Exactness of colimits and amenability are independent for categories of finite substructures*

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Abstract

Let M be a countable homogeneous structure, G its automorphism group and $\mathcal{A}$ the category of its finite substructures and embeddings. The colimit functor on diagrams of R-modules indexed by $\mathcal{A}$ is exact if and only if the functor of coinvariants $N \mapsto N_G$ is exact on the $R[G]$-modules in which every element has an open stabilizer; this is a property of the topos of continuous G-sets. We compare it with amenability of G, the existence of invariant probability measures on compact spaces on which G acts, and show that neither implies the other. For the generic partially ordered set with an independent linear order, colimits are exact over every ring, while the group is not amenable by a result of Kechris and Sokić. For the vector space of countable dimension over the field with two elements, with a generic linear order on its non-zero vectors, the group is amenable and colimits are exact over the rational numbers, but they are exact over no ring in which 2 is not invertible. The first statement rests on a transfer theorem: if a subgroup H has exact coinvariants and finitely many orbits on each G/G_B, then exactness for G over a field k is equivalent to the existence of a G-invariant, finitely additive, k-valued function of total mass 1 on the expansions that H defines; positivity is not required. For posets this function has integer values, given by Euler characteristics of cells of order cones, which are values of the measure of Harman and Snowden for the automorphism group of the ordered real line. A consequence is that the separation property proved by Evans for Ramsey structures holds for a structure that is not a Ramsey structure but is obtained from one by forgetting a relation.

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1 Introduction and results

The setting. We use the setting of [3, §3.1]. Let M be a non-empty countable set and G a group of permutations of M. For $B \subseteq M$ let G_B be the pointwise stabilizer of B. A subgroup of G is *open* if it contains G_B for some finite B; this is the topology of pointwise convergence on G. Let $\mathcal{K}$ be a set of finite subsets of M such that

(K1) $\varnothing \in \mathcal{K}$; $gB \in \mathcal{K}$ for $B \in \mathcal{K}$ and $g \in G$; every finite subset of M is contained in a member of $\mathcal{K}$.

The category $\mathcal{A} = \mathcal{A}(M, G, \mathcal{K})$ has the non-empty members of $\mathcal{K}$ as objects, and its arrows $B \to C$ are the maps $g|_B$ with $g \in G$ and $gB \subseteq C$. If M is the Fraïssé limit of a class of finite structures, $G = \mathrm{Aut}(M)$ and $\mathcal{K}$ is the set of all finite subsets, then every embedding between finite substructures of M extends to an automorphism, so $\mathcal{A}$ is the category of

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non-empty finite substructures of M and embeddings, and it is equivalent to the category of all non-empty structures of the class with embeddings. In Section 5 the members of $\mathcal{K}$ are the finite subspaces of a vector space, without the zero vector.

All rings are commutative with 1. For a ring R, an $R[G]$-module N is *smooth* if every element of N is fixed by an open subgroup. Its *coinvariants* are $N_G = N/I_G N$, where $I_G N$ is the submodule generated by the elements $gn - n$; the functor $N \mapsto N_G$ is right exact.

Definition 1.1. The category $\mathcal{A}$ is *exact over* R if the functor $\operatorname{colim}_{\mathcal{A}}$ from functors $\mathcal{A} \to R$-Mod to R-modules is exact.

Colimits of modules are computed on the underlying abelian groups, so a category that is exact over $\mathbb{Z}$ is exact over every ring; and exactness is invariant under equivalence of categories.

The group G is *amenable* if every continuous action of G on a non-empty compact Hausdorff space has an invariant regular Borel probability measure, and *extremely amenable* if every such action has a fixed point.

Theorem 1.2. *Assume* (K1). *The category $\mathcal{A}(M, G, \mathcal{K})$ is exact over R if and only if the functor of coinvariants preserves monomorphisms of smooth $R[G]$-modules. Equivalently: in the topos of continuous G-sets, the left adjoint of the functor that gives an R-module the trivial action of G is exact on R-module objects. In particular exactness over R depends only on this topos.*

The first sentence is due to the run described in Section 6; the functors in its proof are those of Artin's theorem as written out for modules by Di, Li, Liang and Xu [4, Theorem 5.1]. Section 2 also proves a criterion in terms of the category alone: $\mathcal{A}$ is exact over R if and only if every finite family of parallel arrows is equalized by a combination of arrows with coefficients in R of sum 1 (Proposition 2.4); this is the criterion of Isbell and Mitchell [9] in the present setting.

Theorem 1.3. *Assume* (K1). *Let H be a subgroup of G with finitely many orbits on G/G_B for every $B \in \mathcal{K}$, and let k be a ring such that coinvariants preserve monomorphisms of smooth $k[H]$-modules (smooth for H: every element is fixed by $H \cap G_B$ for some finite B). For $B \in \mathcal{K}$ let $E(B) = G_B \backslash G / H$.*

(a) *If there are functions $\mu_B \colon E(B) \to k$, $B \in \mathcal{K}$, of total sum 1, compatible with the restriction maps $E(C) \to E(B)$ for $B \subseteq C$ and with the action of G (a* normalized invariant measure*, Definition 3.5), then $\mathcal{A}(M, G, \mathcal{K})$ is exact over every k-algebra.*

(b) *If k is a field and $\mathcal{A}(M, G, \mathcal{K})$ is exact over k, there is such a family.*

(c) *If G is amenable and $k \supseteq \mathbb{Q}$, there is such a family; so $\mathcal{A}(M, G, \mathcal{K})$ is then exact over every $\mathbb{Q}$-algebra.*

When H is the automorphism group of an expansion M^* of M, the set $E(B)$ is the set of expansions of B that occur in M^*, and a normalized invariant measure with values in $[0, 1]$ is what is called a consistent random expansion; for excellent pairs of classes its existence is the starting point of Zucker's criterion for amenability [14, §3 and Theorem 3.1]. Theorem 1.3 says that exactness is the same linear condition without positivity (Remark 3.6).

Theorem 1.4. *Let $\mathcal{P}$ be the class of finite structures $(P, \prec, <)$, where $\prec$ is a strict partial order and $<$ a linear order on P, let $M_{\mathcal{P}}$ be its Fraïssé limit and $G = \operatorname{Aut}(M_{\mathcal{P}})$. The category of non-empty members of $\mathcal{P}$ and embeddings is exact over every ring. The group G is not amenable (Kechris and Sokić [11, Remark 3.2]).*

Theorem 1.5. *Let $\mathcal{V}$ be the category of finite-dimensional non-zero vector spaces over $\mathbb{F}_2$ with a linear order on the set of their non-zero vectors, and of injective linear maps that preserve the order. Let $(V_\infty, <)$ be its Fraïssé limit and G the group of linear automorphisms of V_∞ that preserve the order.*

(a) *G is amenable.*

(b) *$\mathcal{V}$ is exact over every $\mathbb{Q}$-algebra.*

(c) *$\mathcal{V}$ is not exact over any ring in which 2 is not invertible; in particular not over $\mathbb{F}_2$ and not over $\mathbb{Z}$.*

So exactness over $\mathbb{Z}$ does not imply amenability, and amenability does not imply exactness over $\mathbb{Z}$ or over a field of characteristic 2. Extreme amenability does imply exactness over $\mathbb{Z}$: this is Lemma 3.7 (i) of Evans [5], in dual form (Remark 3.3).

