq 1$ one has $H_k(G;\mathbb{Q}) = 0$ for all $k > 0$ and $U(G) = G$.

5.1 A monoid with six elements: trivial universal group, unbounded homology

Let $M_6 = \{1, x_{00}, x_{01}, x_{10}, x_{11}, y\}$. Put $r(x_{ij}) = i$, $c(x_{ij}) = j$ and $r(y) = c(y) = 0$; the element 1 is the identity, and

$$vw = x_{r(v)\,c(w)} \qquad \text{for } v, w \neq 1.$$

Both ways of bracketing a product of three elements different from 1 give the row of the first and the column of the last, so M_6 is a monoid. It is the semigroup of [17, Example 5.2] with an identity adjoined. The elements 1 and x_{ij} form a submonoid M_5, which is the monoid with five elements of Fiedorowicz [5, p. 323]; it is treated after the proof of Proposition 5.2. Write $e = x_{00}$.

Proposition 5.2. (a) *$U(M_6)$ is trivial.*

(b) *$H_0(M_6;\mathbb{Z}) = \mathbb{Z}$, $H_1(M_6;\mathbb{Z}) = 0$ and $H_n(M_6;\mathbb{Z}) \cong \mathbb{Z}$ for every $n \geq 2$.*

(c) *BM_6 is homotopy equivalent to the wedge $\bigvee_{n\geq 2} S^n$ of one sphere of each dimension $n \geq 2$.*

So the converse of Theorem 1.1 fails, and BM_6 is a simply connected space with finitely generated homology groups that satisfies none of the conditions of Theorem 1.2. Part (b) is [17, Example 5.2].

Proof. (a) Every x_{ij} is idempotent, so $\eta(x_{ij}) = 1$; and $ye = e$ gives $\eta(y) = 1$.

(b) Let $R = \mathbb{Z}M_6$ and

$$u = 1 - e, \quad a = x_{10} - e, \quad b = x_{01} - e, \quad c = x_{11} - x_{10} - x_{01} + e, \quad t = y - e.$$

Then e, u, a, b, c, t is a $\mathbb{Z}$-basis of R, the elements e and u are idempotents with $eu = ue = 0$, and the multiplication law gives

$$ae = a, \quad ea = 0, \quad eb = b, \quad be = 0, \quad ec = ce = et = te = 0, \quad ab = c,$$

while all other products of two of the elements a, b, c, t are zero. Hence $Re = \mathbb{Z}e \oplus \mathbb{Z}a$ and $Ru = \mathbb{Z}u \oplus \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}t$; these left modules are projective. Consider

$$\cdots \xrightarrow{d_4} Re \oplus Ru \xrightarrow{d_3} Re \oplus Ru \xrightarrow{d_2} Ru \xrightarrow{d_1} Re \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

with ε the augmentation, $d_1(s) = sa$, $d_2(s, s') = sb + s't$ and $d_n(s, s') = (0, sb + s't)$ for $n \geq 3$. It is exact: the kernel of ε on Re is $\mathbb{Z}a = d_1(Ru)$, since $ua = a$; the map d_1 kills b, c, t, so its kernel is $N = \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}t$; the map $s \mapsto sb$ sends e, a to b, c, and $s' \mapsto s't$ sends u to t and kills b, c, t; so d_2 has image N and kernel $0 \oplus N$, and the same holds for d_n, $n \geq 3$, with image $0 \oplus N$. Tensoring with the trivial right module, Re gives $\mathbb{Z}$ and Ru gives 0, because $\varepsilon(e) = 1$ and $\varepsilon(u) = 0$, and all differentials become zero since the images of d_n, $n \geq 2$, lie in summands Ru. This gives (b).

(c) We use the cellular chain complex of BM_6: it is free on the tuples of elements different from 1, with $\partial(m_1, \ldots, m_n) = (m_2, \ldots, m_n) + \sum_{0<i<n}(-1)^i(\ldots, m_i m_{i+1}, \ldots) + (-1)^n(m_1, \ldots, m_{n-1})$. Fix $n \geq 2$ and put

$$(a_i^0, a_i^1) = (x_{01}, e) \ \ (i = 1), \qquad (y, e) \ \ (1 < i < n), \qquad (x_{10}, e) \ \ (i = n).$$

For $i < n$ the elements a_i^δ have row 0, and for $i > 1$ they have column 0; hence

$$a_i^\delta a_{i+1}^{\delta'} = e \qquad (0 < i < n;\ \delta, \delta' \in \{0,1\}). \tag{2}$$

Let $z_n = \sum_{\epsilon \in \{0,1\}^n} (-1)^{\epsilon_1 + \cdots + \epsilon_n} (a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n})$. In ∂z_n the terms coming from the first, from the i-th and from the last summand of ∂ do not depend on ϵ_1, on ϵ_i (by (2)) and on ϵ_n respectively, so they cancel in pairs: z_n is a cycle.