The second statement alone has an elementary witness with non-rigid objects. For the category of finite non-empty sets and injections the group is the symmetric group of a countably infinite set. It is amenable by Lemma 3.4: the uniform distribution ν on the bijections of Q satisfies $\nu a = \nu a'$ for all arrows $a, a' \colon P \to Q$. The average of the bijections of Q is an affine equalizer over $\mathbb{Q}$ of all arrows $P \to Q$, and an affine equalizer of the automorphisms of an n-element set has augmentation in $n!R$; so this category is exact over R exactly when R is a $\mathbb{Q}$-algebra (Proposition 2.4). In Theorem 1.5 every object is rigid and the category is exact over $\mathbb{Q}$. We do not know whether $\mathcal{V}$ is exact over a ring in which 2 is invertible and which is not a $\mathbb{Q}$-algebra. Amenability does not imply exactness over $\mathbb{Q}$ in general: for the group of translations of $M = \mathbb{Z}$, with $\mathcal{K}$ the set of finite subsets, the group is discrete and amenable, the objects are rigid, and the inclusion of $\{0\}$ in $\{0, 1\}$ and the arrow $0 \mapsto 1$ have no affine equalizer over any non-zero ring, because the coefficients of an equalizer would form a finitely supported function on $\mathbb{Z}$ that is invariant under translation. By Theorem 1.3 (c) amenability does imply exactness over $\mathbb{Q}$ when G has a subgroup H as in that theorem.

Corollary 1.6. *Let $M = M_\mathcal{P}$, $G = \mathrm{Aut}(M)$, let W be a set with an action of G in which every point has an open stabilizer, R a ring, Y an $R[G]$-submodule of the free module $R[W]$, and $x \in R[W]$. Let U be an open subgroup of G fixing x, and $\Omega_U \colon R[W] \to R[U\backslash W]$ the map that sends a point to its U-orbit. Then $x \in Y$ if and only if $\Omega_U(x) \in \Omega_U(Y)$. If R is a field and $x \notin Y$, there is a U-invariant function $f \colon W \to R$ whose linear extension vanishes on Y and not on x.*

For ω-categorical Ramsey structures, W a sort of M^{eq} and U the pointwise stabilizer of the support of x, these are Theorem 1.4 and Corollary 1.5 of [5]. In Remarks 2.12 of [5] Evans asks whether the separation property holds for reducts of Ramsey structures. The structure $M_\mathcal{P}$ is a reduct of a Ramsey structure and is not one itself (its group is not amenable); Corollary 1.6 gives the property for it over every field, and Theorem 1.3 (c) gives it in characteristic 0 for every reduct with an amenable automorphism group (Remark 4.3).

What is known and what is new. Known: the non-amenability in Theorem 1.4 [11]; the implication from extreme amenability to exactness [5]; Artin's dictionary between continuous G-sets and sheaves on orbit categories, in the form of [4] and, for Fraïssé classes, of Caramello [2, Theorem 2.9, Proposition 2.10]; the criterion of Isbell and Mitchell [9]; the Euler-characteristic measure of Harman and Snowden [6], of which the weights of Lemma 4.1 are values (Remark 4.2). New, to our knowledge after the search described in Section 6: Theorems 1.3, 1.5, the exactness in Theorem 1.4 and Corollary 1.6. The theory of [6] does

not give these statements: its exactness results concern modules over a completed group algebra, and smoothness of a $k[G]$-module in the sense above is weaker than smoothness over that algebra [6, Remark 10.10]; and by Snowden [12, Proposition 8.2] the class of finite posets has no regular measure in their sense.

2 Coinvariants, the topos and affine equalizers

Throughout, $(M, G, \mathcal{K})$ satisfies (K1) and $\mathcal{A} = \mathcal{A}(M, G, \mathcal{K})$. By (K1) the non-empty members of $\mathcal{K}$ form a directed set under inclusion, inclusions are arrows of $\mathcal{A}$, and every arrow $f\colon B \to C$ factors as $B \xrightarrow{g|_B} gB \subseteq C$ for any $g \in G$ with $g|_B = f$.

Proof of Theorem 1.2. For $V\colon \mathcal{A} \to R$-Mod let $S(V) = \operatorname{colim}_B V(B)$, the colimit over the directed set of non-empty members of $\mathcal{K}$ and inclusions. Write $[B, v]$ for the class of $v \in V(B)$. The rule $g[B, v] = [gB, V(g|_B)v]$ is well defined, because $g|_C$ restricts to $g|_B$ for $B \subseteq C$, and it makes $S(V)$ an $R[G]$-module in which $[B, v]$ is fixed by G_B; so $S(V)$ is smooth. The functor S is exact, because directed colimits of modules are exact.

Claim: $\operatorname{colim}_{\mathcal{A}} V \cong S(V)_G$, *naturally in* V. The module $\operatorname{colim}_{\mathcal{A}} V$ is the quotient of $\bigoplus_B V(B)$ by the relations $v \sim V(f)v$ for arrows f. By the factorization above these relations are generated by those for inclusions, whose quotient is $S(V)$, and those for the bijections $g|_B\colon B \to gB$, which identify $[B, v]$ with $g[B, v]$.

For a smooth $R[G]$-module N define $T(N)\colon \mathcal{A} \to R$-Mod by $T(N)(B) = N^{G_B}$ and $T(N)(g|_B)n = gn$. This is well defined: another element of G with the same restriction to B is gu with $u \in G_B$; and gn is fixed by G_C when $gB \subseteq C$, since $g^{-1}G_C g = G_{g^{-1}C} \subseteq G_B$. Since N is smooth, $S(T(N)) = \bigcup_B N^{G_B} = N$ as $R[G]$-modules, and a monomorphism $N' \to N$ gives a monomorphism $T(N') \to T(N)$.

If coinvariants preserve monomorphisms of smooth modules, then $\operatorname{colim}_{\mathcal{A}} = (-)_G \circ S$ preserves monomorphisms, and it is right exact; so $\mathcal{A}$ is exact over R. Conversely, if $\mathcal{A}$ is exact over R and $N' \subseteq N$ are smooth, then $N'_G = \operatorname{colim}_{\mathcal{A}} T(N') \to \operatorname{colim}_{\mathcal{A}} T(N) = N_G$ is injective.

A continuous G-set is a set with an action of G in which every point has an open stabilizer, so an R-module object of the topos of continuous G-sets is a smooth $R[G]$-module. The functor γ^* from sets to this topos gives a set the trivial action, and for a smooth N and an R-module B we have $\operatorname{Hom}_{R[G]}(N, \gamma^* B) = \operatorname{Hom}_R(N_G, B)$. So the left adjoint of γ^* on R-module objects is $N \mapsto N_G$. An equivalence of toposes commutes with γ^* up to isomorphism, which is unique, and induces an equivalence of the categories of R-module objects; hence the last assertion. □

Remark 2.1. Under the equivalence of Artin between continuous G-sets and sheaves for the atomic topology on an orbit category of G, the functor T is the inclusion of sheaves into presheaves and S is the associated sheaf (when the set of fixed points of G_B in M is B for all $B \in \mathcal{K}$, as in the examples of this paper, so that $\mathcal{A}$ is the opposite of the orbit category on the G/G_B; in general S is exact and left adjoint to the fully faithful T) [4, Theorem 5.1 and the equivalence displayed after it]. When M is the Fraïssé limit of the finitely presentable models of a theory of presheaf type, the topos of continuous $\operatorname{Aut}(M)$-sets is the classifying topos of the theory of homogeneous models [2, Theorem 2.9, Proposition 2.10]. By Theorem 1.2, exactness of colimits over the category of finite models is then an invariant of that theory up to Morita equivalence.

Lemma 2.2. *If coinvariants preserve monomorphisms of smooth $R[G]$-modules, the same holds for every open subgroup U of G. Then for every smooth $R[U]$-module N the map $N^U \to N_U$ is injective.*

Proof. For a smooth $R[U]$-module N the induced module $R[G] \otimes_{R[U]} N$ is smooth, because $g \otimes n$ is fixed by $gU_n g^{-1}$ and the stabilizer U_n of n is open in G. Induction is exact, since $R[G]$ is free as a right $R[U]$-module, and $(R[G] \otimes_{R[U]} N)_G = R \otimes_{R[U]} N = N_U$. So a monomorphism $N' \to N$ of smooth $R[U]$-modules gives a monomorphism $N'_U \to N_U$. For $n \in N^U$ the submodule Rn has trivial action, so $Rn = (Rn)_U \to N_U$ is injective. □

For objects P, C of $\mathcal{A}$ write $\mathcal{A}(P, C)$ for the set of arrows, which is finite, and $R[\mathcal{A}(P, C)]$ for the free module on it. An arrow $a\colon P \to Q$ induces $t \mapsto ta$ from $R[\mathcal{A}(Q, C)]$ to $R[\mathcal{A}(P, C)]$, and aug denotes the sum of the coefficients.