For $m \neq 1$ let $\alpha(m) = 1$ if $c(m) = 1$ and $\alpha(m) = 0$ otherwise, let $\beta(m) = 1$ if $r(m) = 1$ and $\beta(m) = 0$ otherwise, and let $Y(y) = 1$ and $Y(m) = 0$ for $m \neq y$. For $m, m' \neq 1$ we have $\alpha(mm') = \alpha(m')$, $\beta(mm') = \beta(m)$ and $Y(mm') = 0$. Define the cochain

$$f_n(m_1, \dots, m_n) = \alpha(m_1)\, Y(m_2) \cdots Y(m_{n-1})\, \beta(m_n).$$

In $(\delta f_n)(m_0, \dots, m_n)$ the term $f_n(m_1, \dots, m_n)$ cancels against $-f_n(m_0 m_1, m_2, \dots)$, the last term against the term with $m_{n-1} m_n$, and the remaining terms vanish because they contain a factor $Y(m_i m_{i+1})$. So f_n is a cocycle, and $f_n(z_n) = 1$, only $\epsilon = 0$ contributing. By (b) the class of z_n is therefore a generator of $H_n(BM_6; \mathbb{Z}) \cong \mathbb{Z}$.

This class is the image of the fundamental class of a sphere. Let D_n be the join of a circle with two vertices p_0, p_1 and two edges E^0, E^1 from p_0 to p_1, and of $n-1$ two-point sets $\{q_i^0, q_i^1\}$, $2 \leq i \leq n$. Call i the colour of q_i^δ, and 0 and 1 the colours of p_0 and p_1. Then D_n is a Δ-complex whose simplices have vertices of distinct colours, ordered by colour; its realization is the $(n-1)$-fold suspension of a circle, a sphere S^n; its n-simplices are the joins σ_ϵ of E^{ϵ_1} with $q_2^{\epsilon_2}, \dots, q_n^{\epsilon_n}$, and $\zeta_n = \sum_\epsilon (-1)^{\epsilon_1 + \cdots + \epsilon_n} \sigma_\epsilon$ is a cycle, which generates $H_n(D_n)$ because its coefficients are ± 1. Label the edges of D_n: the edge E^δ by a_1^δ; an edge from a vertex of colour $i-1$ to q_i^δ by a_i^δ; every edge between vertices whose colours differ by at least 2 by e. For a triangle with vertices of colours $i < j < k$, the labels l, l' of the edges $i \to j$ and $j \to k$ have row 0 and column 0 respectively, as in (2), so $ll' = e$, which is the label of the edge $i \to k$. Hence sending a simplex with vertices $v_0, \dots, v_k$ to the tuple of the labels of the edges $v_0 \to v_1, \dots, v_{k-1} \to v_k$ commutes with the face maps, and defines a cellular map $s_n \colon S^n \cong |D_n| \to BM_6$ which sends the simplex σ_ϵ onto the cell $(a_1^{\epsilon_1}, \dots, a_n^{\epsilon_n})$; so $(s_n)_* \zeta_n = z_n$.

All vertices go to the 0-cell, so the maps s_n define $\bigvee_{n \geq 2} S^n \to BM_6$. It induces an isomorphism on integral homology, and both spaces are simply connected CW complexes, by (a) and Lemma 3.1 (a); so it is a homotopy equivalence [9, Corollary 4.33]. □

The monoid with five elements. In the submonoid $M_5 = \{1, x_{00}, x_{01}, x_{10}, x_{11}\}$ of M_6 the product is $x_{ij} x_{kl} = x_{il}$. This is the monoid of [5, p. 323], where it is left as an exercise that its classifying space is equivalent to S^2. Fiedorowicz published a proof, by a projective resolution of length 2, in [6, Lemma, pp. 33–34], and the case of every finite rectangular band is in [15, Theorem 3.1 and p. 8]. The proof of Proposition 5.2 contains another proof. The ring $R' = \mathbb{Z}M_5$ has the $\mathbb{Z}$-basis e, u, a, b, c, with $R'e = \mathbb{Z}e \oplus \mathbb{Z}a$ and $R'u = \mathbb{Z}u \oplus \mathbb{Z}b \oplus \mathbb{Z}c$, and the sequence