Definition 2.3. An *affine equalizer over* R of arrows $a_0, \dots, a_k\colon P \to Q$ is an element $t \in R[\mathcal{A}(Q, C)]$, for some object C, with $\operatorname{aug}(t) = 1$ and $ta_i = ta_0$ for all i.

Proposition 2.4. *The category $\mathcal{A}$ is exact over R if and only if every finite family of arrows $a_0, \dots, a_k\colon P \to Q$ of $\mathcal{A}$ has an affine equalizer over R.*

For arbitrary small categories with common successors this is the criterion of Isbell and Mitchell [9, Theorem 3.1, Proposition 3.3 and the remark after it, for a commutative ring in place of $\mathbb{Z}$]. We give a direct proof in the present setting.

Proof. For an object P let E_P be the free R-module on the set of maps $g|_P$, $g \in G$, with G acting by composition; it is the permutation module on G/G_P, it is smooth, and $(E_P)_G = R$ by the augmentation.

Suppose $\mathcal{A}$ is exact over R and let $a_0, \dots, a_k\colon P \to Q$. Let $\phi\colon E_Q \to E_P^k$ have the components $t \mapsto ta_i - ta_0$; it is G-linear. Put $I = \phi(E_Q)$. Coinvariants are right exact, so $I_G = R/\operatorname{aug}(\ker \phi)$. The composite $E_Q \to I \to E_P^k \to (E_P^k)_G = R^k$ is zero, because $\operatorname{aug}(ta_i) = \operatorname{aug}(ta_0)$; so $I_G \to (E_P^k)_G$ is zero. By Theorem 1.2 it is injective. Hence $I_G = 0$, and there is $t \in \ker \phi$ with $\operatorname{aug}(t) = 1$. Its support is a finite set of maps $Q \to M$, whose images lie in some $C \in \mathcal{K}$; so t is an affine equalizer.

Conversely suppose every finite family has an affine equalizer. Let $N' \subseteq N$ be smooth and $n' \in N' \cap I_G N$, say $n' = \sum_{i=1}^k (g_i - 1)n_i$ with $g_i \in G$, $n_i \in N$. Choose $P \in \mathcal{K}$ such that G_P fixes $n_1, \dots, n_k$, and $Q \in \mathcal{K}$ containing P and all $g_i P$. Let $a_0\colon P \to Q$ be the inclusion and $a_i = g_i|_P$, and let $t = \sum_h t_h[h] \in R[\mathcal{A}(Q, C)]$ be an affine equalizer. For each h choose $\hat{h} \in G$ with $\hat{h}|_Q = h$. For $n \in N^{G_P}$ the rule $\Phi_n([g|_P]) = gn$ defines an R-linear map $\Phi_n\colon R[\mathcal{A}(P, C)] \to N$, and $\Phi_n([ha_i]) = \hat{h} g_i n$, because $\hat{h} g_i$ restricts to ha_i on P. Therefore

$$\sum_h t_h \hat{h} g_i n_i = \Phi_{n_i}(ta_i) = \Phi_{n_i}(ta_0) = \sum_h t_h \hat{h} n_i,$$

and summing over i gives $\sum_h t_h \hat{h} n' = 0$. As $\sum_h t_h = 1$,

$$n' = \sum_h t_h n' - \sum_h t_h \hat{h} n' = -\sum_h t_h (\hat{h} - 1) n' \in I_G N'.$$

So $N'_G \to N_G$ is injective, and $\mathcal{A}$ is exact over R by Theorem 1.2. □

3 Ramsey categories, amenability and the transfer

3.1 Ramsey categories

Definition 3.1. A category $\mathcal{C}$ with finite sets of arrows has the *Ramsey property* if for all objects P, Q and every $r \geq 1$ there is an object C such that for every map $\chi\colon \mathcal{C}(P, C) \to \{1, \dots, r\}$ there is $g \in \mathcal{C}(Q, C)$ with χ constant on $\{ga : a \in \mathcal{C}(P, Q)\}$. It has property (Re) if for all arrows $a_0, \dots, a_k\colon P \to Q$ and every $\varepsilon > 0$ there are an object C and a probability vector ν on $\mathcal{C}(Q, C)$ with $\sum_{i=1}^k \|\nu a_i - \nu a_0\|_1 < \varepsilon$.

Definition 2.3 makes sense in any such category.

Lemma 3.2. *In a category with finite sets of arrows and the Ramsey property, every finite family $a_0, \dots, a_k \colon P \to Q$ has an affine equalizer over $\mathbb{Z}$, and property* (Re) *holds.*

Proof. Affine equalizers. Let $N = 3k + 1$ and let C be as in Definition 3.1 for P, Q and $r = N^k$. Let $F = \mathbb{Z}[\mathcal{C}(P, C)]^k \oplus \mathbb{Z}$ and let $L \subseteq F$ be the image of $t \mapsto ((ta_i - ta_0)_i, \operatorname{aug} t)$ on $\mathbb{Z}[\mathcal{C}(Q, C)]$. An affine equalizer with target C exists if and only if $(0, 1) \in L$. Suppose not, and let $x \neq 0$ be the class of $(0, 1)$ in F/L. There is a homomorphism $\psi \colon F/L \to \mathbb{R}/\mathbb{Z}$ such that $c = \psi(x)$ has distance at least $1/3$ from 0: on the cyclic group generated by x send x to $1/2$ if x has infinite order and to $\lfloor m/2 \rfloor / m$ if x has order $m \geq 2$, and extend to F/L, which is possible because $\mathbb{R}/\mathbb{Z}$ is divisible. The homomorphism ψ is given by functions $\theta_1, \dots, \theta_k \colon \mathcal{C}(P, C) \to \mathbb{R}/\mathbb{Z}$ and the value c on $(0, 1)$, and it vanishes on L:

$$\sum_{i=1}^{k} (\theta_i(ga_i) - \theta_i(ga_0)) + c = 0 \qquad \text{for every } g \in \mathcal{C}(Q, C).$$

Divide $\mathbb{R}/\mathbb{Z}$ into N arcs of length $1/N$ and colour $e \in \mathcal{C}(P, C)$ by the k-tuple of the arcs containing $\theta_1(e), \dots, \theta_k(e)$. For a g as in the Ramsey property each difference $\theta_i(ga_i) - \theta_i(ga_0)$ has distance less than $1/N$ from 0, so c has distance less than $k/N < 1/3$ from 0, a contradiction.

Property (Re). Let $N > 2k/\varepsilon$ and let C be as in Definition 3.1 for $r = N^k$. Let Δ be the simplex of probability vectors on $\mathcal{C}(Q, C)$ and $\Lambda = ([-1, 1]^{\mathcal{C}(P,C)})^k$. The function $\Phi(\nu, \lambda) = \sum_i \langle \lambda_i, \nu a_i - \nu a_0 \rangle$ is bilinear on the compact convex sets Δ and Λ, so by the minimax theorem [13] (applied to the matrix game whose columns are the vertices of the cube Λ; a linear function on Λ takes its maximum at a vertex) $\min_\nu \max_\lambda \Phi = \max_\lambda \min_\nu \Phi$. Here $\max_\lambda \Phi(\nu, \lambda) = \sum_i \|\nu a_i - \nu a_0\|_1$, and $\min_\nu \Phi(\nu, \lambda)$ is the minimum over $g \in \mathcal{C}(Q, C)$ of $\sum_i (\lambda_i(ga_i) - \lambda_i(ga_0))$. Given λ, colour $e \in \mathcal{C}(P, C)$ by the k-tuple of the intervals of length $2/N$ of $[-1, 1]$ containing $\lambda_i(e)$; a g as in the Ramsey property gives a sum of absolute value at most $2k/N < \varepsilon$. So the common value is less than ε. □

Remark 3.3. For the category of finite substructures of the Fraïssé limit M^* of a Fraïssé class of finite structures that have a linear order among their relations, the Ramsey property is equivalent to extreme amenability of $\operatorname{Aut}(M^*)$, by the theorem of Kechris, Pestov and Todorcevic [10, Theorem 4.8], which is stated for such order classes and for colourings of substructures; for ordered structures these are the embeddings; and for an extremely amenable group Evans proves that invariants are exact on compact continuous modules [5, Lemma 3.7 (i)], which by Pontryagin duality is the statement that coinvariants are exact on smooth $\mathbb{Z}[G]$-modules. With Proposition 2.4 this gives the first assertion of Lemma 3.2 for such categories. The finite proof above, which comes from the runs, is included because Section 5 applies the lemma to a category for which we want no limit object, and because it is short.

Lemma 3.4. *Assume* (K1). *If $\mathcal{A}(M, G, \mathcal{K})$ has property* (Re)*, then G is amenable.*

Proof. Let G act continuously on a non-empty compact Hausdorff space X. The invariant probability measures are the intersection, in the compact space of all regular Borel probability measures with the weak* topology, of the closed sets $\{\lambda : |\lambda(f \circ g) - \lambda(f)| \leq \eta\}$ for continuous $f \colon X \to [0, 1]$, $g \in G$ and $\eta > 0$. So it is enough to find, for continuous $f_1, \dots, f_r \colon X \to [0, 1]$, elements $g_1, \dots, g_s \in G$ and $\eta > 0$, a probability measure λ with $|\lambda(f_i \circ g_j) - \lambda(f_i)| \leq 3\eta$ for all i, j.

By continuity of the action and compactness of X there is $P \in \mathcal{K}$ with $|f_i(ux) - f_i(x)| < \eta$ for all $x \in X$, $u \in G_P$ and all i. Fix $x_0 \in X$. For each map $e = g|_P$, $g \in G$, choose one such g and put $c_i(e) = f_i(g^{-1}x_0)$; for any other g' with $g'|_P = e$ we have $g' = gu$ with $u \in G_P$, so $|f_i(g'^{-1}x_0) - c_i(e)| < \eta$. Let $Q \in \mathcal{K}$ contain P and all $g_j^{-1}P$, let $a_0 \colon P \to Q$ be the inclusion and $a_j = g_j^{-1}|_P$. By (Re) there are C and a probability vector ν on $\mathcal{A}(Q, C)$ with $\sum_j \|\nu a_j - \nu a_0\|_1 < \eta$. Choose $\hat{h} \in G$ extending each $h \in \mathcal{A}(Q, C)$ and let $\lambda = \sum_h \nu(h)\delta_{\hat{h}^{-1}x_0}$. Since $\hat{h}$ extends ha_0 and $\hat{h}g_j^{-1}$ extends ha_j,

$$|\lambda(f_i) - \textstyle\sum_h \nu(h)c_i(ha_0)| < \eta, \qquad |\lambda(f_i \circ g_j) - \textstyle\sum_h \nu(h)c_i(ha_j)| < \eta,$$

and $|\sum_h \nu(h)(c_i(ha_j) - c_i(ha_0))| = |\langle c_i, \nu a_j - \nu a_0\rangle| \le \|\nu a_j - \nu a_0\|_1 < \eta$. □

3.2 Measures on expansions and the transfer theorem

Let H be a subgroup of G and $E(B) = G_B\backslash G/H$ for $B \in \mathcal{K}$. For $B \subseteq C$ there is the surjection $E(C) \to E(B)$, $\zeta \mapsto \zeta|_B$, induced by $G_C \subseteq G_B$, and for $a \in G$ the bijection $a_* \colon E(B) \to E(aB)$, $G_BgH \mapsto G_{aB}agH$.

Definition 3.5. A *normalized invariant measure* with values in a ring k is a family of functions $\mu_B \colon E(B) \to k$, $B \in \mathcal{K}$, each $E(B)$ being finite, such that

(i) $\sum_{\xi \in E(B)} \mu_B(\xi) = 1$;

(ii) $\mu_B(\xi) = \sum_{\zeta|_B = \xi} \mu_C(\zeta)$ for $B \subseteq C$ and $\xi \in E(B)$;

(iii) $\mu_{aB}(a_*\xi) = \mu_B(\xi)$ for $a \in G$.

Equivalently, it is a G-invariant finitely additive function of total mass 1 on the closed and open subsets of the compact space $X = \varprojlim_B E(B)$, on which G acts continuously by $(ax)_{aB} = a_*(x_B)$.

Proof of Theorem 1.3. (a) Let R be a k-algebra and N a smooth $R[G]$-module. For $n \in N$ choose $B \in \mathcal{K}$ with G_B fixing n, and for $\xi = G_BgH \in E(B)$ let $f_n(\xi)$ be the class of $g^{-1}n$ in N_H. It is well defined: $(ugh)^{-1}n = h^{-1}g^{-1}n$ for $u \in G_B$, $h \in H$, and this has the same class as $g^{-1}n$. Put

$$s_N(n) = \sum_{\xi \in E(B)} \mu_B(\xi) f_n(\xi) \in N_H.$$

By (ii) this does not depend on B: for $B \subseteq C$ the function f_n on $E(C)$ is the composite of $E(C) \to E(B)$ with f_n on $E(B)$, and any two choices are contained in a third. So s_N is R-linear, and it is natural in N. For $a \in G$ the element an is fixed by G_{aB} and $f_{an}(a_*\xi) = f_n(\xi)$, so $s_N(an) = s_N(n)$ by (iii). Hence s_N induces $\bar{s}_N \colon N_G \to N_H$. If $q_N \colon N_H \to N_G$ is the quotient map, then $q_N(f_n(\xi))$ is the class of n, and $q_N\bar{s}_N = \mathrm{id}$ by (i). Now let $N' \subseteq N$ be smooth and let $x \in N'_G$ go to zero in N_G. Then $\bar{s}_{N'}(x)$ goes to zero in N_H; as $N'_H \to N_H$ is injective for $R[H]$-modules that are smooth for H (the hypothesis on H over k gives it over R, since coinvariants do not depend on the ring), $\bar{s}_{N'}(x) = 0$ and $x = q_{N'}\bar{s}_{N'}(x) = 0$. By Theorem 1.2, $\mathcal{A}$ is exact over R.

(b) Let F be the k-vector space of the functions $G/H \to k$ that are invariant under G_B for some $B \in \mathcal{K}$, with $(a\varphi)(x) = \varphi(a^{-1}x)$. It is a smooth $k[G]$-module containing the constants. By exactness and Theorem 1.2, $k = k_G \to F_G$ is injective, so there is a linear form ℓ on F_G with value 1 on the class of the constant function 1. Let $\mu_B(\xi)$ be the value of ℓ on the class of the indicator function 1_ξ of $\xi \subseteq G/H$. Then (i) and (ii) hold because $\sum_\xi 1_\xi = 1$ and $1_\xi = \sum_{\zeta|_B = \xi} 1_\zeta$, and (iii) because $a1_\xi = 1_{a_*\xi}$.