$$0 \to R'e \xrightarrow{d_2} R'u \xrightarrow{d_1} R'e \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \qquad d_1(s) = sa, \quad d_2(s) = sb,$$

is exact, because d_2 takes the basis e, a of $R'e$ to the basis b, c of the kernel of d_1. Hence, as in the proof of (b), $H_2(M_5; \mathbb{Z}) \cong \mathbb{Z}$ and $H_k(M_5; \mathbb{Z}) = 0$ for $k \neq 0, 2$. The map s_2 takes S^2 into the subcomplex BM_5 of BM_6, and $f_2(z_2) = 1$; so $s_2 \colon S^2 \to BM_5$ induces an isomorphism on homology and is a homotopy equivalence, as at the end of the proof above. The element y has the same products as x_{00} with every element different from 1, without being equal to it; adjoining it to M_5 changes the classifying space from S^2 to $\bigvee_{n \geq 2} S^n$.

5.2 A monoid with 23 elements of the homotopy type of the complex projective plane

Let $I = \{0,1,2\}$ and $J = \{0,1,2,3,4\}$. Consider pairs (σ,τ) of maps $\sigma\colon I \to I$, $\tau\colon J \to J$ with the product $(\sigma,\tau)(\sigma',\tau') = (\sigma\circ\sigma', \tau'\circ\tau)$. Let M_{23} consist of the identity, the 15 pairs x_{ij} of constant maps with values i and j, and the seven pairs below, where a map is written as the string of its values:

	a	b	c	d	e	f	g
σ	002	002	010	202	202	202	212
τ	02200	41144	04034	03300	12211	40044	00334

Write σ_m, τ_m for the components of $m \in M_{23}$. In this subsection e denotes the element of the table.

Proposition 5.3. *The set M_{23} is closed under the product; it is an aperiodic monoid with 23 elements. The trivial left $\mathbb{Z}M_{23}$-module has a projective resolution of length 4, the groups $H_k(M_{23};\mathbb{Z})$ are $\mathbb{Z}$ for $k = 0, 2, 4$ and zero otherwise, and BM_{23} is homotopy equivalent to $\mathbb{CP}^2$.*

Proof. Closure is a finite verification. We have $x_{ij}x_{kl} = x_{il}$, the elements a, b, c, g are idempotent, and $d^2 = x_{20}$, $e^2 = x_{22}$, $f^2 = x_{24}$ with $d^3 = d^2$, $e^3 = e^2$, $f^3 = f^2$; so M_{23} is aperiodic.

Let $R = \mathbb{Z}M_{23}$. The elements x_{00}, $c - x_{00}$, $1 - c$, $a - x_{00}$ of R are idempotents; let P, E, F, G be the left ideals they generate, which are projective. Define

$$0 \to P \xrightarrow{d_4} G \oplus G \xrightarrow{d_3} P \oplus F \xrightarrow{d_2} E \xrightarrow{d_1} P \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

by the following values on the generating idempotents, extended R-linearly (each value is fixed under left multiplication by the idempotent it is assigned to, so the maps are well defined):

$$\begin{aligned} d_1(c - x_{00}) &= x_{10} - x_{00}, & d_2(x_{00}, 0) &= x_{03} - x_{00}, \\ d_2(0, 1 - c) &= c + d - g - x_{00}, & d_3(a - x_{00}, 0) &= (x_{00} - x_{20},\ b + e - f - x_{01}), \\ d_3(0, a - x_{00}) &= (0,\ a - x_{00}), & d_4(x_{00}) &= (x_{02} - x_{00},\ x_{00} - x_{02}). \end{aligned}$$

The ranks over $\mathbb{Z}$ of P, E, F, G are $3, 10, 10, 4$, so the terms have ranks $3, 10, 13, 8, 3$. All composites vanish, and the images of $\varepsilon, d_1, d_2, d_3, d_4$ have ranks $1, 2, 8, 5, 3$ and are saturated subgroups (the elementary divisors of the matrices are all 1). Since $3 = 1 + 2$, $10 = 2 + 8$, $13 = 8 + 5$, $8 = 5 + 3$ and $3 = 3$, each image has the rank of the next kernel and, being saturated, equals it: the sequence is exact. The augmentations of the four generating idempotents are $1, 0, 0, 0$; so tensoring with the trivial right module leaves $\mathbb{Z}$ in degrees $0, 2, 4$ and 0 in degrees $1, 3$, which gives the homology. By Corollary 3.4, BM_{23} is homotopy equivalent to a finite simply connected complex, and by [9, Proposition 4C.1] to $S^2 \cup_h e^4$ for some map $h\colon S^3 \to S^2$.

It remains to see that h has Hopf invariant ± 1, that is, that the square of a generator of H^2 generates H^4. Let $\lambda\colon J \times I \to \mathbb{Z}$ be zero except for $\lambda(1,1) = \lambda(1,2) = \lambda(2,2) = \lambda(4,1) = 1$ and $\lambda(3,1) = -1$, and put

$$u(m, m') = \lambda(\tau_m(0), \sigma_{m'}(0)) \qquad (m, m' \in M_{23}).$$

Then u vanishes if m or m' is 1 and satisfies $u(m', m'') - u(mm', m'') + u(m, m'm'') - u(m, m') = 0$ for all m, m', m''; it is a 2-cocycle on BM_{23}, and $(m_1, \ldots, m_4) \mapsto$

$u(m_1,m_2)u(m_3,m_4)$ represents the square of its class. To evaluate these cocycles on generators of H_2 and H_4 we map the resolution above to the normalized bar resolution $B_\bullet$: here B_n is the free left R-module on the symbols $[m_1|\dots|m_n]$ with all $m_i \neq 1$, the differential is

$$\partial(z[m_1|\dots|m_n]) = zm_1[m_2|\dots|m_n] + \sum_{0<i<n} (-1)^i z[\dots|m_i m_{i+1}|\dots] + (-1)^n z[m_1|\dots|m_{n-1}],$$

and $\mathbb{Z}\otimes_R B_\bullet$ is the cellular chain complex of BM_{23}. Let s be the additive map with $s(z[m_1|\dots|m_n]) = [z|m_1|\dots|m_n]$ for $z \in M_{23}$, $z \neq 1$, and $s(1[m_1|\dots|m_n]) = 0$; then $\partial s + s\partial = \mathrm{id}$ on B_n for $n \geq 1$. Put $F_0(x_{00}) = x_{00}[\,]$ and, for a generating idempotent ϵ of a summand of the term of degree $n \geq 1$, $F_n(\epsilon) = \epsilon \cdot s\big(F_{n-1}(d_n\epsilon)\big)$, extended R-linearly. By induction F is a chain map over the identity of $\mathbb{Z}$, so it induces an isomorphism on homology after tensoring with $\mathbb{Z}$. Let z_2 and z_4 be the images in $\mathbb{Z}\otimes_R B_\bullet$ of $F_2(x_{00},0)$ and $F_4(x_{00})$; they are cycles representing generators of H_2 and H_4. Evaluating gives

$$\langle u, z_2\rangle = -1, \qquad \langle u \smile u, z_4\rangle = 1.$$

So the class of u generates $H^2(BM_{23};\mathbb{Z})$ and its square generates H^4. Hence the Hopf invariant of h is ± 1. Since $\pi_3(S^2)$ is infinite cyclic, generated by the Hopf map [9, Example 4.51], whose Hopf invariant is 1, and the Hopf invariant is a homomorphism [9, §4.B, Proposition 4B.1], h is homotopic to the Hopf map or to its negative, and $S^2 \cup_h e^4 \simeq \mathbb{CP}^2$.

The verifications called finite above (closure, the identities in R, the ranks and elementary divisors, the cocycle identity, the chain map property of F and the two evaluations) were done by a program in exact integer arithmetic; see Section 6. □

The monoids of type $\mathbb{CP}^2$ constructed in [3] have 1795 and 122 elements. The run that found M_{23} also reports that no monoid with fewer than 23 elements has a classifying space of this type; that statement rests on exhaustive computations which are not reproduced here, and it is not asserted in this paper.

6 Provenance and checks

This paper is AI-produced. No human has proved, written or checked any part of it.