(c) The space X is non-empty and compact, so there is a G-invariant Borel probability measure λ on it. The measures of the sets $\{x : x_B = \xi\}$ satisfy (i)–(iii) with real values. Composing with a $\mathbb{Q}$-linear map $\mathbb{R} \to \mathbb{Q}$ that sends 1 to 1 gives a normalized invariant measure with values in $\mathbb{Q}$, hence in k, and by (a) the category $\mathcal{A}$ is exact over k. By Proposition 2.4 every finite family of arrows then has an affine equalizer t over k. If $\rho\colon k \to \mathbb{Q}$ is $\mathbb{Q}$-linear with $\rho(1) = 1$, applying ρ to the coefficients of t gives an affine equalizer over $\mathbb{Q}$, because the conditions $\mathrm{aug}(t) = 1$ and $ta_i = ta_0$ are linear with integer coefficients. So $\mathcal{A}$ is exact over $\mathbb{Q}$, hence over every $\mathbb{Q}$-algebra. □

Remark 3.6. Let $H = \mathrm{Aut}(M^*)$ for an expansion M^* of a homogeneous structure M, as in Section 4. Zucker [14, §3] considers the rational vector space on the set Ω of expansions of finite substructures and its subspace V spanned by the vectors "an expansion minus the sum of its extensions along an embedding". His Theorem 3.1 states, for an excellent pair, that $\mathrm{Aut}(M)$ is amenable if and only if every non-zero vector of V has coefficients of both signs. A normalized invariant measure with values in a field k is a linear form on $k\Omega$ that vanishes on V and takes the value 1 on the expansion of the empty structure. So, for H as in Theorem 1.3, the category is exact over k if and only if that vector does not lie in $V \otimes k$.

4 Posets with an independent linear order

4.1 Integer weights on linear extensions

Let $(P, \prec, <)$ be in $\mathcal{P}$; we call $<$ the reference order. A *cell* of P is a sequence $(B_1, \dots, B_b)$ of non-empty pairwise disjoint subsets with union P such that $y \prec z$ implies that the block of y comes before the block of z. A cell *sorts to* the linear order on P that lists $B_1, \dots, B_b$ in this order and each block in the reference order; this is a linear extension of $\prec$. For a linear extension π of $\prec$ put

$$w_P(\pi) = (-1)^{|P|} \sum_{\text{cells sorting to } \pi} (-1)^b,$$

where b is the number of blocks. The empty structure has one cell, with $b = 0$, and $w_\varnothing = 1$.

Lemma 4.1. *The integers $w_P(\pi)$ are invariant under isomorphisms of structures $(P, \prec, <)$, their sum over the linear extensions of P is 1, and for every subset $Q \subseteq P$, with the induced relations, and every linear extension τ of Q,*

$$\sum_{\pi|_Q = \tau} w_P(\pi) = w_Q(\tau).$$

Proof. Invariance is clear. By induction it is enough to prove the identity for $P = Q \cup \{x\}$; the case $Q = \varnothing$ of the general identity is the statement on the sum. Deleting x from a cell of P, and its block if it becomes empty, gives a cell of Q, and the sorting of the new cell is the restriction of the sorting of the old one. Fix a cell $(B_1, \dots, B_b)$ of Q. Let l be the largest index of a block containing an element below x in $\prec$, or 0, and u the smallest index of a block containing an element above x, or $b + 1$. Then $l < u$, since $y \prec x \prec z$ implies $y \prec z$. The cells of P over the given one are obtained by adding x to a block B_j with $l < j < u$, which gives $m = u - l - 1$ cells with b blocks, or by inserting $\{x\}$ as a new block after B_j for some $l \le j < u$, which gives $m + 1$ cells with $b + 1$ blocks. Their signed number is $m(-1)^b + (m+1)(-1)^{b+1} = (-1)^{b+1}$, and $(-1)^{|P|}(-1)^{b+1} = (-1)^{|Q|}(-1)^b$. Summing over the cells of Q that sort to τ gives the identity. □

Remark 4.2. The weights are values of a measure of Harman and Snowden. Let $O(P)$ be the set of maps $t\colon P \to \mathbb{R}$ with $t_y < t_z$ for $y \prec z$, an open convex cone in $\mathbb{R}^P$, and $O_\pi(P)$ the set of those t for which ordering P by the values of t, ties broken by the reference order, gives π. The cells above are the orbits of $\mathrm{Aut}(\mathbb{R}, <)$ on $O(P)$; a cell with b blocks is the set of strictly increasing b-tuples of reals. The compactly supported Euler characteristic is a $\mathbb{Z}$-valued measure μ for $\mathrm{Aut}(\mathbb{R}, <)$ in the sense of [6] (Example 3.5, Theorem 16.7 and Remark 16.8 there), with value $(-1)^b$ on such a cell. So

$$w_P(\pi) = \mu(O_\pi(P))/\mu(O(P)), \qquad \mu(O(P)) = (-1)^{|P|},$$

and the identity of Lemma 4.1 is the transitivity of push-forward [6, Proposition 4.8] for the projections $O(P) \to O(P \setminus \{x\})$, whose fibres are open intervals. The family (w_P) is not itself a measure in the sense of [6] on the class of expanded structures, because it is not multiplicative along embeddings: for the antichain $\{x, y\}$ with $x < y$ one finds $w(x, y) = 0$ and $w(y, x) = 1$, while after adding a point z with $x \prec z$, incomparable to y, and $y < z$, the three linear extensions have weights $w(x, y, z) = -1$, $w(x, z, y) = 1$, $w(y, x, z) = 1$.

4.2 Proof of Theorem 1.4

Let $\mathcal{P}^*$ be the class of finite structures $(P, \prec, <, \lhd)$ with $(P, \prec, <) \in \mathcal{P}$ and $\lhd$ a linear extension of $\prec$. Both $\mathcal{P}$ and $\mathcal{P}^*$ are Fraïssé classes [11, Remark 3.2], and $\mathcal{P}^*$ has the Ramsey property for embeddings (Bodirsky [1, Corollary 1.4 (3)]). Let $M^* = (M, \prec, <, \lhd_0)$ be the Fraïssé limit of $\mathcal{P}^*$. Every linear extension of a subposet of a finite poset extends to the poset, so $(M, \prec, <)$ has the extension property that characterizes the Fraïssé limit of $\mathcal{P}$; we take $M_\mathcal{P} = (M, \prec, <)$. Let $G = \mathrm{Aut}(M_\mathcal{P})$, $H = \mathrm{Aut}(M^*) \subseteq G$ and $\mathcal{K}$ the set of finite subsets of M.

The group H. The category $\mathcal{A}(M, H, \mathcal{K})$ is the category of non-empty finite substructures of M^* and embeddings; it has the Ramsey property because $\mathcal{P}^*$ has it and every member of $\mathcal{P}^*$ embeds in M^*. By Lemma 3.2 and Proposition 2.4 it is exact over $\mathbb{Z}$, so coinvariants preserve monomorphisms of smooth $R[H]$-modules for every ring R (Theorem 1.2).

The sets $E(B)$. For $B \in \mathcal{K}$ the map $gH \mapsto (g\lhd_0)|_B$ induces a bijection from $E(B) = G_B\backslash G/H$ to the set of linear extensions of $(B, \prec)$; here $g\lhd_0$ is the image of $\lhd_0$ under g. Indeed, if $(g\lhd_0)|_B = (g'\lhd_0)|_B$, then $g'^{-1}g$ restricts to an isomorphism of the substructures $g^{-1}B$ and $g'^{-1}B$ of M^*, which extends to some $h \in H$, and $g'hg^{-1} \in G_B$. And every linear extension π occurs: $(B, \prec, <, \pi)$ embeds in M^* by a map e, and by homogeneity of $M_\mathcal{P}$ there is $g \in G$ with $g^{-1}|_B = e$. In particular $E(B)$ is finite. Under these bijections the maps $E(C) \to E(B)$ and a_* are restriction and transport of linear orders.

Exactness. By Lemma 4.1 the functions $\mu_B = w_B$ form a normalized invariant measure with values in $\mathbb{Z}$. By Theorem 1.3 (a) the category $\mathcal{A}(M, G, \mathcal{K})$, which is equivalent to the category of non-empty members of $\mathcal{P}$, is exact over every ring.

Non-amenability is [11, Remark 3.2]: the class called $\mathcal{OP}$ there is $\mathcal{P}$. □

Proof of Corollary 1.6. By Theorems 1.4 and 1.2 and Lemma 2.2, $N^U \to N_U$ is injective for every smooth $R[U]$-module N. Apply this to $N = R[W]/Y$: the map Ω_U identifies $R[W]_U$ with $R[U\backslash W]$, so $N_U = R[U\backslash W]/\Omega_U(Y)$. If $\Omega_U(x) \in \Omega_U(Y)$, the class of x in N is U-invariant and goes to zero in N_U, hence is zero, and $x \in Y$. If R is a field and $x \notin Y$, take a linear form on $R[U\backslash W]$ that vanishes on $\Omega_U(Y)$ and not on $\Omega_U(x)$, and compose it with Ω_U. □

Remark 4.3. The proof uses only that the category is exact over R. So the conclusions of Corollary 1.6 hold over R whenever $\mathcal{A}(M, G, \mathcal{K})$ is exact over R; by Theorem 1.3 (c) and

Remark 3.3 this is the case over every field of characteristic 0 when M is a reduct of an ω-categorical Ramsey structure M^* and $\mathrm{Aut}(M)$ is amenable. Conversely, the first conclusion of Corollary 1.6, required only for $U = G$, implies that $\mathcal{A}(M, G, \mathcal{K})$ is exact over R. Let $N' \subseteq N$ be smooth and $n' \in N' \cap I_G N$. The class of n' in $N/I_G N'$ is G-invariant. Write $N/I_G N' = R[W]/Y$, with W a G-set with open stabilizers and that class the image of a point w of W fixed by G. The class goes to zero in $(N/I_G N')_G = N_G$, so $\Omega_G(w) \in \Omega_G(Y)$; hence $w \in Y$ and $n' \in I_G N'$. So by Theorem 1.5 (c) the membership test with $U = G$ fails for the ordered $\mathbb{F}_2$-space over every ring in which 2 is not invertible. This does not answer the question of [5, Remarks 2.12], which asks only for a function invariant under the pointwise stabilizer of some finite set: for a finite group acting faithfully that weaker property holds trivially, while coinvariants are not exact in the modular case.

5 Ordered vector spaces over the field with two elements

The objects of $\mathcal{V}$ are the pairs $(W, <)$ with W a finite-dimensional non-zero $\mathbb{F}_2$-vector space and $<$ a linear order on $W \setminus \{0\}$; the arrows are the injective linear maps that preserve the order. Two objects embed in a third, and two arrows $W_0 \to W_1$, $W_0 \to W_2$ can be completed to a commutative square in which $W_1 \cap W_2 = W_0$: take the push-out of vector spaces, an order on the union of W_1 and W_2 that extends both orders, and the remaining vectors last. There are countably many objects up to isomorphism. By Fraïssé's theorem for classes of finitely generated structures ([7, §7.1]; see also [10, Theorem 2.2]), applied to the language with addition and the order, there is a countable vector space V_∞ with a linear order on $M = V_\infty \setminus \{0\}$ such that every object of $\mathcal{V}$ embeds in $(V_\infty, <)$ and every isomorphism in $\mathcal{V}$ between finite-dimensional subspaces of V_∞ extends to an element of the group G of order-preserving linear automorphisms. Let $\mathcal{K}$ consist of $\varnothing$ and the sets $W \setminus \{0\}$ for finite-dimensional subspaces $W \neq 0$. Then (K1) holds and $\mathcal{A}(M, G, \mathcal{K})$ is equivalent to $\mathcal{V}$.

5.1 Amenability

Proof of Theorem 1.5(a). By Lemma 3.4 it is enough to prove that the category $\mathcal{V}$ has property (Re). Let $a_0, \dots, a_t$ be arrows from P to Q in $\mathcal{V}$ and let $\varepsilon > 0$. Let c and d be the dimensions of P and Q, and $p = 2^c - 1$, $q = 2^d - 1$ the numbers of their non-zero vectors. The orders identify the non-zero vectors of P with $\{1, \dots, p\}$ and those of Q with $\{1, \dots, q\}$, and they identify a_i with an increasing injection α_i of $\{1, \dots, p\}$ into $\{1, \dots, q\}$.

Step 1. The category of finite linear orders and increasing injections has the Ramsey property, by Ramsey's theorem, hence (Re) by Lemma 3.2. So there are m and a probability vector ν on the increasing injections $\{1, \dots, q\} \to \{1, \dots, m\}$ with $\sum_{i=1}^t \|\nu_i - \nu_0\|_1 < \varepsilon/3$, where ν_i is the image of ν under composition with α_i. Extend ν and the ν_i by zero to all maps into $\{1, \dots, m\}$.

Step 2. Let $n \geq d$ and $V = \mathbb{F}_2^n$. Colour the non-zero vectors of V independently and uniformly by $\chi(v) \in \{1, \dots, m\}$, and order $V \setminus \{0\}$ by colour, ties broken by a fixed order. For an injective linear map $g \colon Q \to V$ put $w(g) = \nu(\chi \circ g)$; it is non-zero only if g preserves the orders. Let $e_r(n) = \prod_{j<r}(2^n - 2^j)$ be the number of injective linear maps $\mathbb{F}_2^r \to V$, and $W = \sum_g w(g)$. The colours of the q points $g(Q \setminus \{0\})$ are independent and uniform, so the expectation of $w(g)$ is m^{-q} and

$$\mathbb{E}\, W = e_d(n)\, m^{-q}.$$

For an injective linear $u \colon P \to V$ let $X_i(u) = \sum_{ga_i = u} w(g)$; there are $N = e_d(n)/e_c(n)$ such g. Given the colours on $u(P)$, the colours of the $q - p$ other points of $g(Q)$ are independent

and uniform, so the conditional expectation of $X_i(u)$ is

$$H_i(u) = N\, m^{-(q-p)}\, \nu_i(\chi \circ u).$$

Two maps g, h with $ga_i = ha_i = u$ and $g(Q) \cap h(Q) = u(P)$ have conditionally independent weights. For fixed g and h uniform among the N maps, each vector of Q outside $a_i(P)$ has an image that is uniform on $V \setminus u(P)$, so the probability that $g(Q) \cap h(Q) \neq u(P)$ is at most $\eta_n = (2^d - 2^c)^2/(2^n - 2^c)$. As $0 \leq w \leq 1$, the conditional variance of $X_i(u)$ is at most $N^2\eta_n$, so $\mathbb{E}|X_i(u) - H_i(u)| \leq N\sqrt{\eta_n}$, and summing over the $e_c(n)$ maps u,

$$\mathbb{E}\|X_i - H_i\|_1 \leq e_d(n)\sqrt{\eta_n}, \qquad \mathbb{E}\|H_i - H_0\|_1 = e_d(n)\, m^{-q}\, \|\nu_i - \nu_0\|_1,$$

the second because the p colours on $u(P)$ are independent and uniform. Let $D = \sum_{i=1}^{t} \|X_i - X_0\|_1$. Then

$$\mathbb{E}\, D \leq e_d(n)\Big(2t\sqrt{\eta_n} + m^{-q} \sum_{i=1}^{t} \|\nu_i - \nu_0\|_1\Big).$$

Choose n with $2tm^q\sqrt{\eta_n} < \varepsilon/3$. Then $\mathbb{E}(D - \varepsilon W) < 0$, so for some colouring $D < \varepsilon W$, and $W > 0$. For this colouring $(V, <)$ is an object of $\mathcal{V}$, w/W is a probability vector on $\mathcal{V}(Q, V)$, its images under the a_i are the X_i/W, and $\sum_i \|X_i/W - X_0/W\|_1 < \varepsilon$. □

5.2 The obstruction in characteristic two

Proof of Theorem 1.5(c). Let $P = \mathbb{F}_2^2$ with non-zero vectors $x_1 < x_2 < x_3$. Write the non-zero vectors of $\mathbb{F}_2^3$ as $1, \ldots, 7$ in binary, so that addition is bitwise addition without carry, and let Q be $\mathbb{F}_2^3$ with the order $1 < 2 < 3 < 4 < 6 < 5 < 7$. Let $a_1, \ldots, a_5 \colon P \to Q$ send (x_1, x_2, x_3) to

$$(1,2,3), \quad (1,6,7), \quad (2,5,7), \quad (3,4,7), \quad (3,6,5).$$

Each triple is the set of non-zero vectors of a plane, listed increasingly, and every bijection between such sets is linear; so these are arrows of $\mathcal{V}$.