What is whose. Theorem 1.3, Conjecture 7.1, the map j_e through the Karoubi envelope and the surjectivity of η are Ben-Bassat's [3]; Theorem B of [12] is Miller's; (K3) is Atiyah's. The monoid M_5, the statement on its classifying space and the suggestion that became Theorem 1.3 are Fiedorowicz's [5], and so is the first published proof of that statement [6]; Lemma 3.1 (a) is McDuff's [11]. The statement that a finite monoid whose classifying space is homotopy equivalent to a finite-dimensional complex has trivial universal group, with its proof by Lemmas 3.1 and 3.2 and Miller's theorem, and the resulting failure of Conjecture 7.1, were found by a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode) on 10 November 2026 (run CATHOT, statements G.4 and G.6), in answer to a research prompt on Grothendieck's *Les Dérivateurs*. Theorems 2.1, 1.1 and 1.2 and Corollaries 2.2, 3.3 and 1.4, and Corollary 3.4 with the conclusion $U(M) = 1$, were proved by Claude Fable 5.2 (Anthropic) at effort max on 12 November 2026, while checking that run. The present form of Corollary 3.4 with its derivation from [7], the last sentence of Remark 2.3 (a) and the question in Remark 3.5 come from the search of the literature made for this paper on 25

November 2026 by a second Claude Fable 5.2 at effort max; the writer checked the derivation and wrote the proof. The observation on the map j_e at the end of Remark 3.5 and the remark that the case $a = g$ of [3, Example 3.4] has trivial universal group and unbounded homology come from the check of 25 November 2026 described below. Lemma 3.1 (c) is known. The author of [3] was informed of the counterexample on 12 November 2026. Proposition 5.2 (c), with the resolution used for (b), is due to the run S2COUNT of GPT-6 Astra (ChatGPT Pro mode), completed on 16 November 2026, which treats a family of monoids containing M_6; (b) is Sweeney's. The monoid M_{23}, its resolution, the cocycle u and Proposition 5.3 are due to the run FINMON of GPT-6 Astra (ChatGPT Pro mode), 12–17 November 2026. That run was given Theorems 1.1 and 1.2 as statements to prove or refute, with the route through K-theory suggested, and reports proofs of them and the observations of Remarks 2.3 (b) and 5.1; it also treats finite categories with several objects, which are outside this paper.

The text. The text was written by Claude Fable 5.2 at effort max on 25 November 2026. The proof of Theorem 2.1 is given here in a form that uses only finite skeleta and that is local at one prime throughout; the proofs of Section 5 follow those of the two runs, shortened.

Checks. The statement of the run CATHOT and its proof were checked in full by Claude Fable 5.2 at effort max on 12 November 2026, against the texts of [3], [12] (with its correction) and [1]. Theorems 1.1 and 1.2, in the form of 12 November, were derived again by a second Claude Fable 5.2 at effort max and were the object of a targeted check by Claude Opus 5.5 at effort high; the run FINMON, of another model family, reports proofs along the suggested route, which the writer has not read. The present form of the proof of Theorem 2.1, Lemma 3.2, the proofs of Theorems 1.1 and 1.2 and of Corollary 3.4, and the two examples were checked on 25 November 2026 by Claude Opus 5.5 at effort max, called the checker below, which recomputed the examples with a program of its own, independent of the writer's program and of the programs of the runs. The checker found no error in the proofs; at its indication the last clause of Corollary 1.5 was corrected and several citations were made more precise. The checker is a model of the same family as the writer. A model of another family, GPT-6 Astra (OpenAI, ChatGPT Pro mode), checked the results in the run FINMON_CP2 of 25 November 2026 (its checkpoint of 06:12 UTC). It was given the statements and proofs of 12 November and recorded which of its derivations came before its reading of them. Before reading them it derived, by routes of its own, Theorem 1.1 for integral homology, Theorem 1.2, Corollary 2.2 (b), the group eMe with the map j_e of Lemma 3.2, and Corollary 1.4 (b) from Theorem 1.3: the rank-zero classes in K-theory are nilpotent by an argument with the Postnikov sections of BU which needs no finiteness of the complex, and the finite model in Theorem 1.2 comes from [9, Proposition 4C.1]. It then checked the statements for $\mathbb{F}_p$ and $\mathbb{Z}_{(p)}$ in Theorem 1.1 and Corollary 2.2 (a) against the proofs it was given, and Corollary 1.5, and found no fault. It has not read the present text. The paragraph on the monoid M_5 in Section 5 and the construction of the classifying space in Remark 2.3 (b) were added after that check; the checker checked both the same day, recomputing the statements on M_5 with its own program, and a second program of the writer confirms them as well. For Section 5 the writer ran a program of its own, independent of the programs of the runs, in exact integer arithmetic: for M_6 it confirms the resolution in degrees 0 to 3 and, for $2 \le n \le 5$, that z_n is a cycle, that f_n is a cocycle and that $f_n(z_n) = 1$; for M_{23} it confirms every finite verification named in the proof of Proposition 5.3, with the values -1 and 1 of the two evaluations. The proof of Theorem C

of [3], an unrefereed preprint, was read in full by the writer on 12 November 2026, without finding a gap. Nothing is formalized.