Let D be an object of $\mathcal{V}$. Choose a basis $b_1, \ldots, b_n$ of D and let $\sqsubset$ be the order on D that compares two vectors at the largest index where their coordinates differ, with 0 before 1; it has nothing to do with the given order of D. For a statement S let $[S]$ be 1 if S holds and 0 otherwise. For $v \neq 0$ let ϵ_v be the linear form that takes the coordinate at the largest index where v has coordinate 1. For $x \neq y$ the vectors x and y differ last at that index of $x + y$, so, with values in $\mathbb{F}_2$,

$$[x \sqsubset y] = \epsilon_{x+y}(y). \tag{1}$$

For distinct non-zero u, v,

$$[u + v \text{ lies between } u \text{ and } v] = \epsilon_u(v) + \epsilon_v(u). \tag{2}$$

Indeed, if the largest indices of u and v agree, then $u + v$ comes before both and the right side is $1 + 1$. If the largest index of u exceeds that of v, then v comes before u and before $u + v$, the vector $u + v$ comes before u exactly when $\epsilon_v(u) = 1$, and $\epsilon_u(v) = 0$.

Let $g \colon Q \to D$ be linear and injective and write i for $g(i)$. By (1), and since $1 + 2 = 5 + 6 = 3$ and $1 + 6 = 2 + 5 = 7$,

$$[2 \sqsubset 1] + [1 \sqsubset 6] + [2 \sqsubset 5] + [5 \sqsubset 6] = \epsilon_3(1) + \epsilon_7(6) + \epsilon_7(5) + \epsilon_3(6) = \epsilon_3(7) + \epsilon_7(3),$$

which by (2) is [4 lies between 3 and 7]. For an arrow $e\colon P \to D$ define, with $y_j = e(x_j)$,

$$f_1(e) = [y_2 \sqsubset y_1], \qquad f_2(e) = f_3(e) = [y_1 \sqsubset y_2],$$
$$f_4(e) = [y_2 \text{ lies between } y_1 \text{ and } y_3], \qquad f_5(e) = [y_3 \sqsubset y_2].$$

The computation says $\sum_j f_j(ga_j) = 0$ for every arrow $g\colon Q \to D$. On the other hand $\sum_j f_j(e) = 1$ for every e: with $\alpha = [y_2 \sqsubset y_1]$ and $\beta = [y_3 \sqsubset y_2]$ one has $f_4 = (1-\alpha)(1-\beta) + \alpha\beta = 1 + \alpha + \beta$ in $\mathbb{F}_2$.

Let R be a ring in which 2 is not invertible, and suppose $t \in R[\mathcal{V}(Q, D)]$ is an affine equalizer of $a_1, \ldots, a_5$. Reduce modulo 2: in the non-zero $\mathbb{F}_2$-algebra $\bar{R} = R/2R$ we get $\bar{t}$ with $\operatorname{aug} \bar{t} = 1$ and $\bar{t}a_j = \bar{m}$ for all j. Extending the f_j linearly,

$$0 = \sum_j f_j(\bar{t}a_j) = \sum_j f_j(\bar{m}) = \operatorname{aug}(\bar{m}) = \operatorname{aug}(\bar{t}) = 1,$$

a contradiction. By Proposition 2.4, applied to $\mathcal{A}(M, G, \mathcal{K}) \simeq \mathcal{V}$, the category is not exact over R. □

5.3 Exactness over the rationals

Let $\mathcal{C}$ be the following category. An object is a finite non-empty linearly ordered set E with a linear order on the set of its non-empty subsets; an arrow is an increasing injection that preserves the order on subsets. The functor $u\colon \mathcal{C} \to \mathcal{V}$ sends E to the vector space $\mathbb{F}_2^E$, whose non-zero vectors are the sums of non-empty sets of basis vectors, with the corresponding order. For a vector of $\mathbb{F}_2^E$, its *support* is the set of basis vectors that occur in it.

Lemma 5.1. *The category $\mathcal{C}$ has the Ramsey property, and it is exact over $\mathbb{Z}$.*

Proof. Let A, B be objects and $b = |B|$; we may assume $|A| \leq b$. For a set S of size at most b in a linearly ordered set, with elements $s_1 < \cdots < s_j$, let $\tau(S) = (s_1, \ldots, s_j, s_j, \ldots, s_j)$, a b-tuple. Regard A and B as structures A', B' with a linear order on the points, the b-ary relation consisting of the tuples $\tau(S)$, and the linear order on this relation given by the order on subsets. By the theorem of Hubička and Nešetřil on structures with linearly ordered points and linearly ordered relations [8, Theorem 4.46] there is, for every r, such a structure C_0 with the Ramsey property for A', B' and r colours, embeddings preserving the relation, its complement and both orders. On the points of C_0 define an object C of $\mathcal{C}$: order the subsets of size at most b so that the order agrees with that of C_0 on those S whose tuple $\tau(S)$ is in the relation of C_0, and put the other subsets anywhere. An embedding $A' \to C_0$ or $B' \to C_0$ maps the tuples $\tau(S)$ to tuples of the same form in the relation of C_0, hence is an arrow of $\mathcal{C}$ into C; and $\mathcal{C}(A, B)$ is the set of embeddings $A' \to B'$. So a colouring of $\mathcal{C}(A, C)$ restricts to a colouring of the embeddings $A' \to C_0$, and an embedding $g\colon B' \to C_0$ with all $g \circ \mathcal{C}(A, B)$ of one colour is the required arrow. This proves the Ramsey property.

The objects of $\mathcal{C}$ are the finite structures in a relational language, with relations saying that one increasing tuple precedes another as a subset, and the arrows are the embeddings. This class is hereditary, has countably many members up to isomorphism, and has the amalgamation and joint embedding properties: on the union of two members over a common part, merge the two orders of the points, extend the two orders on subsets to a common order on the subsets of either member, and put the remaining subsets last. By Fraïssé's theorem $\mathcal{C}$ is equivalent to a category $\mathcal{A}(M', G', \mathcal{K}')$ as in Section 1, and Lemma 3.2 and Proposition 2.4 give exactness over $\mathbb{Z}$. □

Proof of Theorem 1.5(b). Let R be a $\mathbb{Q}$-algebra and $a_0, \dots, a_k \colon P \to Q$ arrows of $\mathcal{V}$. By Proposition 2.4 it is enough to find an affine equalizer over $\mathbb{Q}$. For an object Y of $\mathcal{V}$ let $X_Y : \mathcal{C} \to \mathbb{Q}$-Mod be the functor $C \mapsto \mathbb{Q}[\mathcal{V}(Y, uC)]$; write $[C, h]$ for the class of $h \in \mathcal{V}(Y, uC)$ in $\operatorname{colim}_{\mathcal{C}} X_Y$.

Step 1: classes in $\operatorname{colim}_{\mathcal{C}} X_P$. For $h \in \mathcal{V}(P, uC)$ let $U_h \subseteq C$ be the union of the supports of the vectors of $h(P)$, with the induced orders; it is an object of $\mathcal{C}$, the inclusion $U_h \subseteq C$ is an arrow, and h factors through $u(U_h)$. So $[C, h] = [U_h, h]$, and $[U, h] = [U', h']$ whenever there is an isomorphism $\varphi \colon U \to U'$ in $\mathcal{C}$ with $u(\varphi)h = h'$. Call the isomorphism class of the pair (U_h, h) the *type* of (C, h). Then a finite combination $\sum_j r_j[C_j, h_j]$ depends only on the sums of the coefficients over each type.

Step 2: a class in $\operatorname{colim}_{\mathcal{C}} X_Q$ *with equal images.* Let $Q^\vee$ be the dual space, $E = Q^\vee \setminus \{0\}$, and $j \colon Q \to \mathbb{F}_2^E$, $j(v) = (\ell(v))_{\ell \in E}$, which is linear and injective. Let Ω be the finite set of pairs ω consisting of a linear order on E and a linear order on the non-zero vectors of $\mathbb{F}_2^E$ in which the vectors outside $j(Q)$ come first, in any order, followed by $j(Q \setminus \{0\})$ in the order of Q. Each ω gives an object E_ω of $\mathcal{C}$ and $j \in \mathcal{V}(Q, uE_\omega)$. Put

$$\mu = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} [E_\omega, j] \in \operatorname{colim}_{\mathcal{C}} X_Q, \qquad \operatorname{aug} \mu = 1.$$

The image of μ under a_i is the average of the classes $[E_\omega, ja_i]$. The union of the supports of the vectors $ja_i(x)$ is $E_i = \{\ell \in E : \ell a_i \neq 0\}$, and

$$\mathbb{F}_2^{E_i} \cap j(Q) = ja_i(P) : \tag{3}$$

$j(v)$ has support in E_i exactly when every linear form vanishing on $a_i(P)$ vanishes on v, that is, when $v \in a_i(P)$. So the type of (E_ω, ja_i) is that of the set E_i with the order induced by ω, the order on the non-zero vectors of $\mathbb{F}_2^{E_i}$ in which those outside $ja_i(P)$ come first, in the order induced by ω, followed by $ja_i(P \setminus \{0\})$ in the order of P, and the map ja_i. As ω runs over Ω, the induced order on E_i and the induced order on the vectors of $\mathbb{F}_2^{E_i}$ outside $ja_i(P)$ are independent and uniformly distributed, because restriction of linear orders to a subset has fibres of equal size. Finally, for two indices i, i' there is a bijection $\beta \colon E_i \to E_{i'}$ with $\beta(\ell)a_{i'} = \ell a_i$, because every non-zero linear form on P is ℓa_i for exactly $2^{\dim Q - \dim P}$ elements ℓ of E_i, and likewise for i'; the linear map defined by β carries ja_i to $ja_{i'}$. Hence the distribution of the type of (E_ω, ja_i) is the same for all i, and by Step 1 the images of μ under $a_0, \dots, a_k$ coincide.

Step 3: an affine equalizer. Let $D \colon X_Q \to X_P^k$ be the natural transformation with components $t \mapsto ta_i - ta_0$. By Lemma 5.1 the functor $\operatorname{colim}_{\mathcal{C}}$ is exact, so it commutes with kernels, and μ, which lies in the kernel of $\operatorname{colim}_{\mathcal{C}} D$, is the class of an element of $\operatorname{colim}_{\mathcal{C}} \ker D$. Such an element is a finite sum of classes of elements of $(\ker D)(C_j)$; any two objects of $\mathcal{C}$ map to a third, so it is the class of a single $t \in (\ker D)(C) \subseteq \mathbb{Q}[\mathcal{V}(Q, uC)]$. The augmentation is natural, so $\operatorname{aug} t = \operatorname{aug} \mu = 1$, and $D(C)t = 0$ says $ta_i = ta_0$. □

6 Provenance and checks

This paper is AI-produced. No human has proved, written or checked any part of it.

What is whose. The statements and proofs of Theorems 1.4 and 1.5 (part (c) over $\mathbb{F}_2$ and $\mathbb{Z}$), of the first sentence of Theorem 1.2, of Theorem 1.3 in the form of a group with a closed co-precompact subgroup and a finitely additive invariant function on the closed and open sets of the completion of G/H, of Lemmas 3.4, 4.1 and 5.1 and of the three steps of

Section 5 are due to a run of GPT-6 Astra (OpenAI, ChatGPT Pro mode) that ended on 18 November 2026 (run FREX; in Part A of its text, the statements on coinvariants, on the convex Ramsey property, on signed averaging, on the weights for posets, on ordered posets, and on the amenability, the parity obstruction and the rational exactness of ordered vector spaces, the last with its two lemmas), in answer to a research prompt of this project that followed up the question treated in [3]. The run built on earlier runs of the same model, from one of which the first half of Lemma 3.2 comes. The run cites Kechris and Sokić for the non-amenability in Theorem 1.4 and adds a proof of its own, which is not reproduced here. Due to the writer, Claude Fable 5.2 (Anthropic) at effort max, on 25 November 2026: the direct proof of Proposition 2.4, for which the run cites Isbell and Mitchell; the formulation of Theorem 1.3 by the sets $E(B)$; the second sentence of Theorem 1.2 and Remark 2.1, after a remark in the project's notes by Claude Opus 5.5 at effort max; Theorem 1.5 (c) over every ring in which 2 is not invertible, by reduction to $R/2R$; Lemma 2.2, Corollary 1.6 and Remarks 3.6, 4.2 and 4.3.

The text. Written by Claude Fable 5.2 at effort max on 25 November 2026, in the setting of [3]. Every proof was derived again by the writer from the run's text; the proof of Lemma 3.2 was reconstructed from a one-sentence description in the run.

Checks. The run's text was followed step by step at its intake by Claude Opus 5.5 at effort max (19 and 24 November 2026). The writer derived every proof of this paper again and ran a program of its own, in exact integer arithmetic: it confirms Lemma 4.1 for all 4474 partial orders on at most five labelled points (102085 one-point identities, the number the run reports), the three-point example of Remark 4.2, and the two identities of the proof of Theorem 1.5 (c) for all injective linear maps into $\mathbb{F}_2^n$, $n \leq 5$, with the order $\sqsubset$ of the standard basis. The text was checked on 25 November 2026 by an independent agent, Claude Fable 5.2 at effort max: every proof followed; Theorems 1.2 and 1.3, Proposition 2.4, Theorem 1.5 (c) and the non-amenability in Theorem 1.4 proved again independently, the last two by other routes; every citation but [7] opened; both programs run. Its repairs to Remark 4.3, to the proof of Theorem 1.3 (c) and to Remark 3.3 are included. The sentence of Section 1 on the translations of $\mathbb{Z}$ was added by the writer after that check. Nothing is formalized.

The search of the literature. On 25 November 2026 the writer read [6] (the definitions, Sections 3, 4, 10, 11, 16 in part), [12] (Sections 1–3, 8 and the statements), [5] (Section 1, Remarks 2.9–2.12, Lemma 3.7), [2] (Section 2), [4] (the passage of Theorem 5.1), [11] (Remark 3.2) and [14] (Section 3), and searched the titles and abstracts of arXiv, the web and the Hexagon catalogue for statements relating exactness of colimits or of coinvariants to amenability; none was found. The search does not cover full texts.

Citations. Opened by the writer at the places cited: [2], [3], [4], [5], [6], [11], [12], [14]. Opened by the run and by the independent check of 25 November 2026, at the places cited: [1] (arXiv and journal versions), [8], [9], [10]. Opened by the independent check: [13] (p. 305). Not opened: [7]; the theorem used is [10, Theorem 2.2]. Artin's theorem is quoted from [4, Theorem 5.1], which takes it from Mac Lane and Moerdijk's book; the book was not opened.

Records. This paper is accompanied in its workbench by the file `CHECK_EVENTS.tsv`, which holds every check with model, effort and date, the file `PROVENANCE.md`, which lists

every contributor and the state of every citation, and the program `frex_checks.py` in the folder `checks`. The workbenches are reached from `https://github.com/KokunoYumeto/mathematics-workbenches`.

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