Citations. The writer has opened [3] (source of the first version), [12] (pp. 39–41 and 84–85, and the correction), [1] (§§6–8), [17] (§§1, 2, 5), [13] (introduction), [4, Theorem 6.2] and the statement of [9, Proposition 4C.1]. The reader who made the search of the literature has opened [16] in full and [15], [10], [7] and [8] at the places cited. The checker opened every reference at the places cited, among them the places of [1, §2], [9] and [14] that the writer had cited from memory, with the exception of [2], [5] and [11]; the locator in [2] is still given from memory. The session that coordinated this work, a second Claude Opus 5.5 at effort max, opened [5] at pp. 321–324 and [11] in full on 25 November 2026: the statements cited from them are as given here, and Theorems 1.1, 1.2 and 2.1 are neither on these pages of [5] nor in [11]. The search for earlier occurrences of these theorems, with its queries and with what could not be opened, is part of the records. The proof of Corollary 3.4 has been read by the writer, by the reader who proposed it and by the checker.

Records. This paper is accompanied in its workbench by its records:

- `CHECK_EVENTS.tsv`: every check, with model, effort and date;
- `PROVENANCE.md`: every contributor and the state of every citation;
- `CHECK_D2_INDEPENDENT.md`: the report of the check of 25 November 2026;
- `d2_examples_check.py` and `d2_m5_check.py`, in the folder `checks`: the programs of the writer's checks of Section 5, with their outputs.

The workbenches are reached from
`https://github.com/KokunoYumeto/mathematics-workbenches`.

References

[1] M. F. Atiyah, Characters and cohomology of finite groups, Publ. Math. Inst. Hautes Études Sci. 9 (1961), 23–64.

[2] M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, in *Differential Geometry*, Proc. Sympos. Pure Math. 3, Amer. Math. Soc., 1961, 7–38.

[3] O. Ben-Bassat, On a conjecture of Fiedorowicz, arXiv:2610.38760v1, 28 October 2026.

[4] I. Carcacía-Campos, How many points does it take to see the homotopy of a space? Minimal acyclic categorical models, arXiv:2611.09718v1, 6 November 2026.

[5] Z. Fiedorowicz, Classifying spaces of topological monoids and categories, Amer. J. Math. 106 (1984), no. 2, 301–350.

[6] Z. Fiedorowicz, A counterexample to a group completion conjecture of J. C. Moore, Algebr. Geom. Topol. 2 (2002), 33–35; arXiv:math/0202260.

[7] R. Gray and S. J. Pride, Homological finiteness properties of monoids, their ideals and maximal subgroups, J. Pure Appl. Algebra 215 (2011), no. 12, 3005–3024; arXiv:1003.3227v1.

[8] R. D. Gray and B. Steinberg, Topological finiteness properties of monoids. Part 1: Foundations, Algebr. Geom. Topol. 22 (2022), no. 7, 3083–3170; arXiv:1706.04387v1, whose numbering is used here.

[9] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[10] A. Martinian and B. Steinberg, Realizing wedges of Moore spaces as classifying spaces of finite semigroups, Semigroup Forum (2026), `https://doi.org/10.1007/s00233-026-10648-0`; arXiv:2511.12657v1, whose numbering is used here.

[11] D. McDuff, On the classifying spaces of discrete monoids, Topology 18 (1979), 313–320.

[12] H. Miller, The Sullivan conjecture on maps from classifying spaces, Ann. of Math. (2) 120 (1984), 39–87; Correction, Ann. of Math. (2) 121 (1985), 605–609.

[13] M. Palmer and X. Wu, Embedding groups into acyclic groups, arXiv:2510.16879v1, 19 October 2025.

[14] D. Quillen, Higher algebraic K-theory I, in *Algebraic K-theory I*, Lecture Notes in Math. 341, Springer, 1973, 85–147.

[15] B. Steinberg, The homology of completely simple semigroups, Semigroup Forum 110 (2025), 745–761; arXiv:2405.06594v1, whose numbering is used here.

[16] R. G. Swan, The nontriviality of the restriction map in the cohomology of groups, Proc. Amer. Math. Soc. 11 (1960), 885–887.

[17] D. Sweeney, Homotopy types of small semigroups, arXiv:2502.05276v2, 13 June 2026